General Relativity

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1 Lecture: Introduction

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General Relativity is our best theory of gravitation on the largest scales. It is classical, geometrical and dynamical.

1.1 Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which 'locally looks like $\mathbb{R}^n$', and has enough structure to let us do calculus.

Definition 1.1: A **differentiable manifold** of dimension n is a set M, together with a collection of coordinate charts (O_α, ϕ_α) where

- $O_\alpha \in M$ are subsets of M such that $\cup_\alpha O_\alpha = M$,
- ϕ_α is a bijective map (one to one and onto) from $O_\alpha \to U_\alpha$, an open subset of $\mathbb{R}^n$,
- If $O_\alpha \cap O_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth (infinitely differentiable) map from $\phi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha$ to $\phi_\beta(O_\alpha \cap O_\beta) \subset U_\beta$.

Note. We could replace smooth with finite differentiability (*e.g.* k-differentiable) but it is not particularly interesting.

Further, these charts define a topology of M, $\mathcal{R} \subset M$ is open iff $\phi_\alpha(\mathcal{R} \cap O_\alpha)$ is open in $\mathbb{R}^n$ for all α.

Every open subset of M is itself a manifold (restrict charts to $\mathcal{R}$).

Definition 1.2: The collection $\{(O_\alpha, \phi_\alpha)\}$ is called an **atlas**. Two atlases are **compatible** if their union is an atlas. An atlas A is **maximal** if there exists no atlas B with $A \subsetneq B$.

Every atlas is contained in a maximal atlas (consider the union of all compatible atlases). We can assume without loss of generality we are working with the maximal atlas.

Examples. i) If $U \subset \mathbb{R}^n$ is open, we can take $O = U$ and

$$\phi : O \to U \tag{1}$$
$$\phi\left(x^i\right) = x^i, \tag{2}$$

and $\{(U, \phi)\}$ is an atlas.

ii) $S^1 = \{\mathbf{p} \in \mathbb{R}^2 \mid |p| = 1\}$. If $\mathbf{p} \in S^1 \setminus \{(-1, 0)\} = \mathcal{O}_1$, there is a unique $\theta_1 \in (-\pi, \pi)$ such that $\mathbf{p} = (\cos\theta_1, \sin\theta_1)$.

If $\mathbf{p} \in S^1\{(1, 0)\} = \mathcal{O}_2$, then there is a unique $\theta_2 \in (0, 2\pi)$ such that $\mathbf{p} = (\cos\theta_2, \sin\theta_2)$ such that

$$\phi_1 : \mathbf{p} \to \theta_1, \text{ for } \mathbf{p} \in \mathcal{O}_1, U_1 = (-\pi, \pi), \tag{3}$$
$$\phi_2 : \mathbf{p} \to \theta_2, \text{ for } \mathbf{p} \in \mathcal{O}_2, U_2 = (0, 2\pi). \tag{4}$$

We have that $\phi_1\left(\mathcal{O} \cap \mathcal{O}_2\right) = (-\pi, 0) \cup (0, \pi)$ and

$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta, & \theta \in (0, \pi), \\ \theta + 2\pi, & \theta \in (-\pi, 0). \end{cases} \tag{5}$$

This is smooth where defined and similarly for $\phi_1 \circ \phi_2^{-1}$ and thus S_1 is a 1-manifold.

iii) $S^n = \left\{\mathbf{p} \in \mathbb{R}^{n+1} \middle| \, |\mathbf{p}| = 1\right\}$. We define charts by stereographic projection if $\{\mathbf{E}_1, \cdots, \mathbf{E}_{n+1}\}$ is a standard basis for $\mathbb{R}^{n+1}$ and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a standard basis for $\mathbb{R}^n$, we write

$$\mathbf{p} = p^i \mathbf{e}_i. \tag{6}$$

We set $\mathcal{O}_1 = S^n \setminus \{E_{n+1}\}$ and

$$\phi_1(\mathbf{p}) = \frac{1}{1 - p^{n+1}}\left(p^i \mathbf{e}_i\right), \tag{7}$$

and $\mathcal{O}_2 = S^n \setminus \{-E_{n+1}\}$ such that

$$\phi_2(\mathbf{p}) = \frac{1}{1 + p^{n+1}}\left(p^i \mathbf{e}_i\right). \tag{8}$$

We have $\phi_1\left(\mathcal{O}_1 \cap \mathcal{O}_2\right) = \mathbb{R}^n \setminus \{0\}$ and $\phi_2 \circ \phi_1^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^2}$.

Proof. Take $\mathbf{x} \in \phi_1\left(\mathcal{O}_1 \cap \mathcal{O}_2\right) \subset \mathbb{R}^n$. We have that $\phi_1^{-1}(\mathbf{x}) = \frac{1}{1 + x_j x^j}\left(2x^i, x^j x_j - 1\right)$ which satisfies $\left|\phi_1^{-1}(\mathbf{x})\right| = 1$ and is an inverse as

$$\phi_1 \circ \phi_1^{-1}(x_i) = \frac{1}{1 - \frac{x^j x_j - 1}{1 + x_j x^j}} \frac{2x^i}{1 + x_j x^j} \tag{9}$$
$$= \frac{1 + x_j x^j}{1 + x_j x^j - (x^j x_j - 1)} \frac{2x^i}{1 + x_j x^j} \tag{10}$$

$$= \frac{1}{2} 2x^i = x^i. \tag{11}$$

Similarly, we have

$$\phi_2 \circ \phi_1^{-1}(x_i) = \frac{1}{1 + \frac{x^j x_j - 1}{1 + x_j x^j}} \frac{2x^i}{1 + x_j x^j} \tag{12}$$

$$= \frac{1 + x_j x^j}{1 + x_j x^j + (x^j x_j - 1)} \frac{2x^i}{1 + x_j x^j} \tag{13}$$

$$= \frac{1}{2x_j x^j} 2x^i = \frac{x^i}{|x|^2}, \tag{14}$$

which is well defined on $\mathbb{R}^n \setminus \{0\}$ as desired. □

This is smooth on $\mathbb{R}^n \setminus \{0\}$ and similarly for $\phi_1 \circ \phi_2^{-1}$. Thus S^n is an n-manifold.

2 Lecture

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