

# General Relativity

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## 1 Lecture: Introduction

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General Relativity is our best theory of gravitation on the largest scales. It is classical, geometrical and dynamical.

### 1.1 Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which ‘locally looks like  $\mathbb{R}^n$ ’, and has enough structure to let us do calculus.

**Definition 1.1:** A **differentiable manifold** of dimension  $n$  is a set  $M$ , together with a collection of coordinate charts  $(O_\alpha, \phi_\alpha)$  where

- $O_\alpha \in M$  are subsets of  $M$  such that  $\cup_\alpha O_\alpha = M$ ,
- $\phi_\alpha$  is a bijective map (one to one and onto) from  $O_\alpha \rightarrow U_\alpha$ , an open subset of  $\mathbb{R}^n$ ,
- If  $O_\alpha \cap O_\beta \neq \emptyset$ , then  $\phi_\beta \circ \phi_\alpha^{-1}$  is a smooth (infinitely differentiable) map from  $\phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha$  to  $\phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta$ .

**Note.** We could replace smooth with finite differentiability (*e.g.*  $k$ -differentiable) but it is not particularly interesting.

Further, these charts define a topology of  $M$ ,  $\mathcal{R} \subset M$  is open iff  $\phi_\alpha(\mathcal{R} \cap O_\alpha)$  is open in  $\mathbb{R}^n$  for all  $\alpha$ .

Every open subset of  $M$  is itself a manifold (restrict charts to  $\mathcal{R}$ ).

**Definition 1.2:** The collection  $\{(O_\alpha, \phi_\alpha)\}$  is called an **atlas**. Two atlases are **compatible** if their union is an atlas. An atlas  $A$  is **maximal** if there exists no atlas  $B$  with  $A \subsetneq B$ .

Every atlas is contained in a maximal atlas (consider the union of all compatible atlases). We can assume without loss of generality we are working with the maximal atlas.

**Examples.**

i) If  $U \subset \mathbb{R}^n$  is open, we can take  $O = U$  and

$$\phi : O \rightarrow U \quad (1)$$

$$\phi(x^i) = x^i, \quad (2)$$

and  $\{(U, \phi)\}$  is an atlas.

ii)  $S^1 = \{\mathbf{p} \in \mathbb{R}^2 \mid |\mathbf{p}| = 1\}$ . If  $\mathbf{p} \in S^1 \setminus \{(-1, 0)\} = \mathcal{O}_1$ , there is a unique  $\theta_1 \in (-\pi, \pi)$  such that  $\mathbf{p} = (\cos \theta_1, \sin \theta_1)$ .

If  $\mathbf{p} \in S^1 \setminus \{(1, 0)\} = \mathcal{O}_2$ , then there is a unique  $\theta_2 \in (0, 2\pi)$  such that  $\mathbf{p} = (\cos \theta_2, \sin \theta_2)$  such that

$$\phi_1 : \mathbf{p} \rightarrow \theta_1, \text{ for } \mathbf{p} \in \mathcal{O}_1, U_1 = (-\pi, \pi), \quad (3)$$

$$\phi_2 : \mathbf{p} \rightarrow \theta_2, \text{ for } \mathbf{p} \in \mathcal{O}_2, U_2 = (0, 2\pi). \quad (4)$$

We have that  $\phi_1(\mathcal{O} \cap \mathcal{O}_2) = (-\pi, 0) \cup (0, \pi)$  and

$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta, & \theta \in (0, \pi), \\ \theta + 2\pi, & \theta \in (-\pi, 0). \end{cases} \quad (5)$$

This is smooth where defined and similarly for  $\phi_1 \circ \phi_2^{-1}$  and thus  $S_1$  is a 1-manifold.

iii)  $S^n = \{\mathbf{p} \in \mathbb{R}^{n+1} \mid |\mathbf{p}| = 1\}$ . We define charts by stereographic projection if  $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$  is a standard basis for  $\mathbb{R}^{n+1}$  and  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is a standard basis for  $\mathbb{R}^n$ , we write

$$\mathbf{p} = p^i \mathbf{e}_i. \quad (6)$$

We set  $\mathcal{O}_1 = S^n \setminus \{E_{n+1}\}$  and

$$\phi_1(\mathbf{p}) = \frac{1}{1 - p^{n+1}} (p^i \mathbf{e}_i), \quad (7)$$

and  $\mathcal{O}_2 = S^n \setminus \{-E_{n+1}\}$  such that

$$\phi_2(\mathbf{p}) = \frac{1}{1 + p^{n+1}} (p^i \mathbf{e}_i). \quad (8)$$

We have  $\phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) = \mathbb{R}^n \setminus \{0\}$  and  $\phi_2 \circ \phi_1^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^2}$ .

**Proof.** Take  $\mathbf{x} \in \phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) \subset \mathbb{R}^n$ . We have that  $\phi_1^{-1}(\mathbf{x}) = \frac{1}{1+x_j x^j} (2x^i, x^j x_j - 1)$  which satisfies  $|\phi_1^{-1}(\mathbf{x})| = 1$  and is an inverse as

$$\phi_1 \circ \phi_1^{-1}(x_i) = \frac{1}{1 - \frac{x^j x_j - 1}{1+x_j x^j}} \frac{2x^i}{1+x_j x^j} \quad (9)$$

$$= \frac{1+x_j x^j}{1+x_j x^j - (x^j x_j - 1)} \frac{2x^i}{1+x_j x^j} \quad (10)$$

$$= \frac{1}{2} 2x^i = x^i. \quad (11)$$

Similarly, we have

$$\phi_2 \circ \phi_1^{-1}(x_i) = \frac{1}{1 + \frac{x^j x_j - 1}{1 + x_j x^j}} \frac{2x^i}{1 + x_j x^j} \quad (12)$$

$$= \frac{1 + x_j x^j}{1 + x_j x^j + (x^j x_j - 1)} \frac{2x^i}{1 + x_j x^j} \quad (13)$$

$$= \frac{1}{2x_j x^j} 2x^i = \frac{x^i}{|x|^2}, \quad (14)$$

which is well defined on  $\mathbb{R}^n \setminus \{0\}$  as desired.  $\square$

This is smooth on  $\mathbb{R}^n \setminus \{0\}$  and similarly for  $\phi_1 \circ \phi_2^{-1}$ . Thus  $S^n$  is an  $n$ -manifold.