Quantum Field Theory

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Contents

1 Lecture: Introduction

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Historically, the goal of quantum field theory was to combine quantum mechanics with special relativity. One of the most notable outputs of this study hailed as a success is that the number of particles is not conserved. It is a robust and systematic theory governed by few principles. It concerns itself with locality, symmetries and renormalization which are exceptionally constraining and *almost* uniquely determine what we can study.

In this course we use $c = \hbar = 1$. In these natural units, $E = mc^2$ gives us masses in the units of energy.

For metrics we use $\eta^{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$, $x^\mu = (t, x, y, z)$, $F(t, \vec{x}) \equiv F(x^\mu) \equiv F(\mathbf{x})$

1.1 Classical Field Theory

In classical mechanics, a natural object is the action,

$$S(t_1, t_2) = \int_{t_1}^{t_2} \mathrm{d}t \left(\underbrace{m \sum_{i=1}^{3} \left(\frac{\mathrm{d}x^i}{\mathrm{d}t} \right)^2}_{\text{kinetic term}} - \underbrace{V(x)}_{\text{potential}} \right). \tag{1}$$

This is incredibly useful for us for three main reasons:

- the equations of motion are given for free by extremising S,
- Boundary conditions are supplied externally, and
- S is built on *symmetry* (it is invariant of symmetries of your system).

As we move towards field theory, we no longer want to speak of a single position of a particle $x(t)$.

The fundamental object in field theory is a field $\phi_a(t, \vec{x}) : \mathbb{R}^{3,1} \to \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{R}^n$. Here a labels the type of field we are discussing.

The first consequence is that we are dealing with an infinite number of degrees of freedom as every point in time and space contains some distinct information about the system.

Example. In electromagnetism, as we will discuss in depth later, one has the gauge field $A^\mu(t, \vec{x}) = \left(\phi(x), \vec{A}(x)\right)$ which the electric and magnetic fields can be defined in terms of

$$\vec{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{2}$$

$$\vec{B} = \boldsymbol{\nabla} \cdot \vec{A}, \tag{3}$$

which have equations of motion

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \tag{4}$$

$$\boldsymbol{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}, \tag{5}$$

and two identities

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \tag{6}$$

$$\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} = -\boldsymbol{\nabla} \times \vec{E}. \tag{7}$$

This is a (hopefully) familiar classical field that we will quantise in due time.

1.2 Lagrangians

The Lagrangian in classical mechanics can be written $L = T - V$ and is contained within the action in the form

$$S = \int \mathrm{d}t\, L. \tag{8}$$

We will in QFT concern ourselves with the *Lagrangian density* given by

$$L = \int \mathrm{d}^3x\, \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right), \tag{9}$$

however *everybody* just refers to $\mathcal{L}$ as a Lagrangian as we will here.

The equations of motion are determined by extremizing with respect to the fields.

Note. We assume that the Lagrangian $\mathcal{L}\left[\phi_a, \partial_\mu \phi_a\right]$ is not a function of $\partial^2 \phi_a$ or higher derivatives. This is for complicated reasons related to ghosts that are beyond the scope of this course.

Extremising the action with respect to the field, we want to find the conditions for which $\Delta S = 0$, i.e. the action is at a minima/saddle point. We see that

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\delta\left(\partial_\mu \phi_a\right)\right] \tag{10}$$

$$= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right)\delta\phi_a + \underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\delta\phi_a\right)}_{\text{total derivative}}\right], \tag{11}$$

and by assuming that our fields decay at infinity, the total derivative term vanishes yielding

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right)\right]\delta\phi_a, \tag{12}$$

for which vanishing requires

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{13}$$

Example. A free massive scalar field is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \tag{14}$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 - \frac{1}{2}m^2\phi^2. \tag{15}$$

In traditional classical mechanics, one would have identified $T = \frac{1}{2}\dot{\phi}^2$ and $V = \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2$. In QFT, the 'kinetic terms' sometimes refers to any bilinear combination of fields. For example, $\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is always kinetic and $m^2\phi^2$ is often a (bosonic) mass term.

The equation of motion for the free massive scalar field Lagrangian is

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0. \tag{16}$$

This is the **Klein Gordon equation**. It is also sometimes written with $\partial_\mu\partial^\mu = \Box$.

2 Lecture: Symmetries

14/10/2024

2.1 Hamiltonian Formalism

In a Hamiltonian formalism, one starts by defining the canonical momenta

$$\Pi^a(x) \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_t \phi_a\right)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{17}$$

Definition 2.1: The **Hamiltonian density** is defined by Legendre transform of the Lagrangian density

$$\mathcal{H} = \Pi^a \partial_t \phi_a - \mathcal{L}. \tag{18}$$

The Hamiltonian is given by

$$H = \int \mathrm{d}^3x\, \mathcal{H}. \tag{19}$$

We will (mostly) not abuse notation and always call $\mathcal{H}$ a *Hamiltonian density*, and H a *Hamiltonian.*

Example. For a scalar field with a potential $V(\phi)$, we have

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{20}$$

The canonical momentum is

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \tag{21}$$

and the Hamiltonian is then

$$H = \int \mathrm{d}^3x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + V(\phi)\right). \tag{22}$$

2.2 Symmetry

Symmetries are inseparable from the study of quantum field theory. Most notably they dictate the actions we can write, the class of fields (operators) we can use, and the observables we can compute.

Definition 2.2: The **Lorentz group** has elements $\Lambda^\mu{}_\nu$ such that under Lorentz boosts

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{23}$$

which preserve the spacetime interval $s^2 = x^\mu x^\nu \eta_{\mu\nu} = t^2 - x^i x_i$ such that

$$s^2 \to s'^2 = s^2. \tag{24}$$

This condition implies

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}. \tag{25}$$

In matrix form, this can be written $\Lambda^T \eta \Lambda = \eta$.

Examples. Rotations such as one in the xy plane, leave $t' = t$ and have $\Lambda_1^1 = R_1^1$ such that

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Boosts mix time and space. Boosting in the (t, x) plane, we have

$$\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

where η is the **rapidity** and is given by

$$\cosh\eta = \frac{1}{\sqrt{1-v^2}} \tag{28}$$

$$\sinh\eta = \frac{v}{\sqrt{1-v^2}}. \tag{29}$$

Note. From Eq. (25), we see that in general $\det(\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$.

If $\det\Lambda = 1$, then Λ is called a *proper* Lorentz transformation.

If $\det\Lambda = -1$, then Λ is called a *improper* Lorentz transformation. Parity and time reversal each independently cause $\det\Lambda = -1$. Only proper Lorentz transformations are continuously connected to the identity.

We will assume $\det\Lambda = 1$. We can then expand about the identity infinitesimally and write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \varepsilon^{\mu}{}_{\nu} + \mathcal{O}\left(\varepsilon^2\right). \tag{30}$$

The natural question is what are the properties of $\varepsilon^{\mu}{}_{\nu}$?

Inserting this expression into Eq. (25), we see

$$\begin{aligned} \eta_{\rho\sigma} &= \eta_{\mu\nu}\left(\delta^{\mu}_{\rho} + \varepsilon^{\mu}{}_{\rho} + \cdots\right)\left(\delta^{\nu}_{\sigma} + \varepsilon^{\nu}{}_{\sigma} + \cdots\right) \\ &= \eta_{\mu\nu}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\varepsilon^{\mu}{}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\delta^{\mu}_{\rho}\varepsilon^{\nu}{}_{\sigma} + \mathcal{O}\left(\varepsilon\right)^2 \\ &= \eta_{\rho\sigma} + \varepsilon_{\sigma\rho} + \varepsilon_{\rho\sigma} \\ \Rightarrow \varepsilon_{\sigma\rho} &= -\varepsilon_{\rho\sigma}. \end{aligned} \tag{31}$$

Therefore $\varepsilon_{\sigma\rho}$ is an antisymmetric tensor, which in $d = 4$ has $\frac{d(d-1)}{2} = 6$ independent components.

Therefore we have 6 generators for the Lorentz group:

- 3 rotations, and
- 3 boosts,

as expected.

2.3 Fields Revisited

We can now think of a field as an object which transforms under the Lorentz group. It therefore forms a representation of the algebra.

Definition 2.3: A field is an object that depends on coordinates and has a definite transformation under the action of the Lorentz group,

$$x \to x' = \Lambda x, \tag{32}$$

$$\phi_a(x) \to \phi'_a(x) = D[\Lambda]_a^b \, \phi_b\left(\Lambda^{-1} x\right). \tag{33}$$

$D[\Lambda]$ forms a representation of the Lorentz group as it satisfies

$$D[\Lambda_1] \, D[\Lambda_2] = D[\Lambda_1 \Lambda_2], \tag{34}$$

$$D\left[\Lambda^{-1}\right] = D[\Lambda]^{-1}, \tag{35}$$

$$D[\mathbb{I}] = 1. \tag{36}$$

Examples.

1) Consider the trivial representation $D[\Lambda] = 1$. Then the field transforms as

$$\phi(x) = \phi\left(\Lambda^{-1} x\right), \tag{37}$$

which is an equivalent definition of the *scalar field*. Here we are using active transformations where the coordinates are fixed.

2) We are also familiar with the vector representation given by

$$D[\Lambda]^\mu{}_\nu = \Lambda^\mu{}_\nu. \tag{38}$$

A field transforming under this representation is A^μ such that

$$A^\mu(x) \to A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu\left(\Lambda^{-1} x\right), \tag{39}$$

and similarly,

$$\partial_\mu \phi \to \partial_\mu \phi'(x) = \left(\Lambda^{-1}\right)^\nu{}_\mu \, \partial_\nu \phi\left(\Lambda^{-1} x\right). \tag{40}$$

2.4 Actions Revisited

As we alluded to earlier, actions are also heavily constrained by symmetries. Given the Lagrangian density of the massive scalar field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu} - m^2 \phi^2,$$

we notice that the action is invariant under Lorentz transformations.

3 Lecture: Noether's Theorem

16/10/2024

We can check this transformation explicitly with

$$\begin{aligned}\mathcal{L} &\to \frac{1}{2}\eta^{\mu\nu}\left(\Lambda^{-1}\right)^{\rho}{}_{\mu}\partial_\rho\phi\left(\Lambda^{-1}x\right)\left(\Lambda^{-1}\right)^{\sigma}{}_{\nu}\partial_\sigma\phi\left(\Lambda^{-1}x\right)-\frac{1}{2}m^2\phi^2\left(\Lambda x\right) &(41)\\ &=\frac{1}{2}\eta^{\rho\sigma}\partial_\rho\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2 &(42)\\ &=\mathcal{L}, &(43)\end{aligned}$$

and therefore $\mathcal{L}(x)\to\mathcal{L}'(x)=\mathcal{L}(y)$ gives us

$$S\to\int \mathrm{d}^4x\,\mathcal{L}(y)=\int\mathrm{d}^4y\,\mathcal{L}(y)\,, \tag{44}$$

as $\det(\Lambda)=1$ means the Jacobian is 1. Thus the action is also Lorentz invariant.

Theorem 3.1 (Noether's Theorem):

1) Every **continuous symmetry** of the Lagrangian gives rise to a current j^μ which is conserved $\partial_\mu j^\mu=0$ under the equations of motion.
2) Provided suitable boundary conditions, a conserved current will give rise to a conserved charge Q, where

$$Q=\int\mathrm{d}^3x\,j^0. \tag{45}$$

Proof.

1) We must first define a continuous symmetry.

Definition 3.1: A transformation is continuous if there is an infinitesimal parameter in it. We will see two types:

- *internal* transformations, which do not act on the coordinates, but act on the fields,
- *local* transformations, which act on both the coordinates and the fields.

In both cases, a continuous transformations can be written

$$\delta\phi_a = \phi_a^{'}(x) - \phi_a(x)\,. \tag{46}$$

Such a transformation is a **symmetry** of the system if the **action** is invariant under the transformation.

Namely, under

$$S[\phi] \to S[\phi'] = \int \mathrm{d}^4x\,\mathcal{L}[\phi']\,, \tag{47}$$

we are looking for

$$\delta S = S[\phi'] - S[\phi] = 0, \tag{48}$$

which implies a symmetry. This implies that for the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \partial_\mu \mathcal{J}^\mu, \tag{49}$$

namely, that the Lagrangian can change up to a total derivative without the action changing.

Let's quantify the change in $\mathcal{L}$. We have that

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\partial_\mu\phi_a & (50)\\ &= \left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right)\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) \overset{\text{symm}}{=} \partial_\mu\mathcal{J}^\mu. & (51)\end{aligned}$$

This implies that

$$-\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right)}_{\text{equation of motion}}\delta\phi_a = \partial_\mu\underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - \mathcal{J}^\mu\right)}_{j^\mu}. \tag{52}$$

Therefore if the equation of motion is imposed, one has

$$\partial_\mu j^\mu = 0, \tag{53}$$

for

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - \mathcal{J}^\mu. \tag{54}$$

2) We have

$$Q = \int \mathrm{d}^3 x \, j^0, \tag{55}$$

and

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_V \mathrm{d}^3 x \, \frac{\partial j^0}{\partial t} \tag{56}$$
$$= -\int_V \mathrm{d}^3 x \, \vec{\boldsymbol{\nabla}} \cdot \vec{j} \tag{57}$$
$$= -\int_{\partial V} d\vec{A} \cdot \vec{j} \tag{58}$$
$$= 0, \tag{59}$$

where this last equality holds as the fields decay as $|x| \to \infty$, and thus Q is a conserved quantity.

□

3.1 Energy Momentum Tensor

We consider a local transformation that is a symmetry of almost every theory worthy of study: spatial translations taking

$$x^\mu \to x'^\mu = x^\mu - \varepsilon^\mu, \tag{60}$$

where ε^μ is a constant vector. Under such translations, the fields transform as

$$\phi_a \to \phi'_a(x) = \phi_a(x + \varepsilon), \tag{61}$$

where making this an infinitesimal transformation and expanding in a Taylor series we see

$$\phi'_a(x) = \phi_a(x) + \varepsilon^\mu \partial_\mu \phi_a(x) + \mathcal{O}\left(\varepsilon^2\right) \tag{62}$$
$$\Rightarrow \delta\phi_a = \phi'_a(x) - \phi_a(x) \tag{63}$$
$$= \varepsilon^\mu \partial_\mu \phi(x). \tag{64}$$

The Lagrangian changes by a total derivative under this transformation such that

$$\delta\mathcal{L} = \varepsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu \underbrace{\varepsilon^\mu \mathcal{L}}_{\mathcal{J}^\mu}. \tag{65}$$

Therefore, substituting in $\delta\phi_a$ and $\mathcal{J}^\mu$, our conserved current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \underbrace{\varepsilon^\nu \partial_\nu \phi_a}_{\delta\phi_a} - \underbrace{\varepsilon^\mu \mathcal{L}}_{\mathcal{J}^\mu} \tag{66}$$
$$= \varepsilon^\nu \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \partial_\nu \phi_a - \delta^\mu{}_\nu \mathcal{L}\right) \equiv \varepsilon^\nu T^\mu{}_\nu, \tag{67}$$

where $T^\mu{}_\nu$ is the **energy momentum tensor**.

Using the equation of motion, one can show that

$$\partial_\mu j^\mu = 0 \Rightarrow \partial_\mu T^\mu{}_\nu = 0, \tag{68}$$

namely that the stress energy tensor is conserved on shell.

Further, from $T^{\mu\nu}$ we can construct four conserved charges given by

- the *energy*, $E = \int \mathrm{d}^3x\, T^{00}$ by choosing $\varepsilon^\mu = (1,0,0,0)$,
- the *momenta*, $p^i = \int \mathrm{d}^3x\, T^{0i}$ where $\varepsilon^\mu = (0,1,0,0)$, $(0,0,1,0)$ or $(0,0,0,1)$.

Example (Local Symmetry)**.** For the free massive scalar field

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial_\nu\phi - \delta^\mu_\nu \mathcal{L} \tag{69}$$

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{70}$$

$$T^{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2, \tag{71}$$

where observe that

$$E = \int \mathrm{d}^3x\, T^{00} = H, \tag{72}$$

and

$$p^i = \int \mathrm{d}^3x\, T^{0i} = \int \mathrm{d}^3x\, \dot{\phi}\partial^i\phi. \tag{73}$$

Note. The stress energy tensor

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial_\nu\phi_a - \delta^\mu_\nu \mathcal{L}, \tag{74}$$

is not always symmetric. One can define the *Belifante tensor* given by

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \mathcal{T}^{\rho\mu\nu}, \tag{75}$$

where $\mathcal{T}^{\rho\mu\nu} = -\mathcal{T}^{\mu\rho\nu}$ leads to $\partial_\mu\Theta^{\mu\nu} = 0$.

One can also symmetrize $T^{\mu\nu}$ by coupling fields to $g_{\mu\nu}$ (instead of $\eta^{\mu\nu}$) with

$$\Theta^{\mu\nu} = \left(-\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g_{\mu\nu}}\left(\sqrt{-g}\mathcal{L}\right)\right)\bigg|_{g=\eta}. \tag{76}$$

4 Lecture: Canonical Quantization

18/10/2024

4.1 Internal Symmetry

Example (Internal Symmetry)**.** Internal symmetries do not act on coordinates, only the fields. Consider a complex scalar field

$$\psi\left(x\right) = \frac{1}{\sqrt{2}}\left(\phi_1\left(x\right) + i\phi_2\left(x\right)\right), \tag{77}$$

where ϕ_1, ϕ_2 are real scalar fields. A Lagrangian for this field is

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \psi^* - V\left(|\psi|^2\right). \tag{78}$$

The equations of motion for this theory are

$$\partial_\mu \partial^\mu \psi + \frac{\partial V}{\partial \psi^*} = 0, \qquad \partial_\mu \partial^\mu \psi^* + \frac{\partial V}{\partial \psi} = 0. \tag{79}$$

The internal symmetry of this system, for constant $\alpha \in \mathbb{R}$,

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) \tag{80}$$

$$\psi^*(x) \to (\psi^*(x))' = e^{-i\alpha}\psi^*(x), \tag{81}$$

under which $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$ and $S \to S' = S$. Here α is the continuous parameter of the transformation, such that

$$\delta\psi = \psi'(x) - \psi(x) \tag{82}$$

$$= i\alpha\psi \tag{83}$$

$$\delta\psi^* = -i\alpha\psi^*. \tag{84}$$

We can construct the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}\delta\psi^* - \mathcal{J}^\mu, \tag{85}$$

where there is no total derivative term, $\mathcal{J}^\mu = 0$. We then have

$$j^\mu = i\alpha\left(\psi\partial^\mu\psi^* - \psi^*\partial_\mu\psi\right), \tag{86}$$

which implies a conserved charge

$$Q = \int \mathrm{d}^3x\, j^0, \tag{87}$$

which is in fact the electric charge as we will see.

Observe that it is also possible to view the transformation as

$$\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1' \\ \phi_2'\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \tag{88}$$

which is identical to the previous transformation in Eq. (80).

4.2 Quantum Fields

We will first study the simplest possible theory: a free theory. We will take a Hamiltonian approach and build on the rules of quantum mechanics. Recall the familiar commutation relations of

$$\left[x^i, p^j\right] = i\delta^{ij}. \tag{89}$$

In QFT, we no longer speak of position and momentum variables, but rather a quantum field $\phi_a(x)$ and its conjugate momenta $\Pi^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$ which satisfy

$$\left[\phi_a(\mathbf{x}, t), \Pi^b(\mathbf{y}, t)\right] = i\delta^3(\mathbf{x} - \mathbf{y})\,\delta_a^b, \tag{90}$$

called *equal time* commutation relations. One must make a choice of some kind when transferring from a classical theory to a quantum theory, and this turns out to be one such correct choice.

4.3 Canonical Quantization

Note. In the notes, Tong performs canonical quantization in the Schrödinger picture at $t = 0$. Here we will use the Heisenberg picture.

Our theory of interest is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{91}$$

Its equation of motion is the Klein-Gordon equation, $\partial_\mu\partial^\mu\phi + m^2\phi = 0$.

We know solutions to this equation take the form

$$\phi \sim \exp\left(i\mathbf{k}\cdot\mathbf{x} + i\omega t\right), \tag{92}$$

where $-\omega^2 + \mathbf{k}\cdot\mathbf{k} + m^2 = 0$ which gives us a dispersion relation,

$$\omega\left(k\right) = \pm\sqrt{\mathbf{k}\cdot\mathbf{k} + m^2}. \tag{93}$$

We adopt the notation $\omega \equiv \sqrt{\mathbf{k}\cdot\mathbf{k} + m^2}$. Therefore, taking a linear superposition of fields, one has

$$\phi\left(\mathbf{x}, t\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(a\left(k\right)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + b\left(k\right)e^{i\mathbf{k}\cdot\mathbf{x} + i\omega t}\right). \tag{94}$$

Note. ϕ is real, which imposes restrictions on $a\left(k\right)$ and $b\left(k\right)$. Namely, as $\phi^* = \phi$, we have

$$a^*\left(-k\right) = b\left(k\right) \qquad b^*\left(-k\right) = a\left(k\right), \tag{95}$$

thus we can write

$$\phi\left(x\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(a\left(k\right)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + a^*\left(k\right)e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t}\right). \tag{96}$$

In a more relativistic notation, one has

$$\phi\left(x\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(a\left(k\right)e^{-ik^\mu x_\mu} + a^*\left(k\right)e^{ik^\mu x_\mu}\right), \tag{97}$$

where $k_\mu = \left(\omega, \mathbf{k}\right)$ and $x_\mu = \left(t, \mathbf{x}\right)$ give us $k^\mu x_\mu = \omega t - \mathbf{k}\cdot\mathbf{x}$ and $k^2 = \omega^2 - \mathbf{k}\cdot\mathbf{k} = m^2$.

Note. We will choose to normalize $a\left(k\right)$ and $a^*\left(k\right)$ such that

$$\phi\left(x\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega}}\left(a\left(k\right)e^{-ik_\mu x^\mu} + a^*\left(k\right)e^{ik_\mu x^\mu}\right). \tag{98}$$

Lastly, notice that

$$\Pi\left(x\right) = \dot{\phi} = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(-i\sqrt{\frac{\omega}{2}}\left(a\left(k\right)e^{-ik_\mu x^\mu} - a^*\left(k\right)e^{ik_\mu x^\mu}\right)\right). \tag{99}$$

Next, we ***quantize***, namely, we declare that

$$\left[\phi\left(\vec{x}, t\right), \phi\left(\vec{x}', t\right)\right] = 0 \tag{100}$$
$$\left[\Pi\left(\vec{x}, t\right), \Pi\left(\vec{x}', t\right)\right] = 0 \tag{101}$$
$$\left[\phi\left(\vec{x}, t\right), \Pi\left(\vec{x}', t\right)\right] = i\delta^3\left(\vec{x} - \vec{x}'\right). \tag{102}$$

Claim. These commutation relations promote a to an **operator** such that $a(k)$ becomes $\hat{a}_k$ and $a^*(k)$ becomes $\hat{a}_k^\dagger$. The above commutation relations imply

$$[\hat{a}_k, \hat{a}_{k'}] = 0 \tag{103}$$
$$\left[\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\right] = 0 \tag{104}$$
$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = (2\pi)^3 \delta^3(k-k'). \tag{105}$$

Proof.

1) (*Claim implies declaration*) Taking

$$[\phi(\vec{x},t), \Pi(\vec{y},t)] = \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\frac{1}{2i}\sqrt{\frac{\omega_q}{\omega_p}}\left(\left[a_{\mathbf{p}}e^{i\vec{p}\cdot\vec{x}+i\omega t} + a_{\mathbf{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}+i\omega t}, a_{\mathbf{q}}e^{i\vec{q}\cdot\vec{y}-i\omega t} - a_{\mathbf{q}}^\dagger e^{-i\vec{q}\cdot\vec{y}+i\omega t}\right]\right) \tag{106}$$

$$= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\frac{1}{2i}\sqrt{\frac{\omega_q}{\omega_p}}\left(-\underbrace{\left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger\right]}_{(2\pi)^3\delta^3(\vec{p}-\vec{q})} e^{i\vec{p}\cdot\vec{x}}e^{-i\vec{q}\cdot\vec{y}} + \left[a_{\vec{p}}^\dagger, a_{\mathbf{q}}\right]e^{-i\vec{p}\cdot\vec{x}}e^{i\vec{q}\cdot\vec{y}}\right) \tag{107}$$

$$= i\int \frac{\mathrm{d}^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = i\delta^3(\vec{x}-\vec{y}), \tag{108}$$

as desired.

2) The reverse follows similarly.

□

5 Lecture: Vacuum Energy

21/10/2024

Observe that while the Hamiltonian for the free massive scalar field can be written as

$$H = \int \mathrm{d}^3x\,\mathcal{H} = \frac{1}{2}\int \mathrm{d}^3x\left(\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right), \tag{109}$$

we desire an expression in terms of a and $a^\dagger$. Expanding the fields in terms of these operators, we see

$$\begin{aligned}H = \frac{1}{2}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6}\Bigg(&-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{x}}\right)\\ &-\frac{1}{2}\frac{1}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\left(a_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}\right)\vec{p}\vec{q}\\ &+\frac{m^2}{2}\frac{1}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\left(a_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}\right)\Bigg)\end{aligned} \tag{110}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\Bigg[\underbrace{\left(-\omega_{\mathbf{p}}^2 + \vec{p}^2 + m^2\right)}_{\text{e.o.m. thus vanishes}}\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2i\omega t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega t}\right) \tag{111}$$

$$+\left(\omega_{\mathbf{p}}^2+\vec{p}^2+m^2\right)\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger\right)\Bigg] \tag{112}$$

$$H=\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger\right) \tag{113}$$

$$H=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega\,(2\pi)^3\,\delta^3\,(0)\,. \tag{114}$$

I have skipped some calculation steps, I *highly recommend* one attempts to repeat this calculation and fill them in to make sure you understand what is being done.

This last term is unusual, and appears unphysical as with a vacuum $|0\rangle$ satisfying $a_{\mathbf{p}}|0\rangle=0$, we see

$$H|0\rangle=\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\mathbf{p}}\,(2\pi)^3\,\delta\,(0)\,|0\rangle=E_0\,|0\rangle\to\infty. \tag{115}$$

To understand the nature of this, we need to see the origin of the divergence. There are in fact two divergences here:

- An *infrared divergence*: $(2\pi)^3\,\delta\,(0)$, associated with long distances, as it came from

$$\delta\,(0)=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}^3x\,e^{-i\vec{x}\cdot\vec{p}}\bigg|_{\vec{p}=0}=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}^3x=V, \tag{116}$$

  a diverging volume V. As we are discussing an system with infinite size, we can instead discuss energy *densities* (i.e. per unit volume) such that

$$\varepsilon_0=\frac{E_0}{V}=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2}\omega_{\mathbf{p}}\sim\int\mathrm{d}^3p\,\vec{p}^2, \tag{117}$$

  which is still divergent.

- Namely, it is an *ultraviolet divergence*. Suppose one is performing

$$\int_0^\Lambda\mathrm{d}^3p\,\sqrt{\vec{p}^2+m^2}\overset{\Lambda\to\infty}{\longrightarrow}\infty, \tag{118}$$

  we see that this is a high frequency divergence. It is absurd to think that the theory is valid for arbitrarily high energies, and thus it is valid to consider a maximum energy scale of applicability, a cutoff, Λ.

  The solution here, is to declare that

$$H\equiv\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{119}$$

  One can convince themself that we can only measure energy differences and thus can remove this vacuum energy. However, practically, it is best to just take this H as definition such that it fixes an ambiguity. There is an ambiguity in the *normal ordering* of operators when one converts between classical and quantum field theories. Here it is clear that this H is the correct definition in quantum field theory as it provides $H|0\rangle=0$.

Definition 5.1: If you have a list of fields, we define **normal ordering** as

$$:\phi_1(x_1)\,\phi_2(x_2)\cdots\phi_n(x_n):, \tag{120}$$

where this is the usual product but we put creation operators $a^\dagger_{\mathbf{p}}$ to the left of annihilation operators $a_{\mathbf{p}}$.

5.1 Fock Space

We have the vacuum $|0\rangle$ and want to construct excited states atop it. It is usefully to observe that

$$\left[H, a^\dagger_{\mathbf{p}}\right] = \omega_{\mathbf{p}} a^\dagger_{\mathbf{p}}, \tag{121}$$

and

$$[H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}} a_{\mathbf{p}}. \tag{122}$$

We then aim to construct energy eigenstates by

$$|\vec{p}\rangle = a_{\mathbf{p}}\,|0\rangle\,. \tag{123}$$

This is a single particle state. Observe that then

$$H\,|\vec{p}\rangle = \omega_{\mathbf{p}}\,|\vec{p}\rangle\,. \tag{124}$$

Note. Until now I have noted operators $a^\dagger_{\mathbf{p}}$ and $\omega_{\mathbf{p}}$ with $\mathbf{p}$, however they both only depend on the spatial component $\vec{p}$. I will now switch to this before dropping any decoration on p at all when it is clear from context.

6 Lecture: Relativistic Normalisation

23/10/2024

We can consider the momentum operator represented by

$$\hat{\vec{p}} = -:\int \mathrm{d}^3x\,\Pi\boldsymbol{\nabla}\phi := \int \frac{\mathrm{d}^3k}{(2\pi)^3}\vec{k}a^\dagger_{\vec{k}}a_{\vec{k}}, \tag{125}$$

for which $|\vec{p}\rangle$ is also an eigenstate,

$$\hat{\vec{p}}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle\,. \tag{126}$$

Therefore $|\vec{p}\rangle$ is a momentum and energy eigenstate with

$$E^2 = \omega^2_{\vec{p}} = \vec{p}^2 + m^2. \tag{127}$$

Note. When $\vec{p} = 0$, this particle has no angular momentum such that

$$J^i\,|\vec{p} = 0\rangle = 0, \tag{128}$$

which implies it is a spin 0 particle as we will see later.

Observe that can can construct an n particle state with

$$|\vec{p}_1 \cdots \vec{p}_n\rangle = a^\dagger_{\vec{p}_1} \cdots a^\dagger_{\vec{p}_n} |0\rangle \,. \tag{129}$$

As all $a^\dagger_{\vec{p}}$'s commute, we have

$$|\vec{p}_1 \vec{p}_2\rangle = |\vec{p}_2 \vec{p}_1\rangle \,. \tag{130}$$

Therefore, the Fock space is spanned by all possible combinations of $a^\dagger$ acting on $|0\rangle$. It is interesting then to introduce the **number operator**

$$N \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}}, \tag{131}$$

which tells us the number of particles in a given state. Namely,

$$N |\vec{p}_1 \cdots \vec{p}_n\rangle = n |\vec{p}_1 \cdots \vec{p}_n\rangle \,. \tag{132}$$

For a free theory, we have that

$$[N, H] = 0, \tag{133}$$

which implies that the number of particles is conserved.

Therefore if $\mathscr{H}_n$ denotes the space of n particle states, the Fock space $\mathscr{F}$ can be written

$$\mathscr{F} = \bigoplus_n \mathscr{H}_n. \tag{134}$$

6.1 Relativistic normalization

While we have constructed eigenstates $|\vec{p}_i\rangle$ we have not checked that they are normalized states. To begin, we pick

$$\langle 0|0\rangle = 1. \tag{135}$$

For the 1 particle state,

$$|\vec{p}\rangle = a^\dagger_{\vec{p}} |0\rangle \Rightarrow \langle \vec{p}|\vec{q}\rangle = (2\pi)^3 \delta^3 (\vec{p} - \vec{q}) \,, \tag{136}$$

which is not a Lorenz invariant inner product.

We would hope that under a Lorentz transformation Λ with corresponding unitary transformation $U(\Lambda)$, that $|\vec{p}\rangle$ transforms as

$$\vec{p} \to |\vec{p}'\rangle = U(\Lambda) |\vec{p}\rangle \,. \tag{137}$$

This is not yet the case. To figure out a proper definition of $|\vec{p}\rangle$, we use the identity

$$|\vec{q}\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} |\vec{p}\rangle \langle \vec{p}|\vec{q}\rangle \,, \tag{138}$$

where we have used

$$1 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left|\vec{p}\right\rangle \left\langle\vec{p}\right| , \tag{139}$$

where the measure and hence integral here is clearly not Lorentz invariant. The natural question is how can we alter this identity to make the measure Lorentz invariant. If we instead integrated over

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \to \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \delta\left(p^2 - m^2\right) \Theta\left(p^0\right) , \tag{140}$$

then the measure is now Lorentz invariant, (along with the other functions), where the δ and Heaviside function Θ now enforces the equation of motion. Equivalently, p^0 is not a free parameter, and thus enforcing $p^2 = m^2$ returns us to states in our Fock space, however we must also enforce $p^0 > 0$ as we chose the positive root with $\omega > 0$.

We see that using

$$\int \mathrm{d}x\, \delta\left(f\left(x\right)\right) = \sum_{x_0 | f(x_0)=0} \frac{1}{\left|f'\left(x_0\right)\right|}, \tag{141}$$

we have

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \delta^4\left(p^2 - m^2\right) \Theta\left(p^0\right) = \int \mathrm{d}^3 p \,\frac{1}{2\sqrt{\vec{p}^2 + m^2}} = \int \mathrm{d}^3 p \,\frac{1}{2\omega_{\vec{p}}}. \tag{142}$$

Therefore bringing this measure back to the identity, we see

$$1 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left|\widetilde{\vec{p}}\right\rangle \left\langle\widetilde{\vec{p}}\right| , \tag{143}$$

where we define

$$\left|\widetilde{\vec{p}}\right\rangle = \sqrt{2\omega_{\vec{p}}} a^\dagger_{\vec{p}} \left|0\right\rangle , \tag{144}$$

which is now manifestly Lorentz invariant and thus is called *relativistic normalization*.

6.2 Causality

While we now have Lorentz invariant states, their commutation relations are still at equal time. Is this compatible with special relativity (especially causality)?

We will study causality by determining whether measurements *influence* each other in a time-like fashion. We will do this by finding whether their commutators vanish or not.

We define

$$\Delta\left(x - y\right) = \left[\phi\left(x\right), \phi\left(y\right)\right] , \tag{145}$$

with the interpretation of "measuring" the field at x then y or vice versa.

For the free theory, we see

$$\Delta = \int \frac{\mathrm{d}^3k\,\mathrm{d}^3p}{(2\pi)^6} \frac{1}{2\sqrt{\omega_k\omega_p}} \left(\left[a_k, a_p^\dagger\right] e^{-ikx} e^{ipy} + \left[a_k^\dagger, a_p\right] e^{ikx} e^{-ipy} \right) \tag{146}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right). \tag{147}$$

This integral is also Lorentz invariant immediately by inspection. For the free theory, it is a complex number. Suppose x and y are timelike separated such that without loss of generality, $(x-y)_S = (t, 0, 0, 0)$. This gives us

$$\Delta (x-y)_T = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{-i\omega_p t} - e^{i\omega_p t} \right) \sim e^{-imt} - e^{imt} \neq 0. \tag{148}$$

If we instead look at spacelike separated events, $(x-y)_S = (0, \vec{x} - \vec{y})$,

$$\Delta (x-y)_S = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right) = 0, \tag{149}$$

as one can separate and exchange $\vec{p} \to -\vec{p}$. We already knew that the commutator at equal times vanishes, however as we know this commutator is Lorentz invariant, any spacelike event has zero commutator.

6.3 Propagators

Consider

$$\langle 0| \phi(x) \phi(y) |0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip(x-y)} \equiv D(x-y). \tag{150}$$

For spacelike events,

$$D(x-y) \sim e^{-m(\vec{x}-\vec{y})} \neq 0, \tag{151}$$

but

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0. \tag{152}$$

7 Lecture: Feynman Propagator

25/10/2024

7.1 Feynman Propagator

Definition 7.1: The **Feynman propagator** is given by

$$\Delta_F(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle = \begin{cases} D(x-y), & x^0 > y^0, \\ D(y-x), & y^0 > x^0, \end{cases} \tag{153}$$

where T denotes *time ordering*.

This is motivated by inner products like $\langle f|i\rangle$ where $\langle f|$ is a future final state and $|i\rangle$ is a past initial state.

Claim. We claim that

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2+m^2+i\varepsilon} e^{-ip(x-y)}. \tag{154}$$

Proof. Observe that the time ordering can be captured with

$$\langle 0| T\phi(x)\phi(y)|0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle \Theta\left(x^0-y^0\right) + \langle 0|\phi(y)\phi(x)|0\rangle \Theta\left(y^0-x^0\right). \tag{155}$$

Our claim can be written as

$$\begin{aligned}\Delta_F(x-y) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{2\omega_k} e^{-i\omega_k\left(x^0-y^0\right)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\Theta\left(x^0-y^0\right)\\ &\quad+\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{2\omega_k} e^{-i\omega_k\left(y^0-x_0\right)} e^{i\vec{k}\cdot(\vec{y}-\vec{x})}\Theta\left(y^0-x^0\right) \qquad (156)\\ &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{2\omega_k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(e^{-i\omega_k\tau}\Theta(\tau)+e^{i\omega_k\tau}\Theta(-\tau)\right), \qquad (157)\end{aligned}$$

where $\tau = x^0-y^0$. We focus on the time-dependence and show that

$$e^{i\omega_k\tau}\Theta(\tau)+e^{i\omega_k\tau}\Theta(-\tau) = \lim_{\varepsilon\to 0}\frac{(-2\omega_k)}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{e^{-i\omega\tau}}{\omega^2-\omega_k^2+i\varepsilon}. \tag{158}$$

We begin from the right hand side and observe that

$$\frac{1}{\omega^2-\omega_k^2+i\varepsilon} = \frac{1}{(\omega-(\omega_k-i\widetilde{\varepsilon}))(\omega-(-\omega_k+i\widetilde{\varepsilon}))}, \tag{159}$$

where $\varepsilon = \tau\omega_k\widetilde{\varepsilon}+\cdots$ and we relabel back $\widetilde{\varepsilon}\to\varepsilon$. Thus to leading order in ε we see

$$\frac{1}{\omega^2-\omega_k^2+i\varepsilon} = \frac{1}{2\omega_k}\left[\frac{1}{\omega-(\omega_k-i\varepsilon)}-\frac{1}{\omega-(-\omega_k+i\varepsilon)}\right]+\mathcal{O}\left(\varepsilon^2\right). \tag{160}$$

Consider

$$I_1 = \int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{e^{-i\omega\tau}}{\omega-(\omega_k-i\varepsilon)}. \tag{161}$$

This has a pole at $\omega=\omega_k-i\varepsilon$, below the x-axis. As $e^{-i\omega\tau}=e^{\mathrm{Im}(\omega)\tau}e^{-i\,\mathrm{Re}(\omega)\tau}$, if $\tau<0$, we close the contour with a semicircle above the x-axis where $e^{\mathrm{Im}(\omega)\tau}\sim 0$ for large positive $\mathrm{Im}\,\omega$, and thus $I_1=0$.

If $\tau<0$, we close the contour below the x-axis, which contains the pole, and thus Cauchy's residue theorem gives us

$$I_1 = -2\pi i e^{-i\omega_k\tau}\Theta(\tau)+\mathcal{O}(\varepsilon), \tag{162}$$

where the leading negative is there as the contour is clockwise.

Now consider

$$I_2 = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{e^{-i\omega\tau}}{\omega - (-\omega_k + i\varepsilon)}. \tag{163}$$

If $\tau < 0$, we again close the contour above the x axis, which now contains the pole giving

$$I_2 = 2\pi i e^{i\omega_k \tau} \Theta(-\tau) + \mathcal{O}(\varepsilon). \tag{164}$$

If $\tau > 0$, then the contour can be closed below without any poles implying $I_2 = 0$.

Therefore, gathering our intermediate steps, we see that collecting I_1 and I_2,

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{e^{-i\omega\tau}}{\omega^2 - \omega_k^2 + i\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{2\omega_k} (I_1 - I_2) \tag{165}$$

$$= \frac{1}{2\omega_k} \left(-2\pi i e^{-i\omega\tau} \Theta(\tau) - 2\pi i e^{i\omega_k \tau} \theta(-\tau) \right). \tag{166}$$

Returning this claim to the time ordering expression, we see

$$\langle 0 | T\phi(x) \phi(y) | 0 \rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{i}{2\pi} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{e^{-i\omega\tau}}{\omega^2 - \omega_k^2 + i\varepsilon}, \tag{167}$$

where the $\varepsilon \to 0$ limit is now implicit. Identifying $k^0 = \omega$ and $\tau = t$, this becomes

$$\langle 0 | T\phi(x) \phi(y) | 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)}, \tag{168}$$

where we have used $\omega^2 - \omega_k^2 = \omega^2 - \left(\left| \vec{k} \right|^2 + m^2 \right) = k^2 - m^2$ to give us this result, as desired. □

There are a few comments of note to be made here.

1) Observe that time ordering is equivalently to choosing a contour that weaves between the poles such that one and only one contributes for any given x and y.
2) $\Delta_F(x-y)$ is Lorentz invariant.
3) Observe that

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)}, \tag{169}$$

   we have that $k^2 \neq m^2$ here, namely, it is not *on shell*.
4) The i atop the propagator is important.
5) $\Delta_F(x-y)$ is a Green's function.

Observe that

$$\left(\partial^\mu \partial_\mu + m^2\right) \Delta_F (x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \left(-k^2 + m^2\right) e^{-ik(x-y)} \tag{170}$$

$$= -\int \frac{\mathrm{d}^4 k}{(2\pi)^4} i e^{-ik(x-y)} \tag{171}$$

$$= -i\delta^4 (x-y) , \tag{172}$$

and thus $\Delta_F (x-y)$ is the Greens function associated to the Klein Gordon operator. Propagators are the kernel of the equations of motion.

8 Lecture: Interacting Theories

28/10/2024

Claim. As a last comment of last lecture's digressions, we claim that

$$T\phi (x) \phi (y) =:\phi (x) \phi (y): + \Delta_F (x-y) . \tag{173}$$

Proof. Take $\phi = \phi^+ + \phi^-$ where

$$\phi^+ = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p e^{-ipx} \qquad \phi^- = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p^\dagger e^{ipx}. \tag{174}$$

We then choose $x^0 > y^0$ such that

$$T (\phi (x) \phi (y)) = \phi^+ (x) \phi^+ (y) + \phi^- (x) \phi^+ (y) + \phi^- (y) \phi^+ (x) + \left[\phi^+ (x) , \phi^- (y)\right] + \phi^- (x) \phi^- (y) \tag{175}$$

$$=:\phi (x) \phi (y): + D (x-y) , \tag{176}$$

where $D (x-y) = [\phi^+ (x) , \phi^- (y)]$. For $y^0 > x^0$, one sees instead

$$T (\phi (x) \phi (y)) =:\phi (x) \phi (y): + D (y-x) . \tag{177}$$

□

Theorem 8.1 (Wick's Theorem): The time ordering of a set of fields is equal to the normal ordering plus all possible contractions such that

$$T (\phi (x_1) \cdots \phi (x_n)) =:\phi (x_1) \cdots \phi (x_n): + \text{ all possible contractions.} \tag{178}$$

Example. Given four fields $\phi_i = \phi (x_i)$, we have

$$\begin{aligned} T (\phi (x_1) \phi (x_2) \phi (x_3) \phi (x_4)) =& :\phi (x_1) \phi (x_2) \phi (x_3) \phi (x_4): \\ &+ \overbracket{\phi_1 \phi_2} :\phi_3 \phi_4: + \overbracket{\phi_1 \phi_3} :\phi_2 \phi_4: + \overbracket{\phi_1 \phi_4} :\phi_2 \phi_3: \\ &\overbracket{\phi_2 \phi_3} :\phi_1 \phi_4: + \overbracket{\phi_2 \phi_4} :\phi_1 \phi_4: + \overbracket{\phi_3 \phi_4} :\phi_1 \phi_2: \end{aligned}$$

$$\contraction{}{\phi_1}{}{\phi_2}\contraction{\phi_1\phi_2}{\phi_3}{}{\phi_4}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi_1}{}{\phi_3}\contraction{\phi_1\phi_3}{\phi_2}{}{\phi_4}\phi_1\phi_3\phi_2\phi_4 + \contraction{}{\phi_1}{}{\phi_4}\contraction{\phi_1\phi_4}{\phi_2}{}{\phi_3}\phi_1\phi_4\phi_2\phi_3, \tag{179}$$

where $\contraction{}{\phi_i}{}{\phi_j}\phi_i\phi_j = \Delta_F\left(x_i - x_j\right)$ and this generalises as you would expect.

8.1 Couplings

Free theories are "simple" because we can explicitly construct the Fock space. We want to consider more general Lagrangians but are obstructed in this endeavour as we cannot solve their equations of motion. We do not have access to the Hilbert space of almost any (non-integrable) interacting field theory.

Therefore, we approach QFT perturbatively, splitting our Lagrangian into

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \tag{180}$$

where $\mathcal{L}_0$ is a known free theory that is solvable and $\mathcal{L}_{\text{int}}$ is an unknown interaction term that we treat as a perturbation.

For example, take

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{181}$$

and

$$\mathcal{L}_{\text{int}} = -\sum_{n=0}^{\infty}\frac{\lambda_n}{n!}\phi^n, \tag{182}$$

where $\lambda_n \in \mathbb{R}$.

Naively, one may think the domain of perturbation theory is when $\lambda_n \ll 1$. This is false as we will see.

Namely, to quantify relative "smallness" recall that we are working in natural units where $c = 1 = \hbar$ and thus

$$[L] = [T] = \left[M^{-1}\right], \tag{183}$$

and thus $[M] = 1$. Applying this to the action, we see that

$$S = \int \mathrm{d}^4x\,\mathcal{L}, \tag{184}$$

as $[S] = [\hbar] = 0$, and

$$\left[\int \mathrm{d}^4x\right] = -4 \Rightarrow [\mathcal{L}] = 4. \tag{185}$$

Applying this to $\mathcal{L}_0$, as $[m] = [\partial_\mu] = 1$, we have

$$[\phi] = 1. \tag{186}$$

Therefore

$$[\mathcal{L}_{\text{int}}] = [\lambda_n\phi^n] \Rightarrow [\lambda_n] = 4 - n. \tag{187}$$

Lets assess these cases individually.

1) If $n = 3$, $[\lambda_3] = 1$. More generally, in d dimensions, $[\lambda_3] > 0$.

   As a dimensionless quantity one may compare λ_3 to some energy scale E by considering $\frac{\lambda_3}{E}$.

   If $\lambda_n \ll E$ (high energies), then this is a small perturbation.

   If $\lambda_n \gg E$ (low energies) then this perturbation is large.

   If this holds, we call λ_n a **relevant** coupling.

   In a relativistic theory, $E > m$, so we can treat it perturbatively as $\lambda \ll m$.

2) If $n = 4$, $[\lambda_n] = 0$. As it is a dimensionless coupling, it is meaningful to write $\lambda \ll 1$ or $\lambda \gg 1$.

   If this is the case, λ_n is called a **marginal** coupling.

3) If $n > 4$, then $[\lambda_n] < 0$, and thus our dimensionless combination is $\lambda_n (E)^{n-4}$. This coupling is then not important at low energies but is significant at high energies.

   We then call λ_n an **irrelevant** coupling.

8.2 LSZ reduction formula

The basis quantity to study in QFT is the scattering matrix (S-matrix).

To construct and evaluate the S matrix we can break it down into steps.

1) Define states (asymptotic states)
2) Relate in and out states using the S-matrix
3) Evaluate S using Schwinger-Dyson (which leads to Feynman rules)

8.3 Asymptotic states

Given a system $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ or equivalently, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$, we have some state $|\Omega\rangle$ we call the vacuum state of this interacting theory.

Note. This is distinct from the vacuum of the free theory, $\mathcal{H}_0$, $|0\rangle$.

9 Lecture: Scattering

30/10/2024

9.1 Pictures

One can picture the scattering of one state into another in different pictures,

$$\underbrace{\langle \text{final};\, t_i | \text{initial};\, t_i \rangle}_{\text{Schrödinger}} = \underbrace{\langle f | \, S \, | i \rangle}_{\text{Heisenberg}}, \tag{188}$$

one where the states evolve and one where the operators do.

We assume the Hamiltonian does time evolution such that

$$i\partial_t \phi = [\phi, \mathcal{H}], \tag{189}$$

where ϕ can also be any operator in the theory.

We declare (assume) that at some time $t = t_0$, we can match the Hilbert space of $\mathcal{H}_0$ to that of $\mathcal{H}$ with

$$a_p(t) = e^{iH(t-t_0)} a_p^0 e^{-iH(t-t_0)}. \tag{190}$$

Then for a field in the interacting theory, we can write

$$\phi_{\text{int}}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p(t) e^{-ipx} + a_p^\dagger(t) e^{ipx}\right). \tag{191}$$

9.2 Asymptotic States

With this we can write down states as

$$|\text{initial};\, t_1\rangle = |p_1 p_2\rangle = \sqrt{2\omega_1}\sqrt{2\omega_2} a_{p_1}^\dagger a_{p_2}^\dagger |\Omega\rangle \tag{192}$$

$$|\text{final};\, t_2\rangle = |p_3 p_4\rangle = \sqrt{2\omega_3}\sqrt{2\omega_4} a_{p_3}^\dagger a_{p_4}^\dagger |\Omega\rangle. \tag{193}$$

With the definition of asymptotic states, we will want the interactions to be turned off when $t_i \to -\infty$, $t_f \to \infty$. In this limit

$$\lim_{t\to\pm\infty} a_p^\dagger(t) = a_p^{0\dagger}. \tag{194}$$

Intuitively we are turning off the interactions in our asymptotic states as they are sufficiently far away they are separated from everything.

Naturally we need to figure out how to relate states at $\pm\infty$. We see that

$$\langle f|\, S\, |i\rangle = \langle \text{final};\, t_f | \text{initial};\, t_i\rangle \tag{195}$$

$$= \left(\prod_{i=1}^{4} \sqrt{2\omega_i}\right) \langle\Omega|\, T a_{p_3}(+\infty)\, a_{p_4}(+\infty)\, a_{p_1}^\dagger(-\infty)\, a_{p_2}^\dagger(-\infty)\, |\Omega\rangle. \tag{196}$$

Claim. We claim

$$\sqrt{2\omega_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right) = -i\int \mathrm{d}^4 x\, e^{-ipx}\left(\Box + m^2\right)\phi(x). \tag{197}$$

Proof. In the interacting theory, and here we have $\omega_p = \sqrt{\vec{p}^2 + m^2}$. We begin from the answer on the right. Observe that

$$-i\int \mathrm{d}^4 x\, e^{-ipx}\left(\Box + m^2\right)\phi(x) = -i\int \mathrm{d}^4 x\, e^{-ipx}\left(\partial_t^2 - \boldsymbol{\nabla}^2 + m^2\right)\phi. \tag{198}$$

Using integration by parts, we see that

$$-i\int \mathrm{d}^4 x\, e^{-ipx}\left(\partial_t^2 - \boldsymbol{\nabla}^2 + m^2\right)\phi = -i\int \mathrm{d}^4 x\, e^{-ipx}\,(\partial_t^2 + \underbrace{\vec{p}^2 + m^2}_{\omega_p^2})\,\phi(x) \tag{199}$$

$$= -i\int \mathrm{d}^4 x\, \partial_t\left(e^{-ipx}\partial_t\phi(x) - \left(\partial_t e^{-ipx}\right)\phi(x)\right), \tag{200}$$

which only depends on $t \to \pm\infty$. Recall from the free theory that

$$\sqrt{2\omega_p} a_p^0 = i \int \mathrm{d}^3 x\, e^{ipx} \overleftrightarrow{\partial_t} \phi(x) \tag{201}$$

$$\sqrt{2\omega_p} a_p^{0\dagger} = -i \int \mathrm{d}^3 x\, e^{-ipx} \overleftrightarrow{\partial_t} \phi(x), \tag{202}$$

where $f \overleftrightarrow{\partial_t} g = f \partial_t g - (\partial_t f) g$. This allows us to write

$$\sqrt{2\omega_p} \int_{-\infty}^{\infty} \mathrm{d}t\, \partial_t a_p^\dagger(t) = \sqrt{2\omega_p} \left(a_p^\dagger(\infty) - a_p^\dagger(-\infty) \right), \tag{203}$$

and thus we arrive at the final result, namely the claim,

$$\sqrt{2\omega_p} \left(a_p^\dagger(\infty) - a_p^\dagger(-\infty) \right) = -i \int \mathrm{d}^4 x\, e^{-ipx} \left(\Box + m^2 \right) \phi(x). \tag{204}$$

$\Box$

Analogously, we have

$$\sqrt{2\omega_p} \left(a_p(\infty) - a_p(-\infty) \right) = i \int \mathrm{d}^4 x\, e^{ipx} \left(\Box + m^2 \right) \phi(x). \tag{205}$$

Definition 9.1: With this, we can now write our desired inner product expression as

$$\begin{aligned} \langle f | S | i \rangle = \left(\prod_{i=1}^{4} \sqrt{2\omega_i} \right) & \langle \Omega | T \left(a_{p_3}(+\infty) - a_{p_3}(-\infty) \right) \left(a_{p_4}(+\infty) - a_{p_4}(-\infty) \right) \\ & \times \left(a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(\infty) \right) \left(a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(\infty) \right) | \Omega \rangle \end{aligned} \tag{206}$$

$$\begin{aligned} = \prod_{j=1}^{4} \left(i \int \mathrm{d}^4 x_j \right) & \underbrace{e^{-ip_1 x_1} \left(\Box_1 + m^2 \right) e^{-ip_2 x_2} \left(\Box_2 + m^2 \right)}_{\text{ingoing}} \underbrace{e^{ip_3 x_3} \left(\Box_3 + m^2 \right) e^{ip_4 x_4} \left(\Box_4 + m^2 \right)}_{\text{outgoing}} \\ & \times \underbrace{\langle \Omega | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | \Omega \rangle}_{\text{4 point correlation function}}. \end{aligned} \tag{207}$$

This is the **LSZ reduction formula** for 2-2 scattering.

This has many advantages.

- It is a manifestly Lorentz invariant S-matrix by construction (we don't even have to check).
- It makes clear the relation between n point correlation functions $\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle$ and $\langle f | S | i \rangle$.

Note. This process involved factoring out the operator

$$\langle \Omega | T \left(\Box_x + m^2 \right) \phi(x) \cdots | \Omega \rangle \to \left(\Box_x + m^2 \right) \langle \Omega | T \phi(x) \cdots | \Omega \rangle, \tag{208}$$

which is not strictly equal as there are contact terms unaccounted for. However these are not physically important as they are essentially the identity part of the S matrix, $S \sim \mathbb{I} + iT$ which tells us that it is possible for things not to scatter at all. The **transfer matrix** T is the interesting part of the S matrix.

10 Lecture: Interactions

01/11/2024

10.1 Schwinger-Dyson Formula

It remains for us to figure out a way to evaluate

$$\langle\Omega| T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle. \tag{209}$$

Our strategy is to present a Lagrangian approach to this. We first assume that at any given time, the Hilbert space of the interacting theory is the Hilbert space of the free theory. This implies that

$$[\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] = 0, \tag{210}$$

and

$$[\phi(\mathbf{x},t), \partial_t\phi(\mathbf{x}',t)] = i\delta^3(\mathbf{x}-\mathbf{x}'). \tag{211}$$

We also need to assume that our fields still comply with the Euler-Lagrange equations.

For the free theory, this was the Klein Gordon equation,

$$\left(\Box + m^2\right)\phi = 0, \tag{212}$$

and for the interacting theory, it takes the form

$$\left(\Box + m^2\right)\phi - \frac{\partial\mathcal{L}'_{\text{int}}}{\partial\phi} = 0, \tag{213}$$

as we assume $\mathcal{L}_{\text{int}}$ is a function of ϕ but not $\partial_\mu\phi$.

Note. In a Hamiltonian derivation you would assume

$$\partial_t\phi = i[H,\phi]. \tag{214}$$

Also note that we will use the notation

$$\langle\Omega| T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle \equiv \langle\phi_1\cdots\phi_n\rangle, \tag{215}$$

where we assume the expectation value is taken in a time ordered fashion with respect to the interacting vacuum. We also use $\phi_1 \equiv \phi(x_1)$.

Claim.

$$\left(\Box_x + m^2\right)\langle\phi_x\phi_y\rangle = \left\langle\left(\Box_x + m^2\right)\phi_x\phi_y\right\rangle - i\delta^4(x-y). \tag{216}$$

Proof. As a warm up, lets study the free theory for which

$$\left(\Box_x + m^2\right) \underbrace{\left\langle \phi_x^0 \phi_y^0 \right\rangle}_{\Delta_F(x-y)} = 0 - i\delta^4\left(x-y\right), \tag{217}$$

which we have already established in the free theory. For the interacting theory,

$$\begin{aligned}
\partial_{x^0} \left\langle \phi_x \phi_y \right\rangle &= \partial_{x^0} \left(\left\langle \Omega \right| \phi_x \phi_y \left| \Omega \right\rangle \Theta\left(x^0 - y^0\right) + \left\langle \Omega \right| \phi_y \phi_x \left| \Omega \right\rangle \Theta\left(y^0 - x^0\right) \right) &(218)\\
&= \left\langle \partial_{x^0} \phi_x \phi_y \right\rangle + \left\langle \Omega \right| \phi_x \phi_y \left| \Omega \right\rangle \partial_x \Theta\left(x^0 - y^0\right) + \left\langle \Omega \right| \phi_y \phi_x \left| \Omega \right\rangle \partial_{x^0} \Theta\left(y^0 - x^0\right) &(219)\\
&= \left\langle \partial_{x^0} \phi_x \phi_y \right\rangle + \delta\left(x^0 - y^0\right) \left\langle \Omega \right| \underbrace{\left[\phi_x, \phi_y\right]}_{0} \left| \Omega \right\rangle, &(220)
\end{aligned}$$

where the commutator vanishes as we have equal time. Then notice,

$$\begin{aligned}
\partial_{x^0}^2 \left\langle \phi_x \phi_y \right\rangle &= \left\langle \partial_{x^0}^2 \phi_x \phi_y \right\rangle + \delta\left(x^0 - y^0\right) \left\langle \Omega \right| \underbrace{\left[\partial_{x^0} \phi_x, \phi_y\right]}_{-i\delta^3(\mathbf{x}-\mathbf{y})} \left| \Omega \right\rangle &(221)\\
&= \left\langle \partial_{x^0}^2 \phi_x \phi_y \right\rangle - i\delta^4\left(x-y\right). &(222)
\end{aligned}$$

As the spatial derivatives and mass terms do nothing, we arrive at the claim. □

This is a first example of an expression we will call the *Schwinger-Dyson equation.* It can be generalized such that

$$\left(\Box_x + m^2\right) \left\langle \phi_x \phi_1 \cdots \phi_n \right\rangle = \left\langle \frac{\partial \mathcal{L}_{\text{int}}\left(\phi\left(x\right)\right)}{\partial \phi} \phi_1 \cdots \phi_n \right\rangle - i \sum_{j=1}^{n} \delta^4\left(x - x_j\right) \left\langle \phi_1 \cdots \phi_{j-1} \phi_{j+1} \cdots \phi_n \right\rangle. \tag{223}$$

Example. Observe that for the four point function in the free theory, based on Wick's theorem, we expect

$$\begin{aligned}
\left\langle \phi_1^0 \phi_2^0 \phi_3^0 \phi_4^0 \right\rangle &\underset{\text{Wick's}}{=} \Delta_F\left(x_1 - x_2\right) \Delta_F\left(x_3 - x_4\right) + \Delta_F\left(x_1 - x_3\right) \Delta\left(x_2 - x_4\right) + \Delta_F\left(x_1 - x_4\right) \Delta\left(x_2 - x_3\right)\\
&\equiv \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}. &(224)
\end{aligned}$$

On the contrary, if we derive via Schwinger-Dyson (and dropping the superscripts but still working with the free theory), we see

$$\left\langle \phi_1 \phi_2 \phi_3 \phi_4 \right\rangle = \int \mathrm{d}^3 x \, \delta\left(x - x_1\right) \left\langle \phi_x \phi_2 \phi_3 \phi_4 \right\rangle. \tag{225}$$

As $\delta^4\left(x - x_1\right) = i\left(\Box_x + m^2\right) \Delta_{x1}$,

$$\begin{aligned}
\left\langle \phi_1 \phi_2 \phi_3 \phi_4 \right\rangle &= i \int \mathrm{d}^3 x \left(\left(\Box_x + m^2\right) \Delta_{x1} \right) \left\langle \phi_x \phi_2 \phi_3 \phi_4 \right\rangle &(226)\\
&= i \int \mathrm{d}^3 x \, \Delta_{x1} \left(\left(\Box_x + m^2\right) \left\langle \phi_x \phi_2 \phi_3 \phi_4 \right\rangle \right) &(227)\\
&= i \int \mathrm{d}^3 x \, \Delta_{x1} \left(-i\delta\left(x - x_2\right) \left\langle \phi_3 \phi_4 \right\rangle - i\delta\left(x - x_3\right) \left\langle \phi_2 \phi_4 \right\rangle - i\delta\left(x - x_4\right) \left\langle \phi_2 \phi_3 \right\rangle \right) &(228)
\end{aligned}$$

$$= \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}, \tag{229}$$

which agrees with the expression we obtained using Wick's theorem.

Example. Consider a cubic interaction, $\mathcal{L}_{\text{int}} = \frac{g}{3!}\phi^3$. We will compute the one point function,

$$\langle \phi_x \rangle = \int \mathrm{d}^4 y \, \delta(x-y) \langle \phi_y \rangle \tag{230}$$

$$= i \int \mathrm{d}^4 y \left(\Box_y + m^2\right) \Delta_{xy} \langle \phi_y \rangle \tag{231}$$

Integrating by parts, we see

$$\langle \phi_x \rangle = i \int \mathrm{d}^4 y \, \Delta_{xy} \left(\Box_y + m^2\right) \langle \phi_y \rangle \tag{232}$$

And using the Schwinger-Dyson equation,

$$\langle \phi_x \rangle = i \int \mathrm{d}^4 y \, \Delta_{xy} \frac{g}{2} \langle \phi_y^2 \rangle \tag{233}$$

Expanding perturbatively in g, we see

$$\langle \phi_x \rangle = \frac{ig}{2} \int \mathrm{d}^4 y \, \Delta_{xy} \left\langle \left(\phi_y^0\right)^2 \right\rangle + \mathcal{O}\left(g^3\right) \tag{234}$$

$$\langle \phi_x \rangle = \frac{ig}{2} \int \mathrm{d}^4 y \, \Delta_{xy}\Delta_{yy} + \mathcal{O}\left(g^3\right) \tag{235}$$

$$= \frac{ig}{2} \left(\text{tadpole diagram}\right) + \mathcal{O}\left(g^3\right). \tag{236}$$

11 Lecture: Feynman diagrams

04/11/2024

Example. Calculating the three point function in this ϕ^3 theory, we see

$$\langle \phi_1\phi_2\phi_3 \rangle = \int \mathrm{d}^4 x \, \delta(x - x_1) \langle \phi_x\phi_2\phi_3 \rangle \tag{237}$$

$$= i \int \mathrm{d}^4 x \left(\Box_x + m^2\right) \langle \phi_x\phi_2\phi_3 \rangle \tag{238}$$

$$= \frac{ig}{2} \int \mathrm{d}^4 x \, \Delta_{x1} \langle \phi_x\phi_x\phi_2\phi_3 \rangle + \int \mathrm{d}^4 x \, \Delta_{x1} \left(\delta(x-x_2)\langle x_3 \rangle + \delta(x-x_3)\langle \phi_2 \rangle\right). \tag{239}$$

Using the expression for the four point function in the free theory, and the expression for the one point function, we see

$$\langle \phi_1\phi_2\phi_3 \rangle = \frac{ig}{2} \int \mathrm{d}^4 x \, \Delta_{x1} \left(\Delta_{xx}\Delta_{23} + 2\Delta_{x3}\Delta_{x2}\right) + \frac{ig}{2} \int \mathrm{d}^4 x \, \Delta_{x1}\delta(x-x_2) \int \mathrm{d}^4 y \, \Delta_{3y}\Delta_{yy}$$
$$+ \frac{ig}{2} \int \mathrm{d}^4 x \, \Delta_{x1}\delta(x-x_3) \int \mathrm{d}^4 y \, \Delta_{2y}\Delta_{yy} \tag{240}$$

$$= ig \int \mathrm{d}^4 x \, \Delta_{x1}\Delta_{x3}\Delta_{x2} + \frac{ig}{2} \int \mathrm{d}^4 x \, \Delta_{xx} \left(\Delta_{x1}\Delta_{23} + \Delta_{12}\Delta_{3x} + \Delta_{13}\Delta_{2x}\right) + \mathcal{O}\left(g^2\right) \tag{241}$$

$$= ig\text{diagram} + \frac{ig}{2} \left(\text{diagram} + \cdots\right). \tag{242}$$

Example. Returning to a two point function, we see

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int \mathrm{d}^4x\,\Delta_{1x}\left(\frac{g}{2}\left\langle\phi_x^2\phi_2\right\rangle - i\delta\left(x-x_2\right)\right) &(243)\\ &= \Delta_{12} + \frac{ig}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\delta\left(y-x_2\right)\Delta_{1x}\left\langle\phi_x^2\phi_y\right\rangle &(244)\\ &= \Delta_{12} + \frac{ig}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,i\Delta_{1x}\Delta_{2y}\left(\frac{g}{2}\left\langle\phi_x^2\phi_y^2\right\rangle - 2i\delta\left(x-y\right)\left\langle\phi_x\right\rangle\right) &(245)\\ &= \Delta_{12} + \frac{(ig)^2}{4}\int \mathrm{d}^4x\,\mathrm{d}^4y\left(\Delta_{1x}\Delta_{2y}\Delta_{xx}\Delta_{yy} + 2\Delta_{1x}\Delta_{2y}\Delta_{xy}\Delta_{xy} + 2\Delta_{1x}\Delta_{2x}\Delta_{xy}\Delta_{yy}\right) + \mathcal{O}\left(g^3\right). &(246)\end{aligned}$$

11.1 Feynman Diagrams

The Feynman rules for calculating $\langle\phi_1\cdots\phi_n\rangle$ in Scalar Yukawa Theory are as follows:

1) Start with x_i external points. Draw a line from each point
2) A line can either:
   - contract an existing line giving $\Delta_F\left(x_i-x_j\right)$,
   - split giving a new vertex, where the coefficient will be $i\lambda_n$ for $\mathcal{L}_{\text{int}} = \frac{\lambda_n}{n!}\phi^n$. The number of lines depends on $\mathcal{L}'_{\text{int}}$.
3) At any given order in $i\lambda_n$, the result is the sum of all diagrams with all lines contracted and integrated over vertices.
4) Warning: symmetry factors.

12 Lecture: Scalar Yukawa Theory

05/11/2024

12.1 Quantization

We study scalar Yukawa theory with

$$\mathcal{L}_0 = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2}_{\text{real scalar}} + \underbrace{\partial_\mu\psi^*\partial^\mu\psi - M^2\psi^*\psi}_{\text{complex scalar}}, \tag{247}$$

and interaction term

$$\mathcal{L}_{\text{int}} = -g\psi^*\psi\phi, \tag{248}$$

with $g \ll M, m$.

Our goal is to infer the Feynman rules by evaluating the S matrix. We begin by studying the free theory.

For the free theory, recall that

$$\phi\left(x\right) = \int \frac{\mathrm{d}^3p}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_pe^{-ipx} + a_p^\dagger e^{ipx}\right), \tag{249}$$

with $\omega_p = \sqrt{\vec{p}^2 + m^2}$ and

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\widetilde{\omega}_p}} \left(b_p e^{-ipx} + c_p^\dagger e^{ipx} \right), \tag{250}$$

with $\widetilde{w}_p = \sqrt{\vec{p}^2 + M^2}$. Recall that we impose the canonical quantisation (commutation relations) between each field and its canonical momenta

$$\Bigg[\phi(t,\vec{x}), \underbrace{\partial_t \phi(t,\vec{y})}_{\Pi_\phi} \Bigg] = i\delta(\vec{x}-\vec{y}) \tag{251}$$

$$[\psi(t,\vec{x}), \Pi_\psi(t,\vec{y})] = i\delta(\vec{x}-\vec{y}) \tag{252}$$

$$[\psi^*(t,\vec{x}), \Pi_{\psi^*}(t,\vec{y})] = i\delta(\vec{x}-\vec{y}), \tag{253}$$

where $\Pi_\psi = \partial_t \psi^*$ and $\Pi_{\psi^*} = \partial_t \psi$ which imply

$$\left[a_p, a_{p'}^\dagger \right] = (2\pi)^3 \delta(\vec{p} - \vec{p}') \tag{254}$$

$$\left[b_p, b_{p'}^\dagger \right] = (2\pi)^3 \delta(\vec{p} - \vec{p}') \tag{255}$$

$$\left[c_p, c_{p'}^\dagger \right] = (2\pi)^3 \delta(\vec{p} - \vec{p}'). \tag{256}$$

With these, we can obtain the normal ordered Hamiltonian given by

$$H_0 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\omega_p a_p^\dagger a_p + \widetilde{\omega}_p b_p^\dagger b_p + \widetilde{\omega}_p c_p^\dagger c_p \right). \tag{257}$$

We also have the charge associated to the symmetry $\psi \to e^{i\alpha}\psi$ where

$$Q = i \int \mathrm{d}^3 x : \left(\dot{\psi}^* \psi - \psi^* \dot{\psi} \right): \tag{258}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(c_p^\dagger c_p - b_p^\dagger b_p \right). \tag{259}$$

As this is a conserved charge we have $[H_0, Q] = 0$.

Moving to the Fock space, we once again have a vacuum state defined such that

$$a_p |0\rangle = b_p |0\rangle = c_p |0\rangle = 0. \tag{260}$$

All of which imply $H_0 |0\rangle = 0$. We can then create one particle states, called *meson* states $|\phi\rangle$, with

$$|\phi\rangle \equiv a_p^\dagger |0\rangle \Rightarrow H |\phi\rangle = \omega_p |\phi\rangle \text{ and } Q |\phi\rangle = 0. \tag{261}$$

We also define one particle *nucleon* states $|\psi\rangle$, defined by

$$|\psi\rangle = c_p^\dagger |0\rangle \Rightarrow H |\psi\rangle = \widetilde{\omega}_p |\psi\rangle \text{ and } Q |\psi\rangle = |\psi\rangle. \tag{262}$$

Lastly we define *antinucleon* states $|\psi^\dagger\rangle$ defined by

$$|\psi^\dagger\rangle = b_p^\dagger |0\rangle \Rightarrow H \left|\psi^\dagger = \widetilde{\omega}_p \left|\psi^\dagger\right\rangle\right\rangle \text{ and } Q |\psi^\dagger\rangle = -|\psi^\dagger\rangle. \tag{263}$$

One can of course construct multiparticle states.

12.2 Propagators and Correlation Functions

The Feynman propagator is still

$$\Delta_F(x_1 - x_2) = \langle \phi_1 \phi_2 \rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x_1 - x_2)}, \tag{264}$$

and

$$\begin{aligned} \Delta_F^{\psi}(x_1 - x_2) &= \langle \Omega | T\psi(x_1) \psi^\dagger(x_2) | \Omega \rangle & (265) \\ &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\varepsilon} e^{-ip(x_1 - x_2)} & (266) \\ &\equiv \hat{\Delta}_{12}, & (267) \end{aligned}$$

where only the mass has changed.

Note.

1. In the free theory, $\langle \psi_1 \psi_2 \rangle = \left\langle \psi_1^\dagger \psi_2^\dagger \right\rangle = 0$.

   Proof. Exercise. □

2. Why is $\left\langle \psi_1 \psi_2^\dagger \right\rangle \neq 0$? Notice that

$$\sqrt{2\widetilde{\omega}_p} b_p^\dagger = -i \int \mathrm{d}^3 x \, e^{-ipx} \overleftrightarrow{\partial_t} \psi^\dagger \tag{268}$$

$$\sqrt{2\widetilde{\omega}_p} b_p = i \int \mathrm{d}^3 x \, e^{ipx} \overleftrightarrow{\partial_t} \psi \tag{269}$$

$$\sqrt{2\widetilde{\omega}_p} c_p^\dagger = -i \int \mathrm{d}^3 x \, e^{ipx} \overleftrightarrow{\partial_t} \psi \tag{270}$$

$$\sqrt{2\widetilde{\omega}_p} c_p = -i \int \mathrm{d}^3 x \, e^{-ipx} \overleftrightarrow{\partial_t} \psi^\dagger. \tag{271}$$

Moving to the interacting theory, we apply Schwinger Dyson. Namely, as

$$\left(\Box + m^2\right) \phi - \left(-g\psi^\dagger \psi\right) = 0 \tag{272}$$

$$\left(\Box + M^2\right) \psi - \left(-g\phi\psi\right) = 0 \tag{273}$$

$$\left(\Box + M^2\right) \psi^\dagger - \underbrace{\left(-g\phi\psi^\dagger\right)}_{\mathcal{L}'_{\text{int}} \equiv \frac{\partial \mathcal{L}_{\text{int}}}{\partial \psi}} = 0. \tag{274}$$

Notice that

$$\begin{aligned} \left(\Box_x + m^2\right) \left\langle \phi_x \phi_1 \cdots \phi_n \psi_1 \cdots \psi_m \psi_1^\dagger \cdots \psi_p^\dagger \right\rangle &= -g \left\langle \psi_x \psi_x^\dagger \phi_1 \cdots \phi_n \psi_1 \cdots \psi_m \psi_1^\dagger \cdots \psi_p^\dagger \right\rangle \\ &\quad - i \sum_{j=1}^{n} \delta(x - x_j) \left\langle \phi_1 \cdots \phi_{j-1} \phi_{j+1} \cdots \phi_n \psi_1 \cdots \psi_m \psi_1^\dagger \cdots \psi_p^\dagger \right\rangle. \end{aligned} \tag{275}$$

Similarly,

$$\begin{aligned}\left(\Box_x + M^2\right)\left\langle \phi_1\cdots\phi_n\psi_x\psi_1\cdots\psi_m\psi_1^\dagger\cdots\psi_p^\dagger\right\rangle &= -g\left\langle \phi_1\cdots\phi_n\phi_x\psi_x^\dagger\psi_1\cdots\phi_m\phi_1^\dagger\cdots\phi_p^\dagger\right\rangle \\ &\quad - i\sum_{j=1}^{p}\delta\left(x-x_j\right)\left\langle \phi_1\cdots\phi_n\psi_1\cdots\psi_m\psi_1^\dagger\cdots\psi_{j-1}^\dagger\psi_{j+1}^\dagger\cdots\psi_p^\dagger\right\rangle .\end{aligned} \tag{276}$$

To infer the Feynman rules, we look at the three point function

$$\left\langle \phi_1\psi_2^\dagger\psi_3\right\rangle = -ig\int \mathrm{d}^4x\,\Delta_{1x}\left\langle \psi_x\psi_x^\dagger\psi_2^\dagger\psi_3\right\rangle . \tag{277}$$

We replace this correlator with its free theory variant, giving

$$\left\langle \phi_1\psi_2^\dagger\psi_3\right\rangle = -ig\int \mathrm{d}^4x\,\Delta_{1x}\left(\hat{\Delta}_{xx}\hat{\Delta}_{23}+\hat{\Delta}_{x2}\hat{\Delta}_{3x}\right)+\mathcal{O}\left(g^2\right). \tag{278}$$

We are now equipped to consider scattering in this theory.

We want to evaluate

$$\langle \text{final},\infty|\text{initial},-\infty\rangle = \langle f|\,S\,|i\rangle . \tag{279}$$

We assume as before, that the asymptotic states at $\pm\infty$ exists within the free theory. Then to calculate S we need, for example,

$$\sqrt{2\widetilde{\omega}_p}\left(b_p^\dagger(\infty)-b_p^\dagger(-\infty)\right) = -i\int \mathrm{d}^4x\,e^{-ipx}\left(\Box_x+M^2\right)\psi^\dagger(x), \tag{280}$$

and similar formulas for c_p and $c_p^\dagger$.

13 Lecture: Yukawa Scattering

08/11/2024

We consider the scattering of nucleons, $|\psi\psi\rangle \to |\psi\psi\rangle$ with initial state

$$\lim_{t\to-\infty}\sqrt{2\widetilde{\omega}_{p_1}}\sqrt{2\widetilde{\omega}_{p_2}}c_{p_1}^\dagger(t)\,c_{p_2}^\dagger(t)\,|\Omega\rangle , \tag{281}$$

and final state

$$\lim_{t\to\infty}\sqrt{2\widetilde{\omega}_{p_3}}\sqrt{2\widetilde{\omega}_{p_4}}c_{p_3}^\dagger(t)\,c_{p_4}^\dagger(t)\,|\Omega\rangle . \tag{282}$$

We have that $\langle \text{final}, \ +\infty|\text{initial}, \ -\infty\rangle = \langle f|\,S\,|i\rangle$ gives us

$$\begin{aligned}\langle f|\,S\,|i\rangle = i^4\prod_{j=1}^{4}\int \mathrm{d}^4x_j\,&\underbrace{e^{-ip_1x_1}\left(\Box_1+M^2\right)e^{-ip_2x_2}\left(\Box+M^2\right)}_{\text{ingoing}}\underbrace{e^{ip_3x_3}\left(\Box_3+M^2\right)e^{ip_4x_4}\left(\Box_4+M^2\right)}_{\text{outgoing}} \\ &\langle\Omega|\,T\psi(x_1)\,\psi(x_2)\,\psi^\dagger(x_3)\,\psi^\dagger(x_4)\,|\Omega\rangle .\end{aligned} \tag{283}$$

Proof.

□

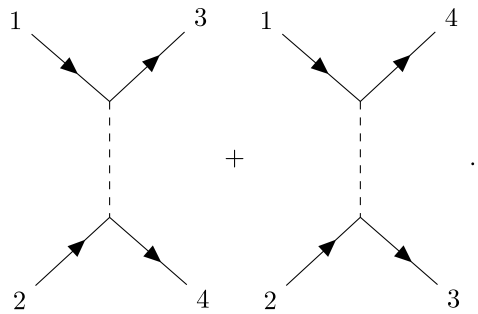
We thus need to find $\left\langle\psi_1\psi_2\psi_3^\dagger\psi_4^\dagger\right\rangle$. We will use Feynman diagrams to leading order g.

To order g^0, we have

$$\begin{array}{c} 3 \rightarrow 4 \\ \\ 1 \rightarrow 2 \end{array} + \begin{array}{c} 4 \rightarrow 3 \\ \\ 1 \rightarrow 2 \end{array} . \tag{284}$$

To order g we have nothing constructible out of this vertex that will leave us with these four external fermions.

To order g^2 we have two disconnected diagrams and two connected ones given by



(285)

The connected components tell us

$$\left\langle\phi_1\phi_2\phi_3^\dagger\phi_4^\dagger\right\rangle_C = (-ig)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\left(\hat{\Delta}_{1x}\hat{\Delta}_{x_3}\Delta_{xy}\hat{\Delta}_{2y}\hat{\Delta}_{y4}+\hat{\Delta}_{1x}\hat{\Delta}_{x4}\Delta_{xy}\hat{\Delta}_{2y}\hat{\Delta}_{y3}\right)+\mathcal{O}\left(g^3\right). \tag{286}$$

We then replace $\langle\Omega|\,T\psi\left(x_1\right)\psi\left(x_2\right)\psi^\dagger\left(x_3\right)\psi^\dagger\left(x_4\right)|\Omega\rangle$ with this expression and see that

$$\left(\Box_4+M^2\right)\left\langle\psi_1\psi_2\psi_3^\dagger\psi_4^\dagger\right\rangle\sim\int\left(\Box_4+M^2\right)\left(\hat{\Delta}_{y4}\cdots\hat{\Delta}_{x4}\cdots\right). \tag{287}$$

We then have

$$\begin{aligned}\langle f|\,S\,|i\rangle &= i^4\left(-ig\right)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\left(-ie^{-ip_1x}\right)\left(-ie^{-ip_2y}\right)\Delta_{xy}\left(-ie^{ip_3x}\right)\left(-ie^{ip_4y}\right)\\ &\quad+\left(-ie^{-ip_1x}\right)\left(-ie^{-ip_2y}\right)\Delta_{xy}\left(-ie^{ip_3y}\right)\left(-ie^{ip_4x}\right)+O\left(g^3\right) \qquad (288)\\ &= i^4\left(-ig\right)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\int\frac{\mathrm{d}^4p}{\left(2\pi\right)^4}\frac{i}{p^2-m^2-i\varepsilon}e^{ix(p_1+p-p_3)}e^{iy(p_2-p-p_4)}+\{3\leftrightarrow 4\} \qquad (289)\end{aligned}$$

$$= (-ig)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 - i\varepsilon} (2\pi)^4 \delta(p_1 + p - p_3) (2\pi)^4 \delta(p_2 - p - p_4) + \{3 \leftrightarrow 4\} \tag{290}$$

$$= (-ig)^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \left(\frac{i}{(p_3 - p_1)^2 - m^2} - \frac{i}{(p_4 - p_1)^2 + m^2} \right) + \mathcal{O}(g^3), \tag{291}$$

where the leading delta function gives us conservation of four momenta.

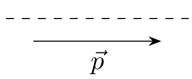
14 Lecture: Feynman Rules

11/11/2024

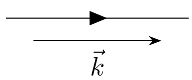
Note. $\Delta_{xy} = \langle 0| T\phi^0(x)\phi^0(y)|0\rangle$ and $\hat{\Delta}_{xy} = \langle 0| T\psi^0(x)\psi^{0\dagger}(y)|0\rangle$

We now state the Feynman rules for scalar Yukawa theory (for the connected S-matrix).

1. Draw all possible connected diagrams for the relevant process (i.e. fixed initial and final states).
2. For each diagram the contribution to the S matrix is:
   i) Assignment momenta to each line.
   ii) Internal lines should be given by

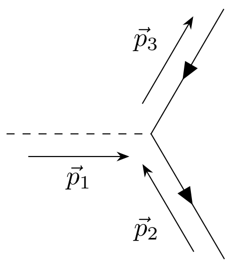


$$= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 - i\varepsilon}, \tag{292}$$



$$= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 - i\varepsilon},. \tag{293}$$

   iii) For external lines do nothing.
   iv) For each vertex, write the factor



$$= (-ig)(2\pi)^4 \delta(p_1 - p_2 + p_3). \tag{294}$$

   v) Sum over all diagrams and integrate over undetermined (loop) momenta.

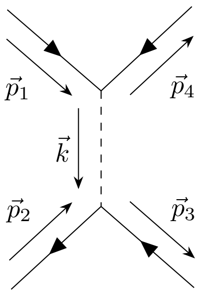
Note. External particles are on shell as $p_i^2 = M^2$. However internal particles are not as $k^2 \neq m^2$. This gives them the name *virtual particles*.

What we computed is a tree-level diagram. This means we do not have any loops in our diagrams. Tree-level diagrams are the leading connected contributions and the emphasis of this course.

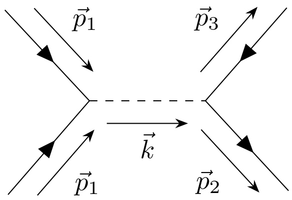
Example. We move to study nucleon–anti nucleon scattering: $\psi^\dagger\psi \to \psi^\dagger\psi$ at leading order (+ connected).

We have that

$$\langle f| S |i\rangle_c = $$



$$+$$



$$+\mathcal{O}\left(g^3\right) \tag{295}$$

$$\begin{aligned}
&= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 - i\varepsilon} (-ig)(2\pi)^4 \delta(p_1 - k - p_3)(-ig)(2\pi)^4 \delta(k + p_2 - p_4) \\
&\quad + \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 - i\varepsilon} (-ig)(2\pi)^4 \delta(p_1 + p_2 + k)(-ig)(2\pi)^4 \delta(-k - p_3 - p_4)
\end{aligned} \tag{296}$$

$$= (-ig)^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \left(\frac{i}{(p_1 - p_3)^2 - m^2 - i\varepsilon} + \frac{i}{(p_1 - p_2)^2 - m^2 - i\varepsilon} \right) + \mathcal{O}\left(g^3\right). \tag{297}$$

This second diagram has an interesting momentum dependence.

In the center of mass frame,

$$(p_1 + p_2)^2 = 4\left(M^2 + \vec{p}_1^{\,2}\right). \tag{298}$$

Then

$$(p_1 + p_2)^2 - m^2 = 4\left(M^2 + \vec{p}_1^{\,2}\right) - m^2. \tag{299}$$

If $m < 2M$, we will never have $(p_1 + p_2)^2 = m^2$ and thus can remove $i\varepsilon$. If $m < 2M$, then for some $\vec{p}_1$ it is possible. This leads to a bump.

14.1 Mandelstam Variables

For $2 \to 2$ scattering the same combinations of momenta appear often. We define

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{300}$$

$$p = (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{301}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2. \tag{302}$$

Where we assume $p_1 + p_2 = p_3 + p_4$.

With these variables we see our two diagrams were representing the t and s channel respectively.

For $\psi\psi \to \psi\psi$ scattering we have t and u channels respectively.

Assuming all particles have the same mass we have that given $p_1 = (E, p\hat{z})$ and $p_2 = (E, -p\hat{z})$, we have $p_3 = (E, \vec{p})$ and $p_4 = (E, -\vec{p})$ with $\vec{p} = (p\sin\theta, p\cos\theta, 0)$ in the center of mass frame.

Notice that

$$s = 4E^2 \qquad t = -2p^2 (1 - \cos\theta) \qquad u = -2p^2 (1 + \cos\theta), \tag{303}$$

which give us $s + t + u = 4m^2$. One can think of s as the total energy of the system in the center of mass frame. t and u are measures of momentum exchange.

15 Lecture: The Dirac Equation

13/11/2024

Recall that the definition of a field is an object which transforms in a representation of the Lorentz group,

$$\phi^a (x) \to D[\Lambda]^a{}_b \, \phi_b \left(\Lambda^{-1} x\right). \tag{304}$$

Our goal is to identity/consider a more interesting representation than the trivial one we used for the scalar field. As we have 6 parameters, we write a generic representation with

$$D[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma} R^{\rho\sigma}\right), \tag{305}$$

where $\Omega_{\rho\sigma} = -\Omega_{\sigma\rho}$ is an antisymmetric tensor containing 6 free parameters, and $R^{\rho\sigma}$ are the generators of the group.

The generators $R^{\rho\sigma}$, by the definition of the Lorentz group, satisfy

$$\left[R^{\rho\sigma}, R^{\delta\nu}\right] = \eta^{\sigma\delta} R^{\rho\nu} - \eta^{\rho\delta} R^{\sigma\nu} + \eta^{\rho\nu} R^{\sigma\delta} - \eta^{\sigma\nu} R^{\rho\delta}. \tag{306}$$

Examples.

1) We have an infinite dimension representation with generators $L^{\rho\sigma} = X^\rho \partial^\sigma - X^\sigma \partial^\rho$ which satisfies the commutation relation, and is infinite as it acts on the space of functions.

2) We also have a 4×4 representation with generators

$$\left(\mathcal{M}^{\rho\sigma}\right)^\mu{}_\nu = \eta^{\rho\mu} \delta^\sigma_\nu - \eta^{\sigma\mu} \delta^\rho_\nu. \tag{307}$$

These also satisfy the commutation relations and when exponentiated,

$$\Lambda = D[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma} \mathcal{M}^{\rho\sigma}\right), \tag{308}$$

which acts on vectors as

$$V^\mu \to D[\Lambda]^\mu{}_\nu \, V^\nu = \Lambda^\mu{}_\nu V^\nu, \tag{309}$$

which makes clearer that we are exponentiating a matrix (which gives us a matrix).

3) Consider a new algebra with generators

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}, \tag{310}$$

Namely, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ if $\mu \neq \nu$ and $\left(\gamma^0\right)^2 = \mathbb{I}$ and $\left(\gamma^i\right)^2 = -\mathbb{I}$. A simple representation of this algebra is called the **chiral representation**, and is written

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{311}$$

where σ^i are the Pauli matrices satisfying $\left[\sigma^i, \sigma^j\right] = 2\delta^{ij}$.

Claim. We claim that

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^\rho, \gamma^\sigma\right], \tag{312}$$

forms a representation of the Lorentz algebra (i.e. complies with the commutation relation).

Proof. First show

$$\left[S^{\mu\nu}, S^{\sigma\rho}\right] = \frac{1}{2}\left[S^{\mu\nu}, \gamma^\rho\gamma^\sigma\right] \tag{313}$$

$$= \frac{1}{2}\left[S^{\mu\nu}, \gamma^\sigma\right] + \frac{1}{2}\gamma^\rho\left[S^{\mu\nu}, \gamma^\sigma\right]. \tag{314}$$

Second show

$$\left[S^{\mu\nu}, \gamma^\rho\right] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}. \tag{315}$$

Combine and recover the commutation relation. □

Therefore, we have

$$S\left[\Lambda\right] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right). \tag{316}$$

Do not forget that $S\left[\Lambda\right]$ here is a 4×4 matrices and can equivalently be written

$$\left(S\left[\Lambda\right]\right)^a{}_b = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\left(S^{\rho\sigma}\right)^a{}_b\right). \tag{317}$$

The natural question is what transforms under $S\left[\Lambda\right]$? We call them **spinors** and write them as $\psi^a\left(x\right)$. As $S\left[\Lambda\right]$ is a 4×4 object, ψ^a must have a 4-element index as well. It transforms as

$$\psi^a\left(x\right) \to S\left[\Lambda\right]^a{}_b\,\psi^b\left(\Lambda^{-1}x\right). \tag{318}$$

How does this differ from the other 4×4 representation we identified? Lets look at the properties of this representation. We consider a rotation, where one along k corresponds to $\Omega_{ij} \neq 0$. Suppose we want to rotate along z-axis, for which we then have $\Omega_{12} \neq 0$. Then we see in the chiral representation

$$S^{12} = -\frac{i}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \tag{319}$$

using $\sigma^i \sigma^j = \delta^{ij} + \varepsilon^{ijk}\sigma^k$. This gives us a group element

$$S[\Lambda] = \exp\left(\frac{1}{2}2\Omega_{12}S^{12}\right) = \exp\left(-i\frac{\Omega_{12}}{2}\begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}\right). \tag{320}$$

We pick $\Omega_{12} = 2\pi$ and see

$$S[\Lambda] = \exp\left(\frac{1}{2}2\Omega_{12}S^{12}\right) = \begin{pmatrix}e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3}\end{pmatrix} \tag{321}$$

$$= -\mathbb{I}_4. \tag{322}$$

Which is not how a vector transforms under a rotation by 2π. This is clearly a distinct representation.

Performing a boost along the x axis, with $\Omega_{i0} = \eta^i$ we see

$$S^{01} = \frac{1}{2}\begin{pmatrix}-\sigma_1 & 0 \\ 0 & \sigma_1\end{pmatrix}, \tag{323}$$

gives us

$$S[\Lambda] = \begin{pmatrix}e^{\eta_1\frac{\sigma^1}{2}} & 0 \\ 0 & e^{-\eta_1\frac{\sigma^1}{2}}\end{pmatrix}, \tag{324}$$

or more generally,

$$S[\Lambda] = \begin{pmatrix}e^{\eta_i\frac{\sigma^i}{2}} & 0 \\ 0 & e^{-\eta_i\frac{\sigma^i}{2}}\end{pmatrix}. \tag{325}$$

This is once again very different from a vector.

15.1 Actions

We then move to construct an action which is Lorentz invariant using ψ and its transformation.

We need some object which transforms like

$$\bullet(x) \to \bullet\left(\Lambda^{-1}x\right) S[\Lambda]^{-1}, \tag{326}$$

to cancel out the transformation of ψ.

A natural object to study is

$$\psi^\dagger(x) = \left(\psi^*(x)\right)^T. \tag{327}$$

We know that

$$\psi^\dagger(x) \to \psi^\dagger\left(\Lambda^{-1}x\right) S[\Lambda]^\dagger. \tag{328}$$

It remains to figure out how $S[\Lambda]^\dagger$ is related to $S[\Lambda]^{-1}$.

In the chiral representation, one can see that

$$\left(\gamma^0\right)^\dagger = \gamma^0 \qquad \left(\gamma^i\right)^\dagger = -\gamma^i. \tag{329}$$

One can write this as

$$\gamma^0\gamma^\mu\gamma^0 = \left(\gamma^\mu\right)^\dagger. \tag{330}$$

Then

$$\left(S^{\rho\sigma}\right)^\dagger = \frac{1}{4}\left[\left(\gamma^\sigma\right)^\dagger, \left(\gamma^\rho\right)^\dagger\right] = -\gamma^0 S^{\rho\sigma}\gamma^0, \tag{331}$$

which implies

$$S\left[\Lambda\right]^\dagger = \exp\left(-\frac{1}{2}\Omega_{\rho\sigma}\gamma^0 S^{\rho\sigma}\gamma^0\right) = \gamma^0 S\left[\Lambda\right]^{-1}\gamma^0. \tag{332}$$

Thus $\psi^\dagger\psi$ is not a Lorentz invariant combination.

16 Lecture: The Adjoint Spinor

15/11/2024

Definition 16.1: We define $\overline{\psi}\left(x\right) = \psi^\dagger\left(x\right)\gamma^0$ to be the **adjoint spinor**.

Claim. $\overline{\psi}\psi$ is Lorentz invariant.

Proof.

$$\overline{\psi}\psi = \psi^\dagger\left(x\right)\gamma^0\psi\left(x\right) \to \psi^\dagger\left(\Lambda^{-1}x\right)\gamma^0 S\left[\Lambda\right]^{-1}\underbrace{\gamma^0\gamma^0}_{\mathbb{I}} S\left[\Lambda\right]\psi\left(\Lambda^{-1}x\right) = \overline{\psi}\psi\left(\Lambda^{-1}x\right). \tag{333}$$

□

Claim. $\overline{\psi}\gamma^\mu\psi$ transforms as a vector.

Proof.

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}S\left[\Lambda\right]^{-1}\gamma^\mu S\left[\Lambda\right]\psi. \tag{334}$$

We want the right hand side to be $\Lambda^\mu{}_\nu\overline{\psi}\gamma^\mu\psi$ as this is how vectors transform. Thus it remains to prove that

$$S\left[\Lambda\right]^{-1}\gamma^\mu S\left[\Lambda\right] = \Lambda^\mu{}_\nu\gamma^\nu. \tag{335}$$

An elegant approach to this proof is to do it infinitesimally with $\Omega \ll 1$. Then

$$S\left[\Lambda\right] = \mathbb{I} + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \cdots \tag{336}$$

$$S\left[\Lambda\right]^{-1} = \mathbb{I} - \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \cdots \tag{337}$$

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) = \mathbb{I} + \frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}. \tag{338}$$

To first order in Ω, this implies that

$$-\left[S^{\rho\sigma}, \gamma^{\mu}\right] = \left(\mathcal{M}^{\rho\sigma}\right)^{\mu}{}_{\nu}\,\gamma^{\nu} \tag{339}$$

$$= \gamma^{\sigma}\eta^{\rho\mu} - \gamma^{\rho}\eta^{\sigma\mu}. \tag{340}$$

□

One can continue and see that $\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi$ transforms as a 2-tensor and $\overline{\psi}\partial_{\mu}\psi$ transforms as a covector.

Then, we want to construct the simplest possible action with the following three ingredients:

- Lorentz invariance,
- As few derivatives as possible,
- reality (i.e. S is a real number).

We see that these give us

$$S = \int \mathrm{d}^4x \left(i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi\right). \tag{341}$$

This is the **Dirac** action.

Note. There is an i in the first term to ensure reality as

$$\left(i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi\right)^{*} = -i\partial_{\mu}\psi^{\dagger}\left(\gamma^{\mu}\right)^{\dagger}\left(\gamma^{0}\right)^{\dagger}\psi \tag{342}$$

$$= -i\partial_{\mu}\psi^{\dagger}\gamma^{0}\gamma^{\mu}\gamma^{0}\gamma^{0}\psi \tag{343}$$

$$= -i\partial_{\mu}\overline{\psi}\gamma^{\mu}\psi \tag{344}$$

$$= -i\,\underbrace{\partial_{\mu}\left(\overline{\psi}\gamma^{\mu}\psi\right)}_{\text{total deriv.}} + i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi \tag{345}$$

$$= i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi, \tag{346}$$

and thus it is real.

This has equations of motion for $\overline{\psi}$,

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0, \tag{347}$$

and for ψ,

$$-i\gamma^{\mu}\partial_{\mu}\overline{\psi} - m\overline{\psi} = 0. \tag{348}$$

These are complex conjugates of each other. This is the **Dirac equation**.

It is useful to note that we can obtain a second order equation of motion by applying a second operator to the Dirac equation such that

$$\left(i\gamma^{\nu}\partial_{\nu} + m\right)\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0 \tag{349}$$

$$\left(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2\right)\psi = 0 \tag{350}$$
$$\left(-\frac{1}{2}\left\{\gamma^\mu,\gamma^\nu\right\}\partial_\mu\partial_\nu - m^2\right) = 0 \tag{351}$$
$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right) = 0, \tag{352}$$

which is the Klein-Gordon equation.

Note. Sometimes one writes $\gamma^\mu\partial_\mu = \not{\partial}$.

This action has three notable symmetries, Lorentz, a $U(1)$ internal one and translations. Noether's theorem gives us conservation of angular momentum, a charge Q and the stress tensors (i.e. energy and momentum) respectively.

Namely, for such translations $x^\mu \to x^\mu + \varepsilon^\mu$, we have

$$\delta\psi = \varepsilon^\mu\partial_\mu\psi, \tag{353}$$

as before.

The stress energy tensor for the Dirac action is then given by

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi, \tag{354}$$

on shell (where the equations of motion have made other terms vanish).

For a Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, the spinor transforms as

$$\delta\psi_a = -\omega^{\mu\nu}\left(x_\nu\partial_\mu\psi_a - \frac{1}{2}\left(S_{\mu\nu}\right)^b{}_a\psi_b\right), \tag{355}$$

where the first part is the same as the scalar, and the second comes from the representation.

The current in this case is

$$\left(J^\mu\right)^{\lambda\nu} = -x^\nu T^{\mu\lambda} + x^\lambda T^{\mu\nu} + i\overline{\psi}\gamma^\mu S^{\lambda\nu}\psi, \tag{356}$$

where the last term is once again from the representation. It will lead to the spin of the particle.

Lastly, we have an internal *vector* symmetry such that under

$$\psi \to e^{i\alpha}\psi \tag{357}$$
$$\overline{\psi} \to e^{-i\alpha}\overline{\psi}, \tag{358}$$

the action is invariant for $\alpha \in \mathbb{R}$. This has an associated current

$$j^\nu_V = \overline{\psi}\gamma^\mu\psi, \tag{359}$$

where the subscript tells us this is a vector symmetry. This current has charge

$$Q = \int \mathrm{d}^3\,\overline{\psi}\gamma^0\psi, \tag{360}$$

which we will see is the *electric charge*.

We move to find simultaneous solutions to the Dirac equation and the Klein-Gordon equation.

From the Klein-Gordon equation, for some vector components $u(\vec{p})$ and $v(\vec{p})$, we know solutions will be of the form

$$\psi(x) \sim u(\vec{p})\, e^{ipx} + v(\vec{p})\, e^{ipx}, \tag{361}$$

with $p^2 = m^2$. Generically this will not solve the Dirac equation. Then, imposing it, we find that

$$(-\not{p} + m)\, u(\vec{p}) = 0 \qquad\qquad (\not{p} + m)\, v(\vec{p}) = 0. \tag{362}$$

In the chiral representation, the first of these becomes

$$\begin{pmatrix} mI_2 & -p\cdot\sigma \\ p\cdot\sigma & mI_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{363}$$

This implies

$$mu_1 = p\cdot\sigma u_2. \tag{364}$$

Note. One should make use of $m^2 = (p\cdot\sigma)(p\cdot\overline{\sigma})$ to recognise the other equation as identical to this one.

These together give us

$$\sqrt{p\cdot\sigma}u_2 = \sqrt{p\cdot\overline{\sigma}}u_1. \tag{365}$$

Then, we see that with ξ^s for $s = 1, 2$ given by $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ allow us to write

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}. \tag{366}$$

Identically for the second of our equations, one finds

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\overline{\sigma}}\eta^s \end{pmatrix}, \tag{367}$$

with η^s defined identically to ξ.

17 Lecture: Weyl Spinors

18/11/2024

Recall that our spinor representation $S[\Lambda]$ admitted a diagonal form in the chiral representation.

This tells us that the Dirac (4-component spinor) representation is reducible into

$$\psi(x) = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_- \end{pmatrix}, \tag{368}$$

where u_+ and u_- are 2 component spinors called **Weyl spinors** or *chiral spinors*.

Under rotations, they transform identically,

$$u_+ \to e^{i\vec{\phi}\cdot\vec{\sigma}/2}u_+ \tag{369}$$

$$u_- \to e^{i\vec{\phi}\cdot\vec{\sigma}/2} u_-. \tag{370}$$

Under boosts however, we have

$$u_+ \to e^{\vec{\chi}\cdot\vec{\sigma}/2} u_+ \tag{371}$$
$$u_- \to e^{-\vec{\chi}\cdot\vec{\sigma}/2} u_-, \tag{372}$$

where they transform with opposite sign.

In the chiral representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \tag{373}$$

it is obvious this splits.

A more general way to identify the two Weyl spinors is by introducing

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\varepsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \tag{374}$$

satisfies $\left\{\gamma^5, \gamma^\mu\right\} = 0$, $\left(\gamma^5\right)^2 = \mathbb{I}$, and

$$\left[S^{\mu\nu}, \gamma^5\right] = 0. \tag{375}$$

In the chiral representation this is

$$\gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{376}$$

We can define the projection operators

$$P_\pm = \frac{1}{2}\left(\mathbb{I} \pm \gamma^5\right), \tag{377}$$

which in the chiral representation are

$$P_+ = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{378}$$

These are projection operators satisfying $P_+P_- = 0$ and $P_+P_- = 0$. $\psi_+ = P_+\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix}$ and $\psi_- = P_-\psi = \begin{pmatrix} 0 \\ u_- \end{pmatrix}$ where these matrices are in the chiral representation.

Looking at the Dirac Lagrangian, in the chiral representation we can write

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{379}$$
$$= iu_-^\dagger\sigma^\mu\partial_\mu u_- + iu_+^\dagger\overline{\sigma}^\mu\partial_\mu u_+ - m\left(u_+^\dagger u_- + u_-^\dagger u_+\right). \tag{380}$$

17.1 Quantizing the Dirac Field

We proceed with the canonical quantization of the Dirac Lagrangian

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi. \tag{381}$$

We follow the principles from the scalar theory and aim to construct a Fock space with positive norm and H bounded from below.

We need to figure out the commutation relations and see that

$$\Pi = \frac{\partial\mathcal{L}}{\partial\left(\partial_t\psi\right)} = i\overline{\psi}\gamma^0 = i\psi^\dagger. \tag{382}$$

One can imagine imposing

$$\left[\psi_a\left(\vec{x},t\right), \psi_b\left(\vec{y},t\right)\right] = 0 \tag{383}$$

$$\left[\psi_a\left(\vec{x},t\right), \Pi_b\left(\vec{y},t\right)\right] = i\delta_{ab}\delta\left(\vec{x}-\vec{y}\right) \tag{384}$$

$$\Rightarrow i\left[\psi_a\left(\vec{x},t\right), \psi_b^\dagger\left(\vec{y},t\right)\right] = i\delta_{ab}\delta\left(\vec{x}-\vec{y}\right). \tag{385}$$

One could also impose

$$\left\{\psi_a\left(\vec{x},t\right), \psi_b^\dagger\left(\vec{y},t\right)\right\} = \delta_{ab}\delta\left(\vec{x}-\vec{y}\right) \tag{386}$$

$$\left\{\psi_a\left(\vec{x},t\right), \psi_b\left(\vec{y},t\right)\right\} = 0. \tag{387}$$

There are some properties that are shared by both routes. Namely, one has

$$\psi\left(x\right) = \sum_s\int\frac{\mathrm{d}^3p}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega}}\left(b_p^s u^s\left(\vec{p}\right)e^{-ipx} + \left(c_p^s\right)^\dagger v^s\left(\vec{p}\right)e^{ipx}\right), \tag{388}$$

where one can equivalently obtain the operators

$$b_p^s = \frac{1}{\sqrt{2\omega_p}}\int\frac{\mathrm{d}^3x}{\left(2\pi\right)^3}e^{ipx}\overline{u}^s\left(\vec{p}\right)\gamma^0\psi\left(x\right) \qquad \left(b_p^s\right)^\dagger = \frac{1}{\sqrt{2\omega_p}}\int\mathrm{d}^3x\,e^{-ipx}\overline{\psi}\left(x\right)\gamma^0v^s\left(\vec{p}\right) \tag{389}$$

$$c_p^s = \frac{1}{\sqrt{2\omega_p}}\int\mathrm{d}^3\,e^{ipx}\overline{\psi}\gamma^0v^s\left(\vec{p}\right) \qquad \left(c_p^s\right)^\dagger = \frac{1}{\sqrt{2\omega_p}}\int\mathrm{d}^3x\,e^{-ipx}\overline{v}^s\left(\vec{p}\right)\gamma^0\psi, \tag{390}$$

where $\overline{u} = u^\dagger\gamma^0$ and $\overline{v} = v^\dagger\gamma^0$.

Further, for both routes, the Hamiltonian

$$H = \int\mathrm{d}^3x\,\Pi\dot{\psi} = i\int\mathrm{d}^3x\,\psi^\dagger\dot{\psi} \qquad = \sum_s\int\frac{\mathrm{d}^3p}{\left(2\pi\right)^3}\omega_p\left(\left(b_p^s\right)^\dagger b_p^s - c_p^s\left(c_p^s\right)^\dagger\right). \tag{391}$$

If one assumes commutation relations, then these imply

$$\left[b_p^s, \left(b_{p'}^{s'}\right)^\dagger\right] = \left(2\pi\right)^3\delta^3\left(\vec{p}-\vec{p}'\right)\delta_{ss'} \tag{392}$$

$$\left[c_p^s, \left(c_{p'}^{s'}\right)^\dagger\right] = -\left(2\pi\right)^3 \delta^3\left(\vec{p} - \vec{p}'\right)\delta_{ss'}. \tag{393}$$

Identically, if one assumes the anti-commutation relations

$$\left\{b_p^s, \left(b_{p'}^{s'}\right)^\dagger\right\} = \left(2\pi\right)^3 \delta^3\left(\vec{p} - \vec{p}'\right)\delta_{ss'} \tag{394}$$

$$\left\{c_p^s, \left(c_{p'}^{s'}\right)^\dagger\right\} = \left(2\pi\right)^3 \delta^3\left(\vec{p} - \vec{p}'\right)\delta_{ss'}. \tag{395}$$

For the commutation relations route, one sees

$$\left[H, \left(b_p^s\right)^\dagger\right] = \omega_p \left(b_p^s\right)^\dagger \qquad \left[H, \left(c_p^s\right)^\dagger\right] = \omega_p \left(c_p^s\right)^\dagger. \tag{396}$$

If $b^\dagger$ and $c^\dagger$ create, then the energies are positive, but the norms of states are negative due to the commutation relations. If $b^\dagger$ and c create instead, then the norm of such states are positive but the energy is negative as $\left[H, c_p^s\right] = -\omega_p c_p^s$. Thus the commutation relations do not give us a well defined Fock space.

We then proceed with the anti-commutation relations and check

$$\left[H, \left(b_p^s\right)^\dagger\right] = \omega_p \left(b_p^s\right)^\dagger \qquad \left[H, \left(c_p^s\right)^\dagger\right] = \omega_p \left(c_p^s\right)^\dagger. \tag{397}$$

With this we can construct a healthy Fock space.

Two comments are in order.

- The choice of commutator or anticommutator is not a choice at all. Depending on the representation of the Lorentz group, one will obtain either *bosonic* or *fermionic* particles with commutation or anti commutation relations respectively.

  We studied the scalar field with spin 0 which followed Bose statistics and have now defined the spinor which has spin $\frac{1}{2}$ which has Fermi statistics.

  There were three ingredients in this process.

  a) Stability: The Hamiltonian being bounded,

  b) Causality: $\left[\Theta\left(x\right), \Theta\left(y\right)\right] = 0$ if $\left(x-y\right)^2 < 0$,

  c) Lorentz invariance of the S-matrix: Such that

  $$\left\langle T\Theta_1 \cdots \Theta_n\right\rangle, \tag{398}$$

  needs to be Lorentz covariant.

- Observe that with the anti-commutation relations the Hamiltonian becomes

  $$H = \sum_s \int \frac{\mathrm{d}^3 p}{\left(2\pi\right)^3}\omega_p \left(\left(b_p^s\right)^\dagger b_p^s + \left(c_p^s\right)^\dagger c_p^s\right) - \int \frac{\mathrm{d}^3 p}{\left(2\pi\right)^3}\omega_p 2\left(2\pi\right)^3 \delta^3\left(\vec{0}\right). \tag{399}$$

  For the scalar field we had exactly $\frac{1}{4}$ of this last term with a positive sign. One can balance scalars (bosons) and spinors (fermions) to get exactly zero vacuum energy. The spinor has four degrees of freedom (two in b_p^s and two in c_p^s) while the real scalar field only has one, thus justifying the coefficient required for cancellation.

18 Lecture: Dirac Interactions

20/11/2024

When normal ordering fermionic operators, note that there is a sign as

$$:b_q c_q b_c^\dagger: = (-1)^2 \, b_c^\dagger b_q c_q \tag{400}$$

$$= (-1) \, b_c^\dagger c_q b_q. \tag{401}$$

We now define the Fock space. We define a vacuum state $|0\rangle$ such that

$$b_p^s |0\rangle = c_p^s |0\rangle = 0, \tag{402}$$

and observe $H|0\rangle = 0$ is a ground state.

We have one particle states

$$|b^s\rangle = \sqrt{2\omega_p} \left(b_p^s\right)^\dagger |0\rangle \qquad H|b^s\rangle = \omega_p |b^s\rangle, \tag{403}$$

$$|c^s\rangle = \sqrt{2\omega_p} \left(c_p^s\right)^\dagger |0\rangle \qquad H|c^s\rangle = \omega_p |c^s\rangle. \tag{404}$$

Recall that we have $U(1)$ charge given by

$$Q = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\left(b_p^s\right)^\dagger b_p^s - \left(c_p^s\right)^\dagger c_p^s \right), \tag{405}$$

where observe

$$Q|b^s\rangle = |b^s\rangle \qquad Q|c^s\rangle = -|c^s\rangle. \tag{406}$$

One can also write down an operator for the angular momentum J_z at $\vec{p} = 0$ and see

$$J_z |b^s\rangle = \pm\frac{1}{2} |b^s\rangle \qquad J_z |c^s\rangle = \pm\frac{1}{2} |c^s\rangle, \tag{407}$$

where $s = 1$ gives $+$ and $s = 2$ gives $-$.

> ***Proof.*** Do this exercise. You need $J_z = \frac{1}{2}\varepsilon_{ij3} \left(j^0\right)^{ij}$ where $j^0 = iS^{ij}\overline{\psi}\gamma^0\psi$, $u_{-p}^{s\dagger} v_p^{s'} = v_p^{s\dagger} u_{-p}^{s'} = 0$ and $u_p^{r\dagger} u_p^s = v_p^{r\dagger} v_p^s = 2\omega_p \delta^{rs}$ as well as the chiral representation of γ^μ. □

We identically have multiparticle states by acting with multiple $b^\dagger$'s and $c^\dagger$'s. For example,

$$|p_1, s_1; p_2, s_2\rangle = \left(b_{p_1}^{s_1}\right)^\dagger \left(b_{p_2}^{s_2}\right)^\dagger |0\rangle = -\left(b_{p_2}^{s_2}\right)^\dagger \left(b_{p_1}^{s_1}\right)^\dagger |0\rangle = -|p_2, s_2; p_1, s_1\rangle. \tag{408}$$

We now move to derive the propagator. We inspect first $\langle 0| \psi_a(x) \overline{\psi}_b(y) |0\rangle$ with the aim of getting $\langle 0| T\psi(x) \overline{\psi}(y) |0\rangle$, the Feynman propagator for the Dirac spinor. This should be Lorentz invariant.

Recall that

$$\psi(x) = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p^s u^s(p) e^{-ipx} + \left(c_p^s\right)^\dagger v^s(p) e^{ipx} \right), \tag{409}$$

and similarly for $\overline{\psi}$. One sees then that

$$\begin{aligned}\langle 0|\,\psi_a(x)\,\overline{\psi}_b(y)\,|0\rangle &= \sum_{s,s'}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}e^{-ipx}e^{ip'y}u_a^s(p)\,\overline{u}_b^{s'}(p')\,\langle 0|\,b_p^s\left(b_{p'}^{s'}\right)^\dagger|0\rangle && (410)\\ &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(\not{p}+m\right)_{ab}e^{-ip(x-y)} && (411)\\ &= \left(i\not{\partial}_x+m\right)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip(x-y)} && (412)\\ &= \left(i\not{\partial}_x+m\right)D(x-y)\,, && (413)\end{aligned}$$

where $D(x-y)$ is the scalar propagator.

Looking at the reverse we see

$$\begin{aligned}\langle 0|\,\overline{\psi}_b(y)\,\psi_a(x)\,|0\rangle &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(\not{p}-m\right)e^{-ip(y-x)} && (414)\\ &= -\left(i\not{\partial}_x+m\right)D(y-x)\,. && (415)\end{aligned}$$

This minus sign is an indication that we should define time ordering such that

$$T\psi_a(x)\,\overline{\psi}_b(y) = \begin{cases}\psi_a(x)\,\overline{\psi}_b(y)\,, & x^0>y^0,\\ -\overline{\psi}_b(y)\,\psi_a(x)\,, & y^0>x^0.\end{cases} \tag{416}$$

With this, we see

$$\langle 0|\,T\psi(x)\,\overline{\psi}(y)\,|0\rangle = \left(i\not{\partial}_x+m\right)\Delta_F(x-y)\equiv S_F(x-y)\,. \tag{417}$$

It follows that

$$\left(i\not{\partial}_x-m\right)S_F(x-y) = i\delta^4(x-y)\,, \tag{418}$$

as $\left(\Box_x+m^2\right)\Delta_F(x-y)=-i\delta^4(x-y)$.

Also,

$$S_F(x-y)\left(i\overleftarrow{\not{\partial}}_x+m\right) = -i\delta^4(x-y)\,. \tag{419}$$

Theorem 18.1 (Wick's Theorem for spinors): Observe

$$T\left(\psi(x)\,\overline{\psi}(y)\right) = :\psi(x)\,\overline{\psi}(y):+\overline{\psi}\psi, \tag{420}$$

where $\overline{\psi}\psi = S_F(x-y)$

Note. If $x_3^0>x_1^0>x_4^0>x_2^0$,

$$T\left(\psi_1\psi_2\psi_3\psi_4\right) = (-1)^3\,\psi_3\psi_1\psi_4\psi_2. \tag{421}$$

18.1 Interactions: Yukawa theory

Given the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi, \tag{422}$$

and the interaction term

$$\mathcal{L}_{\text{int}} = -\lambda\phi\overline{\psi}\psi. \tag{423}$$

Consider nucleon-anti nucleon scattering with

$$|i,-\infty\rangle = \sqrt{2\omega_p}\sqrt{2\omega_q}\left(b_p^s\right)^\dagger(-\infty)\left(c_q^r\right)^\dagger(-\infty)|\Omega\rangle\,, \tag{424}$$

and

$$|f,\infty\rangle = \sqrt{2\omega_{p'}}\sqrt{2\omega_{q'}}\left(b_{p'}^{s'}\right)^\dagger(\infty)\left(b_{q'}^{r'}\right)^\dagger(\infty)|\Omega\rangle\,. \tag{425}$$

Then we have

$$\langle f|\,S\,|i\rangle = \sqrt{2\omega_p}\sqrt{2\omega_{p'}}\sqrt{2\omega_q}\sqrt{2\omega_{q'}}\,\langle\Omega|\,T\left(c_{q'}^{r'}(\infty)\,b_{p'}^{s'}(\infty)\left(b_p^s\right)^\dagger(-\infty)\left(c_q^r\right)^\dagger(-\infty)\right). \tag{426}$$

We want $\left(b_p^s\right)^\dagger(\infty) - \left(b_p^s\right)^\dagger(-\infty)$.

Consider

$$i\int \mathrm{d}^4x\,\overline{\psi}(x)\left(i\overleftarrow{\partial\!\!\!/} + m\right)u_s(p)\,e^{ipx}. \tag{427}$$

Recall that $u_s(p)$ obeys $(p\!\!\!/ - m)\,u_s(p) = 0$ and $\omega_p = \sqrt{\vec{p}^2 + m^2}$.

19 Lecture: Feynman Rules

22/11/2024

We see that

$$i\int \mathrm{d}^4x\,\overline{\psi}(x)\left(i\overleftarrow{\partial\!\!\!/} + m\right)u_s(p)\,e^{ipx} = i\int \mathrm{d}^4x\,\overline{\psi}(x)\left(\gamma^0\overleftarrow{\partial_0} + i\gamma^i\overleftarrow{\gamma_i} + m\right)u_s(p)\,e^{ipx} \tag{428}$$

$$= i\int \mathrm{d}^4x\,\overline{\psi}(x)\left(\gamma^0\overleftarrow{\partial_0} - \underbrace{i\gamma^i\left(ip^i\right) + m}_{p^0\gamma^0}\right)u_s(p)\,e^{ipx} \tag{429}$$

using $\left(p^0\gamma^0 - p^i\gamma^i - m\right)u_s(p) = 0$,

$$= i\int \mathrm{d}^4x\,\overline{\psi}\left(i\gamma^0\overleftarrow{\partial_0} + p_0\gamma^0\right)u_s(p)\,e^{-ipx} \tag{430}$$

$$= i\int \mathrm{d}^4x\,\overline{\psi}\left(i\gamma^0\overleftarrow{\partial_0} + i\gamma^0\partial_0\right)u_s(p)\,e^{-ipx} \tag{431}$$

$$= -\int \mathrm{d}^4 x\, \partial_0 \left(\overline{\psi}\gamma^0 u_s\left(p\right) e^{-ipx}\right) \tag{432}$$

$$= -\sqrt{2\omega_p}\left(\left(b_p^s\right)^\dagger\left(\infty\right) - \left(b_p^s\right)\left(-\infty\right)\right). \tag{433}$$

Repeating this for all of the operators in consideration gives us the following

$$\sqrt{2\omega_p}\left(\left(b_p^s\right)^\dagger\left(-\infty\right) - \left(b_p^s\right)^\dagger\left(\infty\right)\right) = i\int \mathrm{d}^4 x\, \overline{\psi}\left(i\overleftarrow{\partial\!\!\!/} + m\right) u_s\left(p\right) e^{-ipx} \tag{434}$$

$$\sqrt{2\omega_p}\left(b_p^s\left(\infty\right) - b_p^s\left(-\infty\right)\right) = i\int \mathrm{d}^4 x\, e^{ipx}\overline{u}_s\left(p\right)\left(-i\partial\!\!\!/ + m\right)\psi\left(x\right) \tag{435}$$

$$\sqrt{2\omega_p}\left(\left(c_p^s\right)^\dagger\left(-\infty\right) - \left(c_p^s\right)^\dagger\left(\infty\right)\right) = -i\int \mathrm{d}^4 x\, e^{-ipx}\overline{v}_s\left(p\right)\left(-i\partial\!\!\!/ + m\right)\psi\left(x\right) \tag{436}$$

$$\sqrt{2\omega_p}\left(c_p^s\left(\infty\right) - c_p^s\left(-\infty\right)\right) = -i\int \mathrm{d}^4 x\, \overline{\psi}\left(x\right)\left(i\overleftarrow{\partial\!\!\!/} + m\right) v_s\left(p\right) e^{ipx}. \tag{437}$$

Therefore, we finally arrive at

$$\begin{aligned}\langle f|\, S\,|i\rangle = \prod_{j=1}^{4} i\int \mathrm{d}^4 x\, &\underbrace{\left(e^{ip'x_3}\overline{u}_{s'}\left(p'\right)\left(-i\partial\!\!\!/_3 + m\right)\right)}_{\text{outgoing } b^\dagger}\underbrace{\left(e^{-iqx_2}\overline{v}_r\left(q\right)\left(-i\partial\!\!\!/_2 + m\right)\right)}_{\text{ingoing } c^\dagger} \\ &\langle\Omega|\, T\overline{\psi}\left(x_4\right)\psi\left(x_3\right)\overline{\psi}\left(x_1\right)\psi\left(x_2\right)|\Omega\rangle \underbrace{\left(i\overleftarrow{\partial\!\!\!/}_1 + m\right) u_s\left(p\right) e^{-ipx_1}}_{\text{ingoing } b^\dagger}\underbrace{\left(i\overleftarrow{\partial\!\!\!/}_4 + m\right) v_{r'}\left(q'\right) e^{iq'x_4}}_{\text{outgoing } c^\dagger}.\end{aligned} \tag{438}$$

19.1 Feynman Rules

For an incoming fermion,

$$b^\dagger : \xrightarrow[\vec{p},\, s]{u_s\left(p\right)} \tag{439}$$

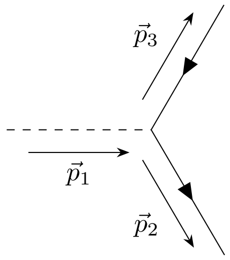
$$c^\dagger : \xleftarrow[\vec{p},\, s \;\rightarrow]{\overline{v}_s\left(p\right)}. \tag{440}$$

For outgoing fermions,

$$b : \xrightarrow[\vec{p},\, s]{\overline{u}_s\left(p\right)} \tag{441}$$

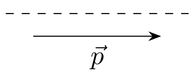
$$c : \xleftarrow[\vec{p},\, s \;\rightarrow]{v_s\left(p\right)}. \tag{442}$$

For the vertex, we see

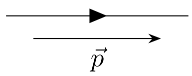


$$= \left(-i\lambda\right)\left(2\pi\right)^4 \delta\left(p_1 - p_2 - p_3\right). \tag{443}$$

And lastly each internal line is given by



$$= \int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4} \frac{i}{p^2 - \mu^2 + i\varepsilon} \tag{444}$$



$$= \int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4} \frac{i\left(\not{p} + m\right)}{p^2 - m^2 + i\varepsilon}. \tag{445}$$

Lastly, indices must be contracted at the vertex with other propagators or spinors. One must also add an extra minus sign appropriate to the statistics of fermions.

Example. For the free theory, take $\left\langle \psi_1 \psi_2 \overline{\psi}_3 \overline{\psi}_4 \right\rangle$ and proceed with Schwinger-Dyson. Namely, one has that

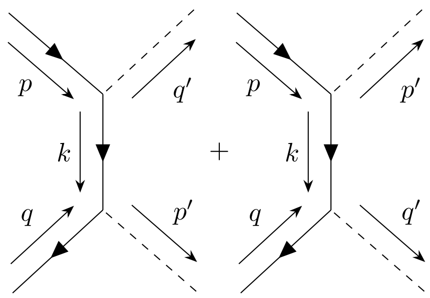
$$\left(i\not{\partial}_x - m\right)\left\langle T\psi\left(x\right)\overline{\psi}\left(x_1\right)\cdots\psi\left(x_n\right)\right\rangle = -i\sum_j \delta\left(x - x_j\right)\left(\text{signed contractions with } \psi\left(x\right)\overline{\psi}\left(x_j\right) \text{ removed}\right). \tag{446}$$

Then observe

$$\begin{aligned}
\left\langle \psi_1 \psi_2 \overline{\psi}_3 \overline{\psi}_4 \right\rangle + 0 &= \int \mathrm{d}^4 y \underbrace{iS_F\left(x - y\right)\left(i\overleftarrow{\not{\partial}}_y + m\right)}_{\delta\left(x_1 - y\right)}\left\langle \psi_y \psi_2 \overline{\psi}_3 \overline{\psi}_4 \right\rangle && (447)\\
&= \int \mathrm{d}^4 y \left(-i\right) S_F\left(x_1 - y\right)\left(i\not{\partial}_y - m\right)\left\langle \psi_y \psi_2 \overline{\psi}_3 \overline{\psi}_4 \right\rangle && (448)\\
&= \left(-1\right)^2 \int \mathrm{d}^4 y\, S_F\left(x_1 - y\right)\left(-\delta\left(x_3 - y\right) S_F\left(x_2 - x_4\right) + \delta\left(x_4 - y\right) S_F\left(x_2 - x_3\right)\right) && (449)\\
&= S_F\left(x_1 - x_4\right) S_F\left(x_2 - x_3\right) - S_F\left(x_1 - x_3\right) S_F\left(x_2 - x_4\right). && (450)
\end{aligned}$$

Example. Nucleon to meson scattering. Suppose one has $\left(b_p^s\right)^\dagger \left(c_q^s\right)^\dagger$ incoming and $a_{p'}^\dagger a_{q'}^\dagger$ going out.

To leading order we have two connected diagrams



(451)

Then we have

$$\langle f|\, S\, |i\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \overline{v}_r(q) \frac{i(\not{k}+m)}{k^2-m^2-i\varepsilon} u_s(p)(-i\lambda)(2\pi)^4 \delta(q+k-q')(-i\lambda)(2\pi)^4\delta(p-p'-k) + \{p' \leftrightarrow q'\} \tag{452}$$

$$= i(2\pi)^4 \delta(p+q-p'-q') \left(\overline{v}_r(q) \frac{(\not{p}-\not{p}'+m)}{(p-p')^2-m^2} u_s(p) + \overline{v}_r(q) \frac{\not{p}-\not{q}'+m}{(p-q')^2-m^2} u_s(p) \right). \tag{453}$$

20 Lecture: Maxwell's Theory

25/11/2024

With scalar and spinor fields established we focus on the classical aspects of Maxwell's field theory. This is described by the Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{454}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and A_μ is the gauge field. The equation of motion for this theory is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \Rightarrow \partial_\mu F^{\mu\nu} = 0. \tag{455}$$

The Bianchi identity also gives us

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \Leftrightarrow \varepsilon_{\mu\nu\rho\tau} \partial_\mu F^{\rho\tau} = 0 \Leftrightarrow \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0. \tag{456}$$

We can write the equation of motion as

$$\partial_\mu \partial^\mu A_\nu - \partial_\mu \partial^\nu A_\mu = 0. \tag{457}$$

Which we can split into when $\nu = i$ and when $\nu = 0$ giving

$$\Box A_j - \partial^i \left(\partial_0 A^0 + \partial_j A^j\right) = 0, \qquad \Box A_0 - \partial_0 \left(\partial_0 A^0 + \partial_j A^j\right) = 0, \tag{458}$$

respectively. The latter can also be written as

$$\left(\boldsymbol{\nabla}^2 A_0 + \partial\right)\left(\boldsymbol{\nabla}\cdot\vec{A}\right) = 0, \tag{459}$$

which has a unique explicit solution for A_0

$$A_0 = \int \mathrm{d}^3x' \, \frac{1}{4\pi\left|\vec{x}-\vec{x}'\right|}\left(\boldsymbol{\nabla}\cdot\frac{\partial\vec{A}}{\partial t}\right)(\vec{x}') \,. \tag{460}$$

Note. We also have a redundancy in this theory as under

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{461}$$

$$\begin{aligned} F_{\mu\nu} \to F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu &(462)\\ &= \partial_\mu\left(A_\nu + \partial_\nu\lambda\right) - \partial_\nu\left(A_\mu + \partial_\mu\lambda\right) &(463)\\ &= F_{\mu\nu}. &(464)\end{aligned}$$

This tells us $F_{\mu\nu}$ is gauge invariant.

This is called a **gauge symmetry**. The existence of this gauge symmetry is our redundancy in our description of light. One way to see this is that

$$\left(\eta_{\mu\nu}\partial_\rho\partial^\rho - \partial_\mu\partial_\nu\right)A^\nu = 0, \tag{465}$$

is a non-invertible operator since $A^\nu = \partial^\nu\lambda$ solves this $\forall\lambda$.

Another reason is that Noether's theorem applied to the gauge symmetry gives us

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\delta A_\nu = -F^{\mu\nu}\partial_\nu\lambda, \tag{466}$$

which subject to the equations of motion becomes

$$j^\mu = -\partial_\nu\left(\lambda F^{\mu\nu}\right), \tag{467}$$

which gives us charge

$$Q = \int \mathrm{d}^3x \, j^0 = \int \mathrm{d}^3x \, \partial_i\left(\lambda F^{-i}\right). \tag{468}$$

For any λ with compact support his vanishes.

All A^ν related by gauge transformations, i.e. $A^\nu + \partial^\nu\lambda$, $\forall\lambda$, are referred to as **gauge orbits**. They partition and span the space of A^ν. Distinct physical states live on different gauge orbits, and states on the same orbit are physically equivalent.

There are many ways to fix this redundancy and gain a unique physical A^ν for each distinct gauge orbit. This is referred to as *fixing the gauge*. Here we list two:

1) The *Lorentz gauge* is the imposition of $\partial_\mu A^\mu = 0$. Say $\partial_\mu A^\mu = f(x)$. One can always find λ such that $\Box\lambda = -f \Rightarrow \partial_\mu A'^\mu = \partial_\mu A^\mu + \Box\lambda = f + (-f) = 0$. The down side that there is still residual freedom as one can further shift by χ if $\Box\chi = 0$. The upside is that this is a Lorentz invariant gauge.

2) The *Coulomb gauge* is when one imposes $\boldsymbol{\nabla} \cdot \vec{A} = 0$. As before, if $\partial_i A^i = f(x)$, we can shift $A_i \to A_i + \partial_i \alpha$ to comply with the condition. There is still the residual $\boldsymbol{\nabla}^2 \chi = 0$ freedom. However, the equation of motion when $\nu = 0$ implies

$$\boldsymbol{\nabla}^2 A_0 + \partial_0 \left(\boldsymbol{\nabla} \cdot \vec{A} \right) = 0, \tag{469}$$

which reduces in the Coulomb gauge to

$$\boldsymbol{\nabla}^2 A_0 = 0. \tag{470}$$

Using λ one can set $A_0 = 0$, with $A_0 = A_0 + \partial_0 \lambda$. The upside is that all freedom is gone, but Lorentz invariance is broken.

21 Lecture: Quantization of Maxwell Theory

29/11/2024

We now move to quantize this theory. We focus on the Coulomb gauge as here we get rid of the redundancy. Recall that $\nabla \cdot \vec{A} = 0$ and $A_0 = 0$ such that $\Box A_i = 0$. Then we must have

$$A_\mu \sim e_\mu(p) \, e^{-ipx}, \tag{471}$$

with $p^2 = 0$.

The choice of gauge restricts $\varepsilon_\mu(p)$. For the coulomb gauge, we see $\varepsilon_0 = 0$ and $\vec{p} \cdot \vec{\varepsilon} = 0$.

Choosing a frame in which $p_\mu = (E, 0, 0, 0)$ gives us two linearly independent solutions

$$\varepsilon^1_\mu = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \tag{472}$$
$$\varepsilon^2_\mu = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \tag{473}$$

which are our two polarization states. We will write in general

$$\vec{e}_r(\vec{p}) \cdot \vec{p} = 0, \tag{474}$$

with $r = 1, 2$ satisfying

$$\vec{\varepsilon}_r \cdot \vec{\varepsilon}_s = \delta_{rs}. \tag{475}$$

We also have the completeness relation

$$\sum_{r=1}^{2} \varepsilon^i_r \varepsilon^j_r = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}. \tag{476}$$

Therefore, the general solution to our gauge field is

$$\vec{A}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{r=1}^{2} \vec{\varepsilon}^{\,r} \left(a^r_p e^{-ipx} + a^{r\dagger}_p e^{ipx} \right). \tag{477}$$

Next, for the commutation relation, we require the conjugate momenta given by

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i. \tag{478}$$

If $\boldsymbol{\nabla}\cdot\vec{A}=0$, then $\boldsymbol{\nabla}\cdot\vec{E}=0$.

Naively, we impose the equal time commutation relation

$$[A_i(\vec{x},t),\Pi_i(\vec{y},t)]=i\delta_{ij}\delta^3(\vec{x}-\vec{y}). \tag{479}$$

This is wrong as it does not comply with the gauge condition. Namely, one can compute

$$0=\left[\boldsymbol{\nabla}\cdot\vec{A},\boldsymbol{\nabla}\cdot\vec{E}\right]=i\boldsymbol{\nabla}^2\delta^3(\vec{x}-\vec{y})\neq 0. \tag{480}$$

A better guess for the commutation relation is

$$[A_i(\vec{x},t),\Pi_j(\vec{y},t)]=i\left(\delta_{ij}-\frac{\partial_i\partial_j}{\boldsymbol{\nabla}^2}\right)\delta^3(\vec{x}-\vec{y}) \tag{481}$$

$$=i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\delta_{ij}-\frac{p_ip_j}{|\vec{p}|^2}\right)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}. \tag{482}$$

We can check if it complies with the gauge conditions with

$$[\partial_iA_i,\Pi_j]=i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\delta_{ij}-\frac{p_ip_j}{|\vec{p}|^2}\right)ip_ie^{i\vec{p}\cdot(\vec{x}-\vec{y})}=0, \tag{483}$$

as desired.

Thus this is the correct commutation relation. We further have

$$[A_i(\vec{x},t),A_j(\vec{y},t)]=0. \tag{484}$$

From here, it is straightforward to see that these imply

$$\left[a_p^r,\left(a_q^s\right)^\dagger\right]=(2\pi)^3\delta^{rs}\delta^3(\vec{p}-\vec{q}), \tag{485}$$

and all others zero.

Anti-commutation relations would not have given us a valid Fock space here.

The Hamiltonian is given by

$$H=\int\mathrm{d}^3x\left(\Pi^i\dot{A}_i-\mathcal{L}\right)=\frac{1}{2}\int\mathrm{d}^3x\left(\vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B}\right), \tag{486}$$

where $E^i=-F^{0i}$ and $\varepsilon^{ijk}B_k=F^{ij}$. After normal ordering, we have

$$H=\int\frac{\mathrm{d}^3p}{(2\pi)^3}|\vec{p}|\sum_{r=1}^{3}\left(a_p^r\right)^\dagger a_p^r. \tag{487}$$

The Coulomb gauge propagator is given by

$$\langle 0|TA_i(x)A_j(y)|0\rangle=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2+i\varepsilon}\left(\delta_{ij}-\frac{p_ip_j}{|\vec{p}|^2}\right)e^{-ip(x-y)}, \tag{488}$$

however we want $\langle 0| T A_\mu (x) A_\nu (y) |0\rangle$. The answer should be Lorentz invariant. Our strategy to obtain this will be to use the fact that it is a Greens function.

Recall that for the scalar field we had

$$\left(\Box + m^2\right) G(x) = J(x)\,, \tag{489}$$

where $J(x)$ is a source.

A fast way to solve this is to go to Fourier space and observe

$$\left(\Box + m^2\right) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} G(p)\, e^{ipx} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} J(p)\, e^{ipx}, \tag{490}$$

implying

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left(\left(-p^2 + m^2\right) G(p) - J(p)\right) e^{ipx}, \tag{491}$$

and thus

$$G(p) = \frac{J(p)}{-p^2 + m^2}. \tag{492}$$

This implies

$$G(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{J(p)}{-p^2 + m^2} e^{ipx}, \tag{493}$$

when $J(x) = -i\delta^4(x)$, $J(p) = -i$ and thus

$$G(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{ipx}. \tag{494}$$

Applying this to Maxwell's equation,

$$\partial^\mu F_{\mu\nu} = J_\nu(x) \tag{495}$$

$$\Rightarrow \Box A_\nu - \partial_\nu \partial^\mu A_\mu = J_\nu(x)\,, \tag{496}$$

which in momentum space is given by

$$\left(-p^2 \eta_{\mu\nu} + p_\mu p_\nu\right) A^\mu(p) = J_\nu(p)\,. \tag{497}$$

One is tempted to apply the scalar field line of logic, but we cannot invert this linear operator as we previously identified that this operator is not invertible as

$$\left(-p^2 \eta^{\mu\nu} + p^\mu p^\nu\right) p_\nu = 0. \tag{498}$$

This is a Fourier space statement that any gauge transformation solves this equation, as we saw before,

$$\left(\eta_{\mu\nu}\Box - \partial_\nu \partial_\mu\right) \partial^\mu \lambda = 0. \tag{499}$$

To get rid of this, we introduce a Lagrange multiplier such that our Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2. \tag{500}$$

Where α is a constant auxiliary variable which acts a Lagrange multiplier (i.e. when the equation of motion with respect to α is imposed) to enforce the gauge condition $\partial_\mu A^\mu = 0$.

If we write down the equation of motion for A_μ we see

$$\left(\eta^{\nu\lambda}\Box + \left(\frac{1}{\alpha} - 1\right)\partial^\nu\partial^\lambda\right)A_\lambda = 0, \tag{501}$$

which is now invertible. Now, going back to the Greens function derivation, we see in Fourier space

$$\underbrace{\left(-\eta_{\lambda\nu}p^2 - \left(\frac{1}{\alpha} - 1\right)p_\nu p_\lambda\right)}_{\hat{\Pi}_{\lambda\nu}}A^\lambda\left(p\right) = J_\nu\left(p\right), \tag{502}$$

that $\hat{\Pi}_{\lambda\nu}$ has an inverse given by

$$\Pi_{\mu\nu} = -\frac{\eta_{\mu\nu} + \left(\alpha - 1\right)\frac{p_\mu p_\nu}{p^2}}{p^2}, \tag{503}$$

where $\hat{\Pi}^{\lambda\nu}\Pi_{\nu\mu} = \delta^\lambda_\mu$ as desired.

Proof. □

22 Lecture: Covariant Derivatives

29/11/2024

We expect that the propagator is the Green's function for the equation of motion and thus acting the equation of motion on the propagator $\langle TA_\mu\left(x\right)A_\nu\left(y\right)\rangle$ should give a delta function. Therefore

$$\langle 0|\,TA_\mu\left(x\right)A_\nu\left(y\right)|0\rangle = i\int\frac{\mathrm{d}^4p}{\left(2\pi\right)^4}e^{-ip(x-y)}\Pi_{\mu\nu} \tag{504}$$

$$= -i\int\frac{\mathrm{d}^4p}{\left(2\pi\right)^4}\frac{e^{-ip(x-y)}}{p^2 + i\varepsilon}\left(\eta_{\mu\nu} + \left(\alpha - 1\right)\frac{p_\mu p_\nu}{p^2}\right). \tag{505}$$

Notes.

- The minus sign is correct. If $\mu = i$ and $\nu = j$, we will see a positive sign for the physical components which is identical to what we saw for a scalar field.
- The presence of α should in principle bother us as it is completely unphysical. The S matrix should not depend on α (and it doesn't as we will see). One can then fix any value of α.
- Common choices for α include

- $\alpha = 1$, called *Feynman-t'Hooft* gauge,
- $\alpha = 0$, called Lorentz (or Landau) gauge, which is a strong enforcement of the Lorentz gauge,
- $\alpha \to \infty$, called the unitary gauge, is useful in non-abelian gauge theories.

22.1 Interactions: couple light to matter

The Maxwell equations, $\partial_\mu F^{\mu\nu} = j^\nu$ have the consequence that

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu \Rightarrow 0 = \partial_\nu j^\nu, \tag{506}$$

namely, that j^ν is a conserved current.

If j^ν is independent of A_ν itself, then we can write

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu. \tag{507}$$

Looking at the action and gauge transforming, we see

$$S = \int \mathrm{d}^4 x \, \mathcal{L} \to \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu \left(A_\mu + \partial_\mu \lambda \right) \right) = \int \mathrm{d}^4 x \, \mathcal{L} - \int \mathrm{d}^4 x \left(\partial_\mu \left(j^\mu \lambda \right) - \partial_\mu j^\mu \right) \lambda. \tag{508}$$

Namely, the action is gauge invariant if the current is conserved.

With this, lets couple light to spinors.

Recall the Dirac Lagrangian $\mathcal{L} = \overline{\psi} \left(i\not{\partial} - m \right) \psi$ has current

$$j^\mu = \overline{\psi} \gamma^\mu \psi, \tag{509}$$

due to the internal symmetry $\psi \to e^{i\alpha} \psi$ (and $\overline{\psi} \to e^{-i\alpha} \psi$).

As this current is conserved, we can couple it to the Maxwell action without breaking gauge invariance. We thus consider Maxwell's theory with spinorial matter and this current coupling them giving

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\not{\partial} - m \right) \psi - e j^\mu A_\mu, \tag{510}$$

where we have introduced a coupling constant e. Notice that we can equivalently write this as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\not{\partial} - e A_\mu \gamma^\mu - m \right) \psi \tag{511}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\not{D} - m \right) \psi, \tag{512}$$

where $D_\mu = \partial_\mu + i\varepsilon A_\mu$ is the **covariant derivative**.

As one has that the gauge field transforms under a local gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda \left(x \right), \tag{513}$$

it is natural to promote the spinorial $U(1)$ symmetry such that under gauge transformation it undergoes

$$\psi \to e^{-ie\lambda(x)}\psi \tag{514}$$
$$\overline{\psi} \to e^{ie\lambda(x)}\overline{\psi}. \tag{515}$$

Then we have

$$\begin{aligned} D_\mu\psi &\to \partial_\mu\left(e^{-ie\lambda}\psi\right) + ie\left(A_\mu + \partial_\mu\lambda\right)e^{-ie\lambda}\psi && (516)\\ &= e^{-ie\lambda}\left(\partial_\mu - ie\partial_\mu\lambda + ieA_\mu + ie\partial_\mu\lambda\psi\right) && (517)\\ &= e^{-ie\lambda}D_\mu\psi, && (518)\end{aligned}$$

which implies

$$\overline{\psi}\not{D}\psi \to \overline{\psi}\not{D}\psi. \tag{519}$$

which makes the action gauge invariant.

Looking at Noether's theorem, we see

$$\begin{aligned} Q &= e\int \mathrm{d}^3x\, F^{0i}\partial_i\lambda && (520)\\ &= -e\int \mathrm{d}^3x\, \partial_i F^{0i}\lambda && (521)\end{aligned}$$

for $\lambda = 1$,

$$\begin{aligned} Q &= -e\int \mathrm{d}^3x\, j^0 && (522)\\ &= -e\int \mathrm{d}^3x\, \overline{\psi}\gamma^0\psi. && (523)\end{aligned}$$

Gauge symmetries, if they contain a *global symmetry* (i.e. internal symmetry) where the parameter is constant, then we have an associated charge.

Note. Constant λ is called a *large gauge transformation* as it does not vanish at infinity.

23 Lecture: QED

02/12/2024

We now have an interacting Lagrangian term

$$\mathcal{L}_{\text{int}} = -e\overline{\psi}A_\mu\gamma^\mu\psi, \tag{524}$$

where e is our (dimensionless) coupling constant, $[e] = 0$ and thus this is a marginal coupling. This also multiplies the $U(1)$ which motivates the name charge. It is traditional to define the *fine structure constant*

$$\alpha = \frac{e^2}{4\pi\hbar c} \sim \frac{1}{137}. \tag{525}$$

Before we dive into full QED, we briefly discuss scalar QED, coupling gauge fields to scalars. We have $\partial_\mu F^{\mu\nu}$ and want to put a conserved current $\partial_\nu j^\nu = 0$ depending on a scalar field ϕ on the right. There are no internal symmetries for real scalars so it is impossible to couple to real scalars. However for complex scalars, we do have such a conserved current. Namely, one can take

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^* D^\mu \phi - m \left|\phi\right|^2, \tag{526}$$

where

$$D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi. \tag{527}$$

The invariance of this action is under

$$A_\mu \to A_\mu + \partial_\mu \lambda \tag{528}$$

$$\phi \to e^{-ie\lambda(x)} \phi \tag{529}$$

$$\phi^* \to e^{ie\lambda(x)} \phi^*, \tag{530}$$

with interaction term

$$\mathcal{L}_{\text{int}} = ie \underbrace{\left(\partial_\mu \phi^* \phi - \phi^* \partial_\mu \phi\right)}_{j_\mu} A^\mu + e^2 A_\mu A^\mu \phi^* \phi, \tag{531}$$

where one sees that the $U(1)$ internal symmetry for the complex scalar field gives us the j_ν term, but also gave us the quadratic term. The current then depends on A_μ as well.

Revisiting Noether's theorem with this $\mathcal{L}_{\text{int}}$,

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^*\right)} \delta\phi^* \tag{532}$$

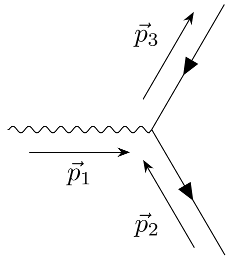
$$= (-ie)\left(\partial_\mu \phi^* \phi - \phi^* \partial_\mu \phi - 2ie\phi^* \phi\right), \tag{533}$$

for constant $\lambda = 1$.

Thus our takeaway is that minimal coupling is done by promoting $\partial_\mu \to D_\mu$.

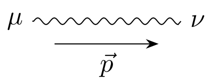
23.1 QED Feynman rules

- For external lines, we first consider photons with have a polarization vector $\varepsilon^s_\mu(p)$ so we write
- For external lines do nothing.
- For each vertex, write the factor



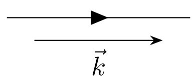
$$= -ie\gamma^\mu (2\pi)^4 \delta\left(p_1 - p_2 - p_3\right). \tag{534}$$

- For the photon propagator, we have



$$= i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2+i\varepsilon}\underbrace{\left(-\eta_{\mu\nu}-(\alpha-1)\frac{p_\mu p_\nu}{p^2}\right)}_{\widetilde{\Pi}_{\mu\nu}}, \tag{535}$$

- and for the fermionic propagator



$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon},. \tag{536}$$

- and we have minus signs if fermions are swapped.

With these we will evaluate the S-matrix

$$\langle f|\, S-\mathbb{I}\,|i\rangle = i\mathcal{A}\,(2\pi)^4\,\delta^4\left(\sum_f p_f - \sum_i p_i\right), \tag{537}$$

where $\mathcal{A}$ is an amplitude.

We have two requirements on the S-matrix:

1. If we have an internal photon, the amplitude should take the structure $\mathcal{A} = \mathcal{A}_{\mu\nu}\widetilde{\Pi}^{\mu\nu}$ where gauge invariance of $\mathcal{A}$ implies $A_{\mu\nu}p^\mu p^\nu = 0$. This is equivalent to saying the answer should be independent of α.
2. If we have an external photon, then the amplitude should take the form $\varepsilon_s^\mu\mathcal{A}_\mu$. Once again the Lorentz invariance of $\mathcal{A}$ gives $A'_\mu = \Lambda_\mu{}^\nu\mathcal{A}_\nu$ and $p'_\mu = \Lambda_\mu{}^\nu p_\nu$.

   Now the polarization vector transforms as

$$\varepsilon_s'^\mu = \Lambda^\mu{}_\nu\varepsilon_s^\nu + cp^\mu. \tag{538}$$

   We have the condition that $\varepsilon\cdot\mathbf{p} = 0$ has a trivial solution of $\varepsilon \propto p$ as $p^2 = 0$.

   Example. Take

$$\Lambda^\mu{}_\nu\begin{pmatrix}\frac{3}{2} & 1 & 0 & -\frac{1}{2}\\ 1 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ \frac{1}{2} & 1 & 0 & \frac{1}{2}\end{pmatrix}, \tag{539}$$

   with $p^\mu = (E,0,0,E)$ and $\varepsilon_1^\mu = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}$, $\varepsilon_2^\mu = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}$ gives

$$\Lambda^\mu{}_\nu p^\nu = p^\nu, \tag{540}$$

   but

$$\Lambda^\mu{}_\nu\varepsilon_1^\nu = \varepsilon_1^\mu + \frac{1}{E}p^\mu. \tag{541}$$

Then, a requirement of Lorentz invariance of $\mathcal{A}$ is

$$p^{\mu}\mathcal{A}_{\mu} = 0. \tag{542}$$

This is a **Ward identity**.

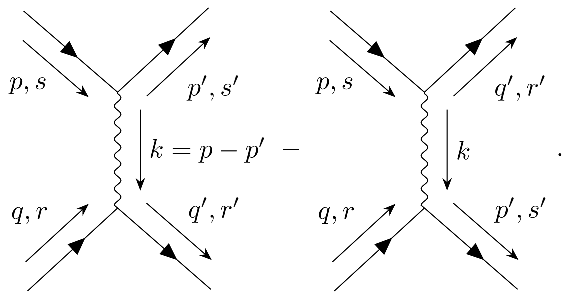
These two conditions are sanity ones as reasonable theories should obey them. They hold non-perturbatively.

23.2 Scattering in QED

Consider $e^- e^- \to e^- e^-$.

One can also consider $e^+ e^- \to e^+ e^-$ or $e^- \gamma \to e^- \gamma$. The last of these requires the Ward identity and the former uses $\mathcal{A}_{\mu\nu} p^{\mu} p^{\nu} = 0$.

We thus consider $b^{\dagger}_{p,s} b_{q,r} \to b^{\dagger}_{p',s'} b^{\dagger}_{q',r'}$. Reordering the initial or final state gives an overall minus sign as these operators anti-commute.



(543)

We can then use our Feynman rules and conclude that the connected diagrams above generate

$$\langle f| S |i\rangle_C = (-ie)^2 \left(\overline{u}^{s'}_{p'} \gamma^{\mu} u^s_p \frac{\widetilde{\Pi}_{\mu\nu}}{(p-p')^2} (p-p') \overline{u}^{r'}_{q'} \gamma^{\nu} u^s_q - \overline{u}^{s'}_{p'} \gamma^{\mu} u^r_q i \frac{\widetilde{\Pi}_{\mu\nu}(p-q')}{(p-q')^2} \overline{u}^{r'}_{q'} \gamma^{\nu} u^s_p \right) (2\pi)^4 \delta(p+q-p'-q'). \tag{544}$$

24 Lecture: Scattering

04/12/2024

Recall that this scattering begins with $\langle \Omega | b^{r'} b^{s'} b^{s\dagger} b^{r\dagger} |\Omega\rangle$ where LSZ gives us $\langle \psi_3 \psi_4 \overline{\psi}_1 \overline{\psi}_2 \rangle$.

Applying Schwinger-Dyson to this gives us leading order contribution $e^2 \langle A_{\mu} A_{\nu} \rangle \langle \psi_3 \psi_4 \overline{\psi}_1 \overline{\psi}_2 \rangle + \cdots$ where one can see

$$\langle \psi_3 \psi_4 \overline{\psi}_1 \overline{\psi}_2 \rangle = S_F^{32} S_F^{41} - S_F^{31} S_F^{42}. \tag{545}$$

We want to see that

$$\overline{u}^{s'}_{p'} \gamma^{\mu} u s^s_p \widetilde{\Pi}_{\mu\nu}(k) \overline{u}^{r'}_{q'} \gamma^{\nu} u^r_q, \tag{546}$$

is independent of α where $k = p - p' = q' - q$. The terms of note are

$$\overline{u}^{s'}_{p'} \gamma^{\mu} u s^s_p k_{\mu} k_{\nu} \overline{u}^{r'}_{q'} \gamma^{\nu} u^r_q, \tag{547}$$

which we need to show vanish. This is equivalent to $\mathcal{A}_{\mu\nu}k^\mu k^\nu = 0$.

We also know that

$$(-\not{p} + m)\, u_p = 0 \qquad \overline{u}_p\,(\not{p} - m) = 0. \tag{548}$$

Then look at

$$\overline{u}_{p'} k_\mu \gamma^\mu u_p = \overline{u}_{p'}\,(\not{p} - \not{p}')\, u_p = \overline{u}_{p'}\,(m - m)\, u_p = 0. \tag{549}$$

Identically,

$$\overline{u} k_\nu \gamma^\nu u_q = \overline{u}_{q'}\,(\not{q} - \not{q}')\, u_q = 0, \tag{550}$$

thus $\langle f|\, S\, |i\rangle$ is independent of α.

Note. In some cases each diagram is separately independent but not always. At the very least one has independence for a given process as a whole.

24.1 Spin sums

A commonly studied quantity in particle physics is the cross section. This is roughly

$$\text{Num. events per unit time} = (\text{incoming particles per unit time}) \times (\text{cross section: fraction of particles that collide})\,. \tag{551}$$

The *differential cross section* is

$$\mathrm{d}C = \frac{\text{Probability per unit time}}{\text{Flux}}, \tag{552}$$

where the probability is

$$P = \frac{|\langle f|\, S\, |i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle}, \tag{553}$$

where

$$\langle f|\, S\, |i\rangle = (2\pi)^4\, \delta\left(\sum p\right) \mathcal{A}, \tag{554}$$

and thus

$$|\langle f|\, S\, |i\rangle|^2 = (2\pi)^4\, \underbrace{\delta\left(\sum 0\right)}_{VT} |\mathcal{A}|^2\,. \tag{555}$$

When we have spin,

- You can't control outgoing spin,
- you don't control the ingoing spin,

Then one defines

$$\mathcal{P} = \left\langle |\mathcal{A}|^2 \right\rangle = \frac{1}{4} \sum_{\text{spins}} |\mathcal{A}|^2 \,, \tag{556}$$

where $\frac{1}{4}$ is included for $2 \to$ anything scattering.

Example. Consider $e^-\mu^- \to e^-\mu^-$ in a theory where they are minimally coupled such that one has vertex $-ie\gamma^\mu A_\mu$ for both the $\gamma\mu^-\mu^-$ and the $\gamma e^- e^-$ interactions. There is no exchange diagram and we have.

Then we see

$$\mathcal{A} = (-ie)^2 \overbrace{\overline{u}^{s'}_{p'}\gamma^\mu u^s_p}^{e^-} \frac{1}{(p-p')^2} \overbrace{\overline{u}^{r'}_{q'}\gamma_\mu u^r_q}^{\mu^-}, \tag{557}$$

which gives us

$$|\mathcal{A}|^2 = \frac{(-ie)^4}{(p-p')^4} \left(\overline{v}^{s'}_{p'}\gamma^\mu u^s_p\right) \left(\overline{u}^{s'}_{p'}\gamma^\nu u^s_p\right)^\dagger \left(\overline{u}^{r'}_{q'}\gamma_\mu u^r_q\right) \left(\overline{u}^{r'}_{q'}\gamma_\nu u^r_q\right)^\dagger \,, \tag{558}$$

as $u^{s\dagger}_p \gamma^{\nu\dagger}\gamma^0 u^{s'}_{p'} = \overline{u}^s_p \gamma^\nu u^{s'}_{p'}$,

We define a spin-averaged amplitude given by

$$\mathcal{P} = \frac{1}{4} \sum_{s,s',r,r'} |\mathcal{A}|^2 \tag{559}$$

$$= (\cdots) \sum_s \sum_{s'} \overline{u}^{s'}_{p'}\gamma^\mu u^s_p \overline{u}^s_p \gamma^\nu u^{s'}_{p'} \sum_r \sum_{r'} (\cdots) \,. \tag{560}$$

Recalling that

$$\sum_s u^s_a \overline{u}^s_b = (\not{p} + m)_{ab} = \sum_s \overline{u}^s_b u^s_a, \tag{561}$$

we see that

$$\mathcal{P} = \frac{(-ie)^4}{4\,(p-p')^4} \operatorname{tr}\left((\not{p}' + m)\,\gamma^\mu\,(\not{p} + m)\,\gamma^\nu\right) \operatorname{tr}\left((\not{q}' + m)\,\gamma_\mu\,(\not{q} + m)\,\gamma_\nu\right) \tag{562}$$

$$= \frac{8e^4}{(p-p')^4} \left(p'\cdot q' p\cdot q + p\cdot q' p'\cdot q - m_e^2 q\cdot q' - m_\mu^2 p\cdot p' + 2m_\mu^2 m_e^2\right), \tag{563}$$

and we are done (with the beginning).