Quantum Field Theory

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2024-10-14

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1 Lecture: Introduction

11/10/2024

Historically, the goal of quantum field theory was to combine quantum mechanics with special relativity. One of the most notable outputs of this study hailed as a success is that the number of particles is not conserved. It is a robust and systematic theory governed by few principles. It concerns itself with locality, symmetries and renormalization which are exceptionally constraining and *almost* uniquely determine what we can study.

In this course we use $c = \hbar = 1$. In these natural units, $E = mc^2$ gives us masses in the units of energy.

For metrics we use $\eta^{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$, $x^\mu = (t, x, y, z)$, $F(t, \vec{x}) \equiv F(x^\mu) \equiv F(\mathbf{x})$

1.1 Classical Field Theory

In classical mechanics, a natural object is the action,

$$S(t_1, t_2) = \int_{t_1}^{t_2} \mathrm{d}t \left(\underbrace{m \sum_{i=1}^{3} \left(\frac{\mathrm{d}x^i}{\mathrm{d}t} \right)^2}_{\text{kinetic term}} - \underbrace{V(x)}_{\text{potential}} \right). \tag{1}$$

This is incredibly useful for us for three main reasons:

- the equations of motion are given for free by extremising S,

- Boundary conditions are supplied externally, and
- S is built on *symmetry* (it is invariant of symmetries of your system).

As we move towards field theory, we no longer want to speak of a single position of a particle $x(t)$.

The fundamental object in field theory is a field $\phi_a(t,\vec{x}) : \mathbb{R}^{3,1} \to \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{R}^n$. Here a labels the type of field we are discussing.

The first consequence is that we are dealing with an infinite number of degrees of freedom as every point in time and space contains some distinct information about the system.

Example. In electromagnetism, as we will discuss in depth later, one has the gauge field $A^\mu(t,\vec{x}) = \left(\phi(x), \vec{A}(x)\right)$ which the electric and magnetic fields can be defined in terms of

$$\vec{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{2}$$

$$\vec{B} = \boldsymbol{\nabla} \cdot \vec{A}, \tag{3}$$

which have equations of motion

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \tag{4}$$

$$\boldsymbol{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}, \tag{5}$$

and two identities

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \tag{6}$$

$$\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} = -\boldsymbol{\nabla} \times \vec{E}. \tag{7}$$

This is a (hopefully) familiar classical field that we will quantise in due time.

1.2 Lagrangians

The Lagrangian in classical mechanics can be written $L = T - V$ and is contained within the action in the form

$$S = \int \mathrm{d}t\, L. \tag{8}$$

We will in QFT concern ourselves with the *Lagrangian density* given by

$$L = \int \mathrm{d}^3x\, \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right), \tag{9}$$

however *everybody* just refers to $\mathcal{L}$ as a Lagrangian as we will here.

The equations of motion are determined by extremizing with respect to the fields.

Note. Note that we assume that the Lagrangian $\mathcal{L}\left[\phi_a, \partial_\mu \phi_a\right]$ is not a function of $\partial^2 \phi_a$ or higher derivatives. This is for complicated reasons related to ghosts that are beyond the scope of this course.

Extremising the action with respect to the field, we want to find the conditions for which $\Delta S = 0$, i.e. the action is at a minima/saddle point. We see that

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\delta\left(\partial_\mu \phi_a\right)\right] \tag{10}$$

$$= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right)\delta\phi_a + \underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\delta\phi_a\right)}_{\text{total derivative}}\right], \tag{11}$$

and by assuming that our fields decay at infinity, the total derivative term vanishes yielding

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right)\right]\delta\phi_a, \tag{12}$$

for which vanishing requires

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{13}$$

Example. A free massive scalar field is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu - \frac{1}{2}m^2\phi^2 \tag{14}$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 - \frac{1}{2}m^2\phi^2. \tag{15}$$

In traditional classical mechanics, one would have identified $T = \frac{1}{2}\dot{\phi}^2$ and $V = \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2$. In QFT, the 'kinetic terms' sometimes refers to any bilinear combination of fields. For example, $\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is always kinetic and $m^2\phi^2$ is often a (bosonic) mass term.

The equation of motion for the free massive scalar field Lagrangian is

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0. \tag{16}$$

This is the **Klein Gordon equation**. It is also sometimes written with $\partial_\mu\partial^\mu = \Box$.

2 Lecture: Symmetries

14/10/2024

2.1 Hamiltonian Formalism

In a Hamiltonian formalism, one starts by defining the canonical momenta

$$\Pi^a\left(x\right) \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_t\phi_a\right)} = \frac{\partial \mathcal{L}}{\partial\dot{\phi}_a}. \tag{17}$$

Definition 2.1: The **Hamiltonian density** is defined by Legendre transform of the Lagrangian density

$$\mathcal{H} = \Pi^a\partial_t\phi_a - \mathcal{L}. \tag{18}$$

The Hamiltonian is given by

$$H = \int \mathrm{d}^3x \, \mathcal{H}. \tag{19}$$

We will not abuse notation and always call $\mathcal{H}$ a *Hamiltonian density*, and H a *Hamiltonian.*

Example. For a scalar field with a potential $V(\phi)$, we have

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{20}$$

The canonical momentum is

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}, \tag{21}$$

and the Hamiltonian is then

$$H = \int \mathrm{d}^3x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V(\phi)\right). \tag{22}$$

2.2 Symmetry

Symmetries are inseparable from the study of quantum field theory. Most notably they dictate the actions we can write, the class of fields (operators) we can use, and the observables we can compute.

Definition 2.2: The **Lorentz group** has elements $\Lambda^\mu{}_\nu$ such that under Lorentz boosts

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{23}$$

which preserve the spacetime interval $s^2 = x^\mu x^\nu \eta_{\mu\nu} = t^2 - x^i x_i$ such that

$$s^2 \to s'^2 = s^2. \tag{24}$$

This condition implies

$$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}. \tag{25}$$

In matrix form, this can be written $\Lambda^T\eta\Lambda = \eta$.

Examples. Rotations such as one in the xy plane, leave $t' = t$ and have $\Lambda^1_1 = R^1_1$ such that

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Boosts mix time and space. Boosting in the (t, x) plane, we have

$$\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

where η is the **rapidity** and is given by

$$\cosh\eta = \frac{1}{\sqrt{1-v^2}} \tag{28}$$

$$\sinh\eta = \frac{v}{\sqrt{1-v^2}}. \tag{29}$$

Note. From 1), we see that in general $\det(\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$.

If $\det\Lambda = 1$, then Λ is called a *proper* Lorentz transformation.

If $\det\Lambda = -1$, then Λ is called a *improper* Lorentz transformation. Parity and time reversal each independently cause $\det\Lambda = -1$. Only proper Lorentz transformations are continuously connected to the identity.

We will assume $\det\Lambda = 1$. We can then expand about the identity infinitesimally and write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \varepsilon^{\mu}{}_{\nu} + \mathcal{O}\left(\varepsilon^2\right). \tag{30}$$

The natural question is what are the properties of $\varepsilon^{\mu}{}_{\nu}$?

Inserting this expression into Eq. (25), we see

$$\begin{aligned} \eta_{\rho\sigma} &= \eta_{\mu\nu}\left(\delta^{\mu}_{\rho} + \varepsilon^{\mu}{}_{\rho} + \cdots\right)\left(\delta^{\nu}_{\sigma} + \varepsilon^{\nu}{}_{\sigma} + \cdots\right) \\ &= \eta_{\mu\nu}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\varepsilon^{\mu}{}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\delta^{\mu}_{\rho}\varepsilon^{\nu}{}_{\sigma} + \mathcal{O}\left(\varepsilon\right)^2 \\ &= \eta_{\rho\sigma} + \varepsilon_{\sigma\rho} + \varepsilon_{\rho\sigma} \\ \Rightarrow \varepsilon_{\sigma\rho} &= -\varepsilon_{\rho\sigma}. \end{aligned} \tag{31}$$

Therefore $\varepsilon_{\sigma\rho}$ is an antisymmetric tensor, which in $d = 4$ has $\frac{d(d-1)}{2} = 6$ independent components.

Therefore we have 6 generators for the Lorentz group:

- 3 rotations, and
- 3 boosts

2.3 Fields Revisited

We can now think of a field as an object which transforms under the Lorentz group. It therefore forms a representation of the algebra.

Definition 2.3: A field is an object that depends on coordinates and has a definite transformation under the action of the Lorentz group,

$$x \to x' = \Lambda x, \tag{32}$$

$$\phi_a(x) \to \phi'_a(x) = D[\Lambda]_a^b\, \phi_b\left(\Lambda^{-1}x\right). \tag{33}$$

$D[\Lambda]$ forms a representation of the Lorentz group as it satisfies

$$D[\Lambda_1]\,D[\Lambda_2] = D[\Lambda_1\Lambda_2]\,, \tag{34}$$
$$D\left[\Lambda^{-1}\right] = D[\Lambda]^{-1}\,, \tag{35}$$
$$D[\mathbb{I}] = 1.. \tag{36}$$

Examples.

1) Consider the trivial representation $D[\Lambda] = 1$. Then the field transforms as

$$\phi(x) = \phi\left(\Lambda^{-1}x\right), \tag{37}$$

which is an equivalent definition of the *scalar field*. Here we are using active transformations where the coordinates are fixed.

2) We are also familiar with the vector representation given by

$$D[\Lambda]^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}. \tag{38}$$

A field transforming under this representation is A^{μ} such that

$$A^{\mu}(x) \to A'^{\mu}(x) = \Lambda^{\mu}{}_{\nu}A^{\nu}\left(\Lambda^{-1}x\right), \tag{39}$$

and similarly,

$$\partial_{\mu}\phi \to \partial_{\mu}\phi'(x) = \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}\,\partial_{\nu}\phi\left(\Lambda^{-1}x\right). \tag{40}$$

2.4 Actions Revisited

As we alluded to earlier, actions are also heavily constrained by symmetries. Given the Lagrangian density of the massive scalar field

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial_{\nu}\phi\eta^{\mu\nu} - m^2\phi^2,$$

we notice that the action is invariant under Lorentz transformations.