Special Topics in Physics

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March 17, 2023

Contents

1 Lecture: Introduction and Schedule

21/02/2023

- Week 2: Path Integral Methods in QFT - Statistical Field Theory (Euclidean QFT) - **James & Cian**
- Week 3- 6: Renormalisation in QFT (φ^4) - **Alex, Achintya, James & Cian**
- Assignment: Renormalisation of $\frac{\lambda}{3!}\varphi^3$ in 6D (1 loop).
- Week 7-9: QED Scattering amplitudes. **TBD.**
- Week 10-13: Renormalisation of QED, SM and QFT on curved spacetime. **TBD.**

2 Lecture: Path Integral Formalism

03/03/2023

This lecture and accompanying notes follow closely LeBellac Chapter 8. - Cian

2.1 Ising Model Example

Consider a single spin-$\frac{1}{2}$ system with Hamiltonian

$$\begin{aligned} H &= -J\sigma_x \\ &= -J\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

which has eigenstates

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

with eigenvalues $E_0 = -J$ and $E_1 = J$.

Definition 2.1: We define the matrix element of the evolution operator $U(t) = \exp(-iHt)$ by

$$F(t, S_b \mid 0, S_a) = \langle S_b| e^{-iHt} |S_a\rangle,$$

where $|S_i\rangle$ are eigenstates of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, (with eigenvalues $S_i = \pm 1$) but in general could be arbitrary states. This matrix element represents the probability amplitude to observe S_b at time t given that the system was in state S_a at $t = 0$.

Next, note that as $\{|S_a\rangle, |S_b\rangle\}$ spans the Hilbert space (i.e. they form a complete basis), we can write

$$\mathbb{I} = \sum_{S_i = \pm 1} |S_i\rangle \langle S_i|.$$

Note. While this expression may be familiar, it is interesting to notice that the $\langle S_i|$ can be considered projection operators onto the state $|S_i\rangle$. For example, in 2D, one may project onto the x and y axes and rewrite the state as $\varphi = (\hat{x}P_x + \hat{y}P_y)\varphi$ where P_x is a projection operator and $\hat{x}$ here is analogous to a ket.

Supposing that $t = N \gg 1$, we can insert an identity operator at every integer t by splitting the evolution operator into N time steps such that

$$\begin{aligned} \langle S_b| e^{-iHt} |S_a\rangle &= \langle S_b| e^{-iH}\mathbb{I}\cdots\mathbb{I}e^{-iH} |S_a\rangle \\ &= \sum_{S_1=\pm 1}\cdots\sum_{S_{N-1}=\pm 1} \langle S_b| e^{-iHt} |S_{N-1}\rangle \\ &\left(\prod_{i=N-1}^{2} \langle S_i| e^{-iH} |S_{i-1}\rangle\right) \langle S_1| e^{-iH} |S_a\rangle. \end{aligned}$$

Then, we take

$$\langle S| e^{-iH} |S'\rangle = e^{-iV(S,S')},$$

which allows us to write

$$\langle S_b| e^{-iH} |S_a\rangle = \sum_{[S_i]} \exp\left(-i\left(V(S_b, S_{N-1}) + \sum_{i=N-1}^{2} V(S_i, S_{i-1}) + V(S_1, S_a)\right)\right).$$

Now, if we take the time to be imaginary with $t = -i\tau$, then the matrix element becomes

$$\langle S_b| e^{-H\tau} |S_a\rangle = \sum_{[S_i]} \exp\left(V(S_b, S_{N-1}) + \sum_{i=N-1}^{2} V(S_i, S_{i-1}) + V(S_1, S_a)\right).$$

Observe that

$$\begin{aligned} e^{-H} &= e^{J\sigma_1} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \\ &= \begin{pmatrix} \cosh J & \sinh J \\ \sinh J & \cosh J \end{pmatrix}, \end{aligned}$$

where if we take

$$Z_N = \sum_{[S_i]} e^{-H}$$

we see that it equals

$$= \operatorname{tr} e^{-iH\tau}.$$

Namely, the partition function of the quantum spin system is equal to that of a classical N spin system.

2.2 Path Integral Derivation

Consider now a particle evolving according to general H with probability amplitude now given by

$$\begin{aligned} F(q', t' \mid q, t) &= \langle q', t' | q, t\rangle \\ &= \langle q'| e^{-iH(t'-t)} |q\rangle. \end{aligned}$$

As before we subdivide the time interval into n intervals of length $\varepsilon = (t' - t)/n$ and we specify

$$H(t' - t) = n\varepsilon \left(\frac{\hat{p}^2}{2m} + V(q)\right).$$

We can then insert complete sets of eigenstates $\{|q_i\rangle\}$ as identity operators yielding

$$F(q', t' \mid q, t) = \int \left(\prod_{l=1}^{n} \mathrm{d}q_i\right) \prod_{l=0}^{n} \langle q_{l+1}| \exp\left(-i\varepsilon\left(\frac{\hat{p}^2}{2m}\right)\right) \exp(-i\varepsilon V(\hat{q})) |q_l\rangle.$$

The matrix element of interest simplifies to

$$
\begin{aligned}
&= \exp\left(-i\varepsilon V\left(q_l\right)\right) \langle q_{l+1}| \exp\left(-i\varepsilon \frac{\hat{p}^2}{2m}\right) |q_l\rangle \\
&= \exp\left(-i\varepsilon V\left(q_l\right)\right) \int \frac{\mathrm{d}p_l}{2\pi} \langle q_{l+1}| \exp\left(-i\varepsilon \frac{\hat{p}^2}{2m}\right) |p_l\rangle \langle p_l|q_l\rangle \\
&= \exp\left(-i\varepsilon V\left(q_l\right)\right) \int \frac{\mathrm{d}p_l}{2\pi} \exp\left(-i\varepsilon \frac{p_l^2}{2m}\right) \langle q_{l+1}|p_l\rangle \langle p_l|q_l\rangle
\end{aligned}
$$

where $\langle p_l|q_l\rangle = \frac{1}{2\pi} e^{-iq_l p_l}$ yields

$$
= \exp\left(-i\varepsilon V\left(q_l\right)\right) \int \frac{\mathrm{d}p_l}{2\pi} \exp\left(-i\varepsilon \frac{p_l^2}{2m}\right) \exp\left(i\left(q_{l+1} - q_l\right) p_l\right).
$$

Returning this matrix element to the probability amplitude expression, we see that

$$
F\left(q', t' \mid q, t\right) = \int \left(\prod_{l=1}^{n} \mathrm{d}q_i\right) \prod_{l=0}^{n} \int \frac{\mathrm{d}p_l}{2\pi} \exp\left(-i\varepsilon\left(V\left(q_l\right) + \frac{p_l^2}{2m}\right)\right) \exp\left(i\left(q_{l+1} - q_l\right) p_l\right).
$$

As V is independent of p_l, we can integrate over p_l and obtain

$$
F\left(q', t' \mid q, t\right) = \lim_{\varepsilon \to 0} \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \int \left(\prod_{l=1}^{n} \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \mathrm{d}q_l\right) \exp\left(i \sum_{l=0}^{n} \varepsilon \frac{m\left(q_l - q_{l+1}\right)^2}{2\varepsilon^2} - \varepsilon V\left(q_l\right)\right)
$$

Where in the limit of $\varepsilon \to 0$,

$$
\begin{aligned}
F\left(q', t' \mid q, t\right) &= \int \mathcal{D}q \exp\left(i \int_t^{t'} \frac{1}{2} m\dot{q}^2 - V\left(q\right)\right) \\
F\left(q', t' \mid q, t\right) &= \int \mathcal{D}q \exp\left(iS\right),
\end{aligned}
$$

and we have denoted the integration measure by

$$
\mathcal{D}q = \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \prod_{l=1}^{n} \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \mathrm{d}q_l\,.
$$

we arrive at the desired path integral expression. Namely, this equation states that it is through summing over all possible paths the particle can take, weighted by their action, that we obtain the amplitude for a given process. The weighting by the action ensures that unlikely paths contribute negligibly (in fact the fast rotation of the action in the complex plane means they cancel themselves). On the other hand paths close to the classical equation of motion (which is stationary with respect to the action) will add constructively and thus contribute measurably to the probability amplitude of the process.

Note. We can then analogously obtain matrix elements of time ordered products such that

$$
\langle q', t'| TQ\left(t_1\right) Q\left(t_2\right) |q, t\rangle = \int \mathcal{D}q q\left(t_1\right) q\left(t_2\right) e^{iS},
$$

namely such that $t_1 > t_2$.

2.3 Functional Derivatives

Definition 2.2: Given a functional $F : M \to \mathbb{R}$ where M is some space (manifold), we have that the functional derivative with respect to some function ρ is given by

$$\begin{aligned}\int \frac{\delta F}{\delta \rho(x)} \varphi(x)\,\mathrm{d}x &= \lim_{\varepsilon \to 0} \frac{F[\rho + \varepsilon\varphi] - F[\rho]}{\varepsilon} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F[\rho + \varepsilon\varphi]\right]_{\varepsilon=0}.\end{aligned}$$

One can consider the functional derivative as the gradient of F at the point ρ in function space in the direction of φ.

Examples. Given

$$F[\varphi(x)] = e^{\int \varphi(x) g(x) \mathrm{d}x},$$

the functional derivative with $\varphi = \delta(x-y)$ is

$$\begin{aligned}\frac{\delta F[\varphi(x)]}{\delta \varphi(y)} &= \frac{\delta F[\varphi(x) + \varepsilon\delta(x-y)] - F[\varphi(x)]}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{e^{\int (\varphi(x) + \varepsilon\delta(x-y)) g(x) \mathrm{d}x} e^{\int \varphi(x) g(x) \mathrm{d}x}}{\varepsilon} \\ &= e^{\int \varphi(x) g(x) \mathrm{d}x} \lim_{\varepsilon \to 0} \frac{e^{\varepsilon \int \delta(x-y) g(x) \mathrm{d}x} - 1}{\varepsilon} \\ &= e^{\int \varphi(x) g(x) \mathrm{d}x} \lim_{\varepsilon \to 0} \frac{e^{\varepsilon g(y)} - 1}{\varepsilon} \\ &= g(y) F[\varphi(x)].\end{aligned}$$

2.4 Generating Functional

We modify the Lagrangian to include a source term

$$L = \frac{1}{2} m\dot{q}^2 - V(q) + j(t) q(t),$$

which results in an external force $j(t)$ which we take to be nonzero on $[t, t']$.

We define the generating functional

$$\begin{aligned}Z(y) &= \lim_{T \to i\infty, T' \to -i\infty} \frac{\langle Q', T' | Q, T\rangle}{e^{-iE_0(T'-T)} \varphi_0^* \varphi_0(Q')} \\ Z(j) &= \int \mathrm{d}q\,\mathrm{d}q'\, \varphi_0^*(q', t') \langle q', t' | q, t\rangle \varphi_0(q, t)\end{aligned}$$

$$Z(j) = \langle 0 | e^{iHt'} U_j(t', t) e^{-iHt} | 0 \rangle,$$

with U_j satisfying

$$i \frac{\mathrm{d}U_j}{\mathrm{d}t} = (H - j(t) Q) U_j(t).$$

Writing the matrix elements as a path integral we have

$$Z(j) = \mathcal{N} \lim_{T^{(\prime)} \to \pm i\infty} \int \mathcal{D}q \exp\left(i \int_T^{T'} \left(L(q, \dot{q}) + j(t)\, q(t) \right) \right),$$

where by using the result derived in the example above, the functional derivative gives

$$\langle 0 | T(Q(t_1)\, Q(t_2)) | 0 \rangle = \frac{(-i)^2}{Z(0)} \frac{\mathrm{d}^2 Z(j)}{\mathrm{d}j(t_1)^2} j(t_2) \bigg|_{j=0}$$

.

3 Lecture: Wick's Theorem

10/03/2023

3.1 Achintya's notes

3.2 James' notes

4 Lecture

16/03/2023

Definition 4.1: We define the **superficial degree (or index) of divergence** of the diagram, which determines the degree of ultraviolet divergence ($q \to \infty$) by

$$\omega(G) = LD - 2I,$$

with $L = I - V + 1$ as the number of independent integration variables/loops, D is the dimension and I is the number of internal lines.

This is motivated by the general dimensional form of the integrand, namely,

$$\int^{\Lambda} \left(\mathrm{d}^D q\right)^L \left(p^2\right)^{-I},$$

has dimension $\omega(G)$.

If $\omega(G) > 0$ then the integral diverges with $\Lambda^{\omega(G)} (\ln \Lambda)^n$ for $n \in \mathbb{Z}$. Similarly, if $\omega(G) = 0$ then the integral diverges with $(\ln \Lambda)^p$. Lastly, if $\omega(G) < 0$ then we call the graph **superficially convergent** (as sub-integrations could still diverge).

4.1 Topological Argument

Alternative expressions for $\omega(G)$ prove useful.

For general derivative type interactions, our dimension now scales with

$$\omega(G) = LD - 2I + \sum_{i=1}^{V} \delta_i$$

where our i interactions each contains δ_i derivatives. With $L = I - V + 1$, we have

$$\omega(G) = I(D-2) - VD + D + D + \sum_{i=1}^{V} \delta_i$$

$$\omega(G) - D = I(D-2) + \sum_{i=1}^{V} (\delta_i - D)$$

Splitting $n_i = n_i^{\text{int}} + n_i^{\text{ext}}$ we have $I = \frac{1}{2}\sum_i n_i^{\text{int}}$ and thus

$$\omega(G) - D = \left(\frac{D}{2} - 1\right) \sum_i n_i^{\text{int}} + \sum_i (\delta_i - D)$$

$$\omega(G) - D = \left(\frac{D}{2} - 1\right) \sum_i \left(n_i - n_i^{\text{ext}}\right) + \sum_i (\delta_i - D)$$

Defining $\omega_i = n_i \left(\frac{D}{2} - 1\right) + \delta_i$ we see with $E = \sum_i n_i^{\text{ext}}$ we have

$$\omega(G) - D = \sum_i (\omega_i - D) = E\left(\frac{D}{2} - 1\right),$$

Examples. For φ^4 theories,

$$\begin{aligned}
\omega_i &= 4\left(\frac{D}{2} - 1\right) + 0 \\
&= 2D - 4 \\
\Rightarrow \omega(G) - D &= V\left(4\frac{D}{2} - 4 - D\right) - E\left(\frac{D}{2} - 1\right) \\
&= V(D-4) - E\left(\frac{D}{2} - 1\right)
\end{aligned}$$

For $E = 2$

$$\omega(G) = V(D-4) + 2$$

For $E = 4$,

$$\omega(G) = V(D-4) + 4 - D$$

For $E = 6$,

$$\omega(G) = V(D-4) + 6 - 2D.$$

Namely, as the number of external lines rises $\omega(G)$ decreases. For such a φ^4 theory with $D = 4$, we see that only $E < 6$ are superficially divergent.

4.2 Dimensional argument

We can write a general interaction term as

$$g_i \int \mathrm{d}^D x\, \boldsymbol{\nabla}^{\delta_i} \varphi^{n_i},$$

which must be dimensionless, and thus

$$[g_i] - D + \delta_i + n_i[\varphi] = 0$$

where $[\varphi] = \frac{D}{2} - 1$. Using the ω_i index definition above, we can write the dimension of the coupling constant as

$$[g_i] = D - \omega_i.$$

One can recover the previous result from this expression by considering $\Gamma^{(E)}(k)$.

4.3 Infrared divergences

While ultraviolet divergences occur independently of the external momenta as they are concerned with internal propagators, **infrared divergences** can arise due to unconstrained external momenta.

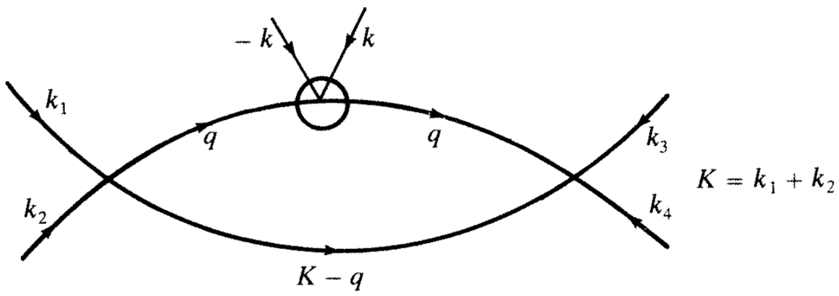

Figure 1: 5.32 from LeBellac p.g. 193

Figure 1 is proportional to

$$\int \frac{\mathrm{d}^4 q}{q^4 \left(k-q\right)^4},$$

which diverges for small $q \to 0$, namely, in the infrared.

Definition 4.2: A configuration of external momenta is called **non-exceptional** if no partial sum of the k_i vanishes,

$$\sum_{i \in I} k_i \neq 0,$$

for any subset I of the external momenta. Likewise, it is called **exceptional** if any partial sum vanishes.

In a non-exceptional configuration, all external momenta can be linked by internal lines with nonzero momentum which we call **hard lines**.

If this was not the case, then we could split the diagram into two by cutting internal lines of zero momentum, appropriately named **soft lines**.

If we contract all hard internal lines to a single vertex, and set n to be the number of soft internal lines, we have the relations

$$\begin{aligned} L &= I - V \\ 2I &= 4V + n \end{aligned}$$

which yields a superficial divergence of

$$\omega\left(G\right) = n + L\left(D-4\right),$$

if all internal loops have negligible momenta.

Namely, we have that the diagram is infrared-convergent when $D = 4$ as we have $n \geq 2 \Rightarrow \omega(G) > 0$.

On the contrary, for $D < 4$, diagrams are commonly infrared divergent even in non-exceptional configurations.

Theorem 4.1 (Weinberg's Theorem): Consider a φ^4 theory in $D = 4$ and assume that J is ultraviolet convergent. If its $m = 0$ limit exists, as it will if the diagram is non-exceptional, then

$$J(\lambda k_i) \underset{\lambda\to\infty}{\sim} \lambda^{\omega(G)}.$$

If J has to be renormalized due to ultraviolet divergences, then we find that

$$J(\lambda k_i) \underset{\lambda\to\infty}{\sim} \lambda^{\omega(G)} (\ln\lambda)^n,$$

for some fixed $n \in \mathbb{Z}$ determined by the graph.

4.4 Classification of Renormalizability

The systematic removal of such divergences is called **renormalization**. In general, this is done by *absorbing* the divergences into the definition of the coupling constant, mass and field normalization.

For a interaction with only a single field, the superficial degree of divergence of a graph G reduces to

$$\omega(G) - D = V(\omega - D) - E\left(\frac{D}{2} - 1\right)$$

,

where $D - \omega = [g]$ is the dimension of the coupling constant. Clearly, we can see that if $\omega > D$, then the degree of divergence increases as V (or equivalently the order of the perturbation) increases. Such a theory is **nonrenormalizable**. We would need to fix infinitely many parameters in order to see finite results. It is possible to have nonrenormalizable interactions in some cases where the divergences cancel for all physical quantities, but this is equivalent to a renormalizable theory.

If $\omega < D$ then we will only have a finite number of divergent graphs as for sufficiently large V, $\omega(G) < 0$. Such theories are called super-renormalizable and despite their appeal, are challenging and have yet to find application.

Lastly, if $\omega = D$ then the degree of divergence is independent of the order of the perturbation and $[g] = 0$. Such theories are **renormalizable**. We can fix the divergence by fixing a finite number of parameters.

Example. For a φ^4 theory, $\omega_i = 2D - 4$, and thus

$$[g_0] = 4 - D.$$

where if $D = 4$ the theory is renormalizable and thus we find that only the two-point and the four-point correlation functions are divergent. For $E \geq 6$ the correlation functions are superficially convergent. Subintegrations can ruin this visage, however we have the following theorem to help.

Theorem 4.2 (First convergence theorem): If all connected 1-PI subdiagrams γ of a diagram G including itself are such that $\omega(\gamma) < 0$, then the Feynman integral of G is absolutely convergent.

4.5 Regularization

There are a number of ways to regularise integrals, namely, to make them finite in the process of a calculation. Note that regularization procedures may differ but should all produce the same renormalized theory.

We have the following common regularization procedures:

a) Brute-force cutoff: restricting integrals to $\|\mathbf{q}\| < \Lambda$. This is impractical beyond one-loop.

b) Schwinger regularization which follows from

$$\frac{1}{q^2+m^2} \to \int_{\Lambda^{-2}}^{\infty} \mathrm{d}\alpha\, e^{-\alpha\left(q^2+m^2\right)}.$$

c) Pauli-Villars regularisation which follows from

$$\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2} - \frac{1}{q^2+\Lambda^2}.$$

d) Dimensional regularization where we take $\mathrm{d}q^{4-\varepsilon}$

e) and lattice regularization where we quantize space and replace fields by a lattice variable

$$\varphi_i \simeq \frac{1}{a^D}\varphi(x)\,\mathrm{d}^D x\,,$$

where we integrate over a volume a^D centered on site i with a cutoff $\Lambda \sim \pi/a$.

Some regularization procedures break symmetries, e.g. lattice regularization breaks rotation and translational invariance.

However, even in this case we can show that the resulting renormalized theory does have the previous symmetries even if they were not present through every step of the calculation.

There also exists cases where the classical symmetry cannot be preserved and this is what is called an **anomaly**.