Concerning the Stability of Complex Time Steppers

Cian J. Duggan

August 23, 2024

Contents

1 Introduction

A paper by Lloyd N. Trefethen [1] offers two definitions of numerical analysis:

> *The study of rounding errors.*
>
> *The study of algorithms for the problems of continuous mathematics.*

Trefethen argues that the first definition, though an accurate summation of the field's history, does not serve to entice the curious to explore the field. He proposes the second as a more compelling definition, one that emphasises the field's role in solving real-world problems, and inspires curiosity.
He offers an optimistic view of the field, one that is not bogged down by the minutiae of rounding errors, but rather one that is focused on the algorithms that make numerical analysis possible. More to ground, the majority of numerical analysis is concerned with the speed of convergence and minimization of error.
Through my work on this paper, I have honed my own definition:

> *Numerical analysis is the field which attempts to straddle the gaps between the discrete and the continuous.*

In this paper, we will take a look at the stability of various numerical methods when applied to problems of differential calculus.
In particular, we wish to:

- Introduce key terminology and concepts in the field of numerical analysis.
- Establish the Exponential Decay Problem as the toy problem for analysis.
- Define stability, and its associated concepts, in the context of numerical analysis.
- Establish the concept of complex time steps for numerical methods.
- Develop a framework for the analysis of the stability of numerical methods.
- Apply this framework to particular numerical methods.
- Expand upon the results of this to contrast the stability regions for real and complex time steps.

2 Definitions and Concepts

2.1 Numerical Methods

Numerical methods are techniques used to approximate solutions to problems that cannot be solved exactly.
Numerical methods are prevalent when working with problems of differential calculus in a computational context, where many problems do not have closed-form solutions.
A numerical method can be interpreted as an algorithm; a series of steps that can be followed to approximate a solution.
Many numerical methods exist, each with their own properties and trade-offs.
In this respect, different methods may be more or less suitable for different problems.
Core to understanding which to use for a particular problem are the concepts of *error* and *stability*.
Before discussing these properties, however, we must understand how a numerical method works.
The basis of many numerical methods is the concept of a *time step*.

2.2 Time Steps

Let's take a step back to when we first studied derivatives.
Newton's Difference Quotient should be familiar: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
Viewing Newton's Difference Quotient in the context of numerical methods, we call h a *time step*.
Due to computational constraints, we cannot take h to be infinitesimal. (A computer's memory is as small as it is big)
Instead, we take h to be a small, finite number.
This defines the concept of a *step size*.

For numerical methods, we use $y(t)$ to denote the solution to a differential equation at time t.
We approximate $y'(t)$ by $\frac{y(t+h)-y(t)}{h}$ using Newton's Difference Quotient.
Note this is an approximation because we have dropped the limit; h is not infinitesimal.
This gives $y(t+h) \approx y(t) + hy'(t)$; a first-order approximation of y a small time step h in the future.
For a numerical method we say $y(t+h) = y(t) + hy'(t)$

2.3 Error

In numerical analysis, *error* is the difference between the numerical solution and the exact solution to a problem.
Error can be caused by many factors, such as the choice of numerical method, the choice of time step, or the precision of the computer.
Error can be classified into two categories: truncation error and rounding error.
Truncation Error is the error introduced by approximating a problem, such as using a finite time step.
Rounding Error is the error introduced by the finite precision of a computer.
Error is a crucial concept in numerical analysis, as it determines the accuracy of a numerical method.
Any error mentioned in this document refers to the truncation error; we won't be looking at how computers run these calculations and the rounding errors that come with it.

2.4 Stability

In numerical analysis, a method is said to be *stable* if small deviations in the input do not lead to large perturbations in the output. In the context of differential equations, a method is said to be stable if the solution does not grow to be unbounded as the number of time steps increases. (This, of course, only applies if the exact solution is bounded. We will restrict ourselves to this with our analysis.)
When solving differential equations numerically, the choice of time step is crucial. If the step size is too large, the solution may become unstable; the numerical solution will diverge from the analytic solution in proportion with the number of steps. If the step size is too small, the solution may be accurate and stable, but the computation may be too slow.
The tradeoff between error, stability and compute is a common theme in numerical analysis.
The sweet spot for maximal efficiency depends entirely on the choice of numerical method.
This marks the core motivation for this report; solidifying an understanding to better inform the choice of method and step size for a given problem.

3 The Exponential Decay Problem

We would like a simple toy problem with which we can build a framework for the analysis of the stability of a numerical method.
We turn to a classic problem in numerical analysis: the Exponential Decay Problem.
Consider a quantity y that decays relative to its current value.
The quantity at time t is $y(t)$ and the rate of decay is λ.
For the solution to actually decay, we require $Re(\lambda) < 0$.

The Exponential Decay Problem is given by the ODE

$y'(t) = \lambda y(t) \quad \text{with} \quad y(0) = y_0$

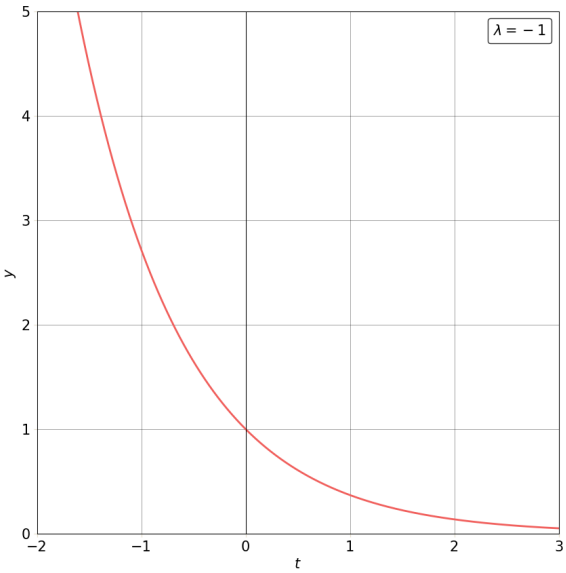
This has the exact solution $y(t) = y_0\, e^{\lambda t}$, hence the name 'Exponential Decay'.

For the sake of keeping things neat, we will take $y_0 = 1$ for the rest of this paper.
This poses no lack of generality, as we can always scale the solution by any initial value, both exact and numerical.

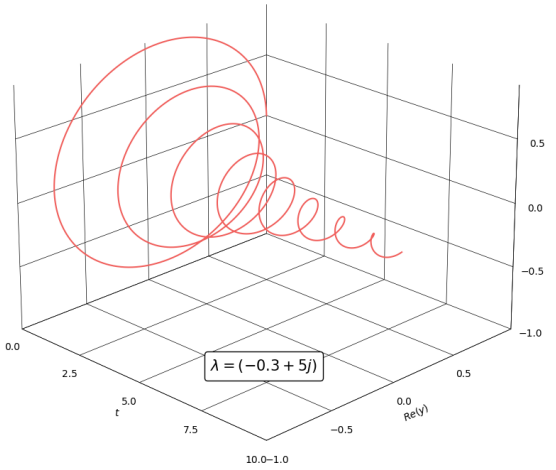
The exact solution we will consider is

$y(t) = e^{\lambda t}$

3.1 A Graphical Representation



If we restrict λ to $\mathbb{R}^-$, we have a simple exponential curve that decays to zero as $t \to \infty$.



If we allow $\lambda \in \mathbb{C}$ and $Re(\lambda) < 0$, we get a spiral in the complex plane, the radius of which decays to zero as $t \to \infty$.

3.2 The Maclaurin Series

The Taylor series is given by:

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!}(t-a)^n$$

Setting $a = 0$, we get the Maclaurin series for the exact solution:

$$y(t) = \sum_{n=0}^{\infty} \frac{(\lambda)^n e^0}{n!} t^n = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \quad \text{which we will denote } M(y)$$

This is a useful form for the exact solution – we will see it pop up again later.

3.3 Proponents of the Exponential Decay Problem

There are a few pros to this choice of problem:

- The exact solution is known and easily computed; it can be used to evaluate a numerical solution.
- The solution is a straightforward and well understood function.
  This simplicity allows us to focus on the numerical method.
- The problem is linear in $y(t)$, making the problem simple to work with.
  Again, our focus can stay on the numerical method.
- The graph of the exact solution is easy to visualise; the more chaotic outputs of the numerical methods can be compared to the smooth curve of the exact solution easily.
- Exponential decay is a model for many physical processes, including radioactive decay, population decline, and capacitor discharge.
  Quantities that decay relative to their current value appear frequently in nature.
  This means readers from many different disciplines may already possess an intuition for the problem.
- The problem is simple to generalise to complex numbers, allowing us to explore the stability of numerical methods in the complex plane.
- The Exponential Decay Problem is *stiff*; for certain method-step pairs, the numerical solution may oscillate wildly.
  This is exactly the kind of behaviour we want to avoid in a numerical method, when we are looking for stability.

Normally, when using the Exponential Decay Problem in a numerical analysis context, we would allow $\lambda \in \mathbb{C}$ with $Re(\lambda) < 0$.
However, for the purposes of this paper, we will restrict λ to $\mathbb{R}$ as we will be looking at $h \in \mathbb{C}$ and we don't want to overcomplicate things.
Unless otherwise mentioned, we will assume that $\lambda \in \mathbb{R}^-$ for the rest of this paper.

We will use the Exponential Decay Problem throughout this paper to derive a useful framework for the analysis of the stability of a numerical method.

4 Stability Analysis

4.1 Introduction

In this section we will lay out a framework for analysing the stability of a numerical method.
To begin, we will introduce the concept of the stability function.
We will show how it can be used to define a stability region.
We will explore the stability region as the set of values for which a numerical method is stable.
Finally, we will extend this framework to view the stability of a numerical method in the context of complex time steps.

4.2 Numerical Method: Forward Euler

Let's consider a simple numerical method applied to the Exponential Decay Problem.
The Forward Euler method is a first-order numerical method for solving ODEs with a given initial condition.
Its algorithm can be defined as:

$$y(t_{j+1}) = y(t_j) + hy'(t_j)$$

Where h is the time step and y_j is the numerical solution at time t_j, j steps on from $t_0 = 0$.
This means $t_j = hj$.

For the Exponential Decay Problem, the Forward Euler method can be written as:

$$y_{j+1} = y_j + h\lambda y_j \quad \text{where} \quad y_0 = 1$$

This gives us the following algorithm:

$$y_{j+1} = (1 + h\lambda)y_j$$

This gives an approximation of the exact solution at time t_j as follows:

$$y_j = (1 + h\lambda)^j y_0 = (1 + h\lambda)^j = y(t_j) = y(hj) \approx e^{\lambda h j}$$

We can see that the Forward Euler method is stable if $|1 + h\lambda| < 1$;
both the exact solution and our approximation will decay to zero as $t \to \infty$.
The stability is dependent on the time step h and the value of λ.
We can write this as $s(\lambda, h) = 1 + h\lambda$.
By analysing s for different values of λ and h,
we can infer the stability of the Forward Euler method for the Exponential Decay Problem.
We call s the *stability function* of the Forward Euler method.

4.3 The Stability Function and corresponding Stability Region

By the same methodology, we can define the stability function for any numerical method by writing the algorithm in the form $y_{j+1} = s(\lambda, h)y_j$.
$s(\lambda, h)$ is the *stability function* of the numerical method with a time step h.
Note: This is equivalent to $y_j = s(\lambda, h)^j y_0$
The *stability region* of a numerical method is the set $S = \left\{ (\lambda, h) \;\middle|\; |s(\lambda, h)| < 1 \right\} \subset \mathbb{C}$
This follows from the definition of stability for our Exponential Decay Problem;
a method is stable if the numerical solution decays to zero as $t \to \infty$.
Clearly, $\lim_{j\to\infty} y_j = \lim_{j\to\infty} s(\lambda, h)^j y_0 = 0 \iff |s(\lambda, h)| < 1$. Now, let's explore some examples of stability regions for different numerical methods.

4.3.1 Stability Region for Euler's Forward Method

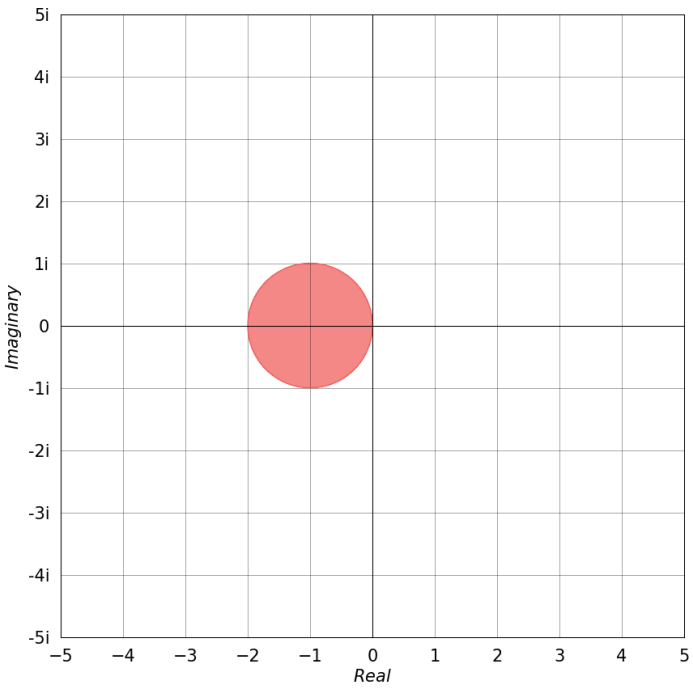
Euler's Forward Method has the stability function

$$s(\lambda, h) = 1 + \lambda h = \sum_{n=0}^{1} \frac{(\lambda h)^n}{n!} = M(y) + \mathcal{O}((\lambda h)^2)$$

The corresponding stability region is

$$S = \left\{ (\lambda, h) \;\middle|\; |1 + \lambda h| < 1 \right\}$$

We can see this region plotted in red on the right; an open unit circle centred at -1.



For a given $\lambda \in \mathbb{C}$, the method is stable for any step-size $h \in \mathbb{R}$ such that $|1 + \lambda h| < 1$.

Expanding $\lambda = a + bi$, we get the restriction

$$|1 + (a + bi)h| < 1 \quad \implies \quad |(1 + ah) + (bh)i| < 1$$

$$\implies \quad a^2h^2 + 2ah + b^2h^2 < 0$$

$$a, b, h \in \mathbb{R} \implies (a^2 + b^2)h^2 > 0$$

$$\implies 2ah < 0 \text{ and } ||2ah|| > (a^2 + b^2)h^2$$

This aligns with our intuition for the problem;

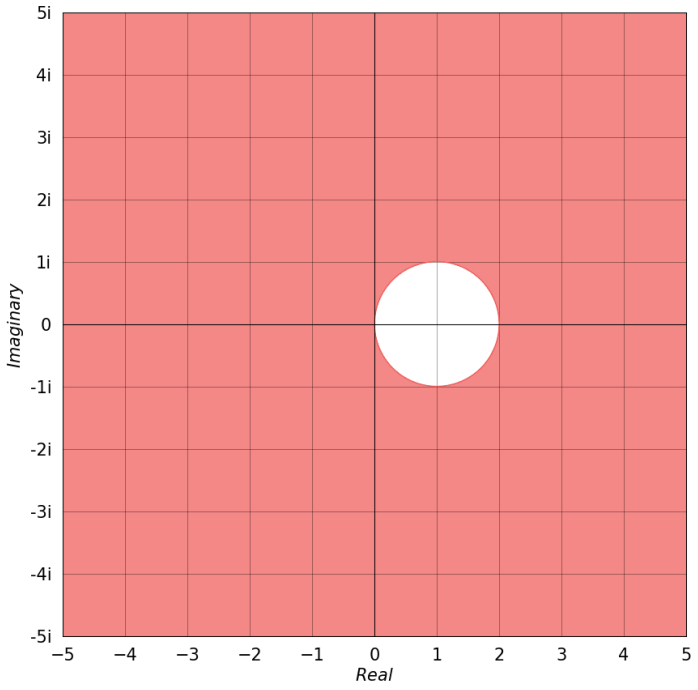
- For the exponential to decay, we must have $Re(\lambda) = a < 0$.
- h must be positive, as it is a time step.

Moreover,

$$0 < (a^2 + b^2)h^2 < ||2ah|| \quad \implies \quad 0 < h < \frac{||2a||}{a^2 + b^2} \quad \implies \quad 0 < h < \frac{2||Re(\lambda)||}{|\lambda|^2}$$

This is also intuitive; if we imagine λ growing linearly, h must shrink quadratically so that their product λh stays inside the stability region.

4.3.2 Stability Region for Euler's Backward Method



Euler's Backward Method can be written as

$$y_{j+1} = y_j + h.y'(t_{j+1})$$

$$\implies y_{j+1} = y_j + h(\lambda y_{j+1}) \implies y_{j+1} = \frac{1}{1 - \lambda h} y_j$$

Thus, the stability function is

$$s(\lambda, h) = \frac{1}{1 - \lambda h}$$

The corresponding stability region is

$$S = \left\{ (\lambda, h) \;\middle|\; \left| \frac{1}{1 - \lambda h} \right| < 1 \right\}$$

This is plotted in red on the left; the region outside a unit circle centred at 1.
The white region of instability is exactly the stability region for Euler's Forward Method, flipped about Imaginary axis.

We could run through the same algebraic proceedure as before, expanding λ and rearranging, and we would find that

$$h > \frac{2Re(\lambda)}{|\lambda|^2}$$

This is a more lax restriction than the $0 < h$ that we have already established.
In fact, this tells us that any positive h will give a stable solution, regardless of the value of λ.
This is a property called *Absolute Stability* or *A-Stability*.
This is often characterised by a stability region that covers the entire left half-plane, as we see here.

4.3.3 Stability Region for Runge-Kutta 4

Runge-Kutta 4 can be written as

$$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where $\phi(t, y) = y'(t) = \lambda y$

$$k_1 = \phi(t_j, y_j) \qquad k_2 = \phi(t_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$$

$$k_3 = \phi(t_j + \frac{h}{2}, y_j + \frac{h}{2}k_2) \qquad k_4 = \phi(t_j + h, y_j + hk_3)$$

For the Exponential Decay Problem, we get

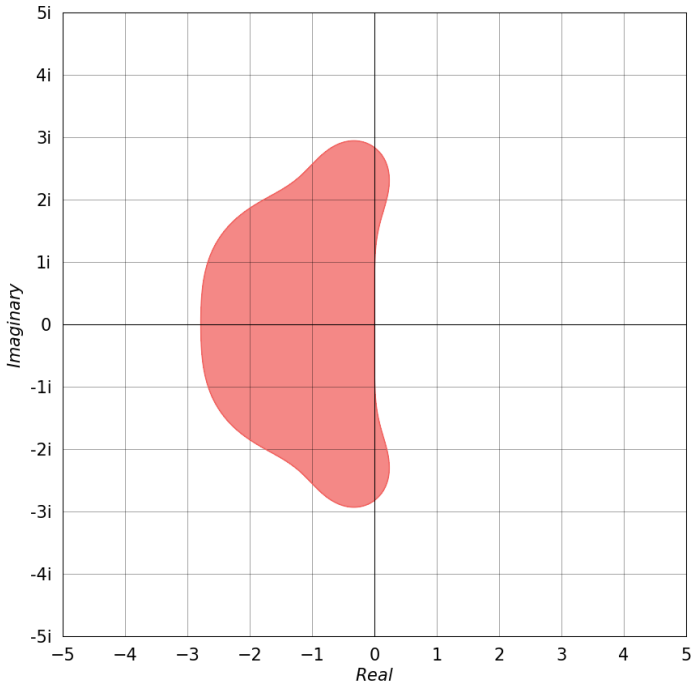
$$y_{j+1} = (1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24})y_j$$

The stability function is

$$s(\lambda, h) = \sum_{n=0}^{4} \frac{(\lambda h)^n}{n!} = M(y) + \mathcal{O}((\lambda h)^5)$$

The corresponding stability region is

$$S = \left\{(\lambda, h) \;\middle|\; \left|\sum_{n=0}^{4} \frac{(\lambda h)^n}{n!}\right| < 1\right\}$$



4.4 Interpretation of Stability Regions

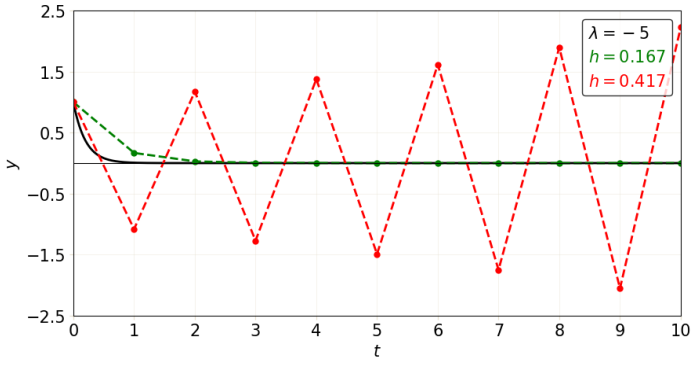
The stability region $S = \left\{(\lambda, h) \;\middle|\; |s(\lambda, h)| < 1\right\} \in \mathbb{C}$ is a set of values for which a numerical method is stable for the Exponential Decay Problem.
We always restrict $h \in \mathbb{R}^+$, as it is a time step.
We have two separate cases for λ:

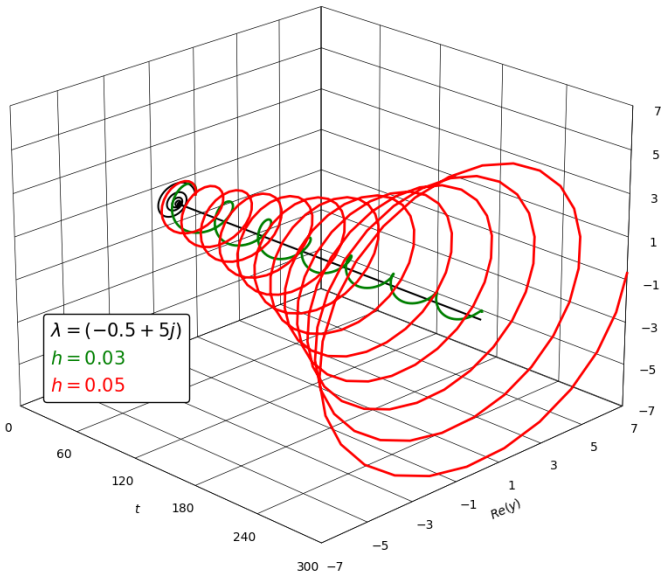
- $\lambda \in \mathbb{R}^-$: S corresponds to $\left\{s(\lambda, h)^2 < 1\right\}$. Of course, $\lambda, h \in \mathbb{R} \implies S \subset \mathbb{R}$.
- $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $Re(\lambda) < 1$: S corresponds to $\left\{\left(s(\lambda, h)\right)\left(\overline{s(\lambda, h)}\right) < 1\right\}$.

From here, one can derive bounds on h for a given value of λ, or vice versa.
We did this with Euler's Forward and Backward methods above.
The Runge-Kutta 4 method is more complicated, as the two cases of λ each give a polynomial of degree 8.

Below we have graphs for the cases where λ is real or complex,
the same as we introduced for the Exponential Decay Problem.
In black is the exact solution. In colour are the numerical solutions due to the method in question.
These (λ, h) pairs were found by first picking a value of λ and then finding h values which met the stability condition.
In green $(\lambda, h) \in S$.
In red $(\lambda, h) \notin S$.

4.4.1 Euler's Forward Method





On the left are the aforementioned graphs for Euler's Forward Method.

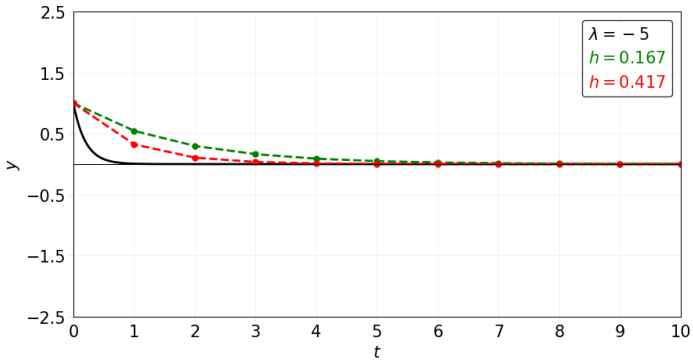
First is the case where $\lambda \in \mathbb{R}^-$.
The exact solution decays quickly in black, very close to $t = 0$. The numerical solution converges to the exact solution for $h = 1/6$ in green.
For $h = 5/12$, the numerical solution oscillates and diverges in red.
The oscillation is due to the stability function
$s(\lambda, h) = 1 + \lambda h = 1 + -5 \cdot \frac{5}{12} = \frac{-13}{12}$
The j^{th} term of the numerical solution is $s(\lambda, h)^j = \left(\frac{-13}{12}\right)^j$.
Of course, natural powers of negative numbers oscillate between positive and negative values.
$|\frac{-13}{12}| > 1 \implies$ the numerical solution diverges.

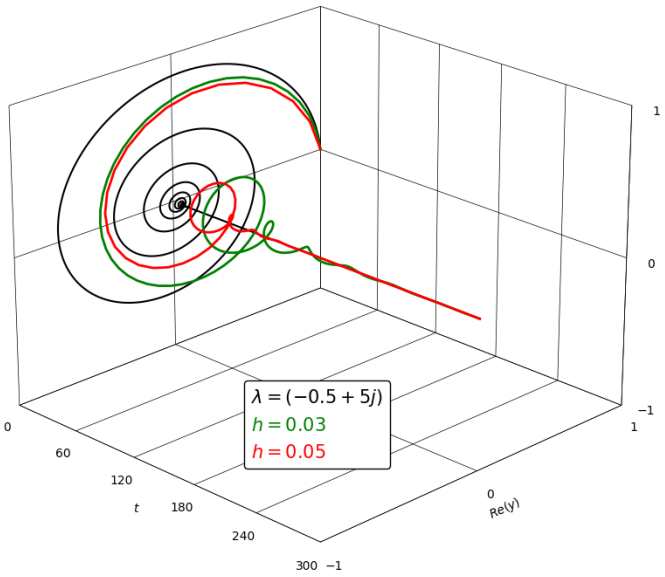
Second is the case where $\lambda \in \mathbb{C}$.
The exact solution decays quickly in black, very close to the $t = 0$ plane.
The green, valid h value converges as t increases, but the red, invalid h value diverges.

While this 3D graph might not be the clearest, it took quite a lot of messing with values of h and λ to find one so demonstrative.
The reader is encouraged to play with values for h and λ in '/Python/Exponential Decay/Exact vs Method.py'
of the GitHub repository [3] to see for themselves.
This has the added benefit of allowing you to rotate the graph to see the behaviour from different angles.

4.4.2 Euler's Backward Method

As mentioned before, Euler's Backward Method is A-Stable. This means that for any $\lambda \in \mathbb{C}$ with $Re(\lambda) < 1$, the method is stable for any $h \in \mathbb{R}^+$.
As a result, there is no choice of (λ, h) that will give us a divergent graph.
For λh to land in the white disk of instability, either h must be negative or $Re(\lambda) > 1$.
Both of these are nonsensical for our Exponential Decay Problem.
Instead, these graphs show two different stable h values for the same λ value.
For $\lambda \in \mathbb{R}^-$ below, the numerical solution converges to the exact solution for both step sizes.

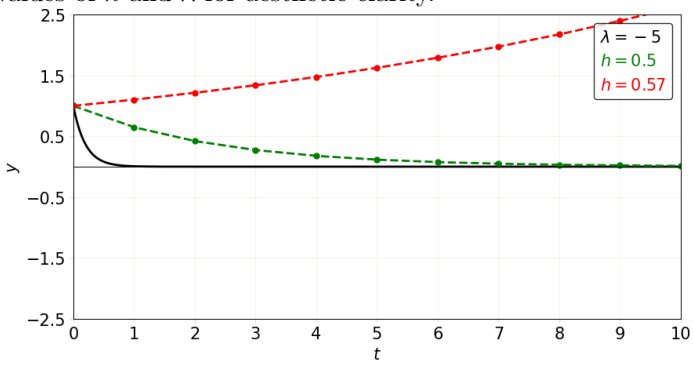


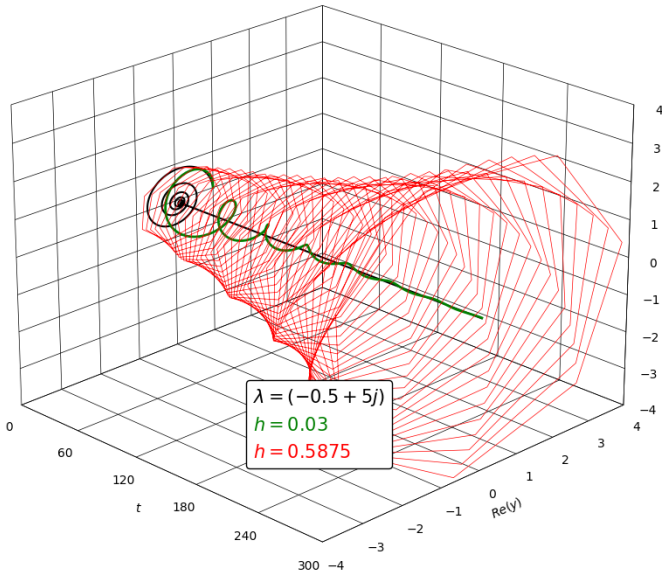


For $\lambda \in \mathbb{C}$ above, the method converges to the exact solution for both step sizes.
There is no divergence, as we have already established.
Interestingly, the method converges faster for the larger step size.
The subsequent section on Stability Magnitudes will explain this behaviour.

4.4.3 Runge-Kutta 4

The same properties hold true for Runge-Kutta 4, as we demonstrated for Euler's Forward, though we choose different values of h and λ for aesthetic clarity.

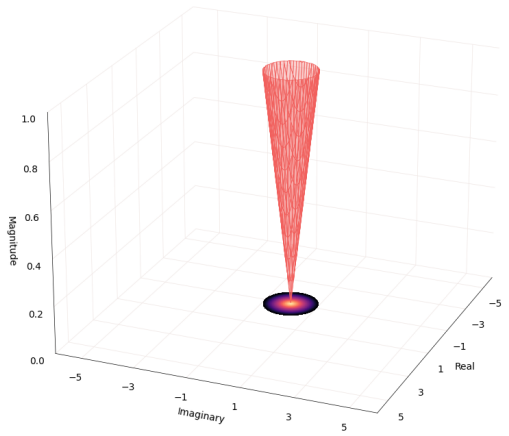


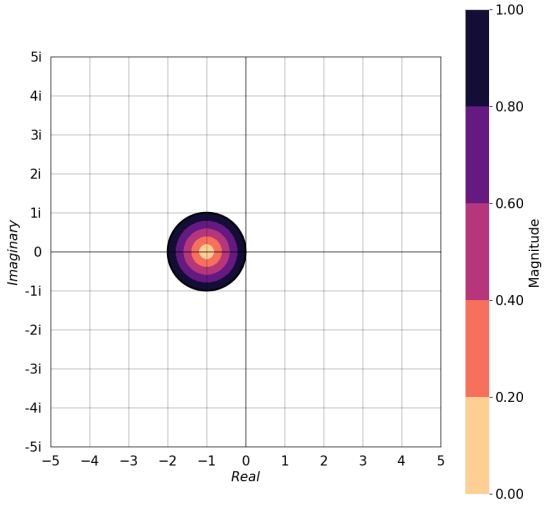


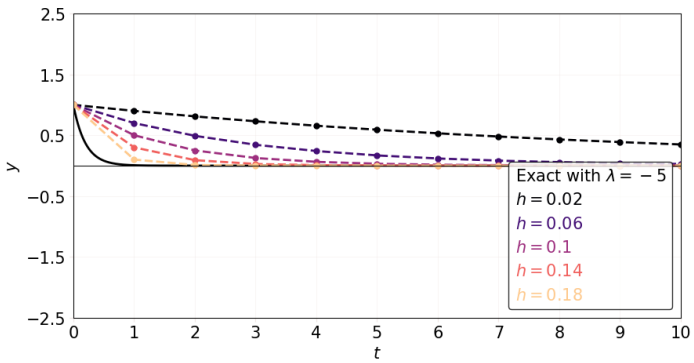
4.5 Stability Magnitudes

So far we have been looking at stability regions given by $|s(\lambda, h)| < 1$.
While this bound is a sufficient condition for stability, we are not truly utilising the metric.
In this section, we'll explore $|s(\lambda, h)|$ in a little more detail.
The plots below show the magnitude of the stability function for different values of λh.
We have limited these to display only the magnitudes within the region S for clarity; $|s(\lambda, h)| < 1$.
The 3D plots show the magnitudes $|s(\lambda, h)|$ above the base plane S.
The magnitudes have been projected down onto the base plane as colour plots.
We show these colour plots separately in the second graphs.
While on the first graphs, these colour gradients are continuous, on the second graphs we have discretised them.
This is so that for the third graphs we can create a correspondance with the plotted numerical solutions.
We'll only present the Exponential Decay graphs for real λ here, as the complex λ graphs are, well, complex. No need to spiral out of control.
Their behavior is the same as for the real λ graphs, with tighter, faster decaying spirals as λh lands in brighter areas of the colour plot.

4.5.1 Euler's Forward Method

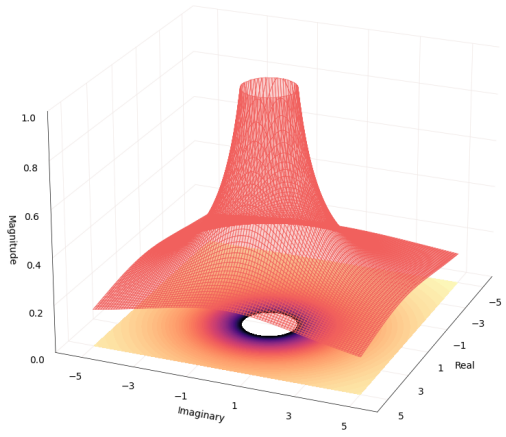


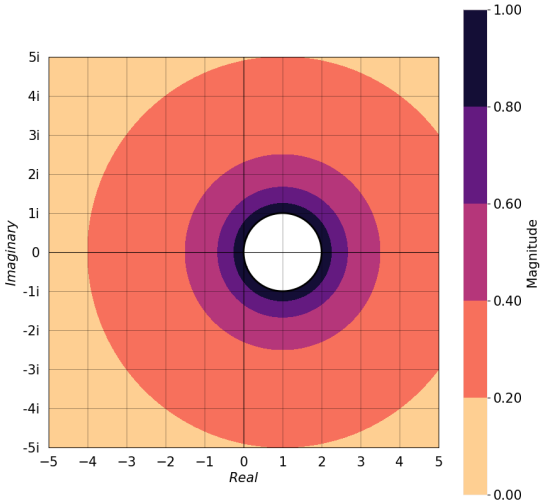


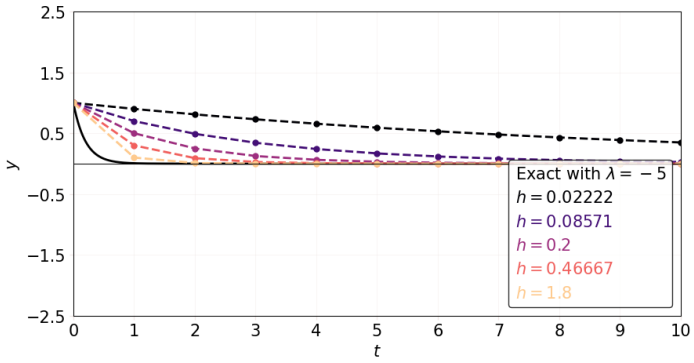


We can see smaller magnitudes correspond to a tighter fit to the exact solution.
This aligns with our intuition; for a derivative,
we allow $h \to 0$ to obtain the exact solution.
Recall, in a computational context, we're looking for the largest h that will give us a stable solution, that has an error within our tolerance.
Between each point on the graph, $\frac{1}{h}$ calculations are done to find the next point.
For larger h, the method is less accurate, but faster.

4.5.2 Euler's Backward Method

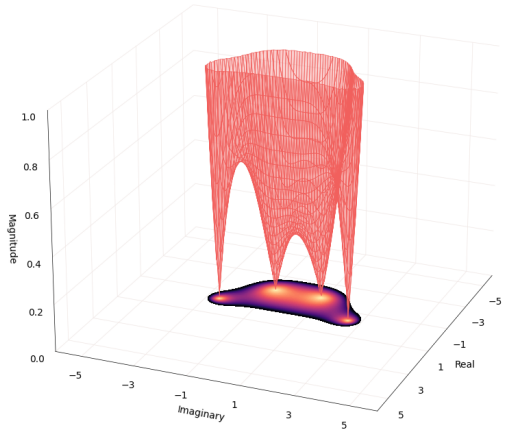


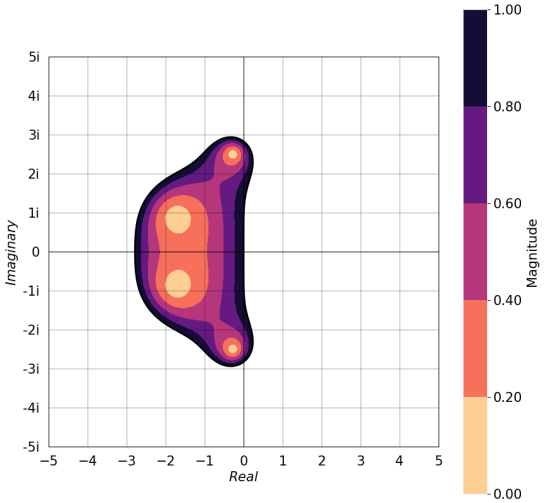


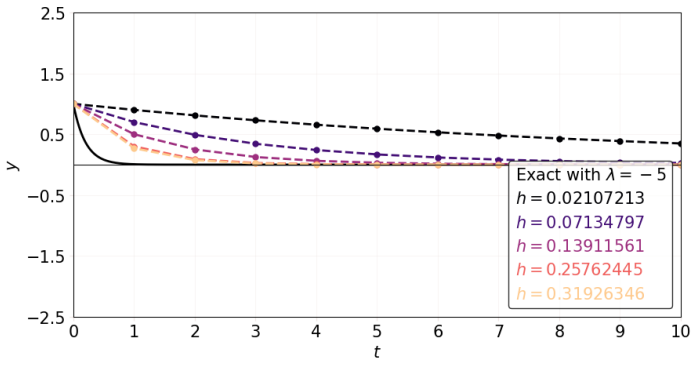


As we have already established, Euler's Backward Method is A-Stable.
Any choice of $h \in \mathbb{R}^+$ will give a stable solution for any $\lambda \in \mathbb{C}$ with $Re(\lambda) < 1$.
It is impossible to pick such (λ, h) values that will give λh in the white circle.
By nature of the stability function $s(\lambda, h) = \frac{1}{1-\lambda h}$, larger values of h will give a smaller magnitudes; a tighter fit to the exact solution.
This indicates why implicit methods like Euler's Backward are often used for stiff problems; they can be more accurate for larger time steps.

4.5.3 Runge-Kutta 4







The same general observations as Euler's Forward Method hold true for Runge-Kutta 4.
The values of h here were calculated numerically, rather than algebraically.
Check out '/Python/Stability Magnitude/Rk4 = c.py' in the GitHub repository [3] for the code.

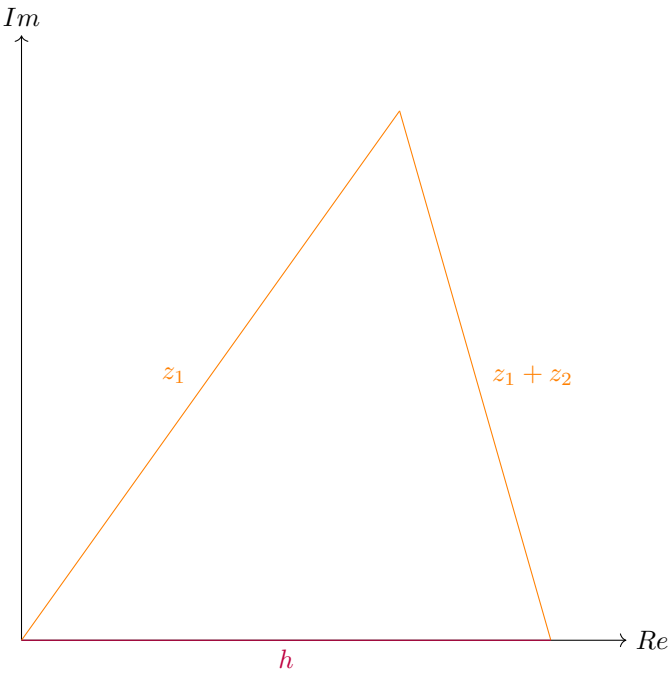
5 Complex Time-Steps

So far, we have restricted our time-steps to the real domain.
It has often proven useful to extend the analysis of a problem to the complex domain.
Motivated by a paper by *George, Yung and Mangan*[2], we will have a look at the stability of numerical methods when the chosen time-step is complex.
They state:

> *Most numerical methods for time integration use real time steps. Complex time steps provide an additional degree of freedom, as we can select the magnitude of the step in both the real and imaginary directions. By time stepping along specific paths in the complex plane, integrators can gain higher orders of accuracy or achieve expanded stability regions.*

5.1 Complex 2-Step

We want to take an overall step of size $h \in \mathbb{R}$ comprised of two steps $z_1, z_2 \in \mathbb{C}$ such that $z_1 + z_2 = h$.
We can set $z_1 = a + bi$ and $z_2 = h - a - bi$.



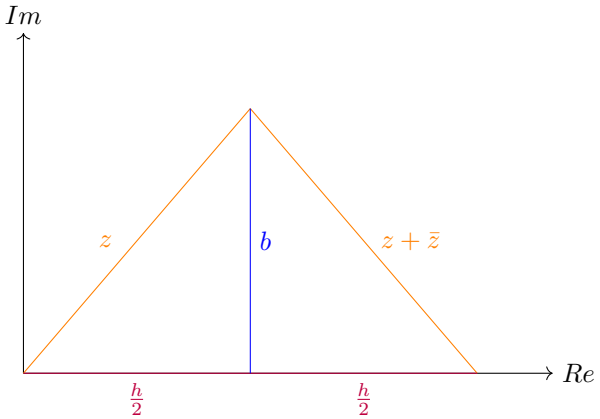
For analysing our numerical methods, we require a comparable implementation of the real step setup.
A real 2-step method where we take a step of size $\frac{h}{2}$ allows us to take two steps and go the same distance as the complex 2-step method.
We can write this real 2-step method as

$$y_{j+1} = s(\lambda, \frac{h}{2})y_{j+\frac{1}{2}} = s(\lambda, \frac{h}{2})s(\lambda, \frac{h}{2})y_j = s(\lambda, \frac{h}{2})^2 y_j$$

We can write the complex 2-step method as

$$y_{j+1} = s(\lambda, z_1)y_{j+\frac{1}{2}} = s(\lambda, z_1)s(\lambda, z_2)y_j$$

5.2 Complex Conjugate Pairs



When we restrict the exponential decay problem by $\lambda \in \mathbb{R}$, we also find that $s(\lambda, z_1)s(\lambda, z_2) \in \mathbb{R}$
This is because any value attained by $e^{\lambda t}$ is real.
This restricts the values of z_1 and z_2, as any imaginary terms in $s(\lambda, z_1)s(\lambda, z_2)$ must cancel out.

One particularly simple family of such complex pairs is defined by setting $a = \frac{h}{2}$
This necessitates $z_2 = \bar{z_1}$
We will refer to this family as the *complex conjugate step pairs*.

5.3 Stability of 2-Step Methods for Complex Conjugate Step Pairs

In this section we will compare the stability regions for the real and complex 2-step methods.
The real step size is given by $\frac{h}{2}$
The complex conjugate step pair is given by $z = \frac{h}{2} + bi$ and $\bar{z} = \frac{h}{2} - bi$
The plots in this section show the regions of stability $S = \{\lambda h : |s(\lambda, h)| \leq 1\}$ for the case where $b = a = \frac{\lambda h}{2}$
We will have a look at varying b in a subsequent section.

5.3.1 Euler's Forward 2-Step

Theorem 5.1. ***Euler's Forward Complex Step Pairs must be Conjugate for*** $\lambda \in \mathbb{R}$

For the exponential decay problem, $y = e^{\lambda t}$*, let* $\lambda \in \mathbb{R}$
Let $z_1 = a_1 + b_1 i$ *and* $z_2 = a_2 + b_2 i$ *be a complex step pair for Euler's Forward 2-Step Method.*
Then we must have $z_2 = \bar{z_1}$

Proof:

$\lambda \in \mathbb{R} \implies y_j, y_{j+1} \in \mathbb{R}$

Euler's Forward 2-Step Method is given by:

$$\begin{aligned} y_{j+1} &= s(\lambda, z_1)s(\lambda, z_2)\ y_j \\ &= \Big(1 + z_1\Big)\Big(1 + z_2\Big)\ y_j \\ &= \Big(1 + a_1 + b_1 i\Big)\Big(1 + a_2 + b_2 i\Big)\ y_j \\ &= \Bigg(\Big(1 + a_1 + a_2 + a_1 a_2 - b_1 b_2\Big) + \Big(b_1 + b_2 + a_1 b_2 + a_2 b_1\Big) i\Bigg)\ y_j \end{aligned}$$

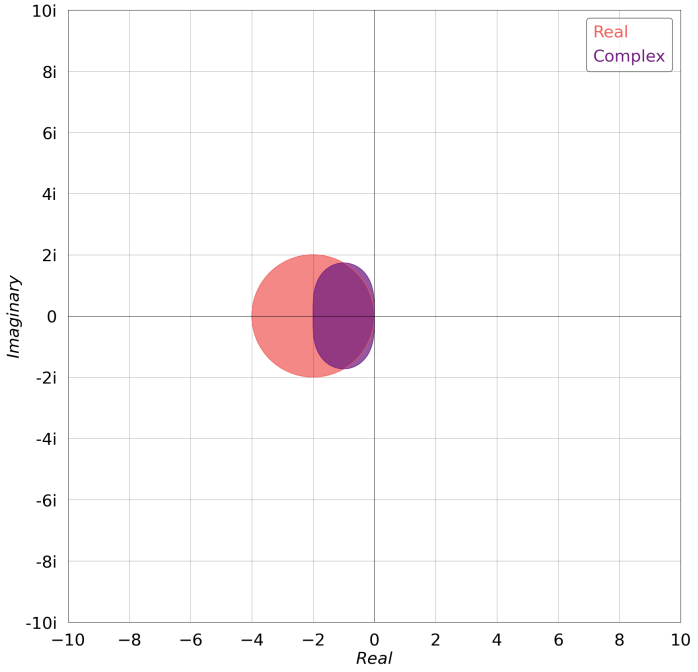
$y_{j+1} \in \mathbb{R} \implies Im(y_{j+1}) = 0 \implies b_1 + b_2 + a_1 b_2 + a_2 b_1 = 0 \implies (1 + a_1)b_2 + (1 + a_2)b_1 = 0$

$z_1 + z_2 = h \in \mathbb{R} \implies a_1 + a_2 + (b_1 + b_2)i = h \implies b_1 + b_2 = 0 \implies b_1 = -b_2$

$(1 + a_1)b_2 + (1 + a_2)b_1 = 0 \implies (1 + a_1)b_2 - (1 + a_2)b_2 = 0 \implies b_2 = 0 = b_1$ *or* $a_1 = a_2$

Thus, $z_1, z_2 \in \mathbb{C} \implies a_1 = a_2$ *and* $b_1 = -b_2$

$\therefore z_1 = a_1 + b_1 i$ *and* $z_2 = a_1 - b_1 i \implies z_2 = \bar{z_1}$ $\square$



Euler's Forward 2-Step $\mathbb{R}$

$$y_{j+1} = \Big(1 + \frac{\lambda h}{2}\Big)^2 y_j$$

$$\implies s_{\mathbb{R}}(\lambda, h) = 1 + \lambda h + \frac{(\lambda h)^2}{4}$$

Euler's Forward 2-Step $\mathbb{C}$

$$\begin{aligned} y_{j+1} &= \Big(1 + z\Big)\Big(1 + \bar{z}\Big) y_j \\ &= \Big(1 + \frac{\lambda h}{2} + bi\Big)\Big(1 + \frac{\lambda h}{2} - bi\Big) y_j \\ &= \Bigg(\Big(1 + \tfrac{\lambda h}{2}\Big)^2 + b^2\Bigg) y_j \end{aligned}$$

$$\implies s_{\mathbb{C}}(\lambda, h) = 1 + \lambda h + \frac{(\lambda h)^2}{4} + b^2$$

$$\implies s_{\mathbb{C}}(\lambda, h) = s_{\mathbb{R}}(\lambda, h) + b^2$$

As $b \in \mathbb{R}$, $s_{\mathbb{C}}(\lambda, h) \geq s_{\mathbb{R}}(\lambda, h)$
This means the stability region for the complex 2-step method is smaller than that of the real 2-step method; there are less values of λh for which the complex 2-step method is stable.
$S_{\mathbb{C}}$ is smaller than $S_{\mathbb{R}}$
The diagram above illustrates the case where $b = a = \frac{\lambda h}{2}$
In this case, we have

$$s_{\mathbb{R}}(\lambda, h) = 1 + \lambda h + \frac{(\lambda h)^2}{4} = M(y) + \mathcal{O}((\lambda h)^2)$$

while

$$s_{\mathbb{C}}(\lambda, h) = 1 + \lambda h + \frac{(\lambda h)^2}{2} = M(y) + \mathcal{O}((\lambda h)^3)$$

The complex 2-step method has a higher order of accuracy than the real 2-step method, with the trade-off of a smaller stability region.

We can do the same kind of analysis for other Numerical Methods.
The same derivations and diagrams for the Backward Euler and Runge-Kutta 4 methods follow.

5.3.2 Euler's Backward 2-Step

Theorem 5.2. ***Euler's Backward Complex Step Pairs must be Conjugate for*** $\lambda \in \mathbb{R}$

For the exponential decay problem, $y = e^{\lambda t}$*, let* $\lambda \in \mathbb{R}$
Let $z_1 = a_1 + b_1 i$ *and* $z_2 = a_2 + b_2 i$ *be a complex step pair for Euler's Backward 2-Step Method.*
Then we must have $z_2 = \bar{z_1}$

Proof:

$\lambda \in \mathbb{R} \implies y_j, y_{j+1} \in \mathbb{R}$

Euler's Backward 2-Step Method is given by:

$$\begin{aligned} y_{j+1} &= s(\lambda, z_1)s(\lambda, z_2)\, y_j \\ &= \left(\frac{1}{1-z_1}\right)\left(\frac{1}{1-z_2}\right) y_j \\ &= \left(\frac{1}{(1-z_1)(1-z_2)}\right) y_j \end{aligned}$$

$$\begin{aligned} (1-z_1)(1-z_2) &= (1-a_1-b_1 i)(1-a_2-b_2 i) \\ &= \big[1-a_1-a_2+a_1a_2-b_1b_2\big] + \big[a_1b_2+a_2b_1-b_1-b_2\big]i \\ &= \big[(a_1-1)(a_2-1)-b_1b_2\big] + \big[b_1(a_2-1)+b_2(a_1-1)\big]i \end{aligned}$$

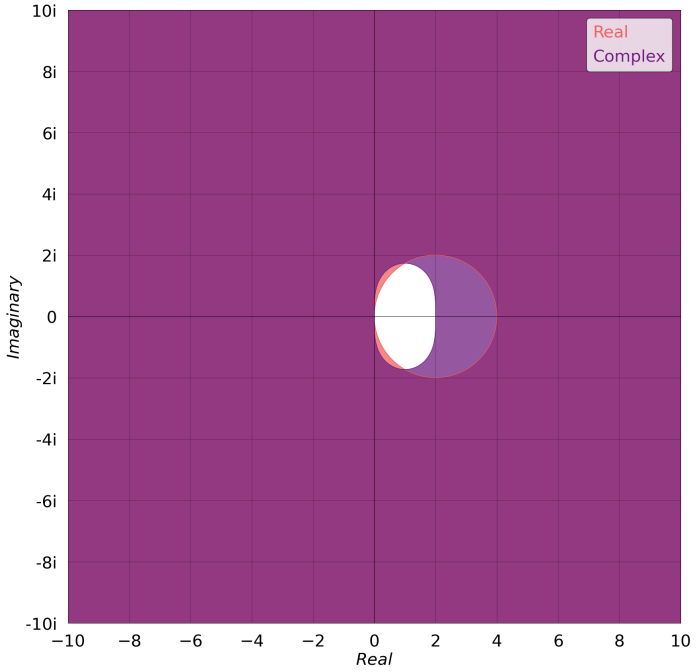
$y_{j+1} \in \mathbb{R} \implies Im(y_{j+1}) = 0 \implies b_1(a_2-1) + b_2(a_1-1) = 0$

$z_1 + z_2 = h \in \mathbb{R} \implies a_1 + a_2 + (b_1+b_2)i = h \implies b_1 + b_2 = 0 \implies b_1 = -b_2$

$b_1(a_2-1) + b_2(a_1-1) = 0 \implies b_1(a_2-1) - b_1(a_1-1) = 0 \implies b_1 = 0 = b_2$ *or* $a_1 = a_2$

Thus, $z_1, z_2 \in \mathbb{C} \implies a_1 = a_2$ *and* $b_1 = -b_2$

$\therefore z_1 = a_1 + b_1 i$ *and* $z_2 = a_1 - b_1 i \implies z_2 = \bar{z_1}$ $\square$



Euler's Backward 2-Step $\mathbb{R}$

$$y_{j+1} = \left(\frac{1}{1-\frac{\lambda h}{2}}\right)^2 y_j$$

$$\implies s_{\mathbb{R}}(\lambda, h) = \frac{1}{1-\lambda h + \frac{(\lambda h)^2}{4}}$$

Euler's Backward 2-Step $\mathbb{C}$

$$y_{j+1} = \left(\frac{1}{1-\frac{\lambda h}{2}+bi}\right)\left(\frac{1}{1-\frac{\lambda h}{2}-bi}\right) y_j$$

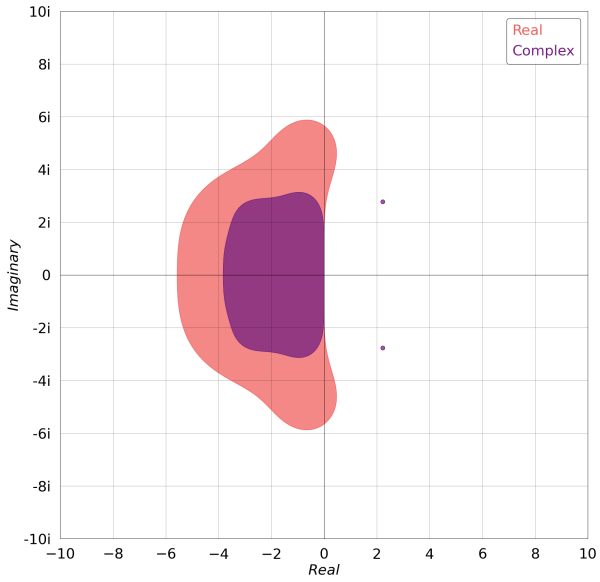
$$\implies s_{\mathbb{C}}(\lambda, h) = \frac{1}{1-\lambda h + \frac{(\lambda h)^2}{4} + b^2}$$

Again, $b \in \mathbb{R}$, so this time, $s_{\mathbb{C}}(\lambda, h) \leq s_{\mathbb{R}}(\lambda, h) \implies S_{\mathbb{C}}$ is larger than $S_{\mathbb{R}}$

5.3.3 Runge-Kutta 4 2-Step

Conjecture 1. ***Runge-Kutta 4 Complex Step Pairs must be Conjugate for*** $\lambda \in \mathbb{R}$

This is left as a conjecture for now, as the expansion we did in the other two proofs is much more complicated for the Runge-Kutta 4 method.
The proof is left as an exercise in futility for the reader.
Have a look at Appendix A for the full expansion of the stability function for the Runge-Kutta 4 method, if long algebraic expressions are your flavour of masochism.



Runge-Kutta 4 2-Step $\mathbb{R}$

$$y_{j+1} = \left(1 + \frac{\lambda h}{2} + \frac{\left(\frac{\lambda h}{2}\right)^2}{2} + \frac{\left(\frac{\lambda h}{2}\right)^3}{6} + \frac{\left(\frac{\lambda h}{2}\right)^4}{24}\right)^2 y_j$$

$$y_{j+1} = \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} + \frac{(\lambda h)^5}{128} + \frac{5(\lambda h)^6}{4608} + \frac{(\lambda h)^7}{9216} + \frac{(\lambda h)^8}{147456}\right) y_j$$

$$\implies s_{\mathbb{R}}(\lambda, h) = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} + \frac{(\lambda h)^5}{128} + \frac{5(\lambda h)^6}{4608} + \frac{(\lambda h)^7}{9216} + \frac{(\lambda h)^8}{147456}$$

Runge-Kutta 4 2-Step $\mathbb{C}$

$$y_{j+1} = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right)\left(1 + \bar{z} + \frac{\bar{z}^2}{2} + \frac{\bar{z}^3}{6} + \frac{\bar{z}^4}{24}\right) y_j$$

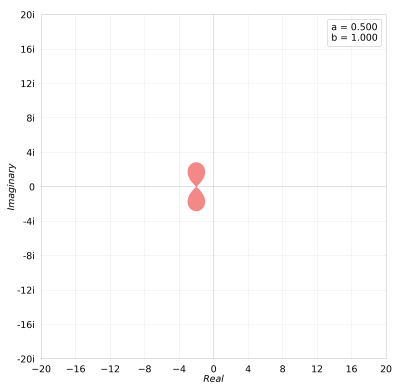
$$\implies s_{\mathbb{C}}(\lambda, h) = 1 + \lambda h + \frac{\lambda h^2}{2} + \frac{\lambda h^3}{6} + \frac{\lambda h^4}{24} + \frac{\lambda h^5}{128} + \frac{5\lambda h^6}{4608} + \frac{\lambda h^7}{9216} + \frac{\lambda h^8}{147456} + \frac{b^2\lambda h^3}{48} + \frac{b^2\lambda h^4}{128} + \frac{b^2\lambda h^5}{768} + \frac{b^2\lambda h^6}{9216} + \frac{b^4\lambda h}{24} - \frac{b^4\lambda h^2}{96} + \frac{b^4\lambda h^3}{192} + \frac{b^4\lambda h^4}{1536} - \frac{b^6}{72} + \frac{b^6\lambda h}{144} + \frac{b^6\lambda h^2}{576} + \frac{b^8}{576}$$

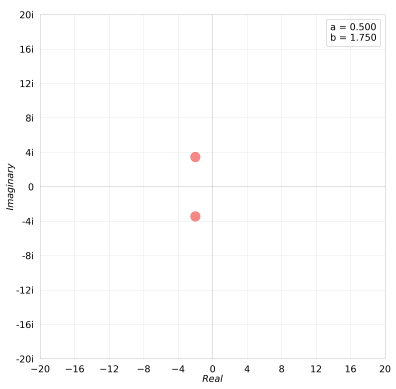
5.4 Varying b for Complex Conjugate Step Pairs

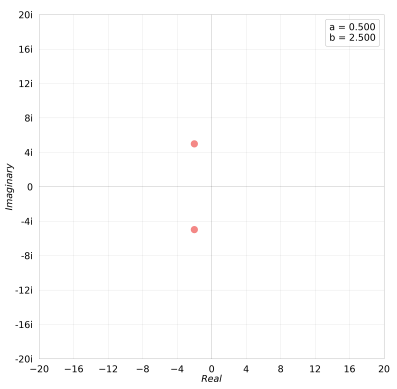
As mentioned earlier, we need not only focus on the case where $b = \frac{\lambda h}{2}$
We can vary b and see how the stability regions change.
In the cases below, we have preserved $a = \frac{\lambda h}{2}$
Consequently, these are still complex conjugate step pairs.
Videos showing the stability regions for varying b values can be found in the *GitHub Repository*[3] for this project.
Below are some of the video frames for each method.
Observations:

- The stability region for complex conjugate pairs is always symmetric accross the real axis.
- As b increases, the stability region shrinks.
  This can be seen quite simply in the $s_{\mathbb{C}}(\lambda, h)$ equation for Euler's Forward and Backward methods.

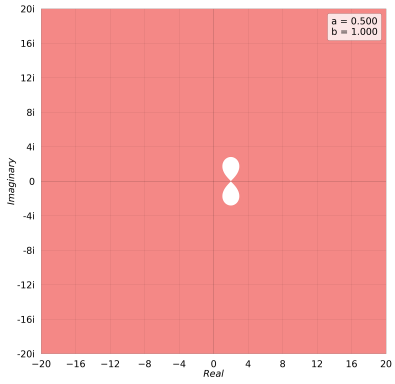
5.4.1 Euler's Forward

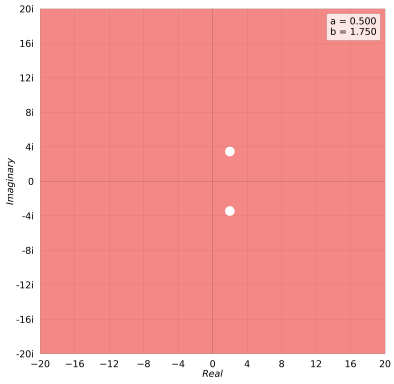


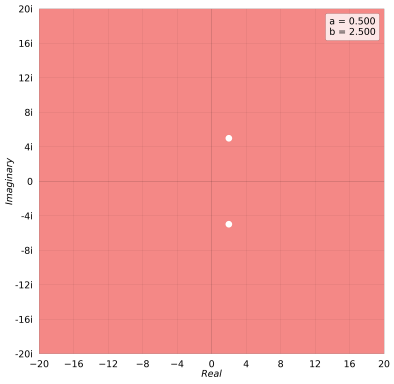




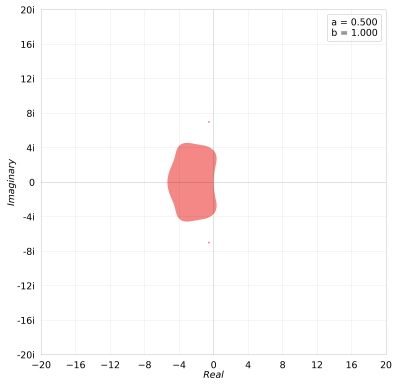
5.4.2 Euler's Backward

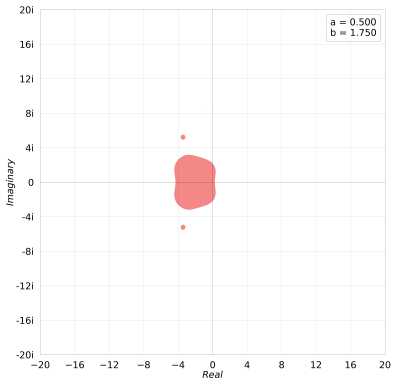


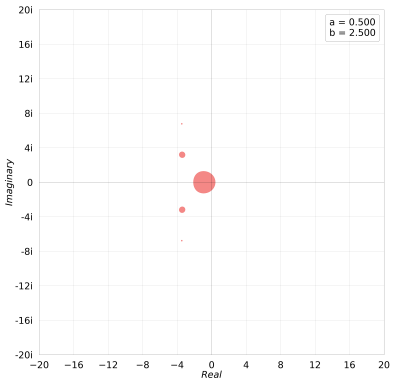




5.4.3 Runga-Kutta 4





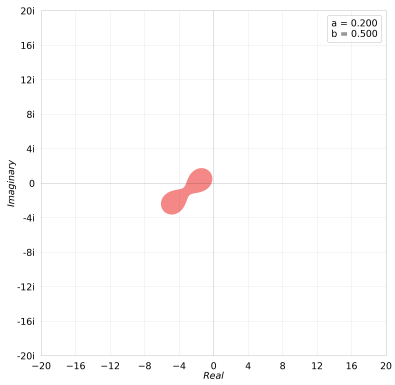


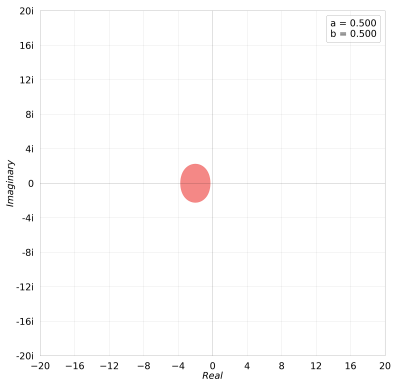
5.5 Varying a

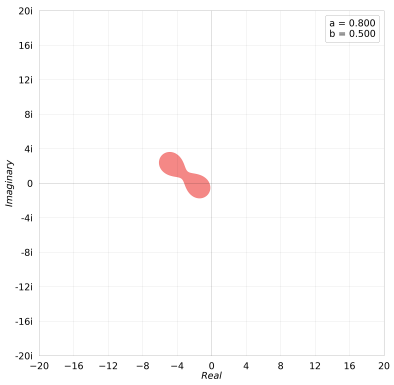
We can also vary a and see how the stability regions change.
This falls outside the scope of complex conjugate step pairs.
Consequently, these stability regions can only be interpreted for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (If you believe Conjecture 1).
In the cases below, we have set $b = \frac{\lambda h}{2}$
Videos showing the stability regions for varying a values can be found in the *GitHub Repository*[3] for this project.
Below are some of the video frames for each method.
Observations:

- The stability region for varying a values is not symmetric across the real axis, except for the case where $a = 0.5$.
- The stability regions for $a = 0.5 \pm \alpha$ are symmetric to each other across the real axis for any $|\alpha| < 0.5$.
- The stability region for Euler's Backward Method is the inverse of that for Euler's Forward Method, after it has been reflected across the imaginary axis.

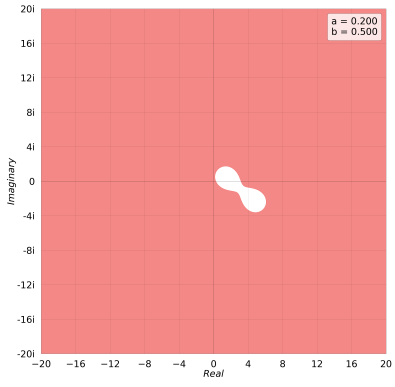
5.5.1 Euler's Forward

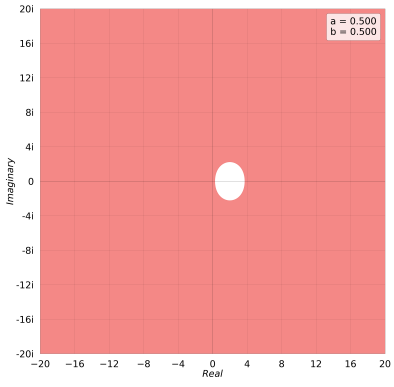


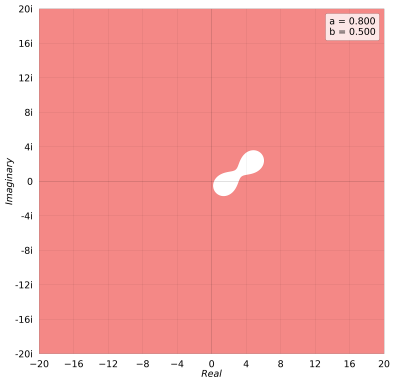




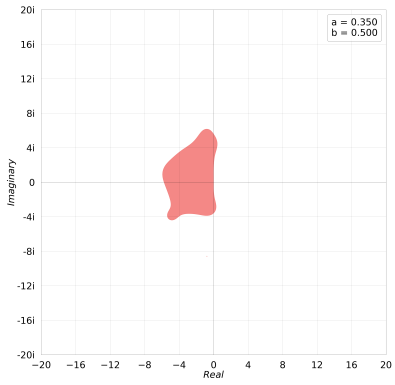
5.5.2 Euler's Backward

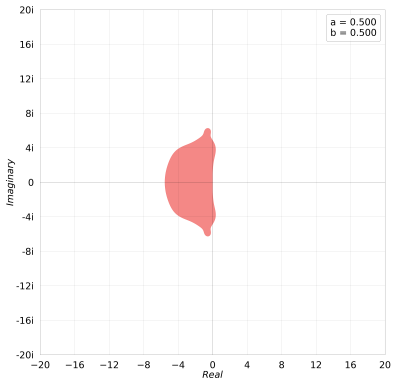


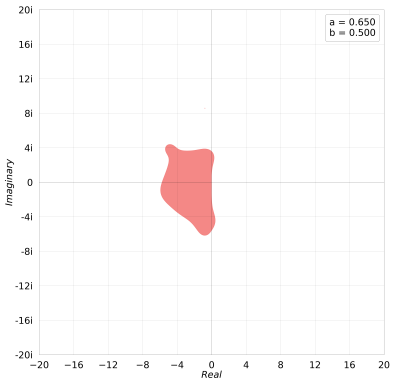




5.5.3 Runga-Kutta 4







References

[1] Llyod N. Trefethen, *The definition of numerical analysis*, Cornell University, 1992.

[2] Jithin D. George, Samuel Y. Jung, and Niall M. Mangan, *Walking into the complex plane to 'order' better time integrators*, https://arxiv.org/abs/2110.04402, 2021.

[3] The GitHub repository for this project, https://github.com/CianJDuggan/Capstone-Project

[4] The Abysmal Kramer-Butler Method, found in the notes of Dr John butler of TU Dublin, https://john-s-butler-dit.github.io/NumericalAnalysisBook/Chapter%2005%20-%20IVP%20Consistent%20Convergence%20Stability/503_Stability.html#step-abysmal-kramer-butler-method

6 Appendix

6.1 Numerical Method: 2-step Abysmal Kramer-Butler Method

A look was taken beyond 1-step numerical methods.
The 2-step Abysmal Kramer-Butler [4] method was designed to be unstable for the sake of demonstration.
The algorithm is as follows:

$$y_{j+1} = y_{j-1} + h(4\phi(t_j, y_j) - 2\phi(t_{j-1}, y_{j-1}))$$

As we know the exact solution to our Exponential Decay problem is $y(t) = e^{\lambda t}$, we know $\phi(t, y) = \lambda y$.
This gives:

$$y_{j+1} = y_{j-1} + h(4\lambda y_j - 2\lambda y_{j-1})$$

Simplified:

$$y_{j+1} = 4\lambda h y_j + (1 - 2\lambda h) y_{j-1}$$

We need this in the form $y_{j+1} = s(\lambda h) y_j$ to find the stability region.
The author could not manage this.
If this is your poison, good luck.

6.2 Runge-Kutta 4 Complex Step Pairs must be Conjugate for $\lambda \in \mathbb{R}$

Here follows the expansion of $s(z_1, h)s(z_2, h)$ for Runge-Kutta 4 with complex step pairs $z_1 = a + bi$ and $z_2 = c + di$:
Real: $1 + \frac{a^4c^4}{576} + \frac{a^4c^3}{144} - \frac{a^4c^2d^2}{96} + \frac{a^4c^2}{48} - \frac{a^4cd^2}{48} + \frac{a^4c}{24} + \frac{a^4d^4}{576} - \frac{a^4d^2}{48} + \frac{a^4}{24} - \frac{a^3bc^3d}{36} - \frac{a^3bc^2d}{12} + \frac{a^3bcd^3}{36} - \frac{a^3bcd}{6} + \frac{a^3bd^3}{36} - \frac{a^3bd}{6} + \frac{a^3c^4}{144} + \frac{a^3c^3}{36} - \frac{a^3c^2d^2}{24} + \frac{a^3c^2}{12} - \frac{a^3cd^2}{12} + \frac{a^3c}{6} + \frac{a^3d^4}{144} - \frac{a^3d^2}{12} + \frac{a^3}{6} - \frac{a^2b^2c^4}{96} - \frac{a^2b^2c^3}{24} + \frac{a^2b^2c^2d^2}{16} - \frac{a^2b^2c^2}{8} + \frac{a^2b^2cd^2}{8} - \frac{a^2b^2c}{4} - \frac{a^2b^2d^4}{96} + \frac{a^2b^2d^2}{8} - \frac{a^2b^2}{4} - \frac{a^2bc^3d}{12} - \frac{a^2bc^2d}{4} + \frac{a^2bcd^3}{12} - \frac{a^2bcd}{2} + \frac{a^2bd^3}{12} - \frac{a^2bd}{2} + \frac{a^2c^4}{48} + \frac{a^2c^3}{12} - \frac{a^2c^2d^2}{8} + \frac{a^2c^2}{4} - \frac{a^2cd^2}{4} + \frac{a^2c}{2} + \frac{a^2d^4}{48} - \frac{a^2d^2}{4} + \frac{a^2}{2} + \frac{ab^3c^3d}{36} + \frac{ab^3c^2d}{12} - \frac{ab^3cd^3}{36} + \frac{ab^3cd}{6} - \frac{ab^3d^3}{36} + \frac{ab^3d}{6} - \frac{ab^2c^4}{48} - \frac{ab^2c^3}{12} + \frac{ab^2c^2d^2}{8} - \frac{ab^2c^2}{4} + \frac{ab^2cd^2}{4} - \frac{ab^2c}{2} - \frac{ab^2d^4}{48} + \frac{ab^2d^2}{4} - \frac{ab^2}{2} - \frac{abc^3d}{6} - \frac{abc^2d}{2} + \frac{abcd^3}{6} - abcd + \frac{abd^3}{6} - abd + \frac{ac^4}{24} + \frac{ac^3}{6} - \frac{ac^2d^2}{4} + \frac{ac^2}{2} - \frac{acd^2}{2} + ac + \frac{ad^4}{24} - \frac{ad^2}{2} + a + \frac{b^4c^4}{576} + \frac{b^4c^3}{144} - \frac{b^4c^2d^2}{96} + \frac{b^4c^2}{48} - \frac{b^4cd^2}{48} + \frac{b^4c}{24} + \frac{b^4d^4}{576} - \frac{b^4d^2}{48} + \frac{b^4}{24} + \frac{b^3c^3d}{36} + \frac{b^3c^2d}{12} - \frac{b^3cd^3}{36} + \frac{b^3cd}{6} - \frac{b^3d^3}{36} + \frac{b^3d}{6} - \frac{b^2c^4}{48} - \frac{b^2c^3}{12} + \frac{b^2c^2d^2}{8} - \frac{b^2c^2}{4} + \frac{b^2cd^2}{4} - \frac{b^2c}{2} - \frac{b^2d^4}{48} + \frac{b^2d^2}{4} - \frac{b^2}{2} - \frac{bc^3d}{6} - \frac{bc^2d}{2} + \frac{bcd^3}{6} - bcd + \frac{bd^3}{6} - bd + \frac{c^4}{24} + \frac{c^3}{6} - \frac{c^2d^2}{4} + \frac{c^2}{2} - \frac{cd^2}{2} + c + \frac{d^4}{24} - \frac{d^2}{2}$
Imaginary:
$\frac{a^4c^3d}{144} + \frac{a^4c^2d}{48} - \frac{a^4cd^3}{144} + \frac{a^4cd}{24} - \frac{a^4d^3}{144} + \frac{a^4d}{24} + \frac{a^3bc^4}{144} + \frac{a^3bc^3}{36} - \frac{a^3bc^2d^2}{24} + \frac{a^3bc^2}{12} - \frac{a^3bcd^2}{12} + \frac{a^3bc}{6} + \frac{a^3bd^4}{144} - \frac{a^3bd^2}{12} + \frac{a^3b}{6} + \frac{a^3c^3d}{36} + \frac{a^3c^2d}{12} - \frac{a^3cd^3}{36} + \frac{a^3cd}{6} - \frac{a^3d^3}{36} + \frac{a^3d}{6} - \frac{a^2b^2c^3d}{24} - \frac{a^2b^2c^2d}{8} + \frac{a^2b^2cd^3}{24} - \frac{a^2b^2cd}{4} + \frac{a^2b^2d^3}{24} - \frac{a^2b^2d}{4} + \frac{a^2bc^4}{48} + \frac{a^2bc^3}{12} - \frac{a^2bc^2d^2}{8} + \frac{a^2bc^2}{4} - \frac{a^2bcd^2}{4} + \frac{a^2bc}{2} + \frac{a^2bd^4}{48} - \frac{a^2bd^2}{4} + \frac{a^2b}{2} + \frac{a^2c^3d}{12} + \frac{a^2c^2d}{4} - \frac{a^2cd^3}{12} + \frac{a^2cd}{2} - \frac{a^2d^3}{12} + \frac{a^2d}{2} - \frac{ab^3c^4}{144} - \frac{ab^3c^3}{36} + \frac{ab^3c^2d^2}{24} - \frac{ab^3c^2}{12} + \frac{ab^3cd^2}{12} - \frac{ab^3c}{6} - \frac{ab^3d^4}{144} + \frac{ab^3d^2}{12} - \frac{ab^3}{6} - \frac{ab^2c^3d}{12} - \frac{ab^2c^2d}{4} + \frac{ab^2cd^3}{12} - \frac{ab^2cd}{2} + \frac{ab^2d^3}{12} - \frac{ab^2d}{2} + \frac{abc^4}{24} + \frac{abc^3}{6} - \frac{abc^2d^2}{4} + \frac{abc^2}{2} - \frac{abcd^2}{2} + abc + \frac{abd^4}{24} - \frac{abd^2}{2} + ab + \frac{ac^3d}{6} + \frac{ac^2d}{2} - \frac{acd^3}{6} + acd - \frac{ad^3}{6} + ad + \frac{b^4c^3d}{144} + \frac{b^4c^2d}{48} - \frac{b^4cd^3}{144} + \frac{b^4cd}{24} - \frac{b^4d^3}{144} + \frac{b^4d}{24} - \frac{b^3c^4}{144} - \frac{b^3c^3}{36} + \frac{b^3c^2d^2}{24} - \frac{b^3c^2}{12} + \frac{b^3cd^2}{12} - \frac{b^3c}{6} - \frac{b^3d^4}{144} + \frac{b^3d^2}{12} - \frac{b^3}{6} - \frac{b^2c^3d}{12} - \frac{b^2c^2d}{4} + \frac{b^2cd^3}{12} - \frac{b^2cd}{2} + \frac{b^2d^3}{12} - \frac{b^2d}{2} + \frac{bc^4}{24} + \frac{bc^3}{6} - \frac{bc^2d^2}{4} + \frac{bc^2}{2} - \frac{bcd^2}{2} + bc + \frac{bd^4}{24} - \frac{bd^2}{2} + b + \frac{c^3d}{6} + \frac{c^2d}{2} - \frac{cd^3}{6} + cd - \frac{d^3}{6} + d$

6.3 Code

6.3.1 Exact Solution for Exponential Decay

Found at '/Python/Exponential Decay/Exact.py' in the repository [3].

```
import os
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl

cmap = mpl.colormaps['magma_r']
colors = cmap(np.linspace(0, 1, 10))

def plot_exponential_function(domain, grid, h, lam):
    x = grid/2
    t = np.arange(0, x, 1/1000)

    y = np.exp(lam * t)

    if domain == 'real':
        plt.figure(figsize=(grid, grid), facecolor='white', edgecolor='black')
```

```
    ax = plt.gca()
    ax.plot(t, y, color=colors[3])
    ax.set_ylabel('$y$', color='black', fontsize=15)
    ax.set_facecolor('white')
    plt.axhline(0, color='black', linewidth=0.75)  # y-axis
    plt.axvline(0, color='black', linewidth=0.75)  # x-axis
    # Set border (spines) linewidth
    for spine in ax.spines.values():
        spine.set_linewidth(0.75)
        spine.set_color('black')

    plt.grid(which='major', linestyle='-', linewidth='0.5', color='black')
    ax.set_xlim(-2, 3)
    ax.set_ylim(0, x)
    ax.set_aspect(aspect='equal')  # Set the aspect ratio to 1:1
    plt.xticks(np.arange(-2, 4, 1), fontsize=15, color='black')
    plt.yticks(np.arange(0, int(x) + 1, 1), fontsize=15, color='black')
    ax.set_xlabel('$t$', color='black', fontsize=15)
    plt.legend(labels=[r'$\lambda = {}$'.format(lam)], labelcolor=['black', 'black'], face


elif domain == 'complex':
    fig = plt.figure(figsize=(10, 10), facecolor='white', edgecolor='black')
    ax = fig.add_subplot(111, projection='3d', facecolor='white')
    ax.view_init(elev=22.5, azim=315, roll=0)
    ax.plot(t, y.real, y.imag, color=colors[3], linewidth=1.5)
    ax.set_xlabel('$t$', color='black')
    ax.set_ylabel('$Re(y)$', color='black')
    ax.set_zlabel('$Im(y)$', color='black')
    plt.legend(labels=[r'$\lambda = {}$'.format(lam)], labelcolor=['black', 'black'], face
    # Set background color for axes planes
    ax.xaxis.set_pane_color('white')
    ax.yaxis.set_pane_color('white')
    ax.zaxis.set_pane_color('white')

    # Customize axis colors
    ax.xaxis.line.set_color('black')
    ax.yaxis.line.set_color('black')
    ax.zaxis.line.set_color('black')

    # Set tick colors
    ax.tick_params(axis='x', colors='black')
    ax.xaxis._axinfo['tick']['inward_factor'] = 0
    ax.xaxis._axinfo['tick']['outward_factor'] = 0
    ax.tick_params(axis='y', colors='black')
    ax.yaxis._axinfo['tick']['inward_factor'] = 0
    ax.yaxis._axinfo['tick']['outward_factor'] = 0
    ax.tick_params(axis='z', colors='black')
    ax.zaxis._axinfo['tick']['inward_factor'] = 0
    ax.zaxis._axinfo['tick']['outward_factor'] = 0

    # Set ranges for axes
    ax.set_xlim([t.min(), t.max()])
    ax.set_ylim([y.real.min(), y.real.max()])
    ax.set_zlim([y.imag.min(), y.imag.max()])

    # Set ticks to create square grid boxes
    tick_space = 4
    ax.set_xticks(np.arange(t.min(), t.max() + 1, (np.ceil(t.max()) - np.floor(t.min()))/t
    ax.set_yticks(np.arange(np.floor(y.real.min()), np.ceil(y.real.max()), (np.ceil(y.real
    ax.set_zticks(np.arange(np.floor(y.imag.min()), np.ceil(y.imag.max()), (np.ceil(y.imag

    # Ensure grid lines are drawn on all planes
    ax.grid(True)
```

```
        ax.grid(color='black')
        ax.xaxis._axinfo["grid"].update({"color": "black", "linewidth": 0.5})
        ax.yaxis._axinfo["grid"].update({"color": "black", "linewidth": 0.5})
        ax.zaxis._axinfo["grid"].update({"color": "black", "linewidth": 0.5})

    # Define output path
    relative_path = os.path.join('..', 'Graphs', 'Exponential-Decay', f'Exact-with-{lam}.png')

    plt.savefig(relative_path, bbox_inches='tight', pad_inches=0)
    plt.show()


#plot_exponential_function(domain = 'real', grid = 10, h = 1/5, lam = -1)
plot_exponential_function(domain = 'complex', grid = 20, h = 1/5, lam = -0.3+5j)
```

6.3.2 Basic Stability Regions

Found at '/Python/Stability Regions/Graphs/Basic.py' in the repository [3].

```
import os
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
from matplotlib.ticker import FuncFormatter as formatter

cmap = mpl.colormaps['magma_r']
# Take colors at regular intervals spanning the colormap
colors = cmap(np.linspace(0, 1, 10))

# Stability Functions
EF = ["Euler's Forward", lambda z: z + 1]
EB = ["Euler's Backward", lambda z: 1/(1 - z)]
RK4 = ["Runge-Kutta 4", lambda z: (z**4)/24 + (z**3)/6 + (z**2)/2 + z + 1]
stability_functions = [EF, EB, RK4]

def plot_Region(stability_function, grid, points):
    # Generate domain (complex grid)
    x = grid / 2
    real_domain = np.linspace(-x, x, points)
    imag_domain = np.linspace(-x, x, points)
    real, imag = np.meshgrid(real_domain, imag_domain)
    domain = real + 1j * imag

    # Evaluate Stability Region
    name = stability_function[0]
    function = stability_function[1]
    Range = function(domain)
    distance = np.abs(Range)

    # Define the Figure
    plt.figure(figsize=(grid, grid), facecolor='white', edgecolor='black')
    ax = plt.gca()

    # Plot the Stability Region
    plt.contour(real, imag, distance, levels=[1], colors=[colors[3]], linewidths=1)  # Bounda
    plt.contourf(real, imag, distance, levels=[0, 1], colors=[colors[3]], alpha=0.75)  # Fill

    # Customize Imaginary axis tick labels to include 'i'
    def im_axis(x, pos):
        return f'{x:.0f}i' if x != 0 else '0'
    ax.yaxis.set_major_formatter(formatter(im_axis))
    ax.set_facecolor('white')

    # Plot bold lines for Real and Imaginary axes
    plt.axhline(0, color='black', linewidth=0.75)  # Imaginary axis
    plt.axvline(0, color='black', linewidth=0.75)  # Real axis

    # Set major grid lines for both Real and Imaginary axes
    plt.grid(which='major', linestyle='-', linewidth='0.5', color='black')
    ax.set_box_aspect(1)
    ax.set_xlim(-x, x)
    ax.set_ylim(-x, x)
    plt.xticks(np.arange(-int(x), int(x) + 1, 1), fontsize=15, color='black')
    plt.yticks(np.arange(-int(x), int(x) + 1, 1), fontsize=15, color='black')

    # Remove tick marks
    plt.tick_params(axis='both', which='both', length=0, pad=10)

    # Set border (spines) linewidth
    for spine in ax.spines.values():
```

```
        spine.set_linewidth(0.75)
        spine.set_color('black')

    # Labelling
    plt.xlabel('$Real$', fontsize=15, color='black')
    plt.ylabel('$Imaginary$', fontsize=15, color='black')

    relative_path = os.path.join('..', '..', 'Graphs', 'Stability-Regions', 'Graphs', 'Real-1
    plt.savefig(relative_path, bbox_inches='tight', pad_inches=0)
    plt.show()

# Iterate through each function in Functions
for s in stability_functions:
     plot_Region(s, 10, 2**13)
```

6.3.3 Exact vs Methods

Found at '/Python/Exponential Decay/Exact Vs Method.py' in the repository [3].

```
import os
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl

cmap = mpl.colormaps['magma']
colors = cmap(np.linspace(0, 1, 10))
background = 'white'
foreground = 'black'

EF = ["Euler's Forward", lambda l, h: (l*h) + 1]
EB = ["Euler's Backward", lambda l, h: 1/(1 - (l*h))]
RK4 = ["Runge–Kutta 4", lambda l, h: ((l*h)**4)/24 + ((l*h)**3)/6 + ((l*h)**2)/2 + (l*h) + 1]

def plot_exponential_function(domain, grid, h_stable, h_unstable, lam, method, pos, red_width)
    # Size and time steps
    t = np.arange(0, grid, 1/1000)

    ## Exact ##
    y = np.exp(lam * t)

    ## Method ##

    # Stable
    t_steps_stable = np.arange(0, grid+1, 1)
    y_steps_stable = method[1](lam, h_stable) ** t_steps_stable
    y_method_stable = np.interp(t, t_steps_stable, y_steps_stable)

    # Unstable
    t_steps_unstable = np.arange(0, grid+1, 1)
    y_steps_unstable = (method[1](lam, h_unstable) ** t_steps_unstable)
    y_method_unstable = np.interp(t, t_steps_unstable, y_steps_unstable)

    ## Plot ##
    if domain == 'real':
        # Instantiate the graph
        plt.figure(figsize=(10, 10), facecolor=background, edgecolor=foreground)
        ax = plt.gca()

        # Exact
        ax.plot(t, y, color=foreground)

        # Stable
        ax.plot(t, y_method_stable, color='green', linestyle='—')
        ax.plot(t_steps_stable, y_steps_stable, color='green', marker='o', markersize=5, lines
        # Unstable
        ax.plot(t, y_method_unstable, color='red', linestyle='—')
        ax.plot(t_steps_unstable, y_steps_unstable, color='red', marker='o', markersize=5, lin

        # Graph settings
        ax.set_ylabel('$y$', color=foreground, fontsize=15)
        ax.set_facecolor(background)
        plt.axhline(0, color=foreground, linewidth=0.75)  # y-axis
        plt.axvline(0, color=foreground, linewidth=0.75)  # x-axis
        for spine in ax.spines.values():
            spine.set_linewidth(0.75)
            spine.set_color(foreground)
        ax.set_xlim(0, grid)
        ax.set_ylim(-grid/4, grid/4)
        ax.set_aspect(aspect='equal')  # Set the aspect ratio to 1:1
        plt.xticks(np.arange(0, grid+1, 1), fontsize=15, color=foreground)
```

```
    plt.yticks(np.arange(-(grid/4), (grid/4)+1, 1), fontsize=15, color=foreground)
    ax.set_xlabel('$t$', color=foreground, fontsize=15)
    h_stable = round(h_stable, 3)
    h_unstable = round(h_unstable, 3)
    plt.legend(labels=[r'$\lambda=-{}$'.format(lam), r'$h=-{}$'.format(h_stable), r'$h=
```

```
elif domain == 'complex':
    # Instantiate the graph
    fig = plt.figure(figsize=(10, 10), facecolor=background, edgecolor=foreground)
    ax = fig.add_subplot(111, projection='3d', facecolor=background)
    ax.view_init(elev=22.5, azim=315, roll=0)

    # Exact
    ax.plot(t, y.real, y.imag, color=foreground, linewidth=1.5)
    # Valid
    ax.plot(t, y_method_stable.real, y_method_stable.imag, color='green', linewidth=2)
    #ax.plot(t_steps_stable, y_steps_stable.real, y_steps_stable.imag, color='green', mar
    # Instable
    ax.plot(t, y_method_unstable.real, y_method_unstable.imag, color='red', linewidth=red
    #ax.plot(t_steps_unstable, y_steps_unstable.real, y_steps_unstable.imag, color='red',

    # Graph settings
    ax.set_xlabel('$t$', color=foreground)
    ax.set_ylabel('$Re(y)$', color=foreground)
    ax.set_zlabel('$Im(y)$', color=foreground)
    plt.legend(labels=[r'$\lambda=-{}$'.format(lam), r'$h=-{}$'.format(h_stable), r'$h=
    ax.xaxis.set_pane_color(background)
    ax.yaxis.set_pane_color(background)
    ax.zaxis.set_pane_color(background)
    ax.xaxis.line.set_color(foreground)
    ax.yaxis.line.set_color(foreground)
    ax.zaxis.line.set_color(foreground)
    ax.tick_params(axis='x', colors=foreground)
    ax.xaxis._axinfo['tick']['inward_factor'] = 0
    ax.xaxis._axinfo['tick']['outward_factor'] = 0
    ax.tick_params(axis='y', colors=foreground)
    ax.yaxis._axinfo['tick']['inward_factor'] = 0
    ax.yaxis._axinfo['tick']['outward_factor'] = 0
    ax.tick_params(axis='z', colors=foreground)
    ax.zaxis._axinfo['tick']['inward_factor'] = 0
    ax.zaxis._axinfo['tick']['outward_factor'] = 0

    ax.set_xlim(t.min(), t.max())

    y_real_max = max(y.real.max(), y_method_stable.real.max(), y_method_unstable.real.max(
    y_real_min = min(y.real.min(), y_method_stable.real.min(), y_method_unstable.real.min(
    ax.set_ylim(y_real_min, y_real_max)

    y_imag_max = max(y.imag.max(), y_method_stable.imag.max(), y_method_unstable.imag.max(
    y_imag_min = min(y.imag.min(), y_method_stable.imag.min(), y_method_unstable.imag.min(
    ax.set_zlim(y_imag_min, y_imag_max)

    ax.set_xticks(np.arange(np.floor(t.min()), np.ceil(t.max()) + 1, grid/5))
    ax.set_yticks(np.arange(np.floor(y_real_min), np.ceil(y_real_max) + 1, np.ceil(y_real_
    ax.set_zticks(np.arange(np.floor(y_imag_min), np.ceil(y_imag_max) + 1, np.ceil(y_imag_

    ax.grid(True)
    ax.grid(color=foreground)
    ax.xaxis._axinfo["grid"].update({"color": foreground, "linewidth": 0.5})
    ax.yaxis._axinfo["grid"].update({"color": foreground, "linewidth": 0.5})
    ax.zaxis._axinfo["grid"].update({"color": foreground, "linewidth": 0.5})
```

```
    relative_path = os.path.join('..', '..', 'Graphs', 'Exponential Decay', 'Exact vs Method',
    plt.savefig(relative_path, bbox_inches='tight', pad_inches=0)

    plt.show()

# Euler's Forward
# Real
#plot_exponential_function(domain = 'real', grid = 10, h_stable = 1/6, h_unstable = 5/12, lam
# Complex
#plot_exponential_function(domain = 'complex', grid = 300, h_stable = 0.03, h_unstable = 0.05,

# Euler's Backward
# Real
#plot_exponential_function(domain = 'real', grid = 10, h_stable = 1/6, h_unstable = 5/12, lam
# Complex
#plot_exponential_function(domain = 'complex', grid = 300, h_stable = 0.03, h_unstable = 0.05,

# Runge-Kutta 4
# Real
#plot_exponential_function(domain = 'real', grid = 10, h_stable = 0.5, h_unstable = 0.57, lam
# Complex
plot_exponential_function(domain = 'complex', grid = 300, h_stable = 0.03, h_unstable = 0.5875
```

6.3.4 Stability Magnitude 3D

Found at '/Python/Stability Magnitude/Stability Magnitude 3D.py' in the repository [3].

```
import os
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.ticker import FuncFormatter as formatter
from matplotlib.colors import LinearSegmentedColormap, Normalize
import matplotlib as mpl

cmap = mpl.colormaps['magma_r']
# Take colors at regular intervals spanning the colormap
colors = cmap(np.linspace(0, 1, 10))

# Define stability functions
EF = ["Euler's Forward", lambda z: z + 1]
EB = ["Euler's Backward", lambda z: 1/(1 - z)]
RK4 = ["Runge–Kutta 4", lambda z: (z**4)/24 + (z**3)/6 + (z**2)/2 + z + 1]

def plot_stability_region(method, grid, points, cmap):
    # Generate domain (complex grid)
    x = grid/2
    real_domain = np.linspace(-x, x, points)
    imag_domain = np.linspace(-x, x, points)
    real, imag = np.meshgrid(real_domain, imag_domain)
    domain = real + 1j * imag

    # Evaluate Stability Region
    name = method[0]
    function = method[1]
    magnitudes = np.abs(function(domain))
    stable_mags = np.where(magnitudes < 1, magnitudes, np.nan)
    unstable_mags = np.where(magnitudes >= 1, magnitudes, np.nan)

    # Define the Figure
    fig = plt.figure(figsize=(grid, grid), facecolor='white', edgecolor='black')
    ax = fig.add_subplot(111, projection='3d', facecolor='white')
    ax.view_init(elev=22.5, azim=22.5, roll=0)

    X, Y, Z = real, imag, stable_mags
    ax.plot_surface(X,Y,Z, edgecolor=colors[3], color=colors[3], linewidth=0.5, alpha=0.3, rco
    ax.contourf(X, Y, Z, zdir='z', offset=0, levels=100, cmap=cmap)
    ax.set_zlim(0, 1)
    ax.set_zticks(np.arange(0, 1.2, 0.2))

    # Set background color for axes planes
    ax.xaxis.set_pane_color('white')
    ax.yaxis.set_pane_color('white')
    ax.zaxis.set_pane_color('white')

    # Customize axis colors
    ax.xaxis.line.set_color('black')
    ax.yaxis.line.set_color('black')
    ax.zaxis.line.set_color('black')

    # Set tick colors
    ax.tick_params(axis='x', colors='black')
    ax.xaxis._axinfo['tick']['inward_factor'] = 0
    ax.xaxis._axinfo['tick']['outward_factor'] = 0
    ax.tick_params(axis='y', colors='black')
    ax.yaxis._axinfo['tick']['inward_factor'] = 0
    ax.yaxis._axinfo['tick']['outward_factor'] = 0
    ax.tick_params(axis='z', colors='black')
    ax.zaxis._axinfo['tick']['inward_factor'] = 0
```

```
    ax.zaxis._axinfo['tick']['outward_factor'] = 0

    # Axes
    ax.set_xlabel('Real', color='black')
    ax.set_ylabel('Imaginary', color='black')
    ax.set_zlabel('Magnitude', color='black')
    ax.set_xlim(-(x+1), x+1)
    ax.set_xticks(np.arange(-x, x+1, 2))
    ax.set_ylim(-(x+1), x+1)
    ax.set_yticks(np.arange(-x, x+1, 2))

    relative_path = os.path.join('..', '..', 'Graphs', 'Stability-Magnitude', '3D', f'{name}.
    plt.savefig(relative_path, bbox_inches='tight', pad_inches=0)
    plt.show()

# Iterate through each function in Functions
plot_stability_region(RK4, 10, 2**12, cmap)
```

6.3.5 Stability Magnitude Contour

Found at '/Python/Stability Magnitude/Stability Magnitude Color.py' in the repository [3].

```
import os
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.ticker import FuncFormatter as formatter
from matplotlib.colors import LinearSegmentedColormap, Normalize
import matplotlib as mpl

cmap = mpl.colormaps['magma_r']
# Take colors at regular intervals spanning the colormap
colors = cmap(np.linspace(0, 1, 10))

# Define stability functions
EF = ["Euler's Forward", lambda z: z + 1]
EB = ["Euler's Backward", lambda z: 1/(1 - z)]
RK4 = ["Runge–Kutta 4", lambda z: (z**4)/24 + (z**3)/6 + (z**2)/2 + z + 1]

def plot_stability_region(method, grid, points, cmap):
    # Generate domain (complex grid)
    x = grid/2
    real_domain = np.linspace(-x, x, points)
    imag_domain = np.linspace(-x, x, points)
    real, imag = np.meshgrid(real_domain, imag_domain)
    domain = real + 1j * imag

    # Evaluate Stability Region
    name = method[0]
    function = method[1]
    magnitudes = np.abs(function(domain))
    stable_mags = np.where(magnitudes < 1, magnitudes, np.nan)
    unstable_mags = np.where(magnitudes > 1, magnitudes, np.nan)

        # Define the Figure
    plt.figure(figsize=(grid, grid), facecolor='white', edgecolor='black')
    ax = plt.gca()

    # Plot the Stability Region
    plt.contourf(real, imag, stable_mags, levels=[0,0.2,0.4,0.6,0.8,1], cmap=cmap)
    cbar = plt.colorbar(label='Magnitude', format='%.2f', ticks=[0, 0.2, 0.4, 0.6, 0.8, 1])
    # Colorbar
    cbar.ax.tick_params(colors='black', labelsize=15)
    cbar.ax.yaxis.label.set_color('black')
    cbar.ax.yaxis.label.set_size(15)
    # Unstable region
    plt.contourf(real, imag, unstable_mags, levels=1, colors='white')
    # Boundary line
    plt.contour(real, imag, magnitudes, levels=[1], colors='black', linewidths=2)

    # Customize Imaginary axis tick labels to include 'i'
    def im_axis(x, pos):
        return f'{x:.0f}i' if x != 0 else '0'
    ax.yaxis.set_major_formatter(formatter(im_axis))
    ax.set_facecolor('white')

    # Plot bold lines for Real and Imaginary axes
    plt.axhline(0, color='black', linewidth=0.75)  # Imaginary axis
    plt.axvline(0, color='black', linewidth=0.75)  # Real axis

    # Set major grid lines for both Real and Imaginary axes
    plt.grid(which='major', linestyle='-', linewidth='0.5', color='black')
    ax.set_box_aspect(1)
    ax.set_xlim(-x, x)
```

```
    ax.set_ylim(-x, x)
    plt.xticks(np.arange(-int(x), int(x) + 1, 1), fontsize=15, color='black')
    plt.yticks(np.arange(-int(x), int(x) + 1, 1), fontsize=15, color='black')

    # Remove tick marks
    plt.tick_params(axis='both', which='both', length=0, pad=10)

    # Set border (spines) linewidth
    for spine in ax.spines.values():
        spine.set_linewidth(0.75)
        spine.set_color('black')

    # Labelling
    plt.xlabel('$Real$', fontsize=15, color='black')
    plt.ylabel('$Imaginary$', fontsize=15, color='black')

    relative_path = os.path.join('..', '..', 'Graphs', 'Stability-Magnitude', 'Color', f'{nam
    plt.savefig(relative_path, bbox_inches='tight', pad_inches=0)
    plt.show()

# Iterate through each function in Functions
plot_stability_region(RK4, 10, 2**12, cmap)
```

6.3.6 Stability Magnitude Solutions

Found at '/Python/Stability Magnitude/Exponential Decay.py' in the repository [3].

```
import os
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl

cmap = mpl.colormaps['magma']
colors = cmap(np.linspace(0, 1, 10))
background = 'white'
foreground = 'black'

EF = ["Euler's Forward", lambda l, h: (l*h) + 1]
EB = ["Euler's Backward", lambda l, h: 1/(1 - (l*h))]
RK4 = ["Runge–Kutta 4", lambda l, h: ((l*h)**4)/24 + ((l*h)**3)/6 + ((l*h)**2)/2 + (l*h) + 1]

def plot_exponential_function(domain, grid, h_values, lam, method):
    # Size and time steps
    t = np.arange(0, grid, 1/1000)

    ## Exact ##
    y = np.exp(lam * t)

    ## Method ##
    t_steps = np.arange(0, grid+1, 1)

    ## Plot ##
    # Instantiate the graph
    plt.figure(figsize=(10, 10), facecolor=background, edgecolor=foreground)
    ax = plt.gca()

    # Exact
    ax.plot(t, y, color=foreground)

    # Method
    for h in h_values:
        index = h_values.index(h)
        y_steps = method[1](lam, h) ** t_steps
        y_method = np.interp(t, t_steps, y_steps)
        ax.plot(t, y_method, color=colors[2*index], linestyle='—')
        ax.plot(t_steps, y_steps, color=colors[2*index], marker='o', markersize=5, linestyle='

    # Graph settings
    ax.set_ylabel('$y$', color=foreground, fontsize=15)
    ax.set_facecolor(background)
    plt.axhline(0, color=foreground, linewidth=0.75)  # y-axis
    plt.axvline(0, color=foreground, linewidth=0.75)  # x-axis
    for spine in ax.spines.values():
        spine.set_linewidth(0.75)
        spine.set_color(foreground)
    ax.set_xlim(0, grid)
    ax.set_ylim(-grid/4, grid/4)
    ax.set_aspect(aspect='equal')  # Set the aspect ratio to 1:1
    plt.xticks(np.arange(0, grid+1, 1), fontsize=15, color=foreground)
    plt.yticks(np.arange(-(grid/4), (grid/4)+1, 1), fontsize=15, color=foreground)
    ax.set_xlabel('$t$', color=foreground, fontsize=15)
    ax.legend([f'Exact with $\lambda={lam}$', f'$h={h_values[0]}$', f'$h={h_values[1]}$',

    relative_path = os.path.join('..', '..', 'Graphs', 'Stability Magnitude', 'Exponential De
    plt.savefig(relative_path, bbox_inches='tight', pad_inches=0)

    plt.show()
```

```
# Euler's Forward
#plot_exponential_function(domain = 'real', grid = 10, lam = -5, method = EF, h_values = [0.02

# Euler's Backward
#plot_exponential_function(domain = 'real', grid = 10, lam = -5, method = EB, h_values = [0.02

# Runge-Kutta 4
plot_exponential_function(domain = 'real', grid = 10, lam = -5, method = RK4, h_values = [0.02
```

6.3.7 Finding h for $|s_{RK4}| = \lambda h$ when given λ

Found at '/Python/Stability Magnitude/Rk4 = c.py' in the repository [3].

```
import numpy as np
from scipy.optimize import fsolve
import math

def objective_function(x, c):
    """
    Objective function to be solved for |sum| = c
    """
    sum_expression = np.sum([(-5*x)**n / math.factorial(n) for n in range(5)])
    return np.abs(sum_expression) - c

def find_solutions(c, initial_guess=0.0):
    """
    Find the roots of the objective function for the given constant c.

    Parameters:
    c (float): The constant value to solve for
    initial_guess (float): Initial guess for the root-finding algorithm

    Returns:
    numpy.ndarray: Array of solutions (x values) for |sum| = c
    """
    solutions = fsolve(objective_function, initial_guess, args=(c,))
    return solutions

# Example usage
c = 0.9
solutions = find_solutions(c)
print(f"Solutions for |sum| = {c}:")
print(solutions)
```

6.3.8 Real VS Complex Stability Regions

Found at '/Python/Stability Regions/Graphs/Real vs Complex Comparison.py' in the repository [3].

```
import os
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
from matplotlib.ticker import FuncFormatter as formatter
import os

cmap = mpl.colormaps['magma_r']
# Take colors at regular intervals spanning the colormap
colors = cmap(np.linspace(0, 1, 10))

# Declare Steps
z_1 = lambda l, a, b: (a + b*1j)*l
z_2 = lambda l, a, b: ((1-a) - b*1j)*l

# Stability Function for Euler's Forward 2-step
EF = ["Euler's Forward", lambda l, a, b: (1 + z_1(l, a, b)) * (1 + z_2(l, a, b))]

EB = ["Euler's Backward", lambda l, a, b: 1/(1 - z_1(l, a, b)) * 1/(1 - z_2(l, a, b))]

rk4_poly = lambda x: 1 + x + (x**2)/2 + (x**3)/6 + (x**4)/24
RK4 = ["Runge-Kutta 4", lambda l, a, b: rk4_poly(z_1(l, a, b)) * rk4_poly(z_2(l, a, b))]


# Plot the Stability Region for a given Stability Function
def plot_Region(stability_function, grid, points):

    # Generate domain (complex grid)
    grid_scale = grid / 10
    x = grid / 2
    real_domain = np.linspace(-x, x, points)
    imag_domain = np.linspace(-x, x, points)
    real, imag = np.meshgrid(real_domain, imag_domain)
    domain = real + 1j * imag

    # Evaluate Stability Region
    name = stability_function[0]
    function = stability_function[1]
    # Real
    a_r = 0.5
    b_r = 0
    realRange = function(domain, a_r, b_r)
    realDistance = np.abs(realRange)
    # Complex
    a_c = 0.5
    b_c = 0.5
    complexRange = function(domain, a_c, b_c)
    complexDistance = np.abs(complexRange)

    # Define the Figure
    plt.figure(figsize=(grid, grid), facecolor='white', edgecolor='black')
    ax = plt.gca()

    # Plot the Stability Region
    # Real
    plt.contour(real, imag, realDistance, levels=[1], colors=[colors[3]], linewidths=1)  # Bo
    plt.contourf(real, imag, realDistance, levels=[0, 1], colors=[colors[3]], alpha=0.75)  #
    # Complex
    plt.contour(real, imag, complexDistance, levels=[1], colors=[colors[6]], linewidths=1)  #
    plt.contourf(real, imag, complexDistance, levels=[0, 1], colors=[colors[6]], alpha=0.75)
```

```
    # Customize Imaginary axis tick labels to include 'i'
    def im_axis(x, pos):
        return f'{x:.0f}i' if x != 0 else '0'
    ax.yaxis.set_major_formatter(formatter(im_axis))
    ax.set_facecolor('white')

    # Plot bold lines for Real and Imaginary axes
    plt.axhline(0, color='black', linewidth=0.75)  # Imaginary axis
    plt.axvline(0, color='black', linewidth=0.75)  # Real axis

    # Set major grid lines for both Real and Imaginary axes
    plt.grid(which='major', linestyle='-', linewidth='0.5', color='black')
    ax.set_box_aspect(1)
    ax.set_xlim(-x, x)
    ax.set_ylim(-x, x)
    plt.xticks(np.arange(-int(x), int(x) + 1, 1*grid_scale), fontsize=15*grid_scale, color='bl
    plt.yticks(np.arange(-int(x), int(x) + 1, 1*grid_scale), fontsize=15*grid_scale, color='bl

    # Remove tick marks
    plt.tick_params(axis='both', which='both', length=0, pad=10*grid_scale)

    # Set border (spines) linewidth
    for spine in ax.spines.values():
        spine.set_linewidth(0.75)
        spine.set_color('black')

    # Labelling
    plt.xlabel('$Real$', fontsize=15*grid_scale, color='black')
    plt.ylabel('$Imaginary$', fontsize=15*grid_scale, color='black')

    # Legend with "Real" and "Complex" labels in colors matching the plot
    plt.legend(labels=['Real', 'Complex'], labelcolor=[colors[3], colors[6]], fontsize=15*grid

    relative_path = os.path.join('..', '..', 'Graphs', 'Stability-Regions', 'Graphs', 'Real-V
    plt.savefig(relative_path, bbox_inches='tight', pad_inches=0)

    plt.show()

plot_Region(EB, 20, 2**13)
```

6.3.9 Video Frame Generation

Found at '/Python/Stability Regions/Videos/Complex Pairs Video Frames.py' in the repository [3].

```
import os
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
from matplotlib.ticker import FuncFormatter as formatter

cmap = mpl.colormaps['magma_r']
# Take colors at regular intervals spanning the colormap
colors = cmap(np.linspace(0, 1, 10))

# Declare Steps
z_1 = lambda l, a, b: a*l + b*1j
z_2 = lambda l, a, b: (1-a)*l - b*1j

# Stability Function for Euler's Forward 2-step
EF = ["Euler's Forward", lambda l, a, b: (1 + z_1(l, a, b)) * (1 + z_2(l, a, b))]

EB = ["Euler's Backward", lambda l, a, b: 1/(1 - z_1(l, a, b)) * 1/(1 - z_2(l, a, b))]

rk4_poly = lambda x: 1 + x + (x**2)/2 + (x**3)/6 + (x**4)/24
RK4 = ["Runge-Kutta 4", lambda l, a, b: rk4_poly(z_1(l, a, b)) * rk4_poly(z_2(l, a, b))]

# Define an array of a values
#varied_a = [[0.537], 0.537] #testing
varied_a = [np.arange(0, 1, 0.001), 0.5] # [a_values, b] where 0<a<1 and b is constant
varied_b = [0.5, np.arange(0, 4, 0.005)] # [a, b_values] where a is constant and 0<b<4

# Plot the Stability Region for a given Stability Function
def plot_Region(stability_function, grid, points, aORb, frame_number):

    # Unpack the stability function and varied values for a or b
    if aORb == 'a':
        a_values = varied_a[0]
        b = varied_a[1]
        a = a_values[frame_number]
        subdir = 'Varied a'
        const = 'b'
        const_val = b
    elif aORb == 'b':
        a = varied_b[0]
        b_values = varied_b[1]
        b = b_values[frame_number]
        subdir = 'Varied b'
        const = 'a'
        const_val = a
    else:
        raise ValueError('Invalid input for aORb')

    # Generate domain (complex grid)
    grid_scale = grid / 10
    x = grid / 2
    real_domain = np.linspace(-x, x, points)
    imag_domain = np.linspace(-x, x, points)
    real, imag = np.meshgrid(real_domain, imag_domain)
    domain = real + 1j * imag

    # Evaluate Stability Region
    name = stability_function[0]
    function = stability_function[1]
    Range = function(domain, a, b)
    distance = np.abs(Range)
```

```
    # Define the Figure
    plt.figure(figsize=(grid, grid), facecolor='white', edgecolor='black')
    ax = plt.gca()

    # Plot the Stability Region
    plt.contour(real, imag, distance, levels=[1], colors=[colors[3]], linewidths=1)  # Bounda
    plt.contourf(real, imag, distance, levels=[0, 1], colors=[colors[3]], alpha=0.75)  # Fill

    # Customize Imaginary axis tick labels to include 'i'
    def im_axis(x, pos):
        return f'{x:.0f}i' if x != 0 else '0'
    ax.yaxis.set_major_formatter(formatter(im_axis))
    ax.set_facecolor('white')

    # Plot bold lines for Real and Imaginary axes
    plt.axhline(0, color='black', linewidth=0.75)  # Imaginary axis
    plt.axvline(0, color='black', linewidth=0.75)  # Real axis

    # Set major grid lines for both Real and Imaginary axes
    plt.grid(which='major', linestyle='-', linewidth='0.5', color='black')
    ax.set_box_aspect(1)
    ax.set_xlim(-x, x)
    ax.set_ylim(-x, x)
    plt.xticks(np.arange(-int(x), int(x) + 1, 1*grid_scale), fontsize=15*grid_scale, color='bl
    plt.yticks(np.arange(-int(x), int(x) + 1, 1*grid_scale), fontsize=15*grid_scale, color='bl

    # Remove tick marks
    plt.tick_params(axis='both', which='both', length=0, pad=10*grid_scale)

    # Set border (spines) linewidth
    for spine in ax.spines.values():
        spine.set_linewidth(0.75)
        spine.set_color('black')

    # Labelling
    plt.xlabel('$Real$', fontsize=15*grid_scale, color='black')
    plt.ylabel('$Imaginary$', fontsize=15*grid_scale, color='black')

    # a & b values legend
    plt.legend([f'a = {a:.3f}\nb = {b:.3f}'], loc='upper right', fontsize=15*grid_scale, faced


    # Save plot as a frame
    relative_path = os.path.join('..', '..', '..', 'Graphs', 'Stability Regions', 'Videos')
    video_path = os.path.join(subdir, name, f"{const}={const_val}", 'frames')
    output_path = os.path.join(relative_path, video_path)
    if not os.path.exists(output_path):
        os.makedirs(output_path)
    frame_path = f'{frame_number:04d}.png'
    path = os.path.join(output_path, frame_path)
    plt.savefig(path, bbox_inches='tight', pad_inches=0)

    output_dir = f'/home/puca/University/Senior Sophister/Capstone/Graphs/Stability Regions/V
    if not os.path.exists(output_dir):
        os.makedirs(output_dir)
    plt.savefig(f'{output_dir}/{frame_number:04d}.png')
    #plt.show()
    plt.close()

# Generate frames for Stability Function with multiple a or b values
for i, b in enumerate(varied_b[1]):
    plot_Region(RK4, 40, 1000, 'b', i)
```

6.3.10 Frame Stitching for Video

Found at '/Python/Stability Regions/Videos/Frames-to-Video.py' in the repository [3].

```
import os
import imageio.v2 as imageio

EF = "Euler's Forward"
EB = "Euler's Backward"
RK4 = "Runge–Kutta 4"

method = RK4
varied = "Varied b"
const = "a=0.5"

# Define the base path using relative paths
base_path = os.path.join('..', '..', '..', 'Graphs', 'Stability Regions', 'Videos')

# Define the frames directory and the output video path using relative paths
frames_dir = os.path.join(base_path, varied, method, const, 'frames')
output_video_path = os.path.join(base_path, varied, method, const, 'video.mp4')

# Ensure the frames directory exists
if not os.path.exists(frames_dir):
    raise ValueError(f"The frames directory {frames_dir} does not exist.")

# Create the video writer
with imageio.get_writer(output_video_path, fps=30) as writer:
    # Iterate through the frames in the directory
    for frame_number in sorted(os.listdir(frames_dir)):
        # Construct the full path to the frame
        frame_path = os.path.join(frames_dir, frame_number)
        # Read and append the frame to the video
        image = imageio.imread(frame_path)
        writer.append_data(image)

print(f"Video created successfully at {output_video_path}")
```