Quantum Field Theory

Cian Luke Martin

2024-10-25

Contents

1 Lecture: Introduction

11/10/2024

Historically, the goal of quantum field theory was to combine quantum mechanics with special relativity. One of the most notable outputs of this study hailed as a success is that the number of particles is not conserved. It is a robust and systematic theory governed by few principles. It concerns itself with locality, symmetries and renormalization which are exceptionally constraining and *almost* uniquely determine what we can study.

In this course we use $c = \hbar = 1$. In these natural units, $E = mc^2$ gives us masses in the units of energy.

For metrics we use $\eta^{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$, $x^\mu = (t, x, y, z)$, $F(t, \vec{x}) \equiv F(x^\mu) \equiv F(\mathbf{x})$

1.1 Classical Field Theory

In classical mechanics, a natural object is the action,

$$S(t_1, t_2) = \int_{t_1}^{t_2} \mathrm{d}t \left(\underbrace{m \sum_{i=1}^{3} \left(\frac{\mathrm{d}x^i}{\mathrm{d}t} \right)^2}_{\text{kinetic term}} - \underbrace{V(x)}_{\text{potential}} \right). \tag{1}$$

This is incredibly useful for us for three main reasons:

- the equations of motion are given for free by extremising S,
- Boundary conditions are supplied externally, and
- S is built on *symmetry* (it is invariant of symmetries of your system).

As we move towards field theory, we no longer want to speak of a single position of a particle $x(t)$.

The fundamental object in field theory is a field $\phi_a(t, \vec{x}) : \mathbb{R}^{3,1} \to \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{R}^n$. Here a labels the type of field we are discussing.

The first consequence is that we are dealing with an infinite number of degrees of freedom as every point in time and space contains some distinct information about the system.

Example. In electromagnetism, as we will discuss in depth later, one has the gauge field $A^\mu(t, \vec{x}) = \left(\phi(x), \vec{A}(x)\right)$ which the electric and magnetic fields can be defined in terms of

$$\vec{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{2}$$

$$\vec{B} = \boldsymbol{\nabla} \cdot \vec{A}, \tag{3}$$

which have equations of motion

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \tag{4}$$

$$\boldsymbol{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}, \tag{5}$$

and two identities

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \tag{6}$$

$$\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} = -\boldsymbol{\nabla} \times \vec{E}. \tag{7}$$

This is a (hopefully) familiar classical field that we will quantise in due time.

1.2 Lagrangians

The Lagrangian in classical mechanics can be written $L = T - V$ and is contained within the action in the form

$$S = \int \mathrm{d}t\, L. \tag{8}$$

We will in QFT concern ourselves with the *Lagrangian density* given by

$$L = \int \mathrm{d}^3x\, \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right), \tag{9}$$

however *everybody* just refers to $\mathcal{L}$ as a Lagrangian as we will here.

The equations of motion are determined by extremizing with respect to the fields.

Note. Note that we assume that the Lagrangian $\mathcal{L}\left[\phi_a, \partial_\mu \phi_a\right]$ is not a function of $\partial^2 \phi_a$ or higher derivatives. This is for complicated reasons related to ghosts that are beyond the scope of this course.

Extremising the action with respect to the field, we want to find the conditions for which $\Delta S = 0$, i.e. the action is at a minima/saddle point. We see that

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta\left(\partial_\mu \phi_a\right)\right] \tag{10}$$

$$= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right) \delta \phi_a + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta \phi_a\right)}_{\text{total derivative}}\right], \tag{11}$$

and by assuming that our fields decay at infinity, the total derivative term vanishes yielding

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right)\right] \delta \phi_a, \tag{12}$$

for which vanishing requires

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{13}$$

Example. A free massive scalar field is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu - \frac{1}{2} m^2 \phi^2 \tag{14}$$

$$= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \left(\boldsymbol{\nabla} \phi\right)^2 - \frac{1}{2} m^2 \phi^2. \tag{15}$$

In traditional classical mechanics, one would have identified $T = \frac{1}{2} \dot{\phi}^2$ and $V = \frac{1}{2} \left(\boldsymbol{\nabla} \phi\right)^2 + \frac{1}{2} m^2 \phi^2$. In QFT, the 'kinetic terms' sometimes refers to any bilinear combination of fields. For example, $\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ is always kinetic and $m^2 \phi^2$ is often a (bosonic) mass term.

The equation of motion for the free massive scalar field Lagrangian is

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{16}$$

This is the **Klein Gordon equation**. It is also sometimes written with $\partial_\mu \partial^\mu = \Box$.

2 Lecture: Symmetries

14/10/2024

2.1 Hamiltonian Formalism

In a Hamiltonian formalism, one starts by defining the canonical momenta

$$\Pi^a(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_a)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{17}$$

Definition 2.1: The **Hamiltonian density** is defined by Legendre transform of the Lagrangian density

$$\mathcal{H} = \Pi^a \partial_t \phi_a - \mathcal{L}. \tag{18}$$

The Hamiltonian is given by

$$H = \int \mathrm{d}^3 x \, \mathcal{H}. \tag{19}$$

We will not abuse notation and always call $\mathcal{H}$ a *Hamiltonian density*, and H a *Hamiltonian.*

Example. For a scalar field with a potential $V(\phi)$, we have

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{20}$$

The canonical momentum is

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \tag{21}$$

and the Hamiltonian is then

$$H = \int \mathrm{d}^3 x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V(\phi)\right). \tag{22}$$

2.2 Symmetry

Symmetries are inseparable from the study of quantum field theory. Most notably they dictate the actions we can write, the class of fields (operators) we can use, and the observables we can compute.

Definition 2.2: The **Lorentz group** has elements $\Lambda^\mu{}_\nu$ such that under Lorentz boosts

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{23}$$

which preserve the spacetime interval $s^2 = x^\mu x^\nu \eta_{\mu\nu} = t^2 - x^i x_i$ such that

$$s^2 \to s'^2 = s^2. \tag{24}$$

This condition implies

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}. \tag{25}$$

In matrix form, this can be written $\Lambda^T\eta\Lambda = \eta$.

Examples. Rotations such as one in the xy plane, leave $t' = t$ and have $\Lambda^1_1 = R^1_1$ such that

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Boosts mix time and space. Boosting in the (t, x) plane, we have

$$\Lambda = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

where η is the **rapidity** and is given by

$$\cosh\eta = \frac{1}{\sqrt{1-v^2}} \tag{28}$$

$$\sinh\eta = \frac{v}{\sqrt{1-v^2}}. \tag{29}$$

Note. From 1), we see that in general $\det(\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$.

If $\det\Lambda = 1$, then Λ is called a *proper* Lorentz transformation.

If $\det\Lambda = -1$, then Λ is called a *improper* Lorentz transformation. Parity and time reversal each independently cause $\det\Lambda = -1$. Only proper Lorentz transformations are continuously connected to the identity.

We will assume $\det\Lambda = 1$. We can then expand about the identity infinitesimally and write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \varepsilon^{\mu}{}_{\nu} + \mathcal{O}\left(\varepsilon^2\right). \tag{30}$$

The natural question is what are the properties of $\varepsilon^{\mu}{}_{\nu}$?

Inserting this expression into Eq. (25), we see

$$\begin{aligned} \eta_{\rho\sigma} &= \eta_{\mu\nu}\left(\delta^{\mu}_{\rho} + \varepsilon^{\mu}{}_{\rho} + \cdots\right)\left(\delta^{\nu}_{\sigma} + \varepsilon^{\nu}{}_{\sigma} + \cdots\right) \\ &= \eta_{\mu\nu}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\varepsilon^{\mu}{}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\delta^{\mu}_{\rho}\varepsilon^{\nu}{}_{\sigma} + \mathcal{O}\left(\varepsilon\right)^2 \\ &= \eta_{\rho\sigma} + \varepsilon_{\sigma\rho} + \varepsilon_{\rho\sigma} \\ \Rightarrow \varepsilon_{\sigma\rho} &= -\varepsilon_{\rho\sigma}. \end{aligned} \tag{31}$$

Therefore $\varepsilon_{\sigma\rho}$ is an antisymmetric tensor, which in $d = 4$ has $\frac{d(d-1)}{2} = 6$ independent components.

Therefore we have 6 generators for the Lorentz group:

- 3 rotations, and
- 3 boosts

2.3 Fields Revisited

We can now think of a field as an object which transforms under the Lorentz group. It therefore forms a representation of the algebra.

Definition 2.3: A field is an object that depends on coordinates and has a definite transformation under the action of the Lorentz group,

$$x \to x' = \Lambda x, \tag{32}$$
$$\phi_a(x) \to \phi'_a(x) = D[\Lambda]_a^b \phi_b\left(\Lambda^{-1}x\right). \tag{33}$$

$D[\Lambda]$ forms a representation of the Lorentz group as it satisfies

$$D[\Lambda_1]\,D[\Lambda_2] = D[\Lambda_1\Lambda_2], \tag{34}$$
$$D\left[\Lambda^{-1}\right] = D[\Lambda]^{-1}, \tag{35}$$
$$D[\mathbb{I}] = 1.. \tag{36}$$

Examples.

1) Consider the trivial representation $D[\Lambda] = 1$. Then the field transforms as

$$\phi(x) = \phi\left(\Lambda^{-1}x\right), \tag{37}$$

which is an equivalent definition of the *scalar field.* Here we are using active transformations where the coordinates are fixed.

2) We are also familiar with the vector representation given by

$$D[\Lambda]^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}. \tag{38}$$

A field transforming under this representation is A^μ such that

$$A^\mu(x) \to A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu\left(\Lambda^{-1}x\right), \tag{39}$$

and similarly,

$$\partial_\mu\phi \to \partial_\mu\phi'(x) = \left(\Lambda^{-1}\right)^\nu{}_\mu \partial_\nu\phi\left(\Lambda^{-1}x\right). \tag{40}$$

2.4 Actions Revisited

As we alluded to earlier, actions are also heavily constrained by symmetries. Given the Lagrangian density of the massive scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi\eta^{\mu\nu} - m^2\phi^2,$$

we notice that the action is invariant under Lorentz transformations.

3 Lecture: Noether's Theorem

16/10/2024

We can check this transformation explicitly with

$$\mathcal{L} \to \frac{1}{2}\eta^{\mu\nu}\left(\Lambda^{-1}\right)^{\rho}{}_{\mu}\,\partial_\rho\phi\left(\Lambda^{-1}x\right)\left(\Lambda^{-1}\right)^{\sigma}{}_{\nu}\,\partial_\sigma\phi\left(\Lambda^{-1}\right) - \frac{1}{2}m^2\phi^2\left(\Lambda x\right) \tag{41}$$

$$= \frac{1}{2}\eta^{\rho\sigma}\partial_\rho\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 = \mathcal{L}, \tag{42}$$

and therefore

$$S \to \int \mathrm{d}^4x\,\mathcal{L}\left(y\right) = \int \mathrm{d}^4y\,\mathcal{L}\left(y\right), \tag{43}$$

as $\det\left(\Lambda\right) = 1$. Thus the action is also Lorentz invariant.

Theorem 3.1 (Noether's Theorem):

1) Every **continuous symmetry** of the Lagrangian gives rise to a current j^μ which is conserved $\partial_\mu j^\mu = 0$ under the equations of motion.
2) Provided suitable boundary conditions, a conserved current will give rise to a conserved charge Q, where

$$Q = \int \mathrm{d}^3x\, j^0. \tag{44}$$

Proof.

1) We must first define a continuous symmetry.

Definition 3.1: A transformation is continuous if there is an infinitesimal parameter in it. We will see two types:

- *internal* transformations, which do not act on the coordinates, but act on the fields,
- *local* transformations, which act on both the coordinates and the fields.

In both cases, a continuous transformations can be written

$$\delta\phi_a = \phi_a^{'}(x) - \phi_a(x)\,. \tag{45}$$

Such a transformation is a **symmetry** of the system if the **action** is invariant under the transformation.

Namely, under

$$S[\phi] \to S[\phi'] = \int \mathrm{d}^4x\,\mathcal{L}[\phi']\,, \tag{46}$$

we are looking for

$$\delta S = S[\phi'] - S[\phi] = 0, \tag{47}$$

which implies a symmetry. This implies that for the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}'(x) - \mathcal{L}(x) = \partial_\mu \mathcal{J}^\mu, \tag{48}$$

namely, that the Lagrangian can change up to a total derivative without the action changing.

Let's quantify the change in $\mathcal{L}$. We have that

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\partial_\mu\phi_a & (49)\\ &= \left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right)\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) \overset{\text{symm}}{=} \partial_\mu\mathcal{J}^\mu. & (50)\end{aligned}$$

This implies that

$$-\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right)}_{\text{equation of motion}}\delta\phi_a = \partial_\mu\underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - \mathcal{J}^\mu\right)}_{j^\mu}. \tag{51}$$

Therefore if the equation of motion is imposed, one has

$$\partial_\mu j^\mu = 0, \tag{52}$$

for

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - \mathcal{J}^\mu. \tag{53}$$

2) We have

$$Q = \int \mathrm{d}^3 x\, j^0, \tag{54}$$

and

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_V \mathrm{d}^3 x\, \frac{\partial j^0}{\partial t} \tag{55}$$
$$= -\int_V \mathrm{d}^3 x\, \vec{\boldsymbol{\nabla}} \cdot \vec{j} \tag{56}$$
$$= -\int_{\partial V} d\vec{A} \cdot \vec{j} \tag{57}$$
$$= 0, \tag{58}$$

where this last equality holds as the fields decay as $|x| \to \infty$, and thus Q is a conserved quantity.

□

3.1 Energy Momentum Tensor

We consider a local transformation that is a symmetry of almost every theory worthy of study: spatial translations taking

$$x^\mu \to x'^\mu = x^\mu - \varepsilon^\mu, \tag{59}$$

where ε^μ is a constant vector. Under such translations, the fields transform as

$$\phi_a \to \phi'_a(x) = \phi_a(x + \varepsilon), \tag{60}$$

where making this an infinitesimal transformation and expanding in a Taylor series we see

$$\phi'_a(x) = \phi_a(x) + \varepsilon^\mu \partial_\mu \phi_a(x) + \mathcal{O}\left(\varepsilon^2\right) \tag{61}$$
$$\Rightarrow \delta\phi_a = \phi'_a(x) - \phi_a(x) \tag{62}$$
$$= \varepsilon^\mu \partial_\mu \phi(x). \tag{63}$$

The Lagrangian changes by a total derivative under this transformation such that

$$\delta\mathcal{L} = e^\mu \partial_\mu \mathcal{L} = \partial_\mu \underbrace{\varepsilon^\mu \mathcal{L}}_{\mathcal{J}^\mu}. \tag{64}$$

Therefore, substituting in $\delta\phi_a$ and $\mathcal{J}^\mu$, our conserved current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi_a)} \underbrace{\varepsilon^\nu \partial_\nu \phi_a}_{\delta\phi_a} - \underbrace{\varepsilon^\mu \mathcal{L}}_{\mathcal{J}^\mu} \tag{65}$$
$$= \varepsilon^\nu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu{}_\nu \mathcal{L} \right) \equiv \varepsilon^\nu T^\mu{}_\nu, \tag{66}$$

where $T^\mu{}_\nu$ is the **energy momentum tensor**.

Using the equation of motion, one can show that

$$\partial_\mu j^\mu = 0 \Rightarrow \partial_\mu T^\mu{}_\nu = 0, \tag{67}$$

namely that the stress energy tensor is conserved on shell.

Further, from $T^{\mu\nu}$ we can construct four conserved charges given by

- the *energy*, $E = \int \mathrm{d}^3x\, T^{00}$ by choosing $e^\mu = (1, 0, 0, 0)$,
- the *momenta*, $p^i = \int \mathrm{d}^3x\, T^{0i}$ where $\varepsilon = (0, 1, 0, 0)$, $(0, 0, 1, 0)$ or $(0, 0, 0, 1)$.

Example (Local Symmetry)**.** For the free massive scalar field

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \tag{68}$$

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu - \eta^{\mu\nu}\mathcal{L} \tag{69}$$

$$T^{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2, \tag{70}$$

where observe that

$$E = \int \mathrm{d}^3x\, T^{00} = H, \tag{71}$$

and

$$p^i = \int \mathrm{d}^3x\, T^{0i} = \int \mathrm{d}^3x\, \dot{\phi}\partial^i\phi. \tag{72}$$

Note. The stress energy tensor

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\nu\phi_a - \delta^\mu_\nu\mathcal{L}, \tag{73}$$

is not always symmetric. One can define the *Belifante tensor* given by

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho\mathcal{T}^{\rho\mu\nu}, \tag{74}$$

where $\mathcal{T}^{\rho\mu\nu} = -\mathcal{T}^{\mu\rho\nu}$ leads to $\partial_\mu\Theta^{\mu\nu} = 0$.

One can also symmetrize $T^{\mu\nu}$ by coupling fields to $g_{\mu\nu}$ (instead of $\eta^{\mu\nu}$) with

$$\Theta^{\mu\nu} = \left(-\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g_{\mu\nu}}\left(\sqrt{-g}\mathcal{L}\right)\right)\bigg|_{g=\eta}. \tag{75}$$

4 Lecture: Canonical Quantization

18/10/2024

4.1 Internal Symmetry

Example (Internal Symmetry)**.** Internal symmetries do not act on coordinates, only the fields. Consider a complex scalar field

$$\psi\left(x\right) = \frac{1}{\sqrt{2}}\left(\phi_1\left(x\right) + i\phi_2\left(x\right)\right), \tag{76}$$

where ϕ_1, ϕ_2 are real scalar fields. A Lagrangian for this field is

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \psi^* - V\left(|\psi|^2\right). \tag{77}$$

The equations of motion for this theory are

$$\partial_\mu \partial^\mu \psi + \frac{\partial V}{\partial \phi^*} = 0, \qquad \partial_\mu \partial^\mu \psi^* + \frac{\partial V}{\partial \phi} = 0. \tag{78}$$

The internal symmetry of this system, for constant $\alpha \in \mathbb{R}$,

$$\phi(x) \to \phi'(x) = e^{i\alpha} \psi(x) \tag{79}$$

$$\phi^*(x) \to (\phi^*(x))' = e^{-i\alpha} \psi^*(x), \tag{80}$$

under which $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$ and $S \to S' = S$. Here α is the continuous parameter of the transformation, such that

$$\delta\phi = \phi'(x) - \phi(x) \tag{81}$$

$$= i\alpha\psi \tag{82}$$

$$\delta\psi^* = -i\alpha\psi^*. \tag{83}$$

We can construct the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \delta\psi^* - \mathcal{J}^\mu, \tag{84}$$

where there is no total derivative term, $\mathcal{J}^\mu = 0$. We then have

$$j^\mu = i\alpha(\psi \partial^\mu \psi^* - \psi^* \partial_\mu \psi), \tag{85}$$

which implies a conserved charge

$$Q = \int \mathrm{d}^3 x\, j^0, \tag{86}$$

which is in fact the electric charge as we will see.

Observe that it is also possible to view the transformation as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{87}$$

which is identical to the previous transformation in Eq. (79).

4.2 Quantum Fields

We will first study the simplest possible theory: a free theory. We will take a Hamiltonian approach and build on the rules of quantum mechanics. Recall the familiar commutation relations of

$$\left[x^i, p^j\right] = i\delta^{ij}. \tag{88}$$

In QFT, we no longer speak of position and momentum variables, but rather a quantum field $\phi_a(x)$ and its conjugate momenta $\Pi^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$ which satisfy

$$\left[\phi_a(\mathbf{x}, t), \Pi^b(\mathbf{y}, t)\right] = i\delta^3(\mathbf{x} - \mathbf{y})\, \delta_a^b, \tag{89}$$

called *equal time* commutation relations. One must make a choice of some kind when transferring from a classical theory to a quantum theory, and this turns out to be one such correct choice.

4.3 Canonical Quantization

Note. In the notes, Tong performs canonical quantization in the Schrödinger picture at $t = 0$. Here we will use the Heisenberg picture.

Our theory of interest is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{90}$$

Its equation of motion is the Klein-Gordon equation, $\partial_\mu\partial^\mu\phi + m^2\phi = 0$.

We know solutions to this equation take the form

$$\phi \sim \exp\left(i\mathbf{k}\cdot\mathbf{x} + i\omega t\right), \tag{91}$$

where $-\omega^2 + \mathbf{k}\cdot\mathbf{k} + m^2 = 0$ which gives us a dispersion relation,

$$\omega\left(k\right) = \pm\sqrt{\mathbf{k}\cdot\mathbf{k} + m^2}. \tag{92}$$

We adopt the notation $\omega \equiv \sqrt{\mathbf{k}\cdot\mathbf{k} + m^2}$. Therefore, taking a linear superposition of fields, one has

$$\phi\left(\mathbf{x}, t\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(a\left(k\right)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + b\left(k\right)e^{i\mathbf{k}\cdot\mathbf{x} + i\omega t}\right). \tag{93}$$

Note. ϕ is real, which imposes restrictions on $a\left(k\right)$ and $b\left(k\right)$. Namely, as $\phi^* = \phi$, we have

$$a^*\left(-k\right) = b\left(k\right) \qquad\qquad b^*\left(-k\right) = a\left(k\right), \tag{94}$$

thus we can write

$$\phi\left(x\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(a\left(k\right)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + a^*\left(k\right)e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t}\right). \tag{95}$$

In a more relativistic notation, one has

$$\phi\left(x\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(a\left(k\right)e^{-ik^\mu x_\mu} + a^*\left(k\right)e^{ik^\mu x_\mu}\right), \tag{96}$$

where $k_\mu = \left(\omega, \mathbf{k}\right)$ and $x_\mu = \left(t, \mathbf{x}\right)$ give us $k^\mu x_\mu = \omega t - \mathbf{k}\cdot\mathbf{x}$ and $k^2 = \omega^2 - \mathbf{k}\cdot\mathbf{k} = m^2$.

Note. We will choose to normalize $a\left(k\right)$ and $a^*\left(k\right)$ such that

$$\phi\left(x\right) = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\frac{1}{\sqrt{2\omega}}\left(a\left(k\right)e^{-ik_\mu x^\mu} + a^*\left(k\right)e^{ik_\mu x^\mu}\right). \tag{97}$$

Lastly, notice that

$$\Pi\left(x\right) = \dot{\phi} = \int \frac{\mathrm{d}^3k}{\left(2\pi\right)^3} - i\sqrt{\frac{\omega}{2}}\left(a\left(k\right)e^{-ik_\mu x^\mu} - a^*\left(k\right)e^{ik_\mu x^\mu}\right). \tag{98}$$

Next, we ***quantize***, namely, we declare that

$$\left[\phi\left(\vec{x}, t\right), \phi\left(\vec{x}', t\right)\right] = 0 \tag{99}$$
$$\left[\Pi\left(\vec{x}, t\right), \Pi\left(\vec{x}', t\right)\right] = 0 \tag{100}$$
$$\left[\phi\left(\vec{x}, t\right), \Pi\left(\vec{x}', t\right)\right] = i\delta^3\left(\vec{x} - \vec{x}'\right). \tag{101}$$

Claim. These commutation relations promote a to an **operator** such that $a(k)$ becomes $\hat{a}_k$ and $a^*(k)$ becomes $\hat{a}_k^\dagger$. The above commutation relations imply

$$[\hat{a}_k, \hat{a}_{k'}] = 0 \tag{102}$$

$$\left[\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\right] = 0 \tag{103}$$

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = (2\pi)^3 \delta^3(k - k'). \tag{104}$$

Proof.

1) (*Claim implies declaration*) Taking

$$[\phi(\vec{x}, t), \Pi(\vec{y}, t)] = \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6} \frac{1}{2i} \sqrt{\frac{\omega_q}{\omega_p}} \left(\left[a_{\mathbf{p}} e^{i\vec{p}\cdot\vec{x}+i\omega t} + a_{\mathbf{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}+i\omega t}, a_{\mathbf{q}} e^{i\vec{q}\cdot\vec{y}-i\omega t} - a_{\mathbf{q}}^\dagger e^{-i\vec{q}\cdot\vec{y}+i\omega t}\right]\right) \tag{105}$$

$$= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6} \frac{1}{2i} \sqrt{\frac{\omega_q}{\omega_p}} \left(- \underbrace{\left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger\right]}_{(2\pi)^3\delta^3(\vec{p}-\vec{q})} e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}} + \left[a_{\vec{p}}^\dagger, a_{\mathbf{q}}\right] e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{y}} \right) \tag{106}$$

$$= i \int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = i\delta^3(\vec{x} - \vec{y}), \tag{107}$$

as desired.

□

5 Lecture: Vacuum Energy

21/10/2024

Observe that while the Hamiltonian for the free massive scalar field can be written as

$$H = \int \mathrm{d}^3x\, \mathcal{H} = \frac{1}{2} \int \mathrm{d}^3x \left(\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right), \tag{108}$$

we desire an expression in terms of a and $a^\dagger$. Expanding the fields in terms of these operators, we see

$$\begin{aligned} H = \frac{1}{2} \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6} \Bigg(&- \frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \left(a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{x}}\right) \\ &- \frac{1}{2} \frac{1}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} \left(a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right) \left(a_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\right) \vec{p}\vec{q} \\ &+ \frac{m^2}{2} \frac{1}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} \left(a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right) \left(a_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\right) \Bigg) \end{aligned} \tag{109}$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \Big[\underbrace{\left(-\omega_{\mathbf{p}}^2 + \vec{p}^2 + m^2\right)}_{\text{e.o.m. thus vanishes}} \left(a_{\mathbf{p}} a_{\mathbf{p}} e^{-2i\omega t} + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger e^{2i\omega t}\right) \tag{110}$$

$$+\left(\omega_{\mathbf{p}}^2+\vec{p}^2+m^2\right)\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger\right)\Bigg] \tag{111}$$

$$H=\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger\right) \tag{112}$$

$$H=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega\,(2\pi)^3\,\delta^3(0)\,. \tag{113}$$

This last term is odd, and appears unphysical as with a vacuum $|0\rangle$ satisfying $a_{\mathbf{p}}|0\rangle=0$, we see

$$H|0\rangle=\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega\,(2\pi)^3\,\delta(0)\,|0\rangle=E_0\,|0\rangle\to\infty. \tag{114}$$

To understand the nature of this, we need to see the origin of the divergence. There are in fact two divergences here:

- An *infrared divergence*: $(2\pi)^3\,\delta(0)$, associated with long distances, as it came from

$$\delta(0)=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}^3x\,e^{-i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=0}=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}^3x=V, \tag{115}$$

  a diverging volume V. As we are discussing an system with infinite size, we can instead discuss energy *densities* (i.e. per unit volume) such that

$$\varepsilon_0=\frac{E_0}{V}=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2}\omega_{\mathbf{p}}\sim\int\mathrm{d}^3p\,\vec{p}^2, \tag{116}$$

  which is still divergent.

- Namely, it is an *ultraviolet divergence*. Suppose one is performing

$$\int_0^\Lambda\mathrm{d}^3p\,\sqrt{\vec{p}^2+m^2}\overset{\Lambda\to\infty}{\longrightarrow}\infty, \tag{117}$$

  we see that this is a high frequency divergence. It is absurd to think that the theory is valid for arbitrarily high energies, and thus it is valid to consider a maximum energy scale of applicability, a cutoff, Λ.

  The solution here, is to declare that

$$H\equiv\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{118}$$

  One can convince themself that we can only measure energy differences and thus can remove this vacuum energy. However, practically, it is best to just take this H as definition such that it fixes an ambiguity. There is an ambiguity in the *normal ordering* of operators when one converts between classical and quantum field theories. Here it is clear that this H is the correct definition in quantum field theory as it provides $H|0\rangle=0$.

Definition 5.1: If you have a list of fields, we define **normal ordering** as

$$:\phi_1(x_1)\,\phi_2(x_2)\cdots\phi_n(x_n):, \tag{119}$$

where this is the usual product but we put creation operators $a_{\mathbf{p}}^\dagger$ to the left of annihilation operators $a_{\mathbf{p}}$.

5.1 Fock Space

We have the vacuum $|0\rangle$ and want to construct excited states atop it. It is usefully to observe that

$$[H, a_{\mathbf{p}}^\dagger] = \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \tag{120}$$

and

$$[H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}} a_{\mathbf{p}}. \tag{121}$$

We then aim to construct energy eigenstates by

$$|\vec{p}\rangle = a_{\vec{p}} |0\rangle . \tag{122}$$

This is a single particle state. Observe that then

$$H |\vec{p}\rangle = \omega_p |\vec{p}\rangle . \tag{123}$$

6 Lecture: Relativistic Normalisation

23/10/2024

We can consider the momentum operator represented by

$$\hat{\vec{p}} = - : \int \mathrm{d}^3x \, \Pi \boldsymbol{\nabla} \phi := \int \frac{\mathrm{d}^3k}{(2\pi)^3} \vec{k} a_{\vec{k}}^\dagger a_{\vec{k}}, \tag{124}$$

for which $|\vec{p}\rangle$ is also an eigenstate,

$$\hat{\vec{p}} |\vec{p}\rangle = \vec{p} |\vec{p}\rangle . \tag{125}$$

Therefore $|\vec{p}\rangle$ is a momentum and energy eigenstate with

$$E^2 = \omega_{\vec{p}}^2 = \vec{p}^2 + m^2. \tag{126}$$

Note. When $\vec{p} = 0$, this particle has no angular momentum such that

$$J^i |\vec{p} = 0\rangle = 0, \tag{127}$$

which implies it is a spin 0 particle as we will see later.

Observe that can can construct an n particle state with

$$|\vec{p}_1 \cdots \vec{p}_n\rangle = a_{\vec{p}_1}^\dagger \cdots a_{\vec{p}_n}^\dagger |0\rangle . \tag{128}$$

As all $a_{\vec{p}}^\dagger$'s commute, we have

$$|\vec{p}_1 \vec{p}_2\rangle = |\vec{p}_2 \vec{p}_1\rangle . \tag{129}$$

Therefore, the Fock space is spanned by all possible combinations of $a^\dagger$ acting on $|0\rangle$. It is interesting then to introduce the **number operator**

$$N \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}}, \tag{130}$$

which tells us the number of particles in a given state. Namely,

$$N \left|\vec{p}_1 \cdots \vec{p}_n\right\rangle = n \left|\vec{p}_1 \cdots \vec{p}_n\right\rangle. \tag{131}$$

For a free theory, we have that

$$[N, H] = 0, \tag{132}$$

which implies that the number of particles is conserved.

Therefore if $\mathscr{H}_n$ denotes the space of n particle states, the Fock space $\mathscr{F}$ can be written

$$\mathscr{F} = \bigoplus_n \mathscr{H}_n. \tag{133}$$

6.1 Relativistic normalization

While we have constructed eigenstates $|\vec{p}_i\rangle$ we have not checked that they are normalized states. To begin, we pick

$$\langle 0|0\rangle = 1. \tag{134}$$

For the 1 particle state,

$$|\vec{p}\rangle = a^\dagger_{\vec{p}} |0\rangle \Rightarrow \langle \vec{p}|\vec{q}\rangle = (2\pi)^3 \delta^3 (\vec{p} - \vec{q}), \tag{135}$$

which is not a Lorenz invariant inner product.

We would hope that under a Lorentz transformation Λ with corresponding unitary transformation $U(\Lambda)$, that $|\vec{p}\rangle$ transforms as

$$\vec{p} \to |\vec{p}'\rangle = U(\Lambda) |\vec{p}\rangle. \tag{136}$$

This is not yet the case. To figure out a proper definition of $|\vec{p}\rangle$, we use the identity

$$|\vec{q}\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} |\vec{p}\rangle \langle \vec{p}|\vec{q}\rangle, \tag{137}$$

where we have used

$$1 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} |\vec{p}\rangle \langle \vec{p}|, \tag{138}$$

where the measure and hence integral here is clearly not Lorentz invariant. The natural question is how can we alter this identity to make the measure Lorentz invariant. If we instead integrated over

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \to \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \delta\left(p^2 - m^2\right) \Theta\left(p^0\right), \tag{139}$$

then the measure is now Lorentz invariant, (along with the other functions), where the δ and Heaviside function Θ now enforces the equation of motion. Equivalently, p^0 is not a free parameter, and thus enforcing $p^2 = m^2$ returns us to states in our Fock space, however we must also enforce $p^0 > 0$ as we chose the positive root with $\omega > 0$.

We see that using

$$\int \mathrm{d}x\, \delta\left(f\left(x\right)\right) = \sum_{x_0 | f(x_0)=0} \frac{1}{\left|f'\left(x_0\right)\right|}, \tag{140}$$

we have

$$\int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4} \delta^4\left(p^2 - m^2\right)\Theta\left(p^2\right) = \int \mathrm{d}^3 p \,\frac{1}{2\sqrt{\vec{p}^2 + m^2}} = \int \mathrm{d}^3 \,\frac{1}{2\omega_{\vec{p}}}. \tag{141}$$

Therefore bringing this measure back to the identity, we see

$$1 = \int \frac{\mathrm{d}^3 p}{\left(2\pi\right)^3} \left|\widetilde{\vec{p}}\right\rangle \left\langle\widetilde{\vec{p}}\right|, \tag{142}$$

where we define

$$\left|\widetilde{\vec{p}}\right\rangle = \sqrt{2\omega_{\vec{p}}} a^\dagger_{\vec{p}} \left|0\right\rangle, \tag{143}$$

which is now manifestly Lorentz invariant and thus is called *relativistic normalization*.

6.2 Causality

While we now have Lorentz invariant states, their commutation relations are still at equal time. Is this compatible with special relativity (especially causality)?

We will study causality by determining whether measurements *influence* each other in a time-like fashion. We will do this by finding whether their commutators vanish or not.

We define

$$\Delta\left(x - y\right) = \left[\phi\left(x\right), \phi\left(y\right)\right], \tag{144}$$

with the interpretation of "measuring" the field at x then y or vice versa.

For the free theory, we see

$$\Delta = \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 p}{\left(2\pi\right)^6} \left(\left[a_k, a^\dagger_p\right] e^{-ikx} e^{ipy} + \left[a^\dagger_k, a_p\right] e^{ikx} e^{-ipy}\right) \tag{145}$$

$$= \int \frac{\mathrm{d}^3 p}{\left(2\pi\right)^3} \frac{1}{2\omega_p} \left(e^{-ip(x-y)} - e^{ip(x-y)}\right). \tag{146}$$

This integral is also Lorentz invariant immediately by inspection. For the free theory, it is a complex number. Suppose x and y are timelike separated such that without loss of generality, $\left(x-y\right)_S = \left(t,0,0,0\right)$. This gives us

$$\Delta\left(x-y\right)_T = \int \frac{\mathrm{d}^3 p}{\left(2\pi\right)^3} \frac{1}{2\omega_p} \left(e^{-i\omega_p t} - e^{i\omega_p t}\right) \sim e^{-imt} - e^{imt} \neq 0. \tag{147}$$

If we instead look at spacelike separated events, $(x-y)_S = (0, \vec{x}-\vec{y})$,

$$\Delta\left(x-y\right)_S = \int \frac{\mathrm{d}^3 p}{\left(2\pi\right)^3} \frac{1}{2\omega_p} \left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right) = 0, \tag{148}$$

as one can separate and exchange $\vec{p} \to -\vec{p}$. We already knew that the commutator at equal times vanishes, however as we know this commutator is Lorentz invariant, any spacelike event has zero commutator.

6.3 Propagators

Consider

$$\langle 0|\,\phi\left(x\right)\phi\left(y\right)|0\rangle = \int \frac{\mathrm{d}^3 p}{\left(2\pi\right)^3} \frac{1}{2\omega_p} e^{-ip(x-y)} \equiv D\left(x-y\right). \tag{149}$$

For spacelike events,

$$D\left(x-y\right) \sim e^{-m(\vec{x}-\vec{y})} \neq 0, \tag{150}$$

but

$$\left[\phi\left(x\right), \phi\left(y\right)\right] = D\left(x-y\right) - D\left(y-x\right) = 0. \tag{151}$$

7 Lecture: Feynman Propagator

25/10/2024

7.1 Feynman Propagator

Definition 7.1: The **Feynman propagator** is given by

$$\Delta_F\left(x-y\right) = \langle 0|\,T\phi\left(x\right)\phi\left(y\right)|0\rangle = \begin{cases} D\left(x-y\right), & x^0 > y^0, \\ D\left(y-x\right), & y^0 > x^0, \end{cases} \tag{152}$$

where T denotes *time ordering.*

This is motivated by inner products like $\langle f|i\rangle$ where $\langle f|$ is a future final state and $|i\rangle$ is a past initial state.

Claim. We claim that

$$\Delta_F\left(x-y\right) = \int \frac{\mathrm{d}^4 p}{\left(2\pi\right)^4} \frac{i}{p^2+m^2+i\varepsilon} e^{-ip(x-y)}. \tag{153}$$

Proof. Observe that the time ordering can be captured with

$$\langle 0|\,T\phi\left(x\right)\phi\left(y\right)|0\rangle = \langle 0|\,\phi\left(x\right)\phi\left(y\right)|0\rangle\,\Theta\left(x^0-y^0\right) + \langle 0|\,\phi\left(y\right)\phi\left(x\right)|0\rangle\,\Theta\left(y^0-x^0\right). \tag{154}$$

Our claim can be written as

$$\Delta_F\left(x-y\right) = \int \frac{\mathrm{d}^3 k}{\left(2\pi\right)^3} \frac{1}{2\omega_k} e^{-i\omega_k\left(x^0-y^0\right)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\Theta\left(x^0-y^0\right)$$

$$+\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\omega_k\left(x^0-y^0\right)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\Theta\left(y^0-x^0\right) \tag{155}$$

$$=\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(e^{-i\omega_k\tau}\Theta\left(\tau\right)+e^{i\omega_k\tau}\Theta\left(-\tau\right)\right), \tag{156}$$

where $\tau = x^0 - y^0$. We focus on the time-dependence and show that

$$e^{i\omega_k\tau}\Theta\left(\tau\right)+e^{i\omega_k\tau}\Theta\left(-\tau\right)=\lim_{\varepsilon\to 0}\frac{(-2\omega_k)}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{e^{-i\omega\tau}}{\omega^2-\omega_k^2+i\varepsilon}. \tag{157}$$

We begin from the right hand side and observe that

$$\frac{1}{\omega^2-\omega_k^2+i\varepsilon}=\frac{1}{\left(\omega-(\omega_k-i\widetilde{\varepsilon})\right)\left(\omega-(-\omega_k+\widetilde{\varepsilon})\right)}, \tag{158}$$

where $\varepsilon = \tau\omega_k\widetilde{\varepsilon}+\cdots$ and we relabel back $\widetilde{\varepsilon}\to\varepsilon$. Thus to leading order in ε we see

$$\frac{1}{\omega^2-\omega_k^2+i\varepsilon}=\frac{1}{2\omega_k}\left[\frac{1}{\omega-(\omega_k-i\varepsilon)}-\frac{1}{\omega-(-\omega_k+i\varepsilon)}\right]+\mathcal{O}\left(\varepsilon^2\right). \tag{159}$$

Consider

$$I_1=\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{e^{-i\omega\tau}}{\omega-(\omega_k-i\varepsilon)}. \tag{160}$$

This has a pole at $\omega=\omega_k-i\varepsilon$, below the x-axis. As $e^{-i\omega\tau}=e^{\mathrm{Im}(\omega)\tau}e^{-i\,\mathrm{Re}(\omega)\tau}$, if $\tau<0$, we close the contour with a semicircle above the x-axis where $e^{\mathrm{Im}(\omega)\tau}\sim 0$ for large positive $\mathrm{Im}\,\omega$, and thus $I_1=0$.

If $\tau<0$, we close the contour below the x-axis, which contains the pole, and thus Cauchy's residue theorem gives us

$$I_1=-2\pi i e^{-i\omega_k\tau}\Theta\left(\tau\right)+\mathcal{O}\left(\varepsilon\right), \tag{161}$$

where the leading negative is there as the contour is clockwise.

Now consider

$$I_2=\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{e^{-i\omega\tau}}{\omega-(-\omega_k+i\varepsilon)}. \tag{162}$$

If $\tau<0$, we again close the contour above the x axis, which now contains the pole giving

$$I_2=2\pi i e^{i\omega_k\tau}\Theta\left(-\tau\right)+\mathcal{O}\left(\varepsilon\right). \tag{163}$$

If $\tau>0$, then the contour can be closed below without any poles implying $I_2=0$.

Therefore, gathering our intermediate steps, we see that collecting I_1 and I_2,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{e^{-i\omega\tau}}{\omega^2-\omega_k^2+i\varepsilon}=\lim_{\varepsilon\to 0}\frac{1}{2\omega_k}\left(I_1-I_2\right) \tag{164}$$

$$= \frac{1}{2\omega_k} \left(-2\pi i e^{-i\omega\tau} \Theta(\tau) - 2\pi i e^{i\omega_k \tau} \theta(-\tau) \right). \tag{165}$$

Returning this claim to the time ordering expression, we see

$$\langle 0 | T\phi(x) \phi(y) | 0 \rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{i}{2\pi} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{e^{-i\omega\tau}}{\omega^2 - \omega_k^2 + i\varepsilon}, \tag{166}$$

where the $\varepsilon \to 0$ limit is now implicit. Identifying $k^0 = \omega$ and $\tau = t$, this becomes

$$\langle 0 | T\phi(x) \phi(y) | 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)}, \tag{167}$$

as desired. □

There are a few comments of note to be made here.

1) Observe that time ordering is equivalently to choosing a contour that weaves between the poles such that one and only one contributes for any given x and y.

2) $\Delta_F(x-y)$ is Lorentz invariant.

3) Observe that

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)}, \tag{168}$$

we have that $k^2 \neq m^2$ here, namely, it is not *on shell*.

4) The i atop the propagator is important.

5) $\Delta_F(x-y)$ is a Green's function.

Observe that

$$\left(\partial^\mu \partial_\mu + m^2\right) \Delta_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 + m^2} \left(-k^2 + m^2\right) e^{-ik(x-y)} \tag{169}$$

$$= -\int \frac{\mathrm{d}^4 k}{(2\pi)^4} i e^{-ik(x-y)} \tag{170}$$

$$= -i\delta^4(x-y), \tag{171}$$

and thus $\Delta_F(x-y)$ is the Greens function associated to the Klein Gordon operator. Propagators are the kernel of the equations of motion.