

# General Relativity

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## 1 Lecture: Introduction

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General Relativity is our best theory of gravitation on the largest scales. It is classical, geometrical and dynamical.

## 1.1 Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which ‘locally looks like  $\mathbb{R}^n$ ’, and has enough structure to let us do calculus.

**Definition 1.1:** A **differentiable manifold** of dimension  $n$  is a set  $M$ , together with a collection of coordinate charts  $(O_\alpha, \phi_\alpha)$  where

- $O_\alpha \subset M$  are subsets of  $M$  such that  $\cup_\alpha O_\alpha = M$ ,
- $\phi_\alpha$  is a bijective map (one to one and onto) from  $O_\alpha \rightarrow U_\alpha$ , an open subset of  $\mathbb{R}^n$ ,
- If  $O_\alpha \cap O_\beta \neq \emptyset$ , then  $\phi_\beta \circ \phi_\alpha^{-1}$  is a smooth (infinitely differentiable) map from  $\phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha$  to  $\phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta$ .

**Note.** We could replace smooth with finite differentiability (*e.g.*  $k$ -differentiable) but it is not particularly interesting.

Further, these charts define a topology of  $M$ ,  $\mathcal{R} \subset M$  is open iff  $\phi_\alpha(\mathcal{R} \cap O_\alpha)$  is open in  $\mathbb{R}^n$  for all  $\alpha$ .

Every open subset of  $M$  is itself a manifold (restrict charts to  $\mathcal{R}$ ).

**Definition 1.2:** The collection  $\{(O_\alpha, \phi_\alpha)\}$  is called an **atlas**. Two atlases are **compatible** if their union is an atlas. An atlas  $A$  is **maximal** if there exists no atlas  $B$  with  $A \subsetneq B$ .

Every atlas is contained in a maximal atlas (consider the union of all compatible atlases). We can assume without loss of generality we are working with the maximal atlas.

**Examples.**

- i) If  $U \subset \mathbb{R}^n$  is open, we can take  $O = U$  and

$$\phi : O \rightarrow U \tag{1}$$

$$\phi(x^i) = x^i, \tag{2}$$

and  $\{(U, \phi)\}$  is an atlas.

- ii)  $S^1 = \{\mathbf{p} \in \mathbb{R}^2 \mid |\mathbf{p}| = 1\}$ . If  $\mathbf{p} \in S^1 \setminus \{(-1, 0)\} = \mathcal{O}_1$ , there is a unique  $\theta_1 \in (-\pi, \pi)$  such that  $\mathbf{p} = (\cos \theta_1, \sin \theta_1)$ .

If  $\mathbf{p} \in S^1 \setminus \{(1, 0)\} = \mathcal{O}_2$ , then there is a unique  $\theta_2 \in (0, 2\pi)$  such that  $\mathbf{p} = (\cos \theta_2, \sin \theta_2)$  such that

$$\phi_1 : \mathbf{p} \rightarrow \theta_1, \text{ for } \mathbf{p} \in \mathcal{O}_1, U_1 = (-\pi, \pi), \tag{3}$$

$$\phi_2 : \mathbf{p} \rightarrow \theta_2, \text{ for } \mathbf{p} \in \mathcal{O}_2, U_2 = (0, 2\pi). \tag{4}$$

We have that  $\phi_1(\mathcal{O} \cap \mathcal{O}_2) = (-\pi, 0) \cup (0, \pi)$  and

$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta, & \theta \in (0, \pi), \\ \theta + 2\pi, & \theta \in (-\pi, 0). \end{cases} \tag{5}$$

This is smooth where defined and similarly for  $\phi_1 \circ \phi_2^{-1}$  and thus  $S_1$  is a 1-manifold.

iii)  $S^n = \{\mathbf{p} \in \mathbb{R}^{n+1} \mid |\mathbf{p}| = 1\}$ . We define charts by stereographic projection if  $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$  is a standard basis for  $\mathbb{R}^{n+1}$  and  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is a standard basis for  $\mathbb{R}^n$ , we write

$$\mathbf{p} = p^i \mathbf{e}_i. \quad (6)$$

We set  $\mathcal{O}_1 = S^n \setminus \{E_{n+1}\}$  and

$$\phi_1(\mathbf{p}) = \frac{1}{1 - p^{n+1}} (p^i \mathbf{e}_i), \quad (7)$$

and  $\mathcal{O}_2 = S^n \setminus \{-E_{n+1}\}$  such that

$$\phi_2(\mathbf{p}) = \frac{1}{1 + p^{n+1}} (p^i \mathbf{e}_i). \quad (8)$$

We have  $\phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) = \mathbb{R}^n \setminus \{0\}$  and  $\phi_2 \circ \phi_1^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^2}$ .

**Proof.** Take  $\mathbf{x} \in \phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) \subset \mathbb{R}^n$ . We have that  $\phi_1^{-1}(\mathbf{x}) = \frac{1}{1+x_j x^j} (2x^i, x^j x_j - 1)$  which satisfies  $|\phi_1^{-1}(\mathbf{x})| = 1$  and is an inverse as

$$\phi_1 \circ \phi_1^{-1}(x_i) = \frac{1}{1 - \frac{x^j x_j - 1}{1+x_j x^j}} \frac{2x^i}{1+x_j x^j} \quad (9)$$

$$= \frac{1+x_j x^j}{1+x_j x^j - (x^j x_j - 1)} \frac{2x^i}{1+x_j x^j} \quad (10)$$

$$= \frac{1}{2} 2x^i = x^i. \quad (11)$$

Similarly, we have

$$\phi_2 \circ \phi_1^{-1}(x_i) = \frac{1}{1 + \frac{x^j x_j - 1}{1+x_j x^j}} \frac{2x^i}{1+x_j x^j} \quad (12)$$

$$= \frac{1+x_j x^j}{1+x_j x^j + (x^j x_j - 1)} \frac{2x^i}{1+x_j x^j} \quad (13)$$

$$= \frac{1}{2x_j x^j} 2x^i = \frac{x^i}{|\mathbf{x}|^2}, \quad (14)$$

which is well defined on  $\mathbb{R}^n \setminus \{0\}$  as desired.  $\square$

This is smooth on  $\mathbb{R}^n \setminus \{0\}$  and similarly for  $\phi_1 \circ \phi_2^{-1}$ . Thus  $S^n$  is an  $n$ -manifold.

## 2 Lecture: Smooth Functions on Manifolds

14/10/2024

### 2.1 Smooth Functions

Suppose  $M, N$  are manifolds of  $\dim n, n'$  respectively. Let  $f : M \rightarrow N$  and  $p \in M$ . We pick charts  $(\mathcal{O}_\alpha, \phi_\alpha)$  for  $M$  and  $(\mathcal{O}'_\beta, \phi'_\beta)$  for  $N$  with  $p \in \mathcal{O}_\alpha$  and  $f(p) \in \mathcal{O}'_\beta$ .

Then  $\phi'_\beta \circ f \circ \phi_\alpha^{-1}$  maps an open neighbourhood of  $\phi_\alpha(p)$  in  $U_\alpha \subset \mathbb{R}^n$  to  $U'_\beta \subset \mathbb{R}^{n'}$ .

**Definition 2.1:** If  $\phi'_\beta \circ f \circ \phi_\alpha^{-1} : (U_\alpha \subset \mathbb{R}^n) \rightarrow (U'_\beta \subset \mathbb{R}^{n'})$  is smooth for all possible choices of charts, we say  $f : M \rightarrow N$  is **smooth**.

**Note.** A smooth map  $\Psi : M \rightarrow N$  which has a smooth inverse  $\Psi^{-1}$  is called a **diffeomorphism** and this implies  $n = n'$ .

Also, if  $N = \mathbb{R}$  or  $\mathbb{C}$ , we sometimes call  $f$  a **scalar field**. Further if  $M$  is an (open) interval such that  $M = I \subset \mathbb{R}$ , then  $f : I \rightarrow N$  is a smooth curve in  $N$ .

Lastly, if  $f$  is smooth in one atlas, it is smooth with respect to all compatible atlases.

### Examples.

- 1) Recall  $S^1 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = 1\}$ . Let  $f(x, y) = x$ ,  $f : S^1 \rightarrow \mathbb{R}$ .

Using previous charts,

$$f \circ \phi_1^{-1} : (-\pi, \pi) \rightarrow \mathbb{R} \quad (15)$$

$$f \circ \phi_1^{-1}(\theta_1) = \cos \theta_1, \quad (16)$$

and similarly,

$$f \circ \phi_2^{-1} : (0, 2\pi) \rightarrow \mathbb{R} \quad (17)$$

$$f \circ \phi_2^{-1}(\theta_2) = \cos \theta_2. \quad (18)$$

In both cases,  $f$  is smooth.

- 2) If  $(\mathcal{O}, \phi)$  is a coordinate chart on  $M$ , write for  $\mathbf{p} \in \mathcal{O}$ ,

$$\phi(\mathbf{p}) = (x^1(\mathbf{p}), x^2(\mathbf{p}), \dots, x^n(\mathbf{p})), \quad (19)$$

then  $x^i(\mathbf{p})$  defines a map from  $\mathcal{O}$  to  $\mathbb{R}$ . This is a smooth map for each  $i = 1, \dots, n$ . If  $(\mathcal{O}', \phi')$  is another overlapping coordinate chart, then  $x^i \circ \phi'^{-1}$  is the  $i$ th component of  $\phi \circ \phi'^{-1}$ , which is smooth.

- 3) We can define a smooth function chart by chart. For simplicity, we take  $N = \mathbb{R}$ . Let  $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$  be an atlas on  $M$ . Define smooth functions  $F_\alpha : U_\alpha \rightarrow \mathbb{R}$ , and suppose that

$$F_\alpha \circ \phi_\alpha = F_\beta \circ \phi_\beta, \quad (20)$$

on  $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$  for all  $\alpha, \beta$ . Then for  $\mathbf{p} \in M$ , we can define  $f(\mathbf{p}) = F_\alpha \circ \phi_\alpha(\mathbf{p})$  where  $(\mathcal{O}_\alpha, \phi_\alpha)$  is any chart with  $\mathbf{p} \in \mathcal{O}_\alpha$  as this is constant by construction of  $F$ .  $f$  is smooth as

$$f \circ \phi_\beta^{-1} = F_\alpha \circ \underbrace{\phi_\alpha \circ \phi_\beta^{-1}}_{\text{always smooth}}. \quad (21)$$

In practice, we often don't distinguish between  $f$  and its **coordinate chart representation**  $F_\alpha$ . This coordinate chart representation  $F_\alpha$  captures  $f$  but maps from  $U_\alpha \subset \mathbb{R}^n$  rather than from subsets of  $M$ . One can think of  $F_\alpha = f \circ \phi_\alpha^{-1}$  as finding the point on  $M$  that  $\phi_\alpha$  mapped from and evaluating  $f$  at that point.

## 2.2 Curves and Vectors

For a surface in  $\mathbb{R}^3$ , we have a notion of a tangent space at a point, consisting of all vectors tangent to the surface. Such tangent spaces are vector spaces (copies of  $\mathbb{R}^2$ ). Different points have different tangent spaces.

In order to define the tangent space for a manifold, we first consider tangent vectors of a curve.

Recall that a smooth map from an interval  $\lambda : I \subset \mathbb{R} \rightarrow M$  is a smooth curve in  $M$ .

If  $\lambda(t)$  is a smooth curve in  $\mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function, then for  $f(\lambda(t)) : \mathbb{R} \rightarrow \mathbb{R}$ , the chain rule gives

$$\frac{d}{dt} [f(\lambda(t))] = \mathbf{X}(t) \cdot \nabla f(\lambda(t)), \quad (22)$$

where  $\mathbf{X}(t) = \frac{d\lambda(t)}{dt}$  is the **tangent vector** to  $\lambda$  at  $t$ . The idea is that we identify the tangent vector  $\mathbf{X}(t)$  with the differential operator  $\mathbf{X}(t) \cdot \nabla$ .

**Definition 2.2:** Let  $\lambda : I \rightarrow M$  be a smooth curve with  $\lambda(0) = \mathbf{p}$ . The **tangent vector** to  $\lambda$  at  $\mathbf{p}$  is the linear map  $X_{\mathbf{p}}$  from the space of smooth functions,  $f : M \rightarrow \mathbb{R}$  given by

$$X_{\mathbf{p}}(f) = \left. \frac{d}{dt} f(\lambda(t)) \right|_{t=0}. \quad (23)$$

We observe a few things of note.

- 1)  $X_{\mathbf{p}}$  is linear such that  $X_{\mathbf{p}}(f + ag) = X_{\mathbf{p}}(f) + aX_{\mathbf{p}}(g)$  for  $f, g$  smooth and  $a \in \mathbb{R}$ .
- 2)  $X_{\mathbf{p}}$  satisfies the Leibniz rule,

$$X_{\mathbf{p}}(fg) = (X_{\mathbf{p}}(f))g + fX_{\mathbf{p}}(g). \quad (24)$$

- 3) If  $(\mathcal{O}, \phi)$  is a chart with  $\mathbf{p} \in \mathcal{O}$ , we write

$$\phi(\mathbf{p}) = (x^1(\mathbf{p}), \dots, x^n(\mathbf{p})). \quad (25)$$

Let  $F = f \circ \phi^{-1}$ ,  $x^i(t) = x^i(\lambda(t))$  and  $\mathbf{x}(t) = \phi(\lambda(t))$ . Then we have

$$f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t) = F \circ \mathbf{x}(t), \quad (26)$$

and thus the tangent vector defined in Eq. (23) can also be written as

$$X_{\mathbf{p}}(f) \equiv \left. \frac{d}{dt} (f(\lambda(t))) \right|_{t=0} = \left. \frac{\partial F(x)}{\partial x^\mu} \frac{dx^\mu}{dt} \right|_{t=0}, \quad (27)$$

where  $\frac{\partial F}{\partial x^\mu}$  depends on  $f$  and  $\phi$  and  $\frac{dx^\mu}{dt}$  depends on  $\lambda$  and  $\phi$ .

## 3 Lecture: Tangent Spaces

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### 3.1 The Tangent Space is a Vector Space

**Proposition 3.1:** The set of tangent vectors to curves at  $\mathbf{p}$  forms a vector space,  $T_{\mathbf{p}}M$  of dimension  $n = \dim M$ . We call  $T_{\mathbf{p}}M$ , the **tangent space** to  $M$  at  $\mathbf{p}$ .

**Proof.** Given  $X_{\mathbf{p}}, Y_{\mathbf{p}}$  are tangent vectors, we need to show that  $\alpha X_{\mathbf{p}} + \beta Y_{\mathbf{p}}$  is a tangent vector for  $\alpha, \beta \in \mathbb{R}$ . Let  $\lambda, \kappa$  be smooth curves with  $\lambda(0) = \kappa(0) = \mathbf{p}$  and whose tangent vectors at  $\mathbf{p}$  are  $X_{\mathbf{p}}$  and  $Y_{\mathbf{p}}$  respectively. Let  $(\mathcal{O}, \phi)$  be a chart with  $p \in \mathcal{O}$  such that  $\phi(\mathbf{p}) = 0$ . We call this a *chart centered at  $\mathbf{p}$* .

Let  $\nu(t) = \phi^{-1}[\alpha\phi(\lambda(t)) + \beta\phi(\kappa(t))]$  where notice  $\nu(0) = \phi^{-1}(0) = \mathbf{p}$ .

From Eq. (27), we have that if  $Z_{\mathbf{p}}$  is the tangent to  $\nu$  at  $\mathbf{p}$ , we have

$$Z_{\mathbf{p}}(f) = \left. \frac{d}{dt} (f(\nu(t))) \right|_0 \quad (28)$$

$$= \left. \frac{\partial F}{\partial x^\mu} \right|_0 \frac{d}{dt} [\alpha x^\mu(\lambda(t)) + \beta x^\mu(\kappa(t))] \Big|_{t=0} \quad (29)$$

$$= \alpha \left. \frac{\partial F}{\partial x^\mu} \right|_0 \frac{d}{dt} x^\mu(\lambda(t)) \Big|_{t=0} + \beta \left. \frac{\partial F}{\partial x^\mu} \right|_0 \frac{d}{dt} x^\mu(\kappa(t)) \Big|_{t=0} \quad (30)$$

$$= \alpha X_{\mathbf{p}}(f) + \beta Y_{\mathbf{p}}(f), \quad (31)$$

as desired. Therefore  $T_{\mathbf{p}}M$  is a vector space.  $\square$

To see that  $T_{\mathbf{p}}M$  is  $n$ -dimensional, consider the curves

$$\lambda_\mu(t) = \phi^{-1} \left( 0, \dots, 0, \underbrace{t}_{\mu\text{th component}}, 0, \dots, 0 \right). \quad (32)$$

We denote the tangent vector to  $\lambda_\mu$  at  $\mathbf{p}$  by  $\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}}$ .

**Note.** This is **not** a differential operator.

However observe that by definition, we have

$$\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} (f) = \left. \frac{\partial F}{\partial x^\mu} \right|_{\phi(\mathbf{p})=0}, \quad (33)$$

and thus it acts like a differential operator in  $\mathbb{R}^n$  on the coordinates of the chart.

The vectors  $\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}}$  are linearly independent. Otherwise  $\exists \alpha^\mu \in \mathbb{R}$  not all zero such that

$$\alpha^\mu \left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} = 0, \quad (34)$$

which implies

$$\alpha^\mu \frac{\partial F}{\partial x^\mu} = 0, \quad (35)$$

for all  $F$ . Setting  $F = x^\nu$  gives  $\alpha^\nu = 0$  and thus these vectors are linearly independent and spanning. They are therefore a basis for the vector space.

Further one can see that  $\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}}$  form a basis for  $T_{\mathbf{p}}M$ , since if  $\lambda$  is any curve with tangent  $X_{\mathbf{p}}$  at  $\mathbf{p}$ , we have

$$X_{\mathbf{p}}(f) = \left. \frac{\partial F}{\partial x^\mu} \right|_{x=0} \frac{d}{dt} x^\mu(\lambda(t)) = X^\mu \left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} (f), \quad (36)$$

where  $X^\mu = \left. \frac{d}{dt} x^\mu(\lambda(t)) \right|_{t=0}$  are the **components** of  $X_{\mathbf{p}}$  with respect to the basis  $\left\{ \left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} \right\}_{\mu=1, \dots, n}$  for  $T_{\mathbf{p}}M$ .

**Note.** The basis  $\left\{ \left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} \right\}_{\mu=1, \dots, n}$  depends on the coordinate chart  $\phi$ .

Suppose we choose another chart  $(\mathcal{O}', \phi')$ , again centered at  $\mathbf{p}$ . We write  $\phi' = \left( (x')^1, \dots, (x')^n \right)$ .

Then if  $F' = f \circ \phi'^{-1}$ , we have

$$F(x) = f \circ \phi^{-1}(x) \quad (37)$$

$$= f \circ \phi'^{-1} \circ \phi' \circ \phi^{-1}(x) \quad (38)$$

$$= F'(x'(x)). \quad (39)$$

Therefore,

$$\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} (f) = \left. \frac{\partial F}{\partial x^\mu} \right|_{\phi(\mathbf{p})} \quad (40)$$

$$= \left( \frac{\partial x'^\nu}{\partial x^\mu} \right) \left. \left( \frac{\partial F'}{\partial x'^\nu} \right) \right|_{\phi'(\mathbf{p})} \quad (41)$$

$$= \left( \frac{\partial x'^\nu}{\partial x^\mu} \right) \left. \left( \frac{\partial}{\partial x'^\nu} \right)_{\mathbf{p}} (f) \right|_{\phi(\mathbf{p})}. \quad (42)$$

We then deduce that

$$\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} = \left( \frac{\partial x'^\nu}{\partial x^\mu} \right) \left. \left( \frac{\partial}{\partial x'^\nu} \right)_{\mathbf{p}} \right|_{\phi(\mathbf{p})}. \quad (43)$$

Let  $X^\mu$  be components of  $X_{\mathbf{p}}$  with respect to the basis  $\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}}$ , and  $X'^\mu$  be components of  $X_{\mathbf{p}}$  with respect to the basis  $\left( \frac{\partial}{\partial x'^\mu} \right)_{\mathbf{p}}$  such that

$$X_{\mathbf{p}} = X^\mu \left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} = X'^\mu \left( \frac{\partial}{\partial x'^\mu} \right)_{\mathbf{p}} \quad (44)$$

$$= X^\mu \left( \frac{\partial x'^\sigma}{\partial x^\mu} \right) \left( \frac{\partial}{\partial x'^\sigma} \right)_{\mathbf{p}}, \quad (45)$$

and therefore

$$X'^\mu = \left( \frac{\partial x'^\mu}{\partial x^\nu} \right) X^\nu. \quad (46)$$

**Note.** We do not have to choose a coordinate basis such as  $\left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}}$ . With respect to a general basis  $\{e_\mu\}$ , for  $T_{\mathbf{p}}M$ , we can write  $X_{\mathbf{p}} = X^\mu e_\mu$  for  $X^\mu \in \mathbb{R}$ .

We always use summation convention, contracting covariant indices with contravariant indices.

### 3.2 Covectors

Recall that if  $V$  is a vector space over  $\mathbb{R}$ , the dual space  $V^*$  is the space of linear maps  $\phi : V \rightarrow \mathbb{R}$ . If  $V$  is  $n$ -dimensional then so is  $V^*$  (the spaces are then isomorphic). Given a basis  $\{e_\mu\}$  for  $V$ , we can define the dual basis  $\{f^\mu\}$  for  $V^*$  by requiring that

$$f^\mu(e_\nu) = \delta^\mu_\nu = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu. \end{cases} \quad (47)$$

If  $V$  is finite dimensional, then  $V^{**} = (V^*)^* \simeq V$ . Namely, to an element  $X \in V$ , we assign the linear map

$$\Lambda_X : V^* \rightarrow \mathbb{R}, \quad (48)$$

$$\Lambda_X(\omega) = \omega(X), \quad (49)$$

for  $\omega \in V^*$ .

**Definition 3.1:** The dual space of  $T_{\mathbf{p}}M$  is denoted  $T_{\mathbf{p}}^*M$  and is called the **cotangent space** to  $M$  at  $\mathbf{p}$ . An element of this space is a **covector** at  $\mathbf{p}$ . If  $\{e_\mu\}$  is a basis for  $T_{\mathbf{p}}M$  and  $\{f^\mu\}$  is the dual basis for  $T_{\mathbf{p}}^*M$ , we can expand a covector  $\eta$  as

$$\eta = \eta_\mu f^\mu, \quad (50)$$

for **components**  $\eta_\mu \in \mathbb{R}$ .

## 4 Lecture: Tensors

18/10/2024

### 4.1 Tangent bundle

Notice that

$$\eta(e_\nu) = \eta_\mu f^\mu(e_\nu) = \eta_\mu \delta^\mu_\nu = \eta_\nu, \quad (51)$$

and thus we can get the components of  $\eta$  by acting it on basis vectors in the tangent space. Further as we have  $X = X^\mu e_\mu$ ,

$$\eta(X) = \eta(X^\mu e_\mu) \quad (52)$$

$$= X^\mu \eta(e_\mu) \quad (53)$$

$$= X^\mu \eta_\mu, \quad (54)$$

and thus the action of the covector  $\eta$  on the vector  $X$  is essentially a contraction between the components.

Recall that a vector  $X$  is defined by its action on a function  $f$ ,  $X : f \rightarrow \mathbb{R}$ , eating a smooth function and returning the rate of change as one moves in the direction of  $X$ .

Analogously, given a function  $f$ , one can consider a linear operator of that function being eaten by a generic vector  $X$ .



**Definition 4.1:** If  $f : M \rightarrow \mathbb{R}$  is a smooth function, then we can define a covector  $(df)_{\mathbf{p}} \in T_{\mathbf{p}}^*M$ , the **differential** of  $f$  at  $\mathbf{p}$ , by

$$(df)_{\mathbf{p}}(X) = X(f), \quad (55)$$

for any  $X \in T_{\mathbf{p}}M$ . This is also sometimes called the **gradient** of  $f$  at  $\mathbf{p}$ .

If  $f$  is constant,  $X(f) = 0$  which implies  $(df)_{\mathbf{p}} = 0$ .

If  $(\mathcal{O}, \phi)$  is a coordination chart with  $\mathbf{p} \in \mathcal{O}$  and  $\phi = (x^1, \dots, x^n)$  then we can set  $f = x^\mu$  to find  $(dx^\mu)_{\mathbf{p}}$ . Observe

$$(dx^\mu)_{\mathbf{p}} \left( \frac{\partial}{\partial x^\nu} \right)_{\mathbf{p}} = \left( \frac{\partial x^\mu}{\partial x^\nu} \right)_{\phi(\mathbf{p})} = \delta^\mu_\nu. \quad (56)$$

Therefore the coordinate differentials  $\{(dx^\mu)_{\mathbf{p}}\}$  is the dual basis to  $\{(\frac{\partial}{\partial x^\mu})_{\mathbf{p}}\}$ .

In this basis, we can compute

$$\left[ (df)_{\mathbf{p}} \right]_\mu = (df)_{\mathbf{p}} \left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} = \left( \frac{\partial}{\partial x^\mu} \right)_{\mathbf{p}} f = \left( \frac{\partial F}{\partial x^\mu} \right)_{\phi(\mathbf{p})}. \quad (57)$$

This justifies the language of *gradient*.

**Exercise 1:** Show that if  $(\mathcal{O}', \phi')$  is another chart with  $\mathbf{p} \in \mathcal{O}'$ , then

$$(dx^\mu)_{\mathbf{p}} = \left( \frac{\partial x^\mu}{\partial (x')^\nu} \right)_{\phi'(\mathbf{p})} (d(x')^\nu)_{\mathbf{p}}, \quad (58)$$

where  $x(x') = \phi \circ (\phi')^{-1}$ , and hence if  $\eta_\mu, \eta'_\mu$  are components with respect to these bases,

$$\eta'_\mu = \left( \frac{\partial x^\nu}{\partial (x')^\mu} \right)_{\phi'(\mathbf{p})} \eta_\nu. \quad (59)$$

**Proof.**

□

**Definition 4.2 (Tangent bundle):** We can glue together the tangent spaces  $T_{\mathbf{p}}M$  as  $\mathbf{p}$  varies to get a new  $2n$  dimensional manifold  $TM$ , the **tangent bundle**. We define the tangent bundle to be

$$TM := \bigcup_{\mathbf{p} \in M} \{\mathbf{p}\} \times T_{\mathbf{p}}M. \quad (60)$$

Namely, it is the set of ordered pairs  $(\mathbf{p}, X)$ , with  $\mathbf{p} \in M$ ,  $X \in T_{\mathbf{p}}M$ .

If  $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$  is an atlas on  $M$ , we obtain an atlas for  $TM$  by setting

$$\mathcal{O}_\alpha = \bigcup_{\mathbf{p} \in \mathcal{O}_\alpha} \{\mathbf{p}\} \times T_{\mathbf{p}}M, \quad (61)$$

and

$$\tilde{\phi}_\alpha((p, X)) = (\phi(\mathbf{p}), X^\mu) \in \mathcal{U}_\alpha \times \mathbb{R}^n = \tilde{\mathcal{U}}_2, \quad (62)$$

where  $X^\mu$  are the components of  $X$  with respect to the coordinate basis of  $\phi_\alpha$ .

**Exercise 2:** If  $(\mathcal{O}, \phi)$  and  $(\mathcal{O}', \phi')$  are two charts on  $M$ , show that on  $\tilde{U} \cap \tilde{U}'$ , if we write  $\phi' \circ \phi^{-1}(x) = x'(x)$ , then

$$\tilde{\phi}' \circ \tilde{\phi}^{-1}(x, X^\mu) = \left( x'(x), \left( \frac{\partial (x')^\mu}{\partial x^\nu} \right)_x X^\nu \right). \quad (63)$$

Deduce that  $TM$  is a (differentiable) manifold.

**Proof.**

□

A similar construction permits us to define the cotangent bundle  $T^*M = \cup_{\mathbf{p} \in M} \{\mathbf{p}\} \times T_{\mathbf{p}}^*M$ .

**Exercise 3:** Show that the map  $\Pi : TM \rightarrow M$  which projects onto the point such that

$$(\mathbf{p}, X) \mapsto \mathbf{p}, \quad (64)$$

is smooth.

**Proof.**

□

## 4.2 Abstract Index Notation

We have used Greek letters  $\mu, \nu$  etc. to label components of vectors (or covectors) with respect to the basis  $\{e_\mu\}$  (respectively  $\{f^\mu\}$ ). Equations involving these quantities refer to a specific basis.

**Example.** Taking  $X^\mu = \delta^\mu$ , this says  $X^\mu$  only has one non-zero component in the current basis. This won't be true in other bases as  $X^\mu$  transforms.

We know some equations do hold in all bases, for example,

$$\eta(X) = X^\mu \eta_\mu. \quad (65)$$

To capture this, we use *abstract index notation*. We denote a vector with  $X^a$ , where the Latin index  $a$  does not denote a component, rather it tells us  $X^a$  is a vector. Similarly, we denote a covector  $\eta$  by  $\eta_a$ .

If an equation is true in all bases, we can replace Greek indices by Latin indices, for example

$$\eta(X) = X^a \eta_a = \eta_a X^a, \quad (66)$$

or

$$X(f) = X^a (df)_a. \quad (67)$$

Such notation tells us that the expression is **independent** of the choice of basis. One can go back to Greek letter by picking a basis and swapping  $a \rightarrow \mu$ .

### 4.3 Tensors

**Definition 4.3:** A tensor of type  $(r, s)$  at  $p$  is a multilinear map

$$T : \underbrace{T_{\mathbf{p}}^*(M) \times \cdots \times T_{\mathbf{p}}^*(M)}_{r \text{ factors}} \times \underbrace{T_{\mathbf{p}}(M) \times \cdots \times T_{\mathbf{p}}(M)}_{s \text{ factors}} \rightarrow \mathbb{R}, \quad (68)$$

where multilinear map means linear in each argument.

**Examples.**

- A tensor of type  $(0, 1)$  is a linear map  $T_{\mathbf{p}}(M) \rightarrow \mathbb{R}$ , i.e. it is a covector.
- A tensor of type  $(1, 0)$  is a linear map from  $T_{\mathbf{p}}^*(M) \rightarrow \mathbb{R}$ , i.e. an element of  $(T_{\mathbf{p}}^*(M))^* \simeq T_{\mathbf{p}}(M)$  thus it is a vector.
- We can define a  $(1, 1)$  tensor,  $\delta$  by  $\delta(\omega, X) = \omega(X)$  for any covector  $\omega$  and vector  $X$ .

**Definition 4.4:** If  $\{e_\mu\}$  is a basis for  $T_{\mathbf{p}}M$  and  $\{f^\mu\}$  is the dual basis, the components of an  $(r, s)$  tensor  $T$  are

$$T^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} = T(f^{\mu_1}, \dots, f^{\mu_r}, e_{\nu_1}, \dots, e_{\nu_s}). \quad (69)$$

In abstract index notation we write  $T$  as  $T^{a_1 \cdots a_r}_{b_1 \cdots b_s}$ .

**Note.** Tensors of type  $(r, s)$  at  $p$  form a vector space over  $\mathbb{R}$  of dimension  $n^{r+s}$ .

**Examples.**

- 1) Consider the  $\delta$  tensor above. It has components

$$\delta^\mu_\nu := \delta(X, \omega) = f^\mu(e_\nu), \quad (70)$$

which recovers our expected Kronecker delta  $\delta^\mu_\nu$ .

- 2) Consider a  $(2, 1)$  tensor  $T$ . If  $\omega, \eta \in T_{\mathbf{p}}^*M$ ,  $X \in T_{\mathbf{p}}M$ ,

$$T(\omega, \eta, X) = T(\omega_\mu f^\mu, \eta_\nu f^\nu, X^\sigma e_\sigma) \quad (71)$$

$$= \omega_\mu \eta_\nu X^\sigma T(f^\mu, f^\nu, e^\sigma) \quad (72)$$

$$= \omega_\mu \eta_\nu X^\sigma T^{\mu\nu}_\sigma. \quad (73)$$

which in abstract index notation is  $T(\omega, \eta, X) = \omega_a \eta_b X^c T^{ab}_c$ . This generalised to higher ranks.

## 5 Lecture: Tensor Fields

21/10/2024

I now drop the bold face on  $\mathbf{p} \in M \rightarrow p \in M$ .

### 5.1 Change of Bases

We've seen how components of  $X$  or  $\eta$  change with respect to a coordinate basis  $(X^\mu, \eta_\nu, \text{respec-})$  tively). Under a change of coordinates, we don't only have to consider coordinate bases.

Suppose  $\{e_\mu\}$  and  $\{e'_\mu\}$  are two bases for  $T_p M$  with dual bases  $\{f^\mu\}$  and  $\{f'^\mu\}$ .

We can expand

$$f'^\mu = A^\mu{}_\nu f^\nu \text{ and } e'_\mu = B^\nu{}_\mu e_\nu, \quad (74)$$

but

$$\delta^\mu{}_\nu = f'^\mu(e'_\nu) \quad (75)$$

$$= A^\mu{}_\tau f^\tau(B^\sigma{}_\nu e_\sigma) \quad (76)$$

$$= A^\mu{}_\tau B^\sigma{}_\nu f^\tau(e_\sigma) \quad (77)$$

$$= A^\mu{}_\tau B^\sigma{}_\nu \delta^\tau{}_\sigma \quad (78)$$

$$= A^\mu{}_\sigma B^\sigma{}_\nu, \quad (79)$$

Thus  $B^\mu{}_\nu = (A^{-1})^\mu{}_\nu$ .

If  $e_\mu = \left(\frac{\partial}{\partial x^\mu}\right)_p$  and  $e'_\mu = \left(\frac{\partial}{\partial x'^\mu}\right)_p$ . We've already seen

$$A^\mu{}_\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} \quad B^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi(p)}. \quad (80)$$

Therefore we see that a change of bases induces a transformation of tensor components. For example, if  $T$  is a  $(1, 1)$ -tensor,

$$T^\mu{}_\nu = T(f^\mu, e_\nu) \quad (81)$$

$$T'^\mu{}_\nu = T(f'^\mu, e'_\nu) \quad (82)$$

$$= T\left(A^\mu{}_\sigma f^\sigma, (A^{-1})^\tau{}_\nu e_\tau\right) \quad (83)$$

$$= A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T(f^\sigma, e_\tau) \quad (84)$$

$$= A^\mu{}_\sigma (A^{-1})^\tau{}_\nu T^\sigma{}_\tau. \quad (85)$$

## 5.2 Tensor operations

**Definition 5.1:** Given an  $(r, s)$  tensor, we can form an  $(r-1, s-1)$  tensor by **contraction**.

For simplicity assume  $T$  is a  $(2, 2)$  tensor. Define a  $(1, 1)$  tensor  $S$  by

$$S(\omega, X) = T(\omega, f^\mu, X, e_\mu). \quad (86)$$

To see that this is independent of the choice of basis, observe that a different basis would give

$$S(\omega, X) = T(\omega, f'^\mu, X, e'_\mu) = T\left(\omega, A^\mu{}_\sigma f^\sigma, X, (A^{-1})^\tau{}_\mu e_\tau\right) \quad (87)$$

$$= A^\mu{}_\sigma (A^{-1})^\tau{}_\mu T(\omega, f^\sigma, X, e_\tau) \quad (88)$$

$$= \delta^\tau{}_\sigma T(\omega, f^\sigma, X, e_\tau) \quad (89)$$

$$= T(\omega, f^\sigma, X, e_\sigma) = S(\omega, X), \quad (90)$$

and thus we have basis independence as desired. Thus we write the components of these tensors as

$$S^\mu{}_\nu = T^{\mu\sigma}{}_{\nu\sigma}, \quad (91)$$

which in abstract index notation, is written

$$S^a_b = T^{ac}_{bc}. \quad (92)$$

This can be generalized to contract any pair of covariant (lower) and contravariant (upper) indices on an arbitrary tensor.

Another way to form new tensors is to use a *tensor product*.

**Definition 5.2:** If  $S$  is a  $(p, q)$  tensor and  $T$  is an  $(r, s)$  tensor then  $S \otimes T$  is a  $(p + r, q + s)$  tensor given by

$$S \otimes T (\omega^1, \dots, \omega^p, \eta^1, \dots, \eta^r, X_1, \dots, X_q, Y_1, \dots, Y_s), \quad (93)$$

which in abstract index notation can be written

$$(S \otimes T)^{a_1 \dots a_p b_1 \dots b_r}_{c_1 \dots c_q d_1 \dots d_s} = S^{a_1 \dots a_p}_{c_1 \dots c_q} T^{b_1 \dots b_r}_{d_1 \dots d_s}. \quad (94)$$

**Exercise 4:** For any  $(1, 1)$  tensor  $T$ , in a basis we have

$$T = T^\mu_\nu e_\mu \otimes f^\nu. \quad (95)$$

**Proof.**

□

The final tensor operations we require are anti-symmetrization and symmetrization.

**Definition 5.3:** If  $T$  is a  $(0, 2)$  tensor, we can define two new tensors

$$S(X, Y) = \frac{1}{2} (T(X, Y) + T(Y, X)) \quad (96)$$

$$A(X, Y) = \frac{1}{2} (T(X, Y) - T(Y, X)), \quad (97)$$

which in abstract index notation become

$$S_{ab} = \frac{1}{2} (T_{ab} + T_{ba}) \quad (98)$$

$$A_{ab} = \frac{1}{2} (T_{ab} - T_{ba}), \quad (99)$$

one also writes  $S_{ab} = T_{(ab)}$  and  $A_{ab} = T_{[ab]}$  to denote symmetrization and antisymmetrization respectively.

These operations can be applied to any pair of matching indices. Similarly, to symmetrize over  $n$  indices we sum over all permutations and divide by  $n!$ , and identically to antisymmetrize, with the addition of a minus sign for odd permutations.

For example,

$$T^{(abc)} = \frac{1}{3!} (T^{abc} + T^{bca} + T^{cab} + T^{acb} + T^{cba} + T^{bac}) \quad (100)$$

$$T^{[abc]} = \frac{1}{3!} (T^{abc} + T^{bca} + T^{cab} - T^{acb} - T^{cba} - T^{bac}). \quad (101)$$

Lastly, to exclude indices from symmetrization, we use vertical lines such that

$$T^{(a|b|c)} = \frac{1}{2} (T^{abc} + T^{cba}). \quad (102)$$

### 5.3 Tensor Bundles

**Definition 5.4:** The space of  $(r, s)$  tensors at a point  $p$  is the vector space  $(T^r_s)_p M$ . These can be glued together to form the **bundle** of  $(r, s)$ -tensors, which we write

$$T^r_s M = \bigcup_{p \in M} \{p\} \times (T^r_s)_p M. \quad (103)$$

If  $(\mathcal{O}, \phi)$  is a coordinate chart on  $M$ , set

$$\tilde{\mathcal{O}} = \bigcup_{p \in \mathcal{O}} \{p\} \times (T^r_s)_p M \subset T^r_s M, \quad (104)$$

where  $\tilde{\phi}(p, S_p) = (\phi(p), S^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s})$

$T^r_s M$  is a manifold with a natural smooth map  $\Pi : T^r_s M \rightarrow M$  such that  $\Pi(p, S_p) = p$ .

**Definition 5.5:** An  $(r, s)$  tensor field is a smooth map  $T : M \rightarrow T^r_s M$  such that  $\Pi \circ T = \text{id}$  (namely, that  $T : p \mapsto (p, S_p)$ ). If  $(\mathcal{O}, \phi)$  is a coordinate chart on  $M$  then

$$\tilde{\phi} \circ T \circ \phi^{-1}(x) = (x, T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x)), \quad (105)$$

which is smooth provided the components  $T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x)$  are smooth functions of  $x$ .

One can think of a tensor field as defining a tensor at every point with respect to the coordinate basis at that point.

If  $T^r_s M = T^1_0 M \sim TM$ , the tensor field is called a **vector field**. In a local coordinate patch, if  $X$  is a vector field, we can write

$$X(p) = (p, X_p), \quad (106)$$

with  $X_p = X^\mu(x) \left( \frac{\partial}{\partial x^\mu} \right)_p$ .

In particular,  $\frac{d}{dx^\mu}$  are always smooth but only defined locally.

## 6 Lecture: The metric tensor

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A vector field can be thought of as we usually do, as placing a vector at every point on the manifold. A vector field can also act on a function  $f : M \rightarrow \mathbb{R}$  to give a new function

$$Xf(p) = X_p(f), \quad (107)$$

which in a coordinate basis becomes

$$Xf(p) = X^\mu(\phi(p)) \frac{\partial F}{\partial x^\mu} \Big|_{\phi(p)}, \quad (108)$$

which we now think of as a function of  $p$  across the manifold.

## 6.1 Integral curves

**Definition 6.1:** Given a vector field  $X$  on  $M$ , we say a curve  $\lambda : I \rightarrow M$ , is an **integral curve** of  $X$  if its tangent vector at every point along it is  $X$ . Namely, denote the tangent vector to  $\lambda$  at  $t$  by  $\frac{d\lambda}{dt}(t)$ , then

$$\frac{d\lambda}{dt}(t) = X_{\lambda(t)}, \quad (109)$$

$\forall t \in I$ .

Through each point  $p$ , an integral curve passes, and is unique up to reparametrization or curve extension.

To see that this is true, pick a chart  $\phi$  with  $\phi = (x^1, \dots, x^n)$  and assume  $\phi(p) = 0$ . In this chart, Eq. (109) becomes

$$\frac{dx^\mu}{dt}(t) = X^\mu(x(t)), \quad (110)$$

where  $x^\mu(t) = x^\mu(\lambda(t))$ . Assuming without loss of generality that  $\lambda(0) = p$ , we get an initial condition that  $x^\mu(0) = 0$ .

Standard ODE theory gives us that Eq. (110) with an initial condition has a solution unique up to extension.

## 6.2 Commutators

Suppose  $X$  and  $Y$  are two vector fields and  $f : M \rightarrow \mathbb{R}$  is smooth. Then  $X(Y(f))$  is a smooth function. Is it of the form  $K(f)$  for some vector field  $K$ ? No, as

$$X(Y(fg)) = X(fX(g) + gY(f)) = X(Y(fg)) \quad (111)$$

$$= X(fX(g)) + X(gY(f)) \quad (112)$$

$$= fX(Y(g)) + gX(Y(f)) + X(f)Y(g) + X(g)Y(f), \quad (113)$$

and thus the Leibniz rule does not hold, implying this cannot be a vector field. However notice that the last two terms that ruin this are symmetric in  $f$  and  $g$ , and thus if we instead consider

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad (114)$$

then the Leibniz rule will hold and (while we have not show it explicitly) this does in fact define a vector field.

To see this, use coordinate bases such that

$$[X, Y](f) = X\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) \quad (115)$$

$$= X^\mu \frac{\partial}{\partial x^\mu} \left( Y^\nu \frac{\partial F}{\partial x^\nu} \right) - Y^\nu \frac{\partial}{\partial x^\nu} \left( X^\mu \frac{\partial F}{\partial x^\mu} \right) \quad (116)$$

$$= X^\mu Y^\nu \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} - Y^\nu X^\mu \frac{\partial^2 F}{\partial x^\nu \partial x^\mu} + X^\mu \frac{\partial Y^\nu}{\partial x^\mu} \frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \frac{\partial F}{\partial x^\mu}. \quad (117)$$

As mixed partials on smooth functions in  $\mathbb{R}^n$  commute, the first two terms cancel leaving

$$[X, Y](f) = X^\mu \frac{\partial Y^\nu}{\partial x^\mu} \frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \frac{\partial F}{\partial x^\mu} \quad (118)$$

$$= \left( X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\nu \frac{\partial X^\mu}{\partial x^\mu} \right) \frac{\partial F}{\partial x^\nu} \quad (119)$$

$$= [X, Y]^\nu \frac{\partial F}{\partial x^\nu}, \quad (120)$$

where  $[X, Y]^\nu = X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}$  are the components of the commutator.

Since  $f$  is arbitrary, the expression

$$[X, Y] = [X, Y]^\nu \frac{\partial}{\partial x^\nu}, \quad (121)$$

is valid only once one has chosen a coordinate basis.

### 6.3 The metric tensor

We're familiar from Euclidean geometry (and special relativity) with the fact that the fundamental object of when discussing distance and angles (time intervals/rapidity) is an inner product between vectors.

**Definition 6.2:** A **metric tensor** at  $p \in M$  is a  $(0, 2)$ -tensor  $g$ , satisfying two conditions:

- i)  $g$  is *symmetric* such that  $g(X, Y) = g(Y, X)$ ,  $\forall X, Y \in T_p M$ , i.e.  $g_{ab} = g_{ba}$ ,
- ii)  $g$  is *non-degenerate*,  $G(X, Y) = 0$ ,  $\forall Y \in T_p M \Leftrightarrow X = 0$ .

Sometimes we write  $g(X, Y) = \langle X, Y \rangle = \langle X, Y \rangle_g = X \cdot Y$ .

By adapting the Gram-Schmidt algorithm, we can always find a basis  $\{e_\mu\}$  for the tangent space at  $p$ ,  $T_p M$ , such that

$$g(e_\mu, e_\nu) = \begin{cases} 0, & \mu \neq \nu, \\ +1 \text{ or } -1, & \mu = \nu. \end{cases} \quad (122)$$

Note this basis is not unique, but the **signature** (the number of  $+1$ 's and  $-1$ 's) does not depend on the choice of basis (Sylvester's Law of inertia).

If  $g$  has signature  $(++ \cdots +)$  we say it is **Riemannian**.

If  $g$  has signature  $(- + \cdots +)$ , we say it is **Lorentzian**.

**Definition 6.3:** A **Riemannian manifold** (or respectively a Lorentzian manifold) is a pair  $(M, g)$  where  $M$  is a manifold and  $g$  is a Riemannian (or respectively Lorentzian) metric tensor field.



On a Riemannian manifold, the norm of a vector is

$$|X| = \sqrt{g(X, X)}, \quad (123)$$

and the angle between  $X, Y \in T_p M$ , is given by

$$\cos \theta = \frac{g(X, Y)}{|X||Y|}. \quad (124)$$

The length  $\ell$  of a curve  $\lambda : (a, b) \rightarrow M$  is given by

$$\ell(\lambda) = \int_a^b \left| \frac{d\lambda}{dt}(t) \right| dt. \quad (125)$$

**Exercise 5:** If  $\tau : (c, d) \rightarrow (a, b)$  with  $\frac{dt}{d\tau} > 0$  and  $\tau(c) = a$ ,  $\tau(d) = b$ , then

$$\tilde{\lambda} = \lambda \circ \tau : (c, d) \rightarrow M, \quad (126)$$

is a reparametrization of  $\lambda$  such that  $\ell(\tilde{\lambda}) = \ell(\lambda)$ .

**Proof.**

□

## 7 Lecture: Proper time

25/10/2024

In a coordinate basis,  $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ . We often write

$$dx^\mu dx^\nu = \frac{1}{2} (dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu), \quad (127)$$

and by convention often write  $g = ds^2$  so that

$$g = ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (128)$$

**Examples.**

i)  $\mathbb{R}^n$  with  $g = ds^2 = (dx^1)^2 + \dots + (dx^n)^2 = \delta_{\mu\nu} dx^\mu dx^\nu$  is called **Euclidean space**. Any chart covering  $\mathbb{R}^n$  in which the metric takes this form is called **Cartesian**.

ii)  $\mathbb{R}^{1+3} = \{(x^0, x^1, x^2, x^3)\}$  with

$$g = ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad (129)$$

$$= \eta_{\mu\nu} dx^\mu dx^\nu, \quad (130)$$

is **Minkowski space**. A coordinate chart covering  $\mathbb{R}^{1+3}$  in which the metric takes this form is called an **inertial frame**.

iii) On  $S^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = 1\}$ . Define a chart by

$$\phi^{-1} : (0, \pi) \times (-\pi, \pi) \rightarrow S^2 \quad (131)$$

$$(\theta, \phi) \mapsto (\cos \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (132)$$

In this chart, the **round metric** is

$$g = ds^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (133)$$

This covers  $S^2 \setminus \{|\mathbf{x}| = 1, x^2 = 0, x' \leq 0\}$ . To cover the rest, let

$$\tilde{\phi}^{-1} : (0, \pi) \times (-\pi, \pi) \rightarrow S^2 \quad (134)$$

$$(\theta', \phi') \mapsto (-\sin \theta' \cos \phi', \cos \theta', \sin \theta' \sin \phi'). \quad (135)$$

This covers  $S^2 \setminus \{|\mathbf{x}| = 1, x^3 = 0, x' \geq 0\}$  and thus setting

$$g = d\theta'^2 + \sin^2 \theta' d\phi'^2. \quad (136)$$

Defines a metric on all of  $S^2$ .

Since  $g_{ab}$  is non-degenerate, it is invertible as a matrix in any basis. We can check that the inverse defines a symmetric  $(2, 0)$  tensor,  $g^{ab}$  satisfying

$$g^{ab} g_{bc} = \delta_c^a. \quad (137)$$

**Example.** In the  $\phi$  coordinates of the  $S^2$  example.

$$g^{\mu\nu} = \left(1, \frac{1}{\sin^2 \theta}\right). \quad (138)$$

An important property of the metric is that it induces a canonical identification of  $T_p M$  and  $T_p^* M$ . Given  $X^a \in T_p M$ , we define a covector  $g_{ab} X^b = X_a$  and given  $\eta_a \in T_p^* M$  we define a vector  $g^{ab} \eta_b = \eta^a$ .

In Euclidean space  $(\mathbb{R}^3, \delta)$  we often do this without realising.

More generally, this allows us to raise tensor indices with  $g^{ab}$  and lower them with  $g_{ab}$ . Namely, if  $T^{ab}_c$  is a  $(2, 1)$  tensor, then  $T_a^{bc}$  is the  $(2, 1)$  tensor given by

$$T_a^{bc} = g_{ad} g^{ce} T_e^{db}. \quad (139)$$

## 7.1 Lorentzian signature

At any point  $p$  in a Lorentzian manifold we can find a basis  $\{e_\mu\}$  such that

$$g(e_\mu, e_\nu) = \eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1). \quad (140)$$

This basis is not unique. Namely, if  $e'_\mu = (A^{-1})^\nu_\mu e_\nu$  is another such basis, then

$$\eta_{\mu\nu} = g(e'_\mu, e'_\nu) = (A^{-1})^\sigma_\mu (A^{-1})^\tau_\nu g(e_\sigma, e_\tau) (A^{-1})^\sigma_\mu (A^{-1})^\tau_\nu \quad (141)$$

$$= (A^{-1})^\sigma_\tau (A^{-1})^\tau_\nu \eta_{\sigma\tau} \quad (142)$$

$$\Rightarrow A^\mu_\kappa A^\nu_\rho \eta_{\mu\nu} = \eta_{\kappa\rho}, \quad (143)$$

which is the condition that  $A^\mu_\nu$  is a **Lorentz transformation**.

The tangent space at  $p$  has  $\eta_{\mu\nu}$  as a metric tensor (in this basis) so has the structure of Minkowski space.

**Definition 7.1:**  $X \in T_p M$  is

$$\begin{cases} \text{spacelike,} & \text{if } g(X, X) > 0, \\ \text{null-like/light-like,} & \text{if } g(X, X) = 0, \\ \text{timelike,} & \text{if } g(X, X) < 0. \end{cases} \quad (144)$$

A curve  $\lambda : I \rightarrow M$  in a Lorentzian manifold is spacelike/timelike/null if the tangent vector is spacelike/timelike/null everywhere respectively.

A spacelike curve has a well-defined **length**, given by the same formula as in the Riemannian case. For a timelike curve  $\lambda : (a, b) \rightarrow M$ , the relevant quantity is the **proper time**

$$\tau(\lambda) = \int_a^b \sqrt{-g_{ab} \frac{d\lambda^a}{du} \frac{d\lambda^b}{du}} du. \quad (145)$$

If  $g_{ab} \frac{d\lambda^a}{du} \frac{d\lambda^b}{du} = -1$  for all  $u$ , then  $\lambda$  is parametrised by proper time.

In this case we call the tangent vector

$$u^a \equiv \frac{d\lambda^a}{du}, \quad (146)$$

the **4-velocity** of  $\lambda$ .

## 7.2 Curves of extremal proper time

Suppose  $\lambda : (0, 1) \rightarrow M$  is timelike, satisfies  $\lambda(0) = p$ ,  $\lambda(1) = q$  and extremizes proper time among all such curves. This is a variational problem, associated to (in a coordinate chart),

$$\tau[\lambda] = \int_0^1 G(x^\mu(u), \dot{x}^\mu(u)) du, \quad (147)$$

with

$$G(x^\mu(u), \dot{x}^\mu(u)) = \sqrt{-g_{\mu\nu}(x(u)) \dot{x}^\mu(u) \dot{x}^\nu(u)}, \quad (148)$$

where  $\dot{x} = \frac{dx}{du}$ . The Euler Lagrange equation is

$$\frac{d}{du} \left( \frac{\partial G}{\partial \dot{x}^\mu} \right) = \frac{\partial G}{\partial x^\mu}. \quad (149)$$

We can compute

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{G} g_{\mu\nu} \dot{x}^\nu \quad (150)$$

$$\frac{\partial G}{\partial x^\mu} = -\frac{1}{2G} \frac{\partial}{\partial x^\mu} (g_{\sigma\tau}) \dot{x}^\sigma \dot{x}^\tau \quad (151)$$

$$= -\frac{1}{2G} g_{\sigma\tau, \mu} \dot{x}^\sigma \dot{x}^\tau. \quad (152)$$

This does not have a unique solution as one can re-parametrize the curve without changing the proper time  $\tau$ .