General Relativity

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Contents

1 Lecture: Introduction

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General Relativity is our best theory of gravitation on the largest scales. It is classical, geometrical and dynamical.

1.1 Differentiable Manifolds

The basic object of study in differential geometry is the (differentiable) manifold. This is an object which 'locally looks like $\mathbb{R}^n$', and has enough structure to let us do calculus.

Definition 1.1: A **differentiable manifold** of dimension n is a set M, together with a collection of coordinate charts (O_α, ϕ_α) where

- $O_\alpha \subset M$ are subsets of M such that $\cup_\alpha O_\alpha = M$,
- ϕ_α is a bijective map (one to one and onto) from $O_\alpha \to U_\alpha$, an open subset of $\mathbb{R}^n$,
- If $O_\alpha \cap O_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth (infinitely differentiable) map from $\phi_\alpha (O_\alpha \cap O_\beta) \subset U_\alpha$ to $\phi_\beta (O_\alpha \cap O_\beta) \subset U_\beta$.

Note. We could replace smooth with finite differentiability (*e.g.* k-differentiable) but it is not particularly interesting.

Further, these charts define a topology of M, $\mathcal{R} \subset M$ is open iff $\phi_\alpha (\mathcal{R} \cap O_\alpha)$ is open in $\mathbb{R}^n$ for all α.

Every open subset of M is itself a manifold (restrict charts to $\mathcal{R}$).

Definition 1.2: The collection $\{(O_\alpha, \phi_\alpha)\}$ is called an **atlas**. Two atlases are **compatible** if their union is an atlas. An atlas A is **maximal** if there exists no atlas B with $A \subsetneq B$.

Every atlas is contained in a maximal atlas (consider the union of all compatible atlases). We can assume without loss of generality we are working with the maximal atlas.

Examples.

i) If $U \subset \mathbb{R}^n$ is open, we can take $O = U$ and

$$\phi : O \to U \tag{1}$$
$$\phi\left(x^i\right) = x^i, \tag{2}$$

and $\{(U, \phi)\}$ is an atlas.

ii) $S^1 = \{\mathbf{p} \in \mathbb{R}^2 \mid |p| = 1\}$. If $\mathbf{p} \in S^1 \setminus \{(-1,0)\} = \mathcal{O}_1$, there is a unique $\theta_1 \in (-\pi, \pi)$ such that $\mathbf{p} = (\cos\theta_1, \sin\theta_1)$.

If $\mathbf{p} \in S^1 \setminus \{(1,0)\} = \mathcal{O}_2$, then there is a unique $\theta_2 \in (0, 2\pi)$ such that $\mathbf{p} = (\cos\theta_2, \sin\theta_2)$ such that

$$\phi_1 : \mathbf{p} \to \theta_1, \text{ for } \mathbf{p} \in \mathcal{O}_1, U_1 = (-\pi, \pi), \tag{3}$$
$$\phi_2 : \mathbf{p} \to \theta_2, \text{ for } \mathbf{p} \in \mathcal{O}_2, U_2 = (0, 2\pi). \tag{4}$$

We have that $\phi_1(\mathcal{O} \cap \mathcal{O}_2) = (-\pi, 0) \cup (0, \pi)$ and

$$\phi_2 \circ \phi_1^{-1}(\theta) = \begin{cases} \theta, & \theta \in (0, \pi), \\ \theta + 2\pi, & \theta \in (-\pi, 0). \end{cases} \tag{5}$$

This is smooth where defined and similarly for $\phi_1 \circ \phi_2^{-1}$ and thus S_1 is a 1-manifold.

iii) $S^n = \{\mathbf{p} \in \mathbb{R}^{n+1} \Big| \, |\mathbf{p}| = 1\}$. We define charts by stereographic projection if $\{\mathbf{E}_1, \cdots, \mathbf{E}_{n+1}\}$ is a standard basis for $\mathbb{R}^{n+1}$ and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a standard basis for $\mathbb{R}^n$, we write

$$\mathbf{p} = p^i \mathbf{e}_i. \tag{6}$$

We set $\mathcal{O}_1 = S^n \setminus \{E_{n+1}\}$ and

$$\phi_1(\mathbf{p}) = \frac{1}{1 - p^{n+1}}\left(p^i \mathbf{e}_i\right), \tag{7}$$

and $\mathcal{O}_2 = S^n \setminus \{-E_{n+1}\}$ such that

$$\phi_2(\mathbf{p}) = \frac{1}{1 + p^{n+1}}\left(p^i \mathbf{e}_i\right). \tag{8}$$

We have $\phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) = \mathbb{R}^n \setminus \{0\}$ and $\phi_2 \circ \phi_1^{-1}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^2}$.

Proof. Take $\mathbf{x} \in \phi_1(\mathcal{O}_1 \cap \mathcal{O}_2) \subset \mathbb{R}^n$. We have that $\phi_1^{-1}(\mathbf{x}) = \frac{1}{1+x_j x^j}\left(2x^i, x^j x_j - 1\right)$ which satisfies $\left|\phi_1^{-1}(\mathbf{x})\right| = 1$ and is an inverse as

$$\phi_1 \circ \phi_1^{-1}(x_i) = \frac{1}{1 - \frac{x^j x_j - 1}{1+x_j x^j}} \frac{2x^i}{1+x_j x^j} \tag{9}$$

$$= \frac{1+x_j x^j}{1+x_j x^j - (x^j x_j - 1)} \frac{2x^i}{1+x_j x^j} \tag{10}$$

$$= \frac{1}{2} 2x^i = x^i. \tag{11}$$

Similarly, we have

$$\phi_2 \circ \phi_1^{-1}(x_i) = \frac{1}{1 + \frac{x^j x_j - 1}{1+x_j x^j}} \frac{2x^i}{1+x_j x^j} \tag{12}$$

$$= \frac{1+x_j x^j}{1+x_j x^j + (x^j x_j - 1)} \frac{2x^i}{1+x_j x^j} \tag{13}$$

$$= \frac{1}{2x_j x^j} 2x^i = \frac{x^i}{|x|^2}, \tag{14}$$

which is well defined on $\mathbb{R}^n \setminus \{0\}$ as desired. □

This is smooth on $\mathbb{R}^n \setminus \{0\}$ and similarly for $\phi_1 \circ \phi_2^{-1}$. Thus S^n is an n-manifold.

2 Lecture: Smooth Functions on Manifolds

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2.1 Smooth Functions

Suppose M, N are manifolds of $\dim n, n'$ respectively. Let $f : M \to N$ and $p \in M$. We pick charts $(\mathcal{O}_\alpha, \phi_\alpha)$ for M and $\left(\mathcal{O}'_\beta, \phi'_\beta\right)$ for N with $p \in \mathcal{O}_\alpha$ and $f(p) \in \mathcal{O}_\beta$.

Then $\phi'_\beta \circ f \circ \phi_\alpha^{-1}$ maps an open neighbourhood of $\phi_\alpha(p)$ in $U_\alpha \subset \mathbb{R}^n$ to $U'_\beta \subset \mathbb{R}^{n'}$.

Definition 2.1: If $\phi'_\beta \circ f \circ \phi_\alpha^{-1} : (U_\alpha \subset \mathbb{R}^n) \to \left(U'_\beta \subset \mathbb{R}^{n'}\right)$ is smooth for all possible choices of charts, we say $f : M \to N$ is **smooth**.

Note. A smooth map $\Psi : M \to N$ which has a smooth inverse Ψ^{-1} is called a **diffeomorphism** and this implies $n = n'$.

Also, if $N = \mathbb{R}$ or $\mathbb{C}$, we sometimes call f a **scalar field**. Further if M is an (open) interval such that $M = I \subset \mathbb{R}$, then $f : I \to N$ is a smooth curve in N.

Lastly, if f is smooth in one atlas, it is smooth with respect to all compatible atlases.

Examples.

1) Recall $S^1 = \{\mathbf{x} \in \mathbb{R}^2 \mid |x| = 1\}$. Let $f(x, y) = x$, $f : S^1 \to \mathbb{R}$.

Using previous charts,

$$f \circ \phi_1^{-1} : (-\pi, \pi) \to \mathbb{R} \tag{15}$$

$$f \circ \phi_1^{-1}(\theta_1) = \cos\theta_1, \tag{16}$$

and similarly,

$$f \circ \phi_2^{-1} : (0, 2\pi) \to \mathbb{R} \tag{17}$$

$$f \circ \phi_2^{-1}(\theta_2) = \cos\theta_2. \tag{18}$$

In both cases, f is smooth.

2) If $(\mathcal{O}, \phi)$ is a coordinate chart on M, write for $\mathbf{p} \in \mathcal{O}$,

$$\phi(\mathbf{p}) = \left(x^1(\mathbf{p}), x^2(\mathbf{p}), \cdots, x^n(\mathbf{p})\right), \tag{19}$$

then $x^i(\mathbf{p})$ defines a map from $\mathcal{O}$ to $\mathbb{R}$. This is a smooth map for each $i = 1, \cdots, n$. If $(\mathcal{O}', \phi')$ is another overlapping coordinate chart, then $x^i \circ \phi'^{-1}$ is the ith component of $\phi \circ \phi'^{-1}$, which is smooth.

3) We can define a smooth function chart by chart. For simplicity, we take $N = \mathbb{R}$. Let $\{(\mathcal{O}_\alpha, \phi_\alpha)\}$ be an atlas on M. Define smooth functions $F_\alpha : U_\alpha \to \mathbb{R}$, and suppose that

$$F_\alpha \circ \phi_\alpha = F_\beta \circ \phi_\beta, \tag{20}$$

on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ for all α, β. Then for $\mathbf{p} \in M$, we can define $f(\mathbf{p}) = F_\alpha \circ \phi_\alpha(\mathbf{p})$ where $(\mathcal{O}_\alpha, \phi_\alpha)$ is any chart with $\mathbf{p} \in \mathcal{O}_\alpha$ as this is constant by construction of F. f is smooth as

$$f \circ \phi_\beta^{-1} = F_\alpha \circ \underbrace{\phi_\alpha \circ \phi_\beta^{-1}}_{\text{always smooth}}. \tag{21}$$

In practice, we often don't distinguish between f and its **coordinate chart representation** F_α. This coordinate chart representation F_α captures f but maps from $U_\alpha \subset \mathbb{R}^n$ rather than from subsets of M. One can think of $F_\alpha = f \circ \phi_\alpha^{-1}$ as finding the point on M that ϕ_α mapped from and evaluating f at that point.

2.2 Curves and Vectors

For a surface in $\mathbb{R}^3$, we have a notion of a tangent space at a point, consisting of all vectors tangent to the surface. Such tangent spaces are vector spaces (copies of $\mathbb{R}^2$). Different points have different tangent spaces.

In order to define the tangent space for a manifold, we first consider tangent vectors of a curve.

Recall that a smooth map from an interval $\lambda : I \subset \mathbb{R} \to M$ is a smooth curve in M.

If $\lambda(t)$ is a smooth curve in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then for $f(\lambda(t)) : \mathbb{R} \to \mathbb{R}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(\lambda(t))\right] = \mathbf{X}(t) \cdot \boldsymbol{\nabla} f(\lambda(t)), \tag{22}$$

where $\mathbf{X}(t) = \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}$ is the **tangent vector** to λ at t. The idea is that we identify the tangent vector $\mathbf{X}(t)$ with the differential operator $\mathbf{X}(t) \cdot \boldsymbol{\nabla}$.

Definition 2.2: Let $\lambda : I \to M$ be a smooth curve with $\lambda(0) = \mathbf{p}$. The **tangent vector** to λ at $\mathbf{p}$ is the linear map $X_{\mathbf{p}}$ from the space of smooth functions, $f : M \to \mathbb{R}$ given by

$$X_{\mathbf{p}}(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(\lambda(t)) \Big|_{t=0}. \tag{23}$$

We observe a few things of note.

1) $X_{\mathbf{p}}$ is linear such that $X_{\mathbf{p}}(f + ag) = X_{\mathbf{p}}(f) + aX_{\mathbf{p}}(g)$ for f, g smooth and $a \in \mathbb{R}$.
2) $X_{\mathbf{p}}$ satisfies the Leibniz rule,
$$X_{\mathbf{p}}(fg) = (X_{\mathbf{p}}(f))\, g + fX_{\mathbf{p}}(g). \tag{24}$$
3) If $(\mathcal{O}, \phi)$ is a chart with $\mathbf{p} \in \mathcal{O}$, we write
$$\phi(\mathbf{p}) = \left(x^1(\mathbf{p}), \cdots, x^n(\mathbf{p})\right). \tag{25}$$
Let $F = f \circ \phi^{-1}$, $x^i(t) = x^i(\lambda(t))$ and $\mathbf{x}(t) = \phi(\lambda(t))$. Then we have
$$f \circ \lambda(t) = f \circ \phi^{-1} \circ \phi \circ \lambda(t) = F \circ \mathbf{x}(t), \tag{26}$$
and thus the tangent vector defined in Eq. (23) can also be written as
$$X_{\mathbf{p}}(f) \equiv \frac{\mathrm{d}}{\mathrm{d}t}(f(\lambda(t)))\Big|_{t=0} = \frac{\partial F(x)}{\partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\Big|_{t=0}, \tag{27}$$
where $\frac{\partial F}{\partial x^\mu}$ depends on f and ϕ and $\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ depends on λ and ϕ.

3 Lecture: Tangent Spaces

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3.1 The Tangent Space is a Vector Space

Proposition 3.1: The set of tangent vectors to curves at $\mathbf{p}$ forms a vector space, $T_{\mathbf{p}}M$ of dimension $n = \dim M$. We call $T_{\mathbf{p}}M$, the **tangent space** to M at $\mathbf{p}$.

Proof. Given $X_{\mathbf{p}}, Y_{\mathbf{p}}$ are tangent vectors, we need to show that $\alpha X_{\mathbf{p}} + \beta Y_{\mathbf{p}}$ is a tangent vector for $\alpha, \beta \in \mathbb{R}$. Let λ, κ be smooth curves with $\lambda(0) = \kappa(0) = \mathbf{p}$ and whose tangent vectors at $\mathbf{p}$ are $X_{\mathbf{p}}$ and $Y_{\mathbf{p}}$ respectively. Let $(\mathcal{O}, \phi)$ be a chart with $p \in \mathcal{O}$ such that $\phi(\mathbf{p}) = 0$. We call this a *chart centered at* $\mathbf{p}$.

Let $\nu(t) = \phi^{-1}[\alpha\phi(\lambda(t)) + \beta\phi(\kappa(t))]$ where notice $\nu(0) = \phi^{-1}(0) = \mathbf{p}$.

From Eq. (27), we have that if Z_p is the tangent to ν at $\mathbf{p}$, we have

$$\begin{aligned} Z_{\mathbf{p}}(f) &= \frac{\mathrm{d}}{\mathrm{d}t}(f(\nu(t)))\Big|_0 & (28)\\ &= \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{\mathrm{d}}{\mathrm{d}t}\left[\alpha x^\mu(\lambda(t)) + \beta x^\mu(\kappa(t))\right]\Big|_{t=0} & (29)\\ &= \alpha \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\lambda(t))\Big|_{t=0} + \beta \frac{\partial F}{\partial x^\mu}\Big|_0 \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\kappa(t))\Big|_{t=0} & (30) \end{aligned}$$

$$= \alpha X_{\mathbf{p}}(f) + \beta X_{\mathbf{p}}(f), \tag{31}$$

as desired. Therefore $T_{\mathbf{p}}M$ is a vector space. □

To see that $T_{\mathbf{p}}M$ is n-dimensional, consider the curves

$$\lambda_\mu(t) = \phi^{-1}\left(0, \cdots, 0, \underbrace{t}_{\mu\text{th component}}, 0, \cdots, 0\right). \tag{32}$$

We denote the tangent vector to λ_μ at $\mathbf{p}$ by $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$.

Note. This is **not** a differential operator.

However observe that by definition, we have

$$\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}(f) = \left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(\mathbf{p})=0}. \tag{33}$$

The vectors $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$ are linearly independent. Otherwise $\exists \alpha^\mu \in \mathbb{R}$ not all zero such that

$$\alpha^\mu \left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}} = 0, \tag{34}$$

which implies

$$\alpha^\mu \frac{\partial F}{\partial x^\mu} = 0, \tag{35}$$

for all F. Setting $F = x^\nu$ gives $\alpha^\nu = 0$ and thus these vectors are linearly independent and spanning. They are therefore a basis for the vector space.

Further one can see that $\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}$ form a basis for $T_{\mathbf{p}}M$, since if λ is any curve with tangent $X_{\mathbf{p}}$ at $\mathbf{p}$, we have

$$X_{\mathbf{p}}(f) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x=0} \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\lambda(t)) = X^\mu \left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}(f), \tag{36}$$

where $X^\mu = \left.\frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\lambda(t))\right|_{t=0}$ are the **components** of $X_{\mathbf{p}}$ with respect to the basis $\{\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}\}_{\mu=1,\cdots,n}$ for $T_{\mathbf{p}}M$.

Note. The basis $\{\left(\frac{\partial}{\partial x^\mu}\right)_{\mathbf{p}}\}_{\mu=1,\cdots,n}$ depends on the coordinate chart ϕ.

Suppose we choose another chart $(\mathcal{O}', \phi')$, again centered at $\mathbf{p}$. We write $\phi' = \left(\left(x'\right)^1, \cdots, (x')^n\right)$. Then if $F' = f \circ \phi'^{-1}$, we have

$$F(x) = f \circ \phi^{-1}(x) \tag{37}$$
$$= f \circ \phi'^{-1} \circ \phi' \circ \phi^{-1}(x) \tag{38}$$

$$= F'\left(x'\left(x\right)\right). \tag{39}$$

Therefore,

$$\left(\frac{\partial}{\partial x^{\mu}}\right)_{\mathbf{p}}(f) = \left.\frac{\partial F}{\partial x^{\mu}}\right|_{\phi(\mathbf{p})} \tag{40}$$

$$= \left.\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right)\right|_{\phi(\mathbf{p})} \left.\left(\frac{\partial F'}{\partial x'^{\nu}}\right)\right|_{\phi'(\mathbf{p})} \tag{41}$$

$$= \left.\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right)\right|_{\phi(\mathbf{p})} \left(\frac{\partial}{\partial x'^{\nu}}\right)_{\mathbf{p}}(f). \tag{42}$$

We then deduce that

$$\left(\frac{\partial}{\partial x^{\mu}}\right)_{\mathbf{p}} = \left.\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right)\right|_{\phi(\mathbf{p})} \left(\frac{\partial}{\partial x'^{\nu}}\right)_{\mathbf{p}}. \tag{43}$$

Let X^{μ} be components of $X_{\mathbf{p}}$ with respect to the basis $\left(\frac{\partial}{\partial x^{\mu}}\right)_{\mathbf{p}}$, and X'^{μ} be components of $X_{\mathbf{p}}$ with respect to the basis $\left(\frac{\partial}{\partial x^{\mu}}\right)_{\mathbf{p}}$ such that

$$X_{\mathbf{p}} = X^{\mu}\left(\frac{\partial}{\partial x^{\mu}}\right)_{\mathbf{p}} = X'^{\mu}\left(\frac{\partial}{\partial x'^{\mu}}\right)_{\mathbf{p}} \tag{44}$$

$$= X^{\mu}\left(\frac{\partial x'^{\sigma}}{\partial x^{\mu}}\right)\left(\frac{\partial}{\partial x'^{\sigma}}\right)_{\mathbf{p}}, \tag{45}$$

and therefore

$$X'^{\mu} = \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) X^{\nu}. \tag{46}$$

Note. We do note have to choose a coordinate basis such as $\left(\frac{\partial}{\partial x^{\mu}}\right)_{\mathbf{p}}$. With respect to a general basis $\{e_{|mu}\}$, for $T_{\mathbf{p}}M$, we can write $X_{\mathbf{p}}X^{\mu}e_{\mu}$ for $X^{\mu} \in \mathbb{R}$.

We always use summation convention, contracting covariant indices with contravariant indices.

3.2 Covectors

Recall that if V is a vector space over $\mathbb{R}$, the dual space V^* is the space of linear maps $\phi : V \to \mathbb{R}$. If V is n-dimensional then so is V^* (the spaces are then isomorphic). Given a basis $\{e_{\mu}\}$ for V, we can define the dual basis $\{f^{\mu}\}$ for V^* by requiring that

$$f^{\mu}\left(e_{\nu}\right) = \delta^{\mu}{}_{\nu} = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu. \end{cases} \tag{47}$$

If V is finite dimensional, then $V^{**} = \left(V^*\right)^* \simeq V$. Namely, to an element $X \in V$, we assign the linear map

$$\Lambda_X : V^* \to \mathbb{R}, \tag{48}$$

$$\Lambda_X\left(\omega\right) = \omega\left(X\right), \tag{49}$$

for $\omega \in V^*$.

Definition 3.1: The dual space of $T_{\mathbf{p}}M$ is denoted $T^*_{\mathbf{p}}M$ and is called the **cotangent space** to M at $\mathbf{p}$. An element of this space is a **covector** at $\mathbf{p}$. If $\{e_\mu\}$ is a basis for $T_{\mathbf{p}}M$ and $\{f^\mu\}$ is the dual basis for $T^*_{\mathbf{p}}M$, we can expand a covector η as

$$\eta = \eta_\mu f^\mu, \tag{50}$$

for **components** $\eta_\mu \in \mathbb{R}$.