The Standard Model

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1 Lecture: Introduction

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The standard model is based on the gauge group

$$G = U(1) \times SU(2) \times SU(3), \tag{1}$$

with $U(1)$ charge called hypercharge, $SU(2)$ mediating the weak force and $SU(3)$ mediating the strong force with electromagnetism hiding inside $U(1) \times SU(2)$.

These forces are coupled to 15 Weyl fermions that, collectively, we call the electron, neutrino and down quark. Moreover these particles have to come in a group of four. Then mysteriously, the pattern repeats twice over to give three generations. Each generation experiences exactly the same forces.

Gen 1	electron	e-neutrino	down	up
mass	1	10^{-6}	9	4
Gen 2	muon	μ-neutrino	strange	charm
mass	207	10^{-6}	186	2495
Gen 3	tau	τ-neutrino	bottom	top
mass	3483	10^{-6}	8180	3×10^5
charge	-1	0	$-\frac{1}{3}$	$-\frac{2}{3}$

We have no explanation for these masses. It is tied up with how these particles interact with the Higgs boson.

1.1 Symmetries

The structure of the standard model is in large part about its symmetries. This involves Lorentz symmetry, gauge symmetries, global symmetries as well as discrete symmetries.

Minkowski space $\mathbb{R}^{1,3}$ has metric $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.

Lorentz transformations map $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ with $\Lambda \subset SO\,(1,3)$ such that

$$\Lambda^T \eta \Lambda = \eta. \tag{2}$$

We write this group element as

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}\right), \tag{3}$$

with $\mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}$ as the generators obeying the algebra commutation relations

$$[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = i\left(\eta^{\nu\rho}\mathcal{M}^{\mu\sigma} - \eta^{\nu\sigma}\mathcal{M}^{\mu\rho} + \eta^{\mu\sigma}\mathcal{M}^{\nu\rho} - \eta^{\mu\rho}\mathcal{M}^{\nu\sigma}\right). \tag{4}$$

Example. Observe we can take

$$\mathcal{M}^{01} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{M}^{12} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

1.2 Dirac vs Weyl Spinors

A Dirac spinor ψ is a 4-component object that transforms in the spinor representation of the Lorentz group. Recall the γ-matrices defined by $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We take the chiral representation in which

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \tag{6}$$

where $\sigma^\mu = \left(\mathbb{I}_2, \sigma^i\right)$ and $\overline{\sigma}^\mu = \left(\mathbb{I}_2, -\sigma^i\right)$.

We construct Lorentz generators

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix}, \tag{7}$$

with $\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right)$ and $\overline{\sigma}^{\mu\nu} = \frac{i}{4}\left(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu\right)$.

These obey the Lorentz algebra such that

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i\left(\eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\nu\sigma}\sigma^{\mu\rho} + \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\mu\rho}\sigma^{\nu\sigma}\right), \tag{8}$$

and similarly for $\overline{\sigma}^{\mu\nu}$.

Note. This is not an irreducible representation of the Lorentz group. It is reducible as we have two block diagonal matrices that form it.

We then notice that as $S^{\mu\nu}$ is block diagonal, we can decompose

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{9}$$

where ψ_L and ψ_R are 2-component *Weyl spinors*. These are irreducible and transform under Lorentz as

$$\psi_L \to S\psi_L \text{ with } S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \tag{10}$$

and similarly for ψ_R.

2 Lecture: Spinors

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Recall that

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix}, \tag{11}$$

where the matrix on the right holds in the chiral representation. As this is block diagonal, both of the 2×2 block matrices also form representations of the Lorentz group.

Note. These two representations are inequivalent. However, if one complex conjugates a spinor, its handedness flips. This follows as

$$\varepsilon^{-1}\left(\sigma^{\mu\nu}\right)^* \varepsilon = \overline{\sigma}^{\mu\nu}, \tag{12}$$

for a similarity matrix

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{13}$$

We notice that we can form a scalar from two left handed spinors (or two right handed

$$\psi_L\chi_L \equiv \varepsilon^{\alpha\beta}\left(\psi_L\right)_\beta \left(\chi_L\right)_\alpha \tag{14}$$
$$= \psi_{L2}\chi_{L1} - \psi_{L1}\chi_{L2}. \tag{15}$$

This is a scalar as

$$\psi_L\chi_L \to \varepsilon^{\alpha\beta} S_\alpha{}^\gamma S_\beta{}^\delta \left(\psi_L\right)_\delta \left(\chi_L\right)_\gamma \tag{16}$$
$$= \det S \varepsilon^{\gamma\delta}\left(\psi_L\right)_\delta \left(\chi_L\right)_\gamma \tag{17}$$
$$= \det S \psi_L\chi_L, \tag{18}$$

where $\det S = 1$ implies this is a scalar.

You can then check that

$$\varepsilon^T\left(\sigma^{\mu\nu}\right)^* \varepsilon = \overline{\sigma}^{\mu\nu}. \tag{19}$$

Note. In QFT, spinors are anti-commuting so

$$\psi_L\chi_L = \psi_{L2}\chi_{L1} - \psi_{L1}\chi_{L2} \tag{20}$$
$$= -\chi_{L1}\phi_{L2} + \chi_{L2}\psi_{L1} \tag{21}$$
$$= \chi_{L1}\phi_L, \tag{22}$$

In particular, $\psi_L\psi_L = 2\psi_{L2}\psi_{L1} \neq 0$.

2.1 Actions

A Dirac spinor has the action

$$S_{\text{Dirac}} = -\int \mathrm{d}^4x \left(i\overline{\psi}\gamma^\mu\partial_\mu\psi - M\overline{\psi}\psi\right) \tag{23}$$

$$= -\int \mathrm{d}^4x \left(i\overline{\psi}_L\overline{\sigma}^\mu\partial_\mu\psi_L + i\overline{\psi}_R\sigma^\mu\partial_\mu\psi_R - M\left(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\right)\right), \tag{24}$$

where $\overline{\psi} = \psi^\dagger\gamma^0$ and $M \in \mathbb{R}$. Note that $\overline{\psi}_L = \psi_L^\dagger$ and thus the mass term has two left-handed and two right handed Weyl spinors as the handedness is changed by conjugation.

Note. M is called a **Dirac mass**. When $M = 0$, the action has $U(1)^2$ global symmetry. When $M \neq 0$, this is just a $U(1)$.

We can also write down an action for a single Weyl fermion

$$S_{\text{Weyl}} = -\int \mathrm{d}^4x \left(i\overline{\psi}_L\overline{\sigma}^\mu\partial_\mu\psi_L + \frac{1}{2}m\psi_L\psi_L + \frac{1}{2}m^*\overline{\psi}_L\overline{\psi}_L\right), \tag{25}$$

for $m \in \mathbb{C}$. This is called a *Majorana mass*. It breaks the $U(1)$ symmetry and so is forbidden if the $U(1)$ is gauged.

2.2 Gauge Invariance

In Maxwell, we have gauge transformations $A_\mu + \partial_\mu\alpha$, with field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{26}$$

which is gauge invariant. The action is

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}. \tag{27}$$

This has equation of motion $\partial_\mu F^{\mu\nu} = 0$ and the Bianchi identity

$$\partial_\mu * F^{\mu\nu} = 0, \tag{28}$$

where $*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\tau}F_{\rho\sigma}$.

Complex scalar fields of charge e transform as

$$\phi(x) \to e^{ie\alpha(x)}\phi(x). \tag{29}$$

We define the covariant derivative to be

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - ieA_\mu\phi, \tag{30}$$

and observe that under gauge transformation it picks up only a phase

$$\mathcal{D}_\mu\phi \to e^{ie\alpha}\mathcal{D}_\mu\phi. \tag{31}$$

Proof. Observe that with $A_\mu \to A_\mu + \partial_\mu \alpha(x)$, we see

$$\begin{aligned}\mathcal{D}_\mu \phi &\to \left(\partial_\mu - ieA_\mu - ie\partial_\mu \alpha(x)\right)\left(e^{ie\alpha(x)}\phi\right) &(32)\\ &= e^{ie\alpha(x)}\left(\partial_\mu + ie\partial_\mu\alpha(x) - ieA_\mu - ie\partial_\mu\alpha(x)\right)\phi &(33)\\ &= e^{ie\alpha(x)}\left(\partial_\mu - ieA_\mu\right)\phi &(34)\\ &= e^{ie\alpha(x)}\mathcal{D}_\mu\phi, &(35)\end{aligned}$$

which is transformation *covariantly* as desired. □

Then we have the action

$$S = \int \mathrm{d}^4x\, \mathcal{D}_\mu\phi^\dagger \mathcal{D}^\mu\phi - V(|\phi|), \tag{36}$$

is *gauge invariant.*

2.3 Yang-Mills Theory

Yang Mills is the extension of Maxwell theory from $G = U(1)$ to an arbitrary simple, compact Lie group G whose algebra has Hermitian generators $T^A = \left(T^A\right)^\dagger$ obeying

$$\left[T^A, T^B\right] = if^{ABC}T^C, \tag{37}$$

with structure constants f^{ABC}.

We will need only $G = SU(N)$ here. The generators in the fundamental representation are $N \times N$ matrices T^A such that $\mathrm{tr}\left(T^A\right) = 0$ and

$$\mathrm{Tr}\left(T^AT^B\right) = \frac{1}{2}\delta^{AB}. \tag{38}$$

Example. For $G = SU(2)$, we have the Pauli matrices $T^A = \frac{1}{2}\sigma^A$ for $A \in \{1, 2, 3\}$.

We have a gauge field A_μ^A for each generator of G. We write

$$A_\mu = A_\mu^A T^A. \tag{39}$$

This is a Lie-algebra valued field (namely, an $N \times N$ matrix).

The gauge symmetry is associated to $\Omega(x) \in G$ under which

$$A_\mu \mapsto \Omega A_\mu \Omega^{-1} + \frac{i}{g}\Omega\partial_\mu\Omega^{-1}, \tag{40}$$

where g is the coupling constant like e in Maxwell theory.

To compare to Maxwell, we write $\Omega(x) = e^{ig\alpha(x)}$ to find

$$\Omega\ A_\mu\Omega^{-1} + \frac{i}{g}\Omega\partial_\mu\Omega^{-1} = A_\mu + \partial_\mu\alpha(x), \tag{41}$$

as before, just now with spatial dependence $\alpha(x)$.

3 Lecture: Discrete Symmetries

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Recall the Yang-Mills gauge field A_μ is an $N \times N$ matrix given by

$$A_\mu = A_\mu^A T^A, \tag{42}$$

for $A = 1, \cdots, \dim G$. The gauge transformation is $A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{i}{g} \Omega \partial_\mu \Omega^{-1}$ where it is parameterized by $\Omega(x) \in G$ and g is the coupling constant.

The field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]. \tag{43}$$

One can check that this transforms as $F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1}$.

Proof.

□

The Yang-Mills action

$$S = -\frac{1}{2} \int \mathrm{d}^4 x \, \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right), \tag{44}$$

where $F^{\mu\nu} = \left(F^A\right)^{\mu\nu} T^A$ and $\mathrm{Tr}\left(T^A T^B\right) = \frac{1}{2} \delta^{AB}$.

The equation of motion is

$$\mathcal{D}_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig [A_\mu, F^{\mu\nu}] = 0. \tag{45}$$

Proof.

□

We also have the Bianchi identity $\mathcal{D}_\mu \star F^{\mu\nu} = 0$. These are non-linear equations.

Matter transforms in some representation of G. We write the generators as $T^A(R)^a{}_b$ for $A = 1, \cdots, \dim G$ and $a, b = 1, \cdots, \dim R$. Then under a gauge transformation,

$$\phi^a \to \Omega(R)^a{}_b \, \phi^b, \tag{46}$$

with $\Omega(R) = e^{ig\alpha^A T^A(R)}$.

We introduce the covariant derivative

$$\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a - ig A_\mu^A T^A (R)^a{}_b \, \phi^b, \tag{47}$$

which transforms as

$$\mathcal{D}_\mu \phi^a \to \Omega(R)^a{}_b \, \mathcal{D}_\mu \phi^b. \tag{48}$$

In the Standard Model, all matter fields live in the fundamental representation.

3.1 Discrete Symmetries

We want to know how parity, charge conjugation and time reversal act.

Parity is an inversion of space, $P : (t, \vec{x}) \mapsto (t, -\vec{x})$. Naturally, one asks how do fields transform under a parity transformation?

The gauge field sits in $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ and we have $\partial_0 \to \partial_0$ and $\partial_i \to -\partial_i$, so we must have

$$P : A_0 (\vec{x}, t) \to A_0 (t, -\vec{x}) \qquad\qquad P : A_i (\vec{x}, t) \to -A_i (t, -\vec{x}) . \tag{49}$$

Then $E_i = F_{0i}$ and $B_i = \frac{1}{2}\varepsilon_{ijk}F^{jk}$ transform as

$$P : \vec{E} (t, \vec{x}) \to -\vec{E} (t, -\vec{x}) \text{ which is a vector,} \quad P : \vec{B} (t, \vec{x}) \to \vec{B} (t, -\vec{x}) \text{ which is a pseudovector.} \tag{50}$$

Spinors are more subtle. Massless Weyl spinors obey

$$\overline{\sigma}^\mu \partial_\mu \psi_L = 0 \qquad\qquad \sigma^\mu \partial_\mu \psi_R . \tag{51}$$

These equations turn into each other under parity and thus a single Weyl fermion is not parity invariant. We need a pair such that

$$P : \psi_L (t, \vec{x}) \mapsto \psi_R (t, -\vec{x}) \tag{52}$$
$$P : \psi_R (t, \vec{x}) \mapsto \psi_L (t, -\vec{x}) . \tag{53}$$

One could have $\pm$ signs here or generically a phase. In terms of a Dirac fermion, $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ we have

$$P\psi (t, \vec{x}) \mapsto \gamma^0 \psi (t, -\vec{x}) , \tag{54}$$

for $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Charge conjugation exchanges particles and anti-particles. On scalars it acts as

$$C : \phi \mapsto \pm\phi^\dagger . \tag{55}$$

As we have $\mathcal{D}_\mu \phi = \partial_\mu \phi - ieA_\mu \phi$,

$$\mathcal{D}_\mu \phi^\dagger = \partial_\mu \phi^\dagger + ieA_\mu \phi^\dagger , \tag{56}$$

tells us that we must have $C : A_\mu \mapsto -A_\mu$ (or for Yang-Mills, $C : A_\mu \mapsto -A_\mu^\dagger$).

Therefore $C : \vec{E} \mapsto -\vec{E}$ and identically for B.

For spinors, the Dirac equation is

$$i\gamma^\mu (\partial_\mu - ieA_\mu) - M\psi = 0. \tag{57}$$

Where taking the complex conjugate gives

$$-i\left(\gamma^{\mu}\right)^{*}\left(\partial_{\mu}+ieA_{\mu}\right)\psi^{*}-M\psi^{*}. \tag{58}$$

Suppose that $C:\psi\mapsto C\psi^{*}$, for some 4×4 matrix C.

Under charge conjugation, the Dirac equation becomes

$$i\gamma^{\mu}\left(\partial_{\mu}+ieA_{\mu}\right)C\psi^{*}-MC\psi^{*}=0 \tag{59}$$

$$\Rightarrow iC^{-1}\gamma^{\mu}C\left(\partial_{\mu}+ieA_{\mu}\right)\psi^{*}-M\psi^{*}=0. \tag{60}$$

Then comparing to the conjugated Dirac equation, we see that we need $C^{-1}\gamma^{\mu}C=-\left(\gamma^{\mu}\right)^{*}$. In the chiral representation, we have $C=\pm i\gamma^{2}$ achieves this. In terms of Weyl spinors, then we see

$$C:\psi_{L}\mapsto\pm i\sigma^{2}\psi_{R}^{*} \tag{61}$$

$$C:\psi_{R}\mapsto\mp i\sigma^{2}\psi_{L}^{*}. \tag{62}$$

Therefore a theory of a single Weyl fermion is not invariant under C either. But it can be invariant under CP acting as

$$CP:\psi_{L}\left(t,\vec{x}\right)\mapsto\mp i\sigma^{2}\psi_{L}^{*}\left(t,-\vec{x}\right). \tag{63}$$