The Standard Model

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1 Lecture: Introduction

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The standard model is based on the gauge group

$$G = U(1) \times SU(2) \times SU(3), \tag{1}$$

with $U(1)$ charge called hypercharge, $SU(2)$ mediating the weak force and $SU(3)$ mediating the strong force with electromagnetism hiding inside $U(1) \times SU(2)$.

These forces are coupled to 15 Weyl fermions that, collectively, we call the electron, neutrino and down quark. Moreover these particles have to come in a group of four. Then mysteriously, the pattern repeats twice over to give three generations. Each generation experiences exactly the same forces.

We have no explanation for these masses. It is tied up with how these particles interact with the Higgs boson.

Gen 1	electron	e-neutrino	down	up
mass	1	10^{-6}	9	4
Gen 2	muon	μ-neutrino	strange	charm
mass	207	10^{-6}	186	2495
Gen 3	tau	τ-neutrino	bottom	top
mass	3483	10^{-6}	8180	3×10^5
charge	-1	0	$-\frac{1}{3}$	$-\frac{2}{3}$

1.1 Symmetries

The structure of the standard model is in large part about its symmetries. This involves Lorentz symmetry, gauge symmetries, global symmetries as well as discrete symmetries.

Minkowski space $\mathbb{R}^{1,3}$ has metric $\eta_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$.

Lorentz transformations map $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ with $\Lambda \subset SO\,(1,3)$ such that

$$\Lambda^T \eta \Lambda = \eta. \tag{2}$$

We write this group element as

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}\right), \tag{3}$$

with $\mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}$ as the generators obeying the algebra commutation relations

$$[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = i\left(\eta^{\nu\rho}\mathcal{M}^{\mu\sigma} - \eta^{\nu\sigma}\mathcal{M}^{\mu\rho} + \eta^{\mu\sigma}\mathcal{M}^{\nu\rho} - \eta^{\mu\rho}\mathcal{M}^{\nu\sigma}\right). \tag{4}$$

Example. Observe we can take

$$\mathcal{M}^{01} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{M}^{12} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

1.2 Dirac vs Weyl Spinors

A Dirac spinor ψ is a 4-component object that transforms in the spinor representation of the Lorentz group. Recall the γ-matrices defined by $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We take the chiral representation in which

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{6}$$

where $\sigma^\mu = \left(\mathbb{I}_2, \sigma^i\right)$ and $\overline{\sigma}^\mu = \left(\mathbb{I}_2, -\sigma^i\right)$.

We construct Lorentz generators

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix}, \tag{7}$$

with $\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right)$ and $\overline{\sigma}^{\mu\nu} = \frac{i}{4}\left(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu\right)$.

These obey the Lorentz algebra such that

$$\left[\sigma^{\mu\nu}, \sigma^{\rho\sigma}\right] = i\left(\eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\nu\sigma}\sigma^{\mu\rho} + \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\mu\rho}\sigma^{\nu\sigma}\right), \tag{8}$$

and similarly for $\overline{\sigma}^{\mu\nu}$.

Note. This is not an irreducible representation of the Lorentz group. It is reducible as we have two block diagonal matrices that form it.

We then notice that as $S^{\mu\nu}$ is block diagonal, we can decompose

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}, \tag{9}$$

where ψ_L and ψ_R are 2-component *Weyl spinors*. These are irreducible and transform under Lorentz as

$$\psi_L \to S\psi_L \text{ with } S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \tag{10}$$

and similarly for ψ_R.

2 Lecture: Spinors

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Recall that

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix}\sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu}\end{pmatrix}, \tag{11}$$

where the matrix on the right holds in the chiral representation. As this is block diagonal, both of the 2×2 block matrices also form representations of the Lorentz group.

Note. These two representations are inequivalent. However, if one complex conjugates a spinor, its handedness flips. This follows as

$$\varepsilon^{-1}\left(\sigma^{\mu\nu}\right)^*\varepsilon = \overline{\sigma}^{\mu\nu}, \tag{12}$$

for a similarity matrix

$$\varepsilon = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}. \tag{13}$$

We notice that we can form a scalar from two left handed spinors (or two right handed

$$\psi_L\chi_L \equiv \varepsilon^{\alpha\beta}\left(\psi_L\right)_\beta\left(\chi_L\right)_\alpha \tag{14}$$

$$= \psi_{L2}\chi_{L1} - \psi_{L1}\chi_{L2}. \tag{15}$$

This is a scalar as

$$\psi_L\chi_L \to \varepsilon^{\alpha\beta}S_\alpha{}^\gamma S_\beta{}^\delta\left(\psi_L\right)_\delta\left(\chi_L\right)_\gamma \tag{16}$$

$$
\begin{aligned}
&= \det S \varepsilon^{\gamma\delta} \left(\psi_L\right)_\delta \left(\chi_L\right)_\gamma && (17)\\
&= \det S \psi_L \chi_L, && (18)
\end{aligned}
$$

where $\det S = 1$ implies this is a scalar.

You can then check that

$$
\varepsilon^T \left(\sigma^{\mu\nu}\right)^* \varepsilon = \overline{\sigma}^{\mu\nu}. \tag{19}
$$

Note. In QFT, spinors are anti-commuting so

$$
\begin{aligned}
\psi_L \chi_L &= \psi_{L2}\chi_{L1} - \psi_{L1}\chi_{L2} && (20)\\
&= -\chi_{L1}\phi_{L2} + \chi_{L2}\psi_{L1} && (21)\\
&= \chi_{L1}\phi_L, && (22)
\end{aligned}
$$

In particular, $\psi_L\psi_L = 2\psi_{L2}\psi_{L1} \neq 0$.

2.1 Actions

A Dirac spinor has the action

$$
\begin{aligned}
S_{\text{Dirac}} &= -\int \mathrm{d}^4x \left(i\overline{\psi}\gamma^\mu \partial_\mu \psi - M\overline{\psi}\psi\right) && (23)\\
&= -\int \mathrm{d}^4x \left(i\overline{\psi}_L \overline{\sigma}^\mu \partial_\mu \psi_L + i\overline{\psi}_R \sigma^\mu \partial_\mu \psi_R - M\left(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\right)\right), && (24)
\end{aligned}
$$

where $\overline{\psi} = \psi^\dagger\gamma^0$ and $M \in \mathbb{R}$. Note that $\overline{\psi}_L = \psi_L^\dagger$ and thus the mass term has two left-handed and two right handed Weyl spinors as the handedness is changed by conjugation.

Note. M is called a **Dirac mass**. When $M = 0$, the action has $U(1)^2$ global symmetry. When $M \neq 0$, this is just a $U(1)$.

We can also write down an action for a single Weyl fermion

$$
S_{\text{Weyl}} = -\int \mathrm{d}^4x \left(i\overline{\psi}_L \overline{\sigma}^\mu \partial_\mu \psi_L + \frac{1}{2} m \psi_L \psi_L + \frac{1}{2} m^* \overline{\psi}_L \overline{\psi}_L\right), \tag{25}
$$

for $m \in \mathbb{C}$. This is called a *Majorana mass*. It breaks the $U(1)$ symmetry and so is forbidden if the $U(1)$ is gauged.

2.2 Gauge Invariance

In Maxwell, we have gauge transformations $A_\mu + \partial_\mu \alpha$, with field strength

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{26}
$$

which is gauge invariant. The action is

$$
S = -\frac{1}{4} \int \mathrm{d}^4x \, F_{\mu\nu} F^{\mu\nu}. \tag{27}
$$

This has equation of motion $\partial_\mu F^{\mu\nu} = 0$ and the Bianchi identity

$$\partial_\mu * F^{\mu\nu} = 0, \tag{28}$$

where $*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\tau}F_{\rho\sigma}$.

Complex scalar fields of charge e transform as

$$\phi(x) \to e^{ie\alpha(x)}\phi(x). \tag{29}$$

We define the covariant derivative to be

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - ieA_\mu\phi, \tag{30}$$

and observe that under gauge transformation it picks up only a phase

$$\mathcal{D}_\mu\phi \to e^{ie\alpha}\mathcal{D}_\mu\phi. \tag{31}$$

Proof. Observe that with $A_\mu \to A_\mu + \partial_\mu\alpha(x)$, we see

$$\begin{aligned}
\mathcal{D}_\mu\phi &\to \left(\partial_\mu - ieA_\mu - ie\partial_\mu\alpha(x)\right)\left(e^{ie\alpha(x)}\phi\right) && (32)\\
&= e^{ie\alpha(x)}\left(\partial_\mu + ie\partial_\mu\alpha(x) - ieA_\mu - ie\partial_\mu\alpha(x)\right)\phi && (33)\\
&= e^{ie\alpha(x)}\left(\partial_\mu - ieA_\mu\right)\phi && (34)\\
&= e^{ie\alpha(x)}\mathcal{D}_\mu\phi, && (35)
\end{aligned}$$

which is transformation *covariantly* as desired. □

Then we have the action

$$S = \int \mathrm{d}^4x\, \mathcal{D}_\mu\phi^\dagger\mathcal{D}^\mu\phi - V(|\phi|), \tag{36}$$

is *gauge invariant*.

2.3 Yang-Mills Theory

Yang Mills is the extension of Maxwell theory from $G = U(1)$ to an arbitrary simple, compact Lie group G whose algebra has Hermitian generators $T^A = \left(T^A\right)^\dagger$ obeying

$$\left[T^A, T^B\right] = if^{ABC}T^C, \tag{37}$$

with structure constants f^{ABC}.

We will need only $G = SU(N)$ here. The generators in the fundamental representation are $N \times N$ matrices T^A such that $\operatorname{tr}\left(T^A\right) = 0$ and

$$\operatorname{Tr}\left(T^AT^B\right) = \frac{1}{2}\delta^{AB}. \tag{38}$$

Example. For $G = SU(2)$, we have the Pauli matrices $T^A = \frac{1}{2}\sigma^A$ for $A \in \{1, 2, 3\}$.

We have a gauge field A_μ^A for each generator of G. We write

$$A_\mu = A_\mu^A T^A. \tag{39}$$

This is a Lie-algebra valued field (namely, an $N \times N$ matrix).

The gauge symmetry is associated to $\Omega(x) \in G$ under which

$$A_\mu \mapsto \Omega A_\mu \Omega^{-1} + \frac{i}{g} \Omega \partial_\mu \Omega^{-1}, \tag{40}$$

where g is the coupling constant like e in Maxwell theory.

To compare to Maxwell, we write $\Omega(x) = e^{ig\alpha(x)}$ to find

$$\Omega\ A_\mu \Omega^{-1} + \frac{i}{g} \Omega \partial_\mu \Omega^{-1} = A_\mu + \partial_\mu \alpha(x), \tag{41}$$

as before, just now with spatial dependence $\alpha(x)$.

3 Lecture: Discrete Symmetries

28/01/2025

Recall the Yang-Mills gauge field A_μ is an $N \times N$ matrix given by

$$A_\mu = A_\mu^A T^A, \tag{42}$$

for $A = 1, \cdots, \dim G$. The gauge transformation is $A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{i}{g} \Omega \partial_\mu \Omega^{-1}$ where it is parameterized by $\Omega(x) \in G$ and g is the coupling constant.

The field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]. \tag{43}$$

One can check that this transforms as $F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1}$.

Proof.

□

The Yang-Mills action

$$S = -\frac{1}{2} \int \mathrm{d}^4 x \, \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right), \tag{44}$$

where $F^{\mu\nu} = \left(F^A\right)^{\mu\nu} T^A$ and $\mathrm{Tr}\left(T^A T^B\right) = \frac{1}{2} \delta^{AB}$.

The equation of motion is

$$\mathcal{D}_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig\left[A_\mu, F^{\mu\nu}\right] = 0. \tag{45}$$

Proof.

□

We also have the Bianchi identity $\mathcal{D}_\mu \star F^{\mu\nu} = 0$. These are non-linear equations.

Matter transforms in some representation of G. We write the generators as $T^A\left(R\right)^a{}_b$ for $A = 1, \cdots, \dim G$ and $a, b = 1, \cdots, \dim R$. Then under a gauge transformation,

$$\phi^a \to \Omega\left(R\right)^a{}_b\,\phi^b, \tag{46}$$

with $\Omega\left(R\right) = e^{ig\alpha^A T^A(R)}$.

We introduce the covariant derivative

$$\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a - igA^A_\mu T^A\left(R\right)^a{}_b\,\phi^b, \tag{47}$$

which transforms as

$$\mathcal{D}_\mu \phi^a \to \Omega\left(R\right)^a{}_b\,\mathcal{D}_\mu \phi^b. \tag{48}$$

In the Standard Model, all matter fields live in the fundamental representation.

3.1 Discrete Symmetries

We want to know how parity, charge conjugation and time reversal act.

Parity is an inversion of space, $P : (t, \vec{x}) \mapsto (t, -\vec{x})$. Naturally, one asks how do fields transform under a parity transformation?

The gauge field sits in $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ and we have $\partial_0 \to \partial_0$ and $\partial_i \to -\partial_i$, so we must have

$$P : A_0\left(\vec{x}, t\right) \to A_0\left(t, -\vec{x}\right) \qquad\qquad P : A_i\left(\vec{x}, t\right) \to -A_i\left(t, -\vec{x}\right). \tag{49}$$

Then $E_i = F_{0i}$ and $B_i = \frac{1}{2}\varepsilon_{ijk}F^{jk}$ transform as

$$P : \vec{E}\left(t, \vec{x}\right) \to -\vec{E}\left(t, -\vec{x}\right) \text{ which is a vector,} \quad P : \vec{B}\left(t, \vec{x}\right) \to \vec{B}\left(t, -\vec{x}\right) \text{ which is a pseudovector.} \tag{50}$$

Spinors are more subtle. Massless Weyl spinors obey

$$\overline{\sigma}^\mu \partial_\mu \psi_L = 0 \qquad\qquad \sigma^\mu \partial_\mu \psi_R. \tag{51}$$

These equations turn into each other under parity and thus a single Weyl fermion is not parity invariant. We need a pair such that

$$P : \psi_L\left(t, \vec{x}\right) \mapsto \psi_R\left(t, -\vec{x}\right) \tag{52}$$
$$P : \psi_R\left(t, \vec{x}\right) \mapsto \psi_L\left(t, -\vec{x}\right). \tag{53}$$

One could have $\pm$ signs here or generically a phase. In terms of a Dirac fermion, $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ we have

$$P\psi\left(t, \vec{x}\right) \mapsto \gamma^0 \psi\left(t, -\vec{x}\right), \tag{54}$$

for $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Charge conjugation exchanges particles and anti-particles. On scalars it acts as

$$C : \phi \mapsto \pm\phi^\dagger. \tag{55}$$

As we have $\mathcal{D}_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$,

$$\mathcal{D}_\mu\phi^\dagger = \partial_\mu\phi^\dagger + ieA_\mu\phi^\dagger, \tag{56}$$

tells us that we must have $C : A_\mu \mapsto -A_\mu$ (or for Yang-Mills, $C : A_\mu \mapsto -A_\mu^\dagger$).

Therefore $C : \vec{E} \mapsto -\vec{E}$ and identically for B.

For spinors, the Dirac equation is

$$i\gamma^\mu \left(\partial_\mu - ieA_\mu\right) - M\psi = 0. \tag{57}$$

Where taking the complex conjugate gives

$$-i\left(\gamma^\mu\right)^* \left(\partial_\mu + ieA_\mu\right)\psi^* - M\psi^*. \tag{58}$$

Suppose that $C : \psi \mapsto C\psi^*$, for some 4×4 matrix C.

Under charge conjugation, the Dirac equation becomes

$$i\gamma^\mu \left(\partial_\mu + ieA_\mu\right) C\psi^* - MC\psi^* = 0 \tag{59}$$

$$\Rightarrow iC^{-1}\gamma^\mu C \left(\partial_\mu + ieA_\mu\right)\psi^* - M\psi^* = 0. \tag{60}$$

Then comparing to the conjugated Dirac equation, we see that we need $C^{-1}\gamma^\mu C = -\left(\gamma^\mu\right)^*$. In the chiral representation, we have $C = \pm i\gamma^2$ achieves this. In terms of Weyl spinors, then we see

$$C : \psi_L \mapsto \pm i\sigma^2\psi_R^* \tag{61}$$

$$C : \psi_R \mapsto \mp i\sigma^2\psi_L^*. \tag{62}$$

Therefore a theory of a single Weyl fermion is not invariant under C either. But it can be invariant under CP acting as

$$CP : \psi_L\left(t, \vec{x}\right) \mapsto \mp i\sigma^2\psi_L^*\left(t, -\vec{x}\right). \tag{63}$$

4 Lecture: CPT

30/01/2025

4.1 Time Reversal

Time reversal acts as $T : (t, \vec{x}) \mapsto (-t, \vec{x})$. In quantum mechanics, time reversal is an *anti-unitary* operator satisfying for $\alpha \in \mathbb{C}$

$$T : \alpha\left|\psi\right\rangle \mapsto \alpha^* T\left|\psi\right\rangle. \tag{64}$$

This follows from the Schrödinger equation, $i\partial_t\Psi = -\boldsymbol{\nabla}^2\Psi + V\Psi$. Comparing this to the heat equation, $\partial_t T = \boldsymbol{\nabla}^2 T$, ones sees that this is crucially not invariant under time reversal while the Schrödinger equation is under time reversal and complex conjugation, $\Psi\left(t\right) \mapsto \Psi^*\left(-t\right)$.

For us, the take away is that time reversal complex conjugates complex numbers.

Example. We look at $D_\mu = \partial_\mu - ieA_\mu$. Under T as $\partial_0 \to -\partial_0$ and $\partial_i \to \partial_i$ but $i \mapsto -i$ so

$$T : A_0 (t, \vec{x}) \mapsto A_0 (-t, \vec{x}) \tag{65}$$
$$T : A_i (t, \vec{x}) \mapsto -A_i (-t, \vec{x}) \,. \tag{66}$$

These imply

$$T : \vec{E} (t, \vec{x}) \mapsto \vec{E} (-t, \vec{x}) \tag{67}$$
$$T : \vec{B} (t, \vec{x}) \mapsto -\vec{B} (-t, \vec{x}) \,. \tag{68}$$

For a Dirac spinor, obeying the Dirac equation $i\gamma^\mu (\partial_\mu - ieA_\mu) \psi - M\psi = 0$, we take

$$T : \psi (t, \vec{x}) \, \Theta \psi (-t, x) \,, \tag{69}$$

where Θ is some 4×4 matrix.

Acting with T on the Dirac equation we get

$$-i \left(- \left(\gamma^0\right)^* D_0 + \left(\gamma^i\right)^* D_i \right) \Theta \phi - M \Theta \psi = 0 \tag{70}$$
$$\Rightarrow i \Theta^{-1} \left(\left(\gamma^0\right)^* D_0 - \left(\gamma^i\right)^* D_i \right) \Theta \psi - M \psi = 0. \tag{71}$$

This gives back the Dirac equation if

$$\Theta^{-1} \left(\gamma^0\right)^* \Theta = \gamma^0 \qquad\qquad \Theta^{-1} \left(\gamma^i\right)^* \Theta = -\gamma^i. \tag{72}$$

In the chiral basis, $\Theta = \gamma^1 \gamma^3$ achieves this. Acting on Weyl spinors,

$$T : \psi_L (t, \vec{x}) \mapsto i\sigma^2 \psi_L (-t, \vec{x}) \tag{73}$$
$$T : \psi_R (t, \vec{x}) \mapsto -i\sigma^2 \psi_R (-t, \vec{x}) \,. \tag{74}$$

Note. Time reversal does not flip handedness unlike parity and charge conjugation.

There are a number of observable consequences of a theory not respecting each of these symmetries.

4.2 CPT

Theories can break C, P or T and we will see that the standard model breaks all of these individually. However, any Lorentz invariant unitary quantum field theory must be invariant under the combination CPT.

We can check that this holds for all fermion bilinears, $\overline{\psi}\psi$, $\overline{\psi}\gamma^5\psi$, $\overline{\psi}\gamma^\mu\psi$ and $\overline{\psi}\gamma^\mu\gamma^5\psi$.

Example. For $CPT : \psi (x) \mapsto -\gamma^5 \psi^* (-x)$ with $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then

$$\overline{\psi}\psi = \overline{\psi}\gamma^0\psi \mapsto \psi^T \gamma^5 \gamma^5 \gamma^0 \gamma^5 \psi^* \tag{75}$$
$$= -\psi^T \gamma^0 \psi^* \tag{76}$$
$$= +\psi^\dagger \gamma^0 \psi = \overline{\psi}\psi, \tag{77}$$

where to arrive at the last line we reordered the fermions.

5 Lecture: Broken Symmetries

01/02/2025

Definition 5.1: A symmetry is said to be *spontaneously broken* when the theory is invariant, but the ground state is not.

Consider the classical system with action

$$S = \int \mathrm{d}t \left(\frac{1}{2} \dot{\phi}^2 - V(\phi) \right), \tag{78}$$

with $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4$.

This has a $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. For $m^2 > 0$, $V(\phi)$ has a single minima at $\phi = 0$ that is preserved by the symmetry $\phi \to -\phi$.

If $m^2 < 0$, then we have two distinct minima at $\phi \equiv \pm v = \pm \sqrt{\frac{-m^2}{\lambda}}$. If one is in one ground state, then the action of the symmetry takes you to the other ground state. Thus, the symmetry is spontaneously broken.

Generally, a $\mathbb{Z}_2$ or discrete symmetry implies that there are multiple degenerate ground states.

One can write the potential as

$$V(\phi) = \frac{1}{4} \lambda \left(\phi^2 - v^2 \right)^2 + \text{const.} \tag{79}$$

and observe immediately that it has two ground states. Writing $\phi(t) = v + \sigma(t)$. Implies

$$V(\sigma) = \lambda \left(v^2 \sigma^2 + v \sigma^3 + \frac{1}{4} \sigma^4 \right). \tag{80}$$

There is no hint of the discrete symmetry here (locally around v) due to the σ^3 term breaking the symmetry.

In quantum mechanics, there is no spontaneous symmetry breaking. As all energy eigenstates are also eigenstates of the generator of $\mathbb{Z}_2$ symmetry,

One can prove that the ground state never crosses the axis and the nth excited state crosses the axis n times.

Take $\langle \pm v |$ to be states sharply peaked in respective shells. Then in the Euclidean path integral

$$\langle v | e^{-H\tau} | -v \rangle = \int \mathcal{D}\phi e^{-S_E[\phi]}, \tag{81}$$

with $S_E[\phi] = \int \mathrm{d}\tau \left(\frac{1}{2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \right)^2 + V(\phi) \right)$. In the saddle point approximation, this is dominated by the classical solution with

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\tau^2} = \lambda \phi \left(\phi^2 - v^2 \right). \tag{82}$$

This is solved by

$$\phi_{\text{classical}}(\tau) = v \tanh \left(\sqrt{\frac{\lambda v^2}{2}} \tau \right). \tag{83}$$

We can evaluate the action of this solution and see

$$S_{\text{classical}} = \int_{-\infty}^{\infty} \mathrm{d}\tau \left(\frac{1}{\varepsilon} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 + \lambda \left(\phi_{\text{classical}}^2 - v^2 \right)^2 \right) = \int_{-\infty}^{\infty} \mathrm{d}\tau \, \frac{\lambda v^4}{2} \frac{1}{\cosh^4 \left(\frac{\sqrt{\lambda v^2}}{2\tau} \right)} = \frac{2}{3} \sqrt{2\lambda} \mathbf{v^3}. \tag{84}$$

We expect $\lim_{\tau \to \infty} \langle v | e^{-H\tau} | -v \rangle = K e^{-S_{\text{classical}}}$ with $K \in \mathbb{R}$ constant. Thus $S_{\text{classical}} \ll 1$ gives us a very small tunelling rate as expected.

Further, one can show $E_{\text{excited}} - E_{\text{ground}} \sim \sqrt{\lambda v^2} e^{-S_{\text{classical}}}$.

Thus while there is no spontaneous symmetry breaking in quantum mechanics, it reappears in quantum field theory. Consider

$$S = \int \mathrm{d}^4 x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right), \tag{85}$$

with $V(\phi) = \frac{1}{4} \lambda \left(\phi^2 - v^2 \right)^2$.

Now there are two ground states, $|\pm v\rangle$. There is no longer tunnelling as the path integral with Euclidean action $S_E[\phi] = \int \mathrm{d}\tau \, \mathrm{d}^3 x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right)$, has the same classical solution,

$$\phi_{\text{classical}}(, \tau, \vec{x}) = v \tanh \left(\sqrt{\frac{\lambda v}{2}} \tau \right), \tag{86}$$

which crucially has no $\vec{x}$ dependence. Then

$$\begin{aligned} S_E &= \int \mathrm{d}\tau \, \mathrm{d}^3 x \left(\frac{1}{2} \partial_\mu \phi_{\mathrm{c}} \partial^\mu \phi_{\mathrm{c}} \right) + V(\phi_{\mathrm{c}}) && (87) \\ &= \mathcal{V} S_{\text{classical}}, && (88) \end{aligned}$$

where $\mathcal{V}$ is the volume of space which diverges in $\mathbb{R}^3$. Therefore as the path integral goes $e^{-\mathcal{V} S_{\text{classical}}} = 0$, tunnelling is suppressed. Note in a compact space it reemerges.

Note. Observe that $m^2 < 0$ tells us that there is an instability of the $\phi = 0$ vacuum.

Also, we can repurpose $\phi_{\text{classical}}$ as a domain wall in a Lorentzian signature. Namely, one finds

$$\phi(t, \vec{x}) = v \tanh \left(\sqrt{\frac{\lambda v}{2}} z \right), \tag{89}$$

which is a spatial domain wall at $z = 0$. This has finite energy density $\frac{2}{3} \sqrt{2\lambda} v^3$.

Last, a natural question is: why are $|\pm v\rangle$ physical and not a linear combination of them? The states $|\pm v\rangle$ alone satisfy cluster decomposition. Namely,

$$\langle \text{vac} | A(x) B(y) | \text{vac} \rangle = \langle \text{vac} | A(x) | \text{vac} \rangle \langle \text{vac} | B(y) | \text{vac} \rangle, \tag{90}$$

holds for $|\pm v\rangle$ but not linear combinations.

6 Lecture: Continuous Symmetries

04/02/2025

We saw that breaking a discrete symmetry gives a finite number of ground states. Breaking a continuous symmetry gives an infinite number of ground states.

In $D = 3 + 1$, consider the complex scalar field ϕ with action

$$S = \int \mathrm{d}^4 x \left(\partial_\mu \phi^\dagger \partial^\mu \phi - V\left(\phi, \phi^\dagger\right)\right), \tag{91}$$

with $V = m^2 |\phi| + \frac{1}{2}\lambda |\phi|^4$.

This theory has a $U(1)$ symmetry given by $\phi \mapsto e^{i\alpha}\phi$.

When $m^2 > 0$, then we get a parabolic cone potential. This has minimum at $\phi = 0$ as expected and the symmetry is unbroken.

If $m^2 < 0$, then we get a Mexican hat potential with minima at $|\phi|^2 = v^2 = -\frac{m^2}{\lambda}$. Thus the symmetry is broken in this theory.

> **Definition 6.1:** The *vacuum manifold* $\mathcal{M}_0$ is the space of constant field configurations such that $V(\phi) = V_{\min}$.

Here $\mathcal{M}_0 = S^1$.

Here we decompose $\phi(x) = r(x) e^{i\Theta(x)}$ and see that

$$S = \int \mathrm{d}^4 x \left(\partial_\mu r \partial^\mu r + r^2 \partial_\mu \theta \partial^\mu \theta - \lambda\left(r^2 - v^2\right)^2\right). \tag{92}$$

The different vacua are labelled by constant θ. Fluctuations of $\Theta(x)$ are massless. This is the **Goldstone boson**.

We write $r = v + \sigma(x)$ and read off the mass of the σ field to be $M_\sigma^2 = 2\lambda v^2$. For energy $E \ll M_\sigma$, we can focus on the $\Theta(x)$ field with $\mathcal{L} = r^2 \partial_\mu \Theta \partial^\mu \phi$.

6.1 The $O(N)$ model

Consider N real scalar fields $\phi^a(x)$ for $a = 1, \cdots, N$ with

$$S = \int \mathrm{d}^4 x \left(\frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - V(\phi)\right), \tag{93}$$

with $V(\phi) = \frac{1}{2}m^2 \phi^a \phi^a + \frac{\lambda}{4}\left(\phi^a \phi^a\right)^2$.

When $m^2 < 0$, $V_{\min}$ sits at $\phi^a \phi^a = v^2 = -\frac{m^2}{\lambda}$. We see this is $\mathcal{M}_0 = S^{N-1}$.

Pick a point in $\mathcal{M}_0$, say $\phi = (0, 0, \cdots, 0, v)$. We write

$$\phi^a(x) = \left(\pi^1(x), \pi^2(x), \cdots, v + \sigma(x)\right). \tag{94}$$

Plugging this into the action, we find

$$S = \int \mathrm{d}^4 x \left(\frac{1}{2}\partial_\mu \pi^i \partial^\mu \pi^i + \partial_\mu \sigma \partial^\mu \sigma - V(\sigma, \pi)\right), \tag{95}$$

and $V = \lambda v^2\sigma^2 + \lambda v\sigma\left(\sigma^2 + \pi^i\pi^i\right) + \frac{1}{4}\lambda\left(\pi^i\pi^i + \sigma^2\right)^2$ where $i = 0, \cdots, N-1$.

Observe σ has mass $M = \sqrt{2\lambda v^2}$ but π^i are massless. These are $N-1$ Goldstone bosons.

One can ask again what happens if we are at energies $E \ll M$? We can ignore σ as before. We insist we remain on the vacuum manifold due to lack of energy to excite out of it, namely,

$$\left(\pi^a\left(x\right)\right)^2 + \left(\phi^N\left(x\right)\right)^2 = v^2. \tag{96}$$

We can then eliminate ϕ^N to get

$$S = \int \mathrm{d}^4x\,\frac{1}{2}\left(\partial_\mu\pi^i\partial^\mu\pi^i + \frac{\left(\pi^i\partial_\mu\pi^i\right)\left(\pi^j\partial^\mu\pi^j\right)}{v^2 - \pi^k\pi^k}\right), \tag{97}$$

where there is implicit sum over i, j and k.

Proof. □

Example. Take $N = 3$. We have $M_0 = S^2$. We can write

$$\pi^1 = v\sin\theta\cos\phi \tag{98}$$
$$\pi^2 = v\sin\theta\sin\phi \tag{99}$$
$$\varphi^3 = v\cos\theta. \tag{100}$$

We get

$$S = \int \mathrm{d}^4x\,\frac{v^2}{2}\left(\partial_\mu\Theta\partial^\mu\Theta + \sin^2\theta\partial_\mu\phi\partial^\mu\phi\right). \tag{101}$$

In general, we have

$$S = \int \mathrm{d}^4x\,\frac{1}{2}g_{ab}\left(\pi\right)\partial_\mu\pi^a\partial^\mu\pi^b. \tag{102}$$

6.2 Goldstone's Theorem (Classical

Consider a theory with scalar ϕ which transforms under a global symmetry G as $\phi \mapsto g\phi$, $g \in G$. The vacuum manifold

$$\mathcal{M}_0 = \{\phi_0 \mid V\left(\phi_0\right) = V_{\min}\}. \tag{103}$$

If $\phi_0 = 0$ then G is unbroken. If $\mathcal{M}_0$ is not a single point, assume that if $\phi_0, \phi_0' \in \mathcal{M}$, then $\phi_0' = g\phi_0$ for some $g \in G$.

The *stability group* H is $H = \{h \in G \mid h\phi_0 = \phi_0\}$.

Note. If H' is the stability group for $\phi_0' = g\phi 0$, then for each $h \in H$, we have $h' = ghg^{-1} \in H' \Rightarrow H \cong H'$.

We say G is spontaneously broken to H and write $G \to H$. We have

$$\mathcal{M}_0 = G/H. \tag{104}$$

Namely, $\mathcal{M}_0$ is the space of equivalence classes $g_1 \sim g_2$ if $g_1 = hg_2$ for some $h \in H$.

Example. If $G = O(N)$, it is broken to $H = O(N-1)$. Thus $\mathcal{M}_0 = O(N)/O(N-1) = S^{N-1}$ as we saw.

Theorem 6.1 (Goldstone's theorem): If $G \to H$, then the number of massless Goldstone bosons is $\dim(G/H) = \dim G - \dim H$.

Example. $\dim(O(N)) = \frac{1}{2}N(N-1)$ and thus $\dim O(N) - \dim O(N-1) = N - 1$.