General Relativity

Cian Luke Martin

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Contents

1 Lecture: Introduction

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General Relativity is our best theory of gravitation on the largest scales. It is classical, geometrical and dynamical.

1.1 Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which 'locally looks like $\mathbb{R}^n$', and has enough structure to let us do calculus.

Definition 1.1: A **differentiable manifold** of dimension n is a set M, together with a collection of coordinate charts (O_α, ϕ_α) where

- $O_\alpha \subset M$ are subsets of M such that $\cup_\alpha O_\alpha = M$,
- ϕ_α is a bijective map (one to one and onto) from $O_\alpha \to U_\alpha$, an open subset of $\mathbb{R}^n$,
- If $O_\alpha \cap O_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth (infinitely differentiable) map from $\phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha$ to $\phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta$.

Note. We could replace smooth with finite differentiability (*e.g.* k-differentiable) but it is not particularly interesting.

Further, these charts define a topology of M, $\mathcal{R} \subset M$ is open iff $\phi_\alpha(\mathcal{R} \cap O_\alpha)$ is open in $\mathbb{R}^n$ for all α.

Every open subset of M is itself a manifold (restrict charts to $\mathcal{R}$).

Definition 1.2: The collection $\{(O_\alpha, \phi_\alpha)\}$ is called an **atlas**. Two atlases are **compatible** if their union is an atlas. An atlas A is **maximal** if there exists no atlas B with $A \subsetneq B$.

Every atlas is contained in a maximal atlas (consider the union of all compatible atlases). We can assume without loss of generality we are working with the maximal atlas.

Examples.

i) If $U \subset \mathbb{R}^n$ is open, we can take $O = U$ and

$$\phi : O \to U \tag{1}$$
$$\phi\left(x^i\right) = x^i, \tag{2}$$

and $\{(U, \phi)\}$ is an atlas.

ii) $S^1 = \{\mathbf{p} \in \mathbb{R}^2 \mid |p| = 1\}$. If $\mathbf{p} \in S^1 \setminus \{(-1,0)\} = \mathcal{O}_1$, there is a unique $\theta_1 \in (-\pi, \pi)$ such that $\mathbf{p} = (\cos\theta_1, \sin\theta_1)$.

If $\mathbf{p} \in S^1 \setminus \{(1,0)\} = \mathcal{O}_2$, then there is a unique $\theta_2 \in (0, 2\pi)$ such that $\mathbf{p} = (\cos\theta_2, \sin\theta_2)$ such that

$$\phi_1 : \mathbf{p} \to \theta_1, \text{ for } \mathbf{p} \in \mathcal{O}_1, U_1 = (-\pi, \pi), \tag{3}$$
$$\phi_2 : \mathbf{p} \to \theta_2, \text{ for } \mathbf{p} \in \mathcal{O}_2, U_2 = (0, 2\pi). \tag{4}$$

We have that $\phi_1(\mathcal{O} \cap \mathcal{O}_2) = (-\pi, 0) \cup (0, \pi)$ and

$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta, & \theta \in (0,\pi), \\ \theta + 2\pi, & \theta \in (-\pi, 0). \end{cases} \tag{5}$$

This is smooth where defined and similarly for $\phi_1 \circ \phi_2^{-1}$ and thus S_1 is a 1-manifold.

iii) $S^n = \left\{\mathbf{p} \in \mathbb{R}^{n+1} \middle| |\mathbf{p}| = 1\right\}$. We define charts by stereographic projection if $\{\mathbf{E}_1, \cdots, \mathbf{E}_{n+1}\}$ is a standard basis for $\mathbb{R}^{n+1}$ and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a standard basis for $\mathbb{R}^n$, we write

$$\mathbf{p} = p^i \mathbf{e}_i. \tag{6}$$

We set $\mathcal{O}_1 = S^n \setminus \{E_{n+1}\}$ and

$$\phi_1(\mathbf{p}) = \frac{1}{1-p^{n+1}}\left(p^i \mathbf{e}_i\right), \tag{7}$$

and $\mathcal{O}_2 = S^n \setminus \{-E_{n+1}\}$ such that

$$\phi_2(\mathbf{p}) = \frac{1}{1+p^{n+1}}\left(p^i \mathbf{e}_i\right). \tag{8}$$

We have $\phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) = \mathbb{R}^n \setminus \{0\}$ and $\phi_2 \circ \phi_1^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^2}$.

Proof. Take $\mathbf{x} \in \phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) \subset \mathbb{R}^n$. We have that $\phi_1^{-1}(\mathbf{x}) = \frac{1}{1+x_j x^j}\left(2x^i, x^j x_j - 1\right)$ which satisfies $\left|\phi_1^{-1}(\mathbf{x})\right| = 1$ and is an inverse as

$$\begin{aligned}
\phi_1 \circ \phi_1^{-1}(x_i) &= \frac{1}{1-\frac{x^j x_j - 1}{1+x_j x^j}} \frac{2x^i}{1+x_j x^j} &(9)\\
&= \frac{1+x_j x^j}{1+x_j x^j - (x^j x_j - 1)} \frac{2x^i}{1+x_j x^j} &(10)\\
&= \frac{1}{2} 2x^i = x^i. &(11)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\phi_2 \circ \phi_1^{-1}(x_i) &= \frac{1}{1+\frac{x^j x_j - 1}{1+x_j x^j}} \frac{2x^i}{1+x_j x^j} &(12)\\
&= \frac{1+x_j x^j}{1+x_j x^j + (x^j x_j - 1)} \frac{2x^i}{1+x_j x^j} &(13)\\
&= \frac{1}{2x_j x^j} 2x^i = \frac{x^i}{|x|^2}, &(14)
\end{aligned}$$

which is well defined on $\mathbb{R}^n \setminus \{0\}$ as desired. □

This is smooth on $\mathbb{R}^n \setminus \{0\}$ and similarly for $\phi_1 \circ \phi_2^{-1}$. Thus S^n is an n-manifold.

2 Lecture: Smooth Functions on Manifolds

14/10/2024

2.1 Smooth Functions

Suppose M, N are manifolds of $\dim n, n'$ respectively. Let $f : M \to N$ and $p \in M$. We pick charts $(\mathcal{O}_\alpha, \phi_\alpha)$ for M and $\left(\mathcal{O}'_\beta, \phi'_\beta\right)$ for N with $p \in \mathcal{O}_\alpha$ and $f(p) \in \mathcal{O}_\beta$.

Then $\phi'_\beta \circ f \circ \phi_\alpha^{-1}$ maps an open neighbourhood of $\phi_\alpha(p)$ in $U_\alpha \subset \mathbb{R}^n$ to $U'_\beta \subset \mathbb{R}^{n'}$.

Definition 2.1: If $\phi'_\beta \circ f \circ \phi_\alpha^{-1} : (U_\alpha \subset \mathbb{R}^n) \to \left(U'_\beta \subset \mathbb{R}^{n'}\right)$ is smooth for all possible choices of charts, we say $f : M \to N$ is **smooth**.

Note. A smooth map $\Psi : M \to N$ which has a smooth inverse Ψ^{-1} is called a **diffeomorphism** and this implies $n = n'$.

Also, if $N = \mathbb{R}$ or $\mathbb{C}$, we sometimes call f a **scalar field**. Further if M is an (open) interval such that $M = I \subset \mathbb{R}$, then $f : I \to N$ is a smooth curve in N.

Lastly, if f is smooth in one atlas, it is smooth with respect to all compatible atlases.

Examples.

1) Recall $S^1 = \{\mathbf{x} \in \mathbb{R}^2 \mid |x| = 1\}$. Let $f(x, y) = x,\ f : S^1 \to \mathbb{R}$.

   Using previous charts,

$$f \circ \phi_1^{-1} : (-\pi, \pi) \to \mathbb{R} \tag{15}$$
$$f \circ \phi_1^{-1}(\theta_1) = \cos\theta_1, \tag{16}$$

   and similarly,

$$f \circ \phi_2^{-1} : (0, 2\pi) \to \mathbb{R} \tag{17}$$
$$f \circ \phi_2^{-1}(\theta_2) = \cos\theta_2. \tag{18}$$

   In both cases, f is smooth.

2) If $(\mathcal{O}, \phi)$ is a coordinate chart on M, write for $\mathbf{p} \in \mathcal{O}$,

$$\phi(\mathbf{p}) = \left(x^1(\mathbf{p}), x^2(\mathbf{p}), \cdots, x^n(\mathbf{p})\right), \tag{19}$$

   then $x^i(\mathbf{p})$ defines a map from $\mathcal{O}$ to $\mathbb{R}$. This is a smooth map for each $i = 1, \cdots, n$. If $(\mathcal{O}', \phi')$ is another overlapping coordinate chart, then $x^i \circ \phi'^{-1}$ is the ith component of $\phi \circ \phi'^{-1}$, which is smooth.

3) We can define a smooth function chart by chart. For simplicity, we take $N = \mathbb{R}$. Let $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$ be an atlas on M. Define smooth functions $F_\alpha : U_\alpha \to \mathbb{R}$, and suppose that

$$F_\alpha \circ \phi_\alpha = F_\beta \circ \phi_\beta, \tag{20}$$

   on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ for all α, β. Then for $\mathbf{p} \in M$, we can define $f(\mathbf{p}) = F_\alpha \circ \phi_\alpha(\mathbf{p})$ where $(\mathcal{O}_\alpha, \phi_\alpha)$ is any chart with $\mathbf{p} \in \mathcal{O}_\alpha$ as this is constant by construction of F. f is smooth as

$$f \circ \phi_\beta^{-1} = F_\alpha \circ \underbrace{\phi_\alpha \circ \phi_\beta^{-1}}_{\text{always smooth}}. \tag{21}$$

   In practice, we often don't distinguish between f and its **coordinate chart representation** F_α. This coordinate chart representation F_α captures f but maps from $U_\alpha \subset \mathbb{R}^n$ rather than from subsets of M. One can think of $F_\alpha = f \circ \phi_\alpha^{-1}$ as finding the point on M that ϕ_α mapped from and evaluating f at that point.

2.2 Curves and Vectors

For a surface in $\mathbb{R}^3$, we have a notion of a tangent space at a point, consisting of all vectors tangent to the surface. Such tangent spaces are vector spaces (copies of $\mathbb{R}^2$). Different points have different tangent spaces.

In order to define the tangent space for a manifold, we first consider tangent vectors of a curve.

Recall that a smooth map from an interval $\lambda : I \subset \mathbb{R} \to M$ is a smooth curve in M.

If $\lambda(t)$ is a smooth curve in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then for $f(\lambda(t)) : \mathbb{R} \to \mathbb{R}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(\lambda(t))\right] = \mathbf{X}(t) \cdot \boldsymbol{\nabla} f(\lambda(t)), \tag{22}$$

where $\mathbf{X}(t) = \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}$ is the **tangent vector** to λ at t. The idea is that we identify the tangent vector $\mathbf{X}(t)$ with the differential operator $\mathbf{X}(t) \cdot \boldsymbol{\nabla}$.

Definition 2.2: Let $\lambda : I \to M$ be a smooth curve with $\lambda(0) = \mathbf{p}$. The **tangent vector** to λ at $\mathbf{p}$ is the linear map $X_\mathbf{p}$ from the space of smooth functions, $f : M \to \mathbb{R}$ given by

$$X_\mathbf{p}(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(\lambda(t))\Big|_{t=0}. \tag{23}$$

We observe a few things of note.

1) $X_\mathbf{p}$ is linear such that $X_\mathbf{p}(f + ag) = X_\mathbf{p}(f) + aX_\mathbf{p}(g)$ for f, g smooth and $a \in \mathbb{R}$.
2) $X_\mathbf{p}$ satisfies the Leibniz rule,

$$X_\mathbf{p}(fg) = (X_\mathbf{p}(f))\, g + f X_\mathbf{p}(g). \tag{24}$$

3) If $(\mathcal{O}, \phi)$ is a chart with $\mathbf{p} \in \mathcal{O}$, we write

$$\phi(\mathbf{p}) = \left(x^1(\mathbf{p}), \cdots, x^n(\mathbf{p})\right). \tag{25}$$

Let $F = f \circ \phi^{-1}$, $x^i(t) = x^i(\lambda(t))$ and $\mathbf{x}(t) = \phi(\lambda(t))$. Then we have

$$f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t) = F \circ \mathbf{x}(t), \tag{26}$$

and thus the tangent vector defined in Eq. (23) can also be written as

$$X_\mathbf{p}(f) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(f(\lambda(t))\right)\Big|_{t=0} = \frac{\partial F(x)}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\Big|_{t=0}, \tag{27}$$

where $\frac{\partial F}{\partial x^\mu}$ depends on f and ϕ and $\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ depends on λ and ϕ.

3 Lecture: Tangent Spaces

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3.1 The Tangent Space is a Vector Space

Proposition 3.1: The set of tangent vectors to curves at $\mathbf{p}$ forms a vector space, $T_\mathbf{p}M$ of dimension $n = \dim M$. We call $T_\mathbf{p}M$, the **tangent space** to M at $\mathbf{p}$.

Proof. Given $X_\mathbf{p}, Y_\mathbf{p}$ are tangent vectors, we need to show that $\alpha X_\mathbf{p} + \beta Y_\mathbf{p}$ is a tangent vector for $\alpha, \beta \in \mathbb{R}$. Let λ, κ be smooth curves with $\lambda(0) = \kappa(0) = \mathbf{p}$ and whose tangent vectors at $\mathbf{p}$ are $X_\mathbf{p}$ and $Y_\mathbf{p}$ respectively. Let $(\mathcal{O}, \phi)$ be a chart with $p \in \mathcal{O}$ such that $\phi(\mathbf{p}) = 0$. We call this a *chart centered at* $\mathbf{p}$.

Let $\nu(t) = \phi^{-1}[\alpha\phi(\lambda(t)) + \beta\phi(\kappa(t))]$ where notice $\nu(0) = \phi^{-1}(0) = \mathbf{p}$.

From Eq. (27), we have that if Z_p is the tangent to ν at $\mathbf{p}$, we have

$$\begin{aligned} Z_{\mathbf{p}}(f) &= \frac{\mathrm{d}}{\mathrm{d}t}(f(\nu(t)))\bigg|_0 && (28)\\ &= \frac{\partial F}{\partial x^\mu}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}t}[\alpha x^\mu(\lambda(t)) + \beta x^\mu(\kappa(t))]\bigg|_{t=0} && (29)\\ &= \alpha\frac{\partial F}{\partial x^\mu}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}t}x^\mu(\lambda(t))\bigg|_{t=0} + \beta\frac{\partial F}{\partial x^\mu}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}t}x^\mu(\kappa(t))\bigg|_{t=0} && (30)\\ &= \alpha X_{\mathbf{p}}(f) + \beta X_{\mathbf{p}}(f), && (31) \end{aligned}$$

as desired. Therefore $T_{\mathbf{p}}M$ is a vector space. □

To see that $T_{\mathbf{p}}M$ is n-dimensional, consider the curves

$$\lambda_\mu(t) = \phi^{-1}\left(0, \cdots, 0, \underbrace{t}_{\mu\text{th component}}, 0, \cdots, 0\right). \tag{32}$$

We denote the tangent vector to λ_μ at $\mathbf{p}$ by $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$.

Note. This is **not** a differential operator.

However observe that by definition, we have

$$\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}(f) = \frac{\partial F}{\partial x^\mu}\bigg|_{\phi(\mathbf{p})=0}, \tag{33}$$

and thus it acts like a differential operator in $\mathbb{R}^n$ on the coordinates of the chart.

The vectors $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$ are linearly independent. Otherwise $\exists\alpha^\mu \in \mathbb{R}$ not all zero such that

$$\alpha^\mu\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}} = 0, \tag{34}$$

which implies

$$\alpha^\mu\frac{\partial F}{\partial x^\mu} = 0, \tag{35}$$

for all F. Setting $F = x^\nu$ gives $\alpha^\nu = 0$ and thus these vectors are linearly independent and spanning. They are therefore a basis for the vector space.

Further one can see that $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$ form a basis for $T_{\mathbf{p}}M$, since if λ is any curve with tangent $X_{\mathbf{p}}$ at $\mathbf{p}$, we have

$$X_{\mathbf{p}}(f) = \frac{\partial F}{\partial x^\mu}\bigg|_{x=0}\frac{\mathrm{d}}{\mathrm{d}t}x^\mu(\lambda(t)) = X^\mu\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}(f), \tag{36}$$

where $X^\mu = \frac{\mathrm{d}}{\mathrm{d}t}x^\mu(\lambda(t))\Big|_{t=0}$ are the **components** of $X_{\mathbf{p}}$ with respect to the basis $\{\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}\}_{\mu=1,\cdots,n}$ for $T_{\mathbf{p}}M$.

Note. The basis $\{\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}\}_{\mu=1,\cdots,n}$ depends on the coordinate chart ϕ.

Suppose we choose another chart $(\mathcal{O}', \phi')$, again centered at $\mathbf{p}$. We write $\phi' = \left(\left(x'\right)^1, \cdots, (x')^n\right)$. Then if $F' = f \circ \phi'^{-1}$, we have

$$F(x) = f \circ \phi^{-1}(x) \tag{37}$$
$$= f \circ \phi'^{-1} \circ \phi' \circ \phi^{-1}(x) \tag{38}$$
$$= F'(x'(x)). \tag{39}$$

Therefore,

$$\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}(f) = \left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(\mathbf{p})} \tag{40}$$
$$= \left.\left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\right|_{\phi(\mathbf{p})} \left.\left(\frac{\partial F'}{\partial x'^\nu}\right)\right|_{\phi'(\mathbf{p})} \tag{41}$$
$$= \left.\left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\right|_{\phi(\mathbf{p})} \left(\frac{\partial}{\partial x'^\nu}\right)_{\mathbf{p}}(f). \tag{42}$$

We then deduce that

$$\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}} = \left.\left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\right|_{\phi(\mathbf{p})} \left(\frac{\partial}{\partial x'^\nu}\right)_{\mathbf{p}}. \tag{43}$$

Let X^μ be components of $X_{\mathbf{p}}$ with respect to the basis $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$, and X'^μ be components of $X_{\mathbf{p}}$ with respect to the basis $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$ such that

$$X_{\mathbf{p}} = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}} = X'^\mu \left(\frac{\partial}{\partial x'^\mu}\right)_{\mathbf{p}} \tag{44}$$
$$= X^\mu \left(\frac{\partial x'^\sigma}{\partial x^\mu}\right) \left(\frac{\partial}{\partial x'^\sigma}\right)_{\mathbf{p}}, \tag{45}$$

and therefore

$$X'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right) X^\nu. \tag{46}$$

Note. We do note have to choose a coordinate basis such as $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$. With respect to a general basis $\{e_\mu\}$, for $T_{\mathbf{p}}M$, we can write $X_{\mathbf{p}} = X^\mu e_\mu$ for $X^\mu \in \mathbb{R}$.

We always use summation convention, contracting covariant indices with contravariant indices.

3.2 Covectors

Recall that if V is a vector space over $\mathbb{R}$, the dual space V^* is the space of linear maps $\phi : V \to \mathbb{R}$. If V is n-dimensional then so is V^* (the spaces are then isomorphic). Given a basis $\{e_\mu\}$ for V, we can define the dual basis $\{f^\mu\}$ for V^* by requiring that

$$f^\mu(e_\nu) = \delta^\mu{}_\nu = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu. \end{cases} \tag{47}$$

If V is finite dimensional, then $V^{**} = (V^*)^* \simeq V$. Namely, to an element $X \in V$, we assign the linear map

$$\Lambda_X : V^* \to \mathbb{R}, \tag{48}$$
$$\Lambda_X(\omega) = \omega(X), \tag{49}$$

for $\omega \in V^*$.

Definition 3.1: The dual space of $T_{\mathbf{p}}M$ is denoted $T^*_{\mathbf{p}}M$ and is called the **cotangent space** to M at $\mathbf{p}$. An element of this space is a **covector** at $\mathbf{p}$. If $\{e_\mu\}$ is a basis for $T_{\mathbf{p}}M$ and $\{f^\mu\}$ is the dual basis for $T^*_{\mathbf{p}}M$, we can expand a covector η as

$$\eta = \eta_\mu f^\mu, \tag{50}$$

for **components** $\eta_\mu \in \mathbb{R}$.

4 Lecture: Tensors

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4.1 Tangent bundle

Notice that

$$\eta(e_\nu) = \eta_\mu f^\mu(e_\nu) = \eta_\mu \delta^\mu{}_\nu = \eta_\nu, \tag{51}$$

and thus we can get the components of η by acting it on basis vectors in the tangent space. Further as we have $X = X^\mu e_\mu$,

$$\begin{aligned} \eta(X) &= \eta(X^\mu e_\mu) &(52)\\ &= X^\mu \eta(e_\mu) &(53)\\ &= X^\mu \eta_\mu, &(54) \end{aligned}$$

and thus the action of the covector η on the vector X is essentially a contraction between the components.

Recall that a vector X is defined by its action on a function f, $X : f \to \mathbb{R}$, eating a smooth function and returning the rate of change as one moves in the direction of X.

Analogously, given a function f, one can consider a linear operator of that function being eaten by a generic vector X.

Definition 4.1: If $f : M \to \mathbb{R}$ is a smooth function, then we can define a covector $(df)_{\mathbf{p}} \in T^*_{\mathbf{p}}M$, the **differential** of f at $\mathbf{p}$, by

$$(df)_{\mathbf{p}}(X) = X(f), \tag{55}$$

for any $X \in T_{\mathbf{p}}M$. This is also sometimes called the **gradient** of f at $\mathbf{p}$.

If f is constant, $X(f) = 0$ which implies $(df)_{\mathbf{p}} = 0$.

If $(\mathcal{O}, \phi)$ is a coordination chart with $\mathbf{p} \in \mathcal{O}$ and $\phi = (x^1, \cdots, x^n)$ then we can set $f = x^\mu$ to find $(dx^\mu)_{\mathbf{p}}$. Observe

$$(dx^\mu)_{\mathbf{p}} \left(\frac{\partial}{\partial x^\nu}\right)_{\mathbf{p}} = \left(\frac{\partial x^\mu}{\partial x^\nu}\right)_{\phi(\mathbf{p})} = \delta^\mu{}_\nu. \tag{56}$$

Therefore the coordinate differentials $\{(dx^\mu)_{\mathbf{p}}\}$ is the dual basis to $\{\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}\}$.

In this basis, we can compute

$$\left[(df)_{\mathbf{p}}\right]_\mu = (df)_{\mathbf{p}} \left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}} = \left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}} f = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(\mathbf{p})}. \tag{57}$$

This justifies the language of *gradient*.

Exercise 1: Show that if $(\mathcal{O}', \phi')$ is another chart with $\mathbf{p} \in \mathcal{O}'$, then

$$(dx^\mu)_{\mathbf{p}} = \left(\frac{\partial x^\mu}{\partial (x')^\nu}\right)_{\phi'(\mathbf{p})} \left(d (x')^\nu\right)_{\mathbf{p}}, \tag{58}$$

where $x(x') = \phi \circ (\phi')^{-1}$, and hence if η_μ, η'_μ are components with respect to these bases,

$$\eta'_\mu = \left(\frac{\partial x^\nu}{\partial (x')^\mu}\right)_{\phi'(\mathbf{p})} \eta_\nu. \tag{59}$$

Proof.

□

Definition 4.2 (Tangent bundle): We can glue together the tangent spaces $T_{\mathbf{p}}M$ as $\mathbf{p}$ varies to get a new $2n$ dimensional manifold TM, the **tangent bundle**. We define the tangent bundle to be

$$TM := \bigcup_{\mathbf{p} \in M} \{\mathbf{p}\} \times T_{\mathbf{p}}M. \tag{60}$$

Namely, it is the set of ordered pairs $(\mathbf{p}, X)$, with $\mathbf{p} \in M$, $X \in T_{\mathbf{p}}M$.

If $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$ is an atlas on M, we obtain an atlas for TM by setting

$$\mathcal{O}_\alpha = \bigcup_{\mathbf{p} \in \mathcal{O}_\alpha} \{\mathbf{p}\} \times T_{\mathbf{p}}M, \tag{61}$$

and

$$\widetilde{\phi}_\alpha ((p, X)) = (\phi(\mathbf{p}), X^\mu) \in \mathcal{U}_\alpha \times \mathbb{R}^n = \widetilde{\mathcal{U}}_2, \tag{62}$$

where X^μ are the components of X with respect to the coordinate basis of ϕ_α.

Exercise 2: If $(\mathcal{O}, \phi)$ and $(\mathcal{O}', \phi')$ are two charts on M, show that on $\widetilde{U} \cap \widetilde{U}'$, if we write $\phi' \circ \phi^{-1}(x) = x'(x)$, then

$$\widetilde{\phi}' \circ \widetilde{\phi}^{-1}(x, X^\mu) = \left(x'(x), \left(\frac{\partial (x')^\mu}{\partial x^\nu}\right)_x X^\nu\right). \tag{63}$$

Deduce that TM is a (differentiable) manifold.

Proof.

□

A similar construction permits us to define the cotangent bundle $T^*M = \cup_{\mathbf{p}\in M}\{\mathbf{p}\} \times T^*_{\mathbf{p}}M$.

Exercise 3: Show that the map $\Pi : TM \to M$ which projects onto the point such that

$$(\mathbf{p}, X) \mapsto \mathbf{p}, \tag{64}$$

is smooth.

Proof.

□

4.2 Abstract Index Notation

We have used Greek letters μ, ν etc. to label components of vectors (or covectors) with respect to the basis $\{e_\mu\}$ (respectively $\{f^\mu\}$). Equations involving these quantities refer to a specific basis.

Example. Taking $X^\mu = \delta^\mu$, this says X^μ only has one non-zero component in the current basis. This won't be true in other bases as X^μ transforms.

We know some equations do hold in all bases, for example,

$$\eta\left(X\right) = X^\mu \eta_\mu. \tag{65}$$

To capture this, we use *abstract index notation.* We denote a vector with X^a, where the Latin index a does not denote a component, rather it tells us X^a is a vector. Similarly, we denote a covector η by η_a.

If an equation is true in all bases, we can replace Greek indices by Latin indices, for example

$$\eta\left(X\right) = X^a \eta_a = \eta_a X^a, \tag{66}$$

or

$$X\left(f\right) = X^a \left(df\right)_a. \tag{67}$$

Such notation tells us that the expression is **independent** of the choice of basis. One can go back to Greek letter by picking a basis and swapping $a \to \mu$.

4.3 Tensors

Definition 4.3: A **tensor** of type (r, s) at p is a multilinear map

$$T : \underbrace{T^*_{\mathbf{p}}\left(M\right) \times \cdots \times T^*_{\mathbf{p}}\left(M\right)}_{r \text{ factors}} \times \underbrace{T_{\mathbf{p}}\left(M\right) \times \cdots \times T_{\mathbf{p}}\left(M\right)}_{s \text{ factors}} \to \mathbb{R}, \tag{68}$$

where multilinear map means linear in each argument.

Examples.

- A tensor of type $(0,1)$ is a linear map $T_{\mathbf{p}}(M) \to \mathbb{R}$, i.e. it is a covector.
- A tensor of type $(1,0)$ is a linear map from $T^*_{\mathbf{p}}(M) \to \mathbb{R}$, i.e. an element of $\left(T^*_{\mathbf{p}}(M)\right)^* \simeq T_{\mathbf{p}}(M)$ thus it is a vector.
- We can define a $(1,1)$ tensor, δ by $\delta(\omega, X) = \omega(X)$ for any covector ω and vector X.

Definition 4.4: If $\{e_\mu\}$ is a basis for $T_{\mathbf{p}}M$ and $\{f^\mu\}$ is the dual basis, the components of an (r,s) tensor T are

$$T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = T\left(f^{\mu_1}, \cdots, f^{\nu_r}, e_{\nu_1} \cdots e_{\nu_s}\right). \tag{69}$$

In abstract index notation we write T as $T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}$.

Note. Tensors of type (r,s) at p form a vector space over $\mathbb{R}$ of dimension n^{r+s}.

Examples.

1) Consider the δ tensor above. It has components

$$\delta^\mu{}_\nu := \delta(X, \omega) = f^\mu(e_\nu), \tag{70}$$

which recovers our expected Kronecker delta δ^μ_ν.

2) Consider a $(2,1)$ tensor T. If $\omega, \eta \in T^*_{\mathbf{p}}M$, $X \in T_{\mathbf{p}}M$,

$$\begin{aligned} T(\omega, \eta, X) &= T\left(\omega_\mu f^\mu, \eta_\nu f^\nu, X^\sigma e_\sigma\right) && (71) \\ &= \omega_\mu \eta_\nu X^\sigma T\left(f^\mu, f^\nu, e^\sigma\right) && (72) \\ &= \omega_\mu \eta_\nu X^\sigma T^{\mu\nu}{}_\sigma. && (73) \end{aligned}$$

which in abstract index notation is $T(\omega, \eta, X) = \omega_a \eta_b X^c T^{ab}{}_c$. This generalised to higher ranks.

5 Lecture: Tensor Fields

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I now drop the bold face on $\mathbf{p} \in M \to p \in M$.

5.1 Change of Bases

We've seen how components of X or η change with respect to a coordinate basis (X^μ, η_ν, respectively). Under a change of coordinates, we don't only have to consider coordinate bases.

Suppose $\{e_\mu\}$ and $\{e'_\mu\}$ are two bases for T_pM with dual bases $\{f^\mu\}$ and $\{f'^\mu\}$.

We can expand

$$f'^\mu = A^\mu{}_\nu f^\nu \text{ and } e'_\mu = B^\nu{}_\mu e_\nu, \tag{74}$$

but

$$\begin{aligned} \delta^\mu_\nu &= f'^\mu\left(e'_\nu\right) && (75) \\ &= A^\mu{}_\tau f^\tau\left(B^\sigma{}_\nu e_\sigma\right) && (76) \end{aligned}$$

$$= A^\mu{}_\tau B^\sigma{}_\nu f^\tau (e_\sigma) \tag{77}$$
$$= A^\mu{}_\tau B^\sigma{}_\nu \delta^\tau{}_\sigma \tag{78}$$
$$= A^\mu{}_\sigma B^\sigma{}_\nu , \tag{79}$$

Thus $B^\mu{}_\nu = \left(A^{-1}\right)^\mu{}_\nu$.

If $e_\mu = \left(\frac{\partial}{\partial x^\mu}\right)_p$ and $e'_\mu = \left(\frac{\partial}{\partial x^\mu}\right)_p$. We've already seen

$$A^\mu{}_\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} \qquad B^\mu{}_\nu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_{\phi(p)} . \tag{80}$$

Therefore we see that a change of bases induces a transformation of tensor components. For example, if T is a $(1,1)$-tensor,

$$T^\mu{}_\nu = T\left(f^\mu, e_\nu\right) \tag{81}$$
$$T'^\mu{}_\nu = T\left(f'^\mu, e'_\nu\right) \tag{82}$$
$$= T\left(A^\mu{}_\sigma f^\sigma, \left(A^{-1}\right)^\tau{}_\nu e_\tau\right) \tag{83}$$
$$= A^\mu{}_\sigma \left(A^{-1}\right)^\tau{}_\nu T\left(f^\sigma, e_\tau\right) \tag{84}$$
$$= A^\mu{}_\sigma \left(A^{-1}\right)^\tau{}_\nu T^\sigma{}_\tau . \tag{85}$$

5.2 Tensor operations

Definition 5.1: Given an (r, s) tensor, we can form an $(r-1, s-1)$ tensor by **contraction**.

For simplicity assume T is a $(2,2)$ tensor. Define a $(1,1)$ tensor S by

$$S(\omega, X) = T\left(\omega, f^\mu, X, e_\mu\right) . \tag{86}$$

To see that this is independent of the choice of basis, observe that a different basis would give

$$S(\omega, X) = T\left(\omega, f'^\mu, X, e'_\mu\right) = T\left(\omega, A^\mu{}_\sigma f^\sigma, X, \left(A^{-1}\right)^\tau{}_\mu e_\tau\right) \tag{87}$$
$$= A^\mu{}_\sigma \left(A^{-1}\right)^\tau{}_\mu T\left(\omega, f^\sigma, X, e_\tau\right) \tag{88}$$
$$= \delta^\tau_\sigma T\left(\omega, f^\sigma, X, e_\tau\right) \tag{89}$$
$$= T\left(\omega, f^\sigma, X, e_\sigma\right) = S(\omega, X) , \tag{90}$$

and thus we have basis independence as desired. Thus we write the components of these tensors as

$$S^\mu{}_\nu = T^{\mu\sigma}{}_{\nu\sigma} , \tag{91}$$

which in abstract index notation, is written

$$S^a{}_b = T^{ac}{}_{bc} . \tag{92}$$

This can be generalized to contract any pair of covariant (lower) and contravariant (upper) indices on an arbitrary tensor.

Another way to form new tensors is to use a *tensor product*.

Definition 5.2: If S is a (p,q) tensor and T is an (r,s) tensor then $S \otimes T$ is a $(p+r, q+s)$ tensor given by

$$S \otimes T\left(\omega^1, \cdots, \omega^p, \eta^1, \cdots, \eta^r, X_1, \cdots, X_q, Y_1, \cdots, Y_s\right), \tag{93}$$

which in abstract index notation can be written

$$(S \otimes T)^{a_1 \cdots a_p b_1 \cdots b_r}{}_{c_1 \cdots c_q d_1 \cdots d_s} = S^{a_1 \cdots a_p}{}_{c_1 \cdots c_q} T^{b_1 \cdots b_r}{}_{d_1 \cdots d_s}\,. \tag{94}$$

Exercise 4: For any $(1,1)$ tensor T, in a basis we have

$$T = T^{\mu}{}_{\nu}\, e_\mu \otimes f^\nu. \tag{95}$$

Proof.

□

The final tensor operations we require are anti-symmetrization and symmetrization.

Definition 5.3: If T is a $(0,2)$ tensor, we can define two new tensors

$$S(X,Y) = \frac{1}{2}\left(T(X,Y) + T(Y,X)\right) \tag{96}$$

$$A(X,Y) = \frac{1}{2}\left(T(X,Y) - T(Y,X)\right), \tag{97}$$

which in abstract index notation become

$$S_{ab} = \frac{1}{2}\left(T_{ab} + T_{ba}\right) \tag{98}$$

$$A_{ab} = \frac{1}{2}\left(T_{ab} - T_{ba}\right), \tag{99}$$

one also writes $S_{ab} = T_{(ab)}$ and $A_{ab} = T_{[ab]}$ to denote symmetrization and antisymmetrization respectively.

These operations can be applied to any pair of matching indices. Similarly, to symmetrize over n indices we sum over all permutations and divide by $n!$, and identically to antisymmetrize, with the addition of a minus sign for odd permutations.

For example,

$$T^{(abc)} = \frac{1}{3!}\left(T^{abc} + T^{bca} + T^{cab} + T^{acb} + T^{cba} + T^{bac}\right) \tag{100}$$

$$T^{[abc]} = \frac{1}{3!}\left(T^{abc} + T^{bca} + T^{cab} - T^{acb} - T^{cba} - T^{bac}\right). \tag{101}$$

Lastly, to exclude indices from symmetrization, we use vertical lines such that

$$T^{(a|b|c)} = \frac{1}{2}\left(T^{abc} + T^{cba}\right). \tag{102}$$

5.3 Tensor Bundles

Definition 5.4: The space of (r, s) tensors at a point p is the vector space $\left(T^r{}_s\right)_p M$. These can be glued together to form the **bundle** of (r, s)-tensors, which we write

$$T^r{}_s M = \bigcup_{p \in M} \{p\} \times \left(T^r{}_s\right)_p M. \tag{103}$$

If $(\mathcal{O}, \phi)$ is a coordinate chart on M, set

$$\widetilde{\mathcal{O}} = \bigcup_{p \in \mathcal{O}} \{p\} \times \left(T^r{}_s\right)_p M \subset T^r{}_s M, \tag{104}$$

where $\widetilde{\phi}\left(p, S_p\right) = \left(\phi\left(p\right), S^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}\right)$

$T^r{}_s M$ is a manifold with a natural smooth map $\Pi : T^r{}_s M \to M$ such that $\Pi\left(p, S_p\right) = p$.

Definition 5.5: An (r, s) tensor field is a smooth map $T : M \to T^r{}_s M$ such that $\Pi \circ T = \mathrm{id}$ (namely, that $T : p \mapsto (p, S_p)$). If $(\mathcal{O}, \phi)$ is a coordinate chart on M then

$$\widetilde{\phi} \circ T \circ \phi^{-1}\left(x\right) = \left(x, T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}\left(x\right)\right), \tag{105}$$

which is smooth provided the components $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}\left(x\right)$ are smooth functions of x.

One can think of a tensor field as defining a tensor at every point with respect to the coordinate basis at that point.

If $T^r{}_s M = T^1{}_0 M \sim TM$, the tensor field is called a **vector field**. In a local coordinate patch, if X is a vector field, we can write

$$X\left(p\right) = \left(p, X_p\right), \tag{106}$$

with $X_p = X^\mu\left(x\right)\left(\frac{\partial}{\partial x^\mu}\right)_p$.

In particular, $\frac{\mathrm{d}}{\mathrm{d}x^\mu}$ are always smooth but only defined locally.

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A vector field can be thought of as we usually do, as placing a vector at every point on the manifold. A vector field can also act on a function $f : M \to \mathbb{R}$ to give a new function

$$Xf\left(p\right) = X_p\left(f\right), \tag{107}$$

which in a coordinate basis becomes

$$Xf\left(p\right) = X^\mu\left(\phi\left(p\right)\right)\left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(p)}, \tag{108}$$

which we now think of as a function of p across the manifold.

6.1 Integral curves

Definition 6.1: Given a vector field X on M, we say a curve $\lambda : I \to M$, is an **integral curve** of X if its tangent vector at every point along it is X. Namely, denote the tangent vector to λ at t by $\frac{\mathrm{d}\lambda}{\mathrm{d}t}(t)$, then

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t}(t) = X_{\lambda(t)}, \tag{109}$$

$\forall t \in I$.

Through each point p, an integral curve passes, and is unique up to reparametrization or curve extension.

To see that this is true, pick a chart ϕ with $\phi = \left(x^1, \cdots, x^n\right)$ and assume $\phi(p) = 0$. In this chart, Eq. (109) becomes

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t}(t) = X^\mu(x(t)), \tag{110}$$

where $x^\mu(t) = x^\mu(\lambda(t))$. Assuming without loss of generality that $\lambda(0) = p$, we get an initial condition that $x^\mu(0) = 0$.

Standard ODE theory gives us that Eq. (110) with an initial condition has a solution unique up to extension.

6.2 Commutators

Suppose X and Y are two vector fields and $f : M \to \mathbb{R}$ is smooth. Then $X(Y(f))$ is a smooth function. Is it of the form $K(f)$ for some vector field K? No, as

$$\begin{aligned} X(Y(fg)) = X(fX(g) + gY(f)) &= X(Y(fg)) && (111)\\ &= X(fX(g)) + X(gY(f)) && (112)\\ &= fX(Y(g)) + gX(Y(f)) + X(f)Y(g) + X(g)Y(f), && (113) \end{aligned}$$

and thus the Leibniz rule does not hold, implying this cannot be a vector field. However notice that the last two terms that ruin this are symmetric in f and g, and thus if we instead consider

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \tag{114}$$

then the Leibniz rule will hold and (while we have not show it explicitly) this does in fact define a vector field.

To see this, use coordinate bases such that

$$\begin{aligned} [X, Y](f) &= X\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) && (115)\\ &= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y^\nu \frac{\partial}{\partial x^\nu}\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) && (116)\\ &= X^\mu Y^\mu \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} - Y^\nu X^\mu \frac{\partial^2 F}{\partial x^\nu \partial x^\mu} + X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu}. && (117) \end{aligned}$$

As mixed partials on smooth functions in $\mathbb{R}^n$ commute, the first two terms cancel leaving

$$[X, Y](f) = X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu} \tag{118}$$

$$= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\nu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial F}{\partial x^\nu} \tag{119}$$

$$= [X, Y]^\nu \frac{\partial F}{\partial x^\nu}, \tag{120}$$

where $[X, Y]^\nu = X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}$ are the components of the commutator.

Since f is arbitrary, the expression

$$[X, Y] = [X, Y]^\nu \frac{\partial}{\partial x^\nu}, \tag{121}$$

is valid only once one has chosen a coordinate basis.

6.3 The metric tensor

We're familiar from Euclidean geometry (and special relativity) with the fact that the fundamental object of when discussing distance and angles (time intervals/rapidity) is an inner product between vectors.

Definition 6.2: A **metric tensor** at $p \in M$ is a $(0, 2)$-tensor g, satisfying two conditions:

i) g is *symmetric* such that $g(X, Y) = g(Y, X)$, $\forall X, Y \in T_pM$, i.e. $g_{ab} = g_{ba}$,

ii) g is *non-degenerate*, $G(X, Y) = 0$, $\forall Y \in T_pM \Leftrightarrow X = 0$.

Sometimes we write $g(X, Y) = \langle X, Y \rangle = \langle X, Y \rangle_g = X \cdot Y$.

By adapting the Gram-Schmidt algorithm, we can always find a basis $\{e_\mu\}$ for the tangent space at p, T_pM, such that

$$g(e_\mu, e_\nu) = \begin{cases} 0, & \mu \neq \nu, \\ +1 \text{ or } -1, & \mu = \nu. \end{cases} \tag{122}$$

Note this basis is not unique, but the **signature** (the number of $+1$'s and -1's) does not depend on the choice of basis (Sylvestre's Law of inertia).

If g has signature $(+ + \cdots +)$ we say it is **Riemannian**.

If g has signature $(- + \cdots +)$, we say it is **Lorentzian**.

Definition 6.3: A **Riemannian manifold** (or respectively a Lorentzian manifold) is a pair (M, g) where M is a manifold and g is a Riemannian (or respectively Lorentzian) metric tensor field.

On a Riemannian manifold, the norm of a vector is

$$|X| = \sqrt{g(X, X)}, \tag{123}$$

and the angle between $X, Y \in T_pM$, is given by

$$\cos\theta = \frac{g(X, Y)}{|X| |Y|}. \tag{124}$$

The length ℓ of a curve $\lambda : (a,b) \to M$ is given by

$$\ell(\lambda) = \int_a^b \left| \frac{\mathrm{d}\lambda}{\mathrm{d}t}(t) \right| \mathrm{d}t\,. \tag{125}$$

Exercise 5: If $\tau : (c,d) \to (a,b)$ with $\frac{\mathrm{d}t}{\mathrm{d}\tau} > 0$ and $\tau(c) = a$, $\tau(d) = b$, then

$$\widetilde{\lambda} = \lambda \circ \tau : (c,d) \to M, \tag{126}$$

is a reparametrization of λ such that $\ell\left(\widetilde{\lambda}\right) = \ell(\lambda)$.

Proof.

□