Statistical Field Theory

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1 Lecture: Introduction

Motivation

This course concerns itself with **universality**: the idea that different physical systems can exhibit the same behaviour.

For example, a liquid gas system has a critical temperature, below which a first order phase transition can be found between liquid and gas. Above it there is no such distinction and one can only move continuously.

Experiments suggest that as a function of the temperature, the density is given by $|\rho^{\pm} - \rho_c| \propto |T - T_C|^{\beta}$ with $\beta \approx 0.327$ for $T \approx T_C$.

Another example is ferromagnets for which one also has a **critical temperature** called the Curie temperature T_C.

For $T > T_C$, $M = 0$, and for $T \leq T_C$ $M \propto (T_C - T)^{\beta}$ where the critical exponent β is as observed for gases.

In this course we will be studying the **classical statistical mechanics of fields**.

1.1   The Ising Model

The Ising model is a simple model for a magnet. In d spatial dimensions, consider a lattice with N sites.

On the ith site we have a "spin" $S_i \in \{-1, 1\}$. The configuration $\{s_i\}$ has energy

$$E = -B \sum_i^{s_i} -J \sum_{\langle ij \rangle} s_i s_j, \tag{1}$$

where B represents an external magnetic field and J represents an interaction strength. Naturally, one should ask how the physics depends on B, J and the temperature T.

For:

- $J > 0$, the spins prefer to be aligned: $\uparrow\uparrow$ or $\downarrow\downarrow$. This is a **ferromagnet**.

- $J < 0$, the spins prefer to be anti-aligned: $\uparrow\downarrow$ or $\downarrow\uparrow$. This is an **anti-ferromagnet**.

  We will assume and fix $J > 0$.

- For $B > 0$ spins prefer $\uparrow$

- For $B < 0$ spins prefer $\downarrow$

- At low temperature, the system wants to minimise energy. This leads to an ordered state where all spins will align due to the lower energy preferenced state.

- At high temperature, entropy dominates which leads to a disordered state.

Recall that in the **canonical ensemble**, the probability of a given configuration $\{s_i\}$ is

$$p[s_i] = \frac{1}{Z} e^{-\beta E[s_i]}, \tag{2}$$

where $\beta = \frac{1}{T}$, $k_B = 1$ and Z is the **partition function**.

Definition 1.1: The **partition function** is given by

$$Z(T, B) = \sum_{\{s_i\}} e^{-\beta E[s_i]}. \tag{3}$$

Definition 1.2: The **thermodynamic free energy** $F_{\text{thermo}}(T, B) = \langle E \rangle - TS = -T \log Z$

Definition 1.3: The **magnetisation** is

$$m = \frac{1}{N} \left\langle \sum_{n=1}^{N} s_i \right\rangle \in [-1, 1]. \tag{4}$$

Note. The average $\langle X \rangle$ is an average over the probability distribution $p[s_i]$ at a fixed T.

Magnetisation is an **order parameter** as it can distinguish between ordered phases ($m \neq 0$) and disordered phases where $m \approx 0$. We can expand the expectation value in m such that

$$m = \sum_{\{s_i\}} \frac{1}{Z} e^{-\beta[E_i]} \frac{1}{N} \sum_i s_i = \frac{1}{N\beta} \frac{\partial}{\partial \beta} \log Z. \tag{5}$$

Therefore we want to compute the partition function as it will allow us to derive these observables.

This is easy in $d = 1$. This is hard in $d = 2$ and there is no general solution for generic lattices. For a square lattice with $B = 0$ it is possible and was famously solved by Onsager (winning him some famous prize). $d \geq 3$ are intractable analytically.

Our aim is to approximate in a way that correctly captures the long-distance behaviour.

We can define m for any $\{s_i\}$ by $m = \frac{1}{N} \sum s_i$, now no longer taking a statistical average. We can then write the partition function as

$$Z = \sum_m \sum_{\{s_i\}|m} e^{-\beta E[s_i]} =: \sum_m e^{-\beta F(m)}. \tag{6}$$

The spacing in allowed values of m is $\frac{2}{N}$. For $N \gg 1$ we can approximate m as continuous however and thus write the partition function as

$$Z \approx \frac{N}{2} \int_{-1}^{1} \mathrm{d}m \, e^{-\beta F(m)}. \tag{7}$$

We call $F(m)$ the **effective free energy** which depends on T, B and critically, m. This contains more information than F_{thermo}.

Let $f(m) = \frac{1}{N} F(m)$ and thus

$$Z \propto \int_{-1}^{1} \mathrm{d}m \, e^{-\beta N f(m)}. \tag{8}$$

For N large, $\beta f(m) \sim \mathcal{O}(1)$ as it is an **intensive** property (doesn't scale with the system). Performing a saddle point approximation, we can replace $f(m)$ by it's minimum where

$$\left. \frac{\partial f}{\partial m} \right|_{m=m_{\min}} = 0, \tag{9}$$

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which gives us a partition function of

$$Z \propto e^{-\beta N f(M_{\min})}, \tag{10}$$

which gives us a thermodynamic free energy of

$$F_{\text{thermo}}(T, B) \approx F(M_{\min}(T, B), T, B). \tag{11}$$

Even with the saddle point approximation, computing $F(m)$ is hard. One such approach is to use the **mean field approximation** where we replace each spin by the average of the field, $s_i \to m$. This gives

$$E = -\beta \sum_i m - J \sum_{\langle ij \rangle} m^2 = -BNm - \frac{1}{2} NJqm^2, \tag{12}$$

where q is the number of nearest neighbours. $q = 2d + 2$ for a square lattice in d dimensions.

Therefore in the mean field approximation, the partition function becomes

$$Z \approx \sum_m \Omega(m) e^{-\beta E[n]}, \tag{13}$$

where $\Omega(m)$ is the number of configurations with $\frac{1}{N} \sum s_i = m$.

2   Lecture: Phase Transitions

Let $N_\uparrow$ be the number of up spins and thus $N_\downarrow = N - N_\uparrow$ down spins. The magnetisation is therefore

$$m = \frac{N^\uparrow - N_\downarrow}{N} = \frac{2N^\uparrow - N}{N}. \tag{14}$$

The number of configurations with this magnetisation is

$$\Omega(m) = \frac{N!}{N_\uparrow!(N - N_\uparrow!)} = \frac{N}{N_\uparrow! N_\downarrow!}. \tag{15}$$

Using the Stirling approximation,

$$\log n! = n \log n - n, \tag{16}$$

we have

$$\log \Omega \approx N(\log N - 1) - N_\uparrow (\log N^\uparrow - 1) - N_\downarrow (\log N_\downarrow - 1), \tag{17}$$

and thus

$$\frac{\log \Omega}{N} \approx \log 2 - \frac{1}{2}(1 + m) \log(1 + m) - \frac{1}{2}(1 - m) \log(1 - m). \tag{18}$$

Therefore, we have in the mean field approximation that

$$e^{-\beta N f(m)} \approx \Omega(m) e^{-\beta E(m)} \tag{19}$$

$$\Rightarrow f(m) \approx -Bm - \frac{1}{2}Jqm^2 - T\left[\log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)\right]. \quad (20)$$

We now move to minimise f such that $\frac{\partial f}{\partial m} = 0$. This implies

$$\beta(B + Jqm) = \frac{1}{2}\log\left(\frac{1+m}{1-m}\right) \quad (21)$$

$$\Rightarrow m = \tanh\left[\beta\underbrace{(B + Jqm)}_{B_{\text{eff}}}\right]. \quad (22)$$

Each spin feels an *effective B* field $B_{\text{eff}}(m)$.

2.1   Landau Theory of phase transitions

At a phase transition some quantity, called the **order parameter**, is not smooth. For us this is m. For small m, we have

$$f(m) = -T\log 2 - Bm + \frac{1}{2}\left(T - \underbrace{Jq}_{T_C}\right)m^2 + \frac{1}{12}Tm^4 + \cdots. \quad (23)$$

Notice we have introduced the critical temperature, $T_C = Jq$.

The natural question is: In equilibrium at $m = m_{\min}$, how does this behave as we vary the temperature T or the external magnetic field B.

For $B = 0$ this reduces to

$$f(m) = \frac{1}{2}(T - T_C)m^2 + \frac{1}{2}Tm^4 + \cdots. \quad (24)$$

For $T > T_C$, $m_{\min} = 0$, and for $T < T_C$, $m_{\min} = \pm m_0 = \pm\sqrt{\frac{3(T_C - T)}{T}}$. This is valid for $m_0 \ll 1$ i.e. T close to T_C.

Plotting the minima as a function of T, we see a branching at $T = T_C$ and while continuous the function is not smooth and thus we have a phase transition.

- For $T > T_C : m = 0$ is called the *disordered phase*,

- For $T < T_C : m \neq 0$ is called the *ordered phase*.

This is an example of a *continuous phase transition*, or in modern language *second order phase transition*.

Notes.

- Notice that F is invariant under $\mathbb{Z}_2$ symmetry with simultaneous $m \rightarrow -m$ and $B \rightarrow -B$ (inherited from the Ising model's symmetry of $s_i \rightarrow -s_i$ and $B \rightarrow -B$).

- For $T < T_C$, either $m = +m_0$ or $m = -m_0$ so the $\mathbb{Z}_2$ symmetry of the system does not preserve the ground state. This is called **spontaneous symmetry breaking**.

- At finite N, Z is analytic in T and B. Therefore, phase transitions can only occur in the limit $N \to \infty$.

- Spontaneous symmetry breaking also only occurs for $N \to \infty$. Namely,

$$m = \lim_{B \to 0} \lim_{N \to \infty} \frac{1}{N} \sum \langle s_i \rangle, \tag{25}$$

if the order of these limits is reversed we would get $m = 0$ for all temperatures T. This follows from the $\mathbb{Z}_2$ symmetry as at finite N, $\mathbb{Z}_2$ symmetry implies Z is an even function in B and thus F_{thermo} is as well. Then,

$$\langle m \rangle = -\frac{1}{N} \frac{\partial F_{\text{thermo}}}{\partial B} \bigg|_{B=0} = 0, \tag{26}$$

as the derivative of an even function is an odd function which necessarily vanishes at zero.

Definition 2.1: The **heat capacity** is defined

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial}{\partial T} \left(-\frac{\partial \log Z}{\partial \beta} \right) = \beta^2 \frac{\partial^2 \log Z}{\partial \beta^2}, \tag{27}$$

This implies

$$\log Z = -\beta N f (m_{\min}) \tag{28}$$

$$= \begin{cases} \text{const}, & T > T_C, \\ \frac{3N}{4} \frac{(T_C - T)^2}{T^2} + \text{const}, & T < T_C. \end{cases} \tag{29}$$

It is more fruitful to work with the **specific heat capacity** given by

$$c = \frac{C}{N} = \begin{cases} 0, & T \to T_C \text{ from above}, \\ \frac{3}{2}, & T \to T_C \text{ from below}. \end{cases} \tag{30}$$

Therefore c is *discontinuous* at $T = T_C$.

So far our discussion has assumed $B = 0$ in the limit. We now work with finite $B > 0$. This gives us

$$f(m) \approx -Bm + \frac{1}{2} (T - T_C) m^2 + \frac{1}{12} T m^4 + \cdots . \tag{31}$$

For $T > T_C$ we have a concave curve with a single positive minima $m_{\min} \approx \frac{B}{T}$ for $T \to \infty$.

For $T < T_C$, we still have a double well with two local minima, but now they are no longer degenerate and there is a true global minimum. The other is referred to as a *metastable state*, as it may be stable on short time scales, but given sufficient time and perturbations, would decay to the true minimum $m_{\min}$.

As a function of T the global minimum $m_{\min}(T)$ is now a smooth function of temperature which implies we see no phase transition if T is varied at fixed $B \neq 0$.

However, we can now instead vary B at fixed $T < T_C$. For $B > 0$ we see the weighted double well as we did before and for $B = 0$ the degenerate double well. For $B < 0$ we see the global minima discontinuously shifts (from m_0 to $-m_0$) as B decreases from $B > 0$ to $B < 0$.

This is an example of a **discontinuous phase transition** which is also called a **first order phase transition** as our order parameter is the first derivative of $m_{\text{equil}} = -\frac{\partial f_{\text{thermo}}}{\partial B}$. As this is discontinuous, we call it a first order phase transition.

Therefore, as a function of B **and** T for $B = 0$ and $T < T_C$ we see a line of first order phase transitions. For $B = 0$ and $T = T_C$, this becomes a second order phase transition.

3 Lecture: Critical Exponents

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If one fixes $T = T_C$ we see

$$f \approx -Bm + \frac{1}{12}T_C m^4 + \cdots, \tag{32}$$

where minimizing leads to $m^3 \sim B \Rightarrow m \sim B^{\frac{1}{3}}$ for small m.

It is natural then to define $\frac{\partial m}{\partial B}$.

Definition 3.1: We call ξ the **magnetic susceptibility** and is given by

$$\xi = \left(\frac{\partial m}{\partial B} \right)_T. \tag{33}$$

For $T > T_C$, we see

$$f(m) \approx -Bm + \frac{1}{2}(T - T_C) m^2 + \cdots, \tag{34}$$

which implies

$$m \approx \frac{B}{T - T_C} \Rightarrow \xi = \frac{1}{T - T_C}. \tag{35}$$

For $T < T_C$, we write $m = m_0 + \delta m$ and solve for δm to leading order in B. This leads to

$$m = m_0 + \frac{B}{2(T_C - T)}, \tag{36}$$

which implies

$$\xi \approx \frac{1}{2(T_C - T)}, \tag{37}$$

and thus

$$\xi \sim \frac{1}{|T - T_C|}, \tag{38}$$

which diverges at T_C from both sides.

3.1 Validity of Mean Field Theory

Naturally, given all these approximations, one may ask if they, especially mean field theory, give correct results. This is a function of dimension it turns out.

- In $d = 1$, MFT is not valid as there are no phase transitions.

- In $d = 2, 3$, the phase diagram is qualitatively correct, but qualitative predictions at the critical point are incorrect.

- In $d \geq 4$, MFT both gives the correct phase diagram and qualitative predictions of behaviour.

This is similar for other systems: Mean field theory gets the phase structure wrong for $d \leq d_L$ for some *lower critical dimension* and mean field theory is correct for $d > d_c$, for some *upper critical dimension*. For Ising one has $d_c = 1$ and $d_L = 4$. For $d_c < d < d_L$ there is interesting disagreement.

3.2   Critical Exponents

The qualitative predictions that are often disagreeing in MFT are *critical exponents*. Near the critical point, MFT predicts if $B = 0$,

$$
\begin{cases}
m \sim (T_C - T)^{\beta}, & \beta = \frac{1}{2}, \text{ as } T \to T_C^-, \\
c \sim C_{\pm} |T - T_C|^{\alpha}, & \alpha = 0, \text{ as } T \to T_C^+, \\
\xi \sim |T - T_C|^{-\xi}, & \xi = 1.
\end{cases}
\tag{39}
$$

If $T = T_C$ as $B \to 0$, $m \sim B^{\frac{1}{\delta}}$ where $\delta = 3$. This behaviour is correct, and agrees with the exact solution of the Ising model in $d = 2$, but the values of $\alpha, \beta, \gamma, \delta$ are **not**.

Table 1: Critical Exponents

	MFT	$d = 2$	$d = 3$
α	0	0	0.1101
β	$\frac{1}{2}$	$\frac{1}{8}$	0.3264
γ	1	$\frac{7}{4}$	0.3264
δ	3	15	4.7898

The $d = 2$ data here comes from the exact solution to the Ising model and the $d = 3$ data come from simulations as there is no exact solution known.

There is a notion of universality here, in that for any normal material we see a similar Liquid-gas phase transition. Namely a line of first order phase transitions ending at a critical point with a second order phase transition.

The external magnetic field is instead the pressure $B \leftrightarrow p$ and our order parameter, which was magnetisation, is now $m \leftrightarrow v \equiv \frac{V}{N}$.

Similar to before, we have

$$
\begin{cases}
v_{\text{gas}} - v_{\text{liquid}} \sim (T_C - T)^{\beta}, & \beta = \frac{1}{2}, \text{ as } T \to T_C^- \text{ and } p = p_C, \\
v_{\text{gas}} - v_{\text{liquid}} \sim (p - p_C)^{\frac{1}{2}}, & \delta = 3, \text{ as } p \to p_C \text{ and at } T = T_C.
\end{cases}
\tag{40}
$$

3.3   Isothermal compressibility

One has

$$
\kappa = \frac{1}{v} \left(\frac{\partial v}{\partial p} \right)_T \sim \frac{1}{|T - T_C|^{\xi}},
\tag{41}
$$

where $\xi = 1$. The heat capacity is

$$C_V \sim C_\pm \left| T - T_C \right|^{-\alpha}, \tag{42}$$

where $\alpha = 0$. The predictions for $\alpha, \beta, \xi, \delta$ are the same as the MFT for the Ising model.

This is surprising and experiment verifies that the true values of these critical exponents are the same as the critical values for $d = 3$ Ising theory.

This is an example of **universality**: different physical systems can exhibit the same behaviour at the critical point. This suggests that the microscopic physics is unimportant at a critical point. Systems governed by the same critical point belong to the same **universality class**.

3.4   Landau-Ginzburg theory

Therefore, if we can describe one such theory in a universality class, we can describe them all. Therefore our aim becomes: find one model that correctly describes the *long-distance* physics near the critical point and use it to compute the critical exponents for all theories in the same universality class.

Landau-Ginzburg theory, also called *Landau theory* generalizes MFT to allow for spatial variations in m. $M(\mathbf{x})$ is now a field.

The field $M(\mathbf{x})$ is produced from a microscopic model by **course-graining**, where we average over scales small compared to the length scales we are interested in but large compared to the microscopic physics.

In the Ising model, for example, one can course grain the lattice over boxes of side length a each with $N' \ll N$ sites. We define

$$M(\mathbf{x}) = \frac{1}{N'} \sum \text{ spins in box with center } \mathbf{x}. \tag{43}$$

Take $N' \gg 1$ so disorderedness of $M(\mathbf{x})$ can be ignored, $M(\mathbf{x}) \in [-1, 1]$. We also assume $a \ll \xi$, the length scale over which physics varies (i.e. Temperature etc.).

We then treat $M(\mathbf{x})$ as a smooth function that doesn't rely on scales less than a. We then proceed as before.

The partition function can be written

$$Z = \sum_{M(\mathbf{x})} \sum_{\{s_i\}|M(\mathbf{x})} e^{-\beta E[s]} \tag{44}$$

$$= \sum_{M(\mathbf{x})} e^{-\beta F[M(\mathbf{x})]}, \tag{45}$$

where $F[M(\mathbf{x})]$ is a *functional* called the **Landau-Ginzburg free energy**.

Definition 3.2: We convert this sum over all possible functions to a **path integral**, written and defined such that

$$Z = \int \mathcal{D}[M(\mathbf{x})] e^{-\beta F[M(\mathbf{x})]}. \tag{46}$$

This is a *functional integral* which sums over all possible fields $M(\mathbf{x})$ that don't vary on scales less than a.

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4 Lecture: The path integral

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We can interpret this as the probability of a field configuration, $m(\mathbf{x})$,

$$P\left[m\left(\mathbf{x}\right)\right] = \frac{1}{Z}e^{-\beta F\left[m\left(\mathbf{x}\right)\right]}. \tag{47}$$

The form of $F\left[m\left(\mathbf{x}\right)\right]$ is constrained by the following

- *locality*, spins only influence nearby spins, which leads to

$$F\left[m\left(\mathbf{x}\right)\right] = \int \mathrm{d}^d x\, f\left(m\left(\mathbf{x}\right), \boldsymbol{\nabla}_i m\left(\mathbf{x}\right), \boldsymbol{\nabla}_i \boldsymbol{\nabla}_j m\left(\mathbf{x}\right), \cdots\right), \tag{48}$$

  i.e. not $\int \mathrm{d}^d x\, \mathrm{d}^d y\, f\left(m\left(\mathbf{x}\right), m\left(\mathbf{y}\right)\right)$ which would be a non-local interaction.

- *Translational symmetry*, which is inherited from the discrete translational symmetry of the lattice.

- *Rotational symmetry*, which also follows from the lattice however we strengthen this symmetry and assume it holds for *any* rotation.

- $\mathbb{Z}_2$ *symmetry*, $B = 0$ Ising model has $s_i \to -s_i$ and if $B \neq 0$, one can flip $s_i \to -s_i$ and $B \to -B$ which leaves the theory invariant. We then assume F is invariant $m\left(\mathbf{x}\right) \to -m\left(\mathbf{x}\right)$ and $B \to -B$.

- and finally, we assume *analyticity*: $F\left[m\left(\mathbf{x}\right)\right]$ defined by this course graining over a finite number of spins, suggests that F is going to be analytic which lets us Taylor expand it near $m = 0$.

By dimensional analysis, one can see that each successive derivative contributes less as we assumed that m only varies on a scale $\xi \gg a$. Therefore, for $B = 0$, Taylor expanding m,

$$F\left[m\left(\mathbf{x}\right)\right] = \int \mathrm{d}^d x \left(\frac{1}{2}\alpha_2\left(T\right)m^2 + \frac{1}{4}\alpha^4\left(T\right)m^4 + \frac{1}{2}\gamma\left(T\right)\left(\boldsymbol{\nabla}m\right)^2 + \cdots\right). \tag{49}$$

We could include $F_0\left(T\right)$ however this does not contribute a physically significant term in mean field theory. When $B \neq 0$ one includes odd powers of m.

The coefficients $\alpha_{2i}\left(T\right)$ are hard to compute from first principles, however from mean field theory we expect

$$\alpha_2\left(T\right) \sim T - T_C, \qquad\qquad \alpha_4\left(T\right) \sim \frac{1}{3}T, \tag{50}$$

however all we'll assume is that the coefficients are analytic in T, that $\alpha_2\left(T\right)$ has a simple zero (i.e. it crosses) at some $T = T_C$ and that $\gamma\left(T\right) > 0$.

We once again use a *saddle point approximation*. We assume the functional integral is dominated by a saddle point, i.e. some $m\left(\mathbf{x}\right)$ that minimises $F\left[m\left(\mathbf{x}\right)\right]$. To do this, we vary $m\left(\mathbf{x}\right) \to m\left(\mathbf{x}\right) + \delta m\left(\mathbf{x}\right)$, and inspect

$$\delta F = \int \mathrm{d}^d x \left(\alpha_2 m \delta m + \alpha_4 m^3 \delta m + \gamma \boldsymbol{\nabla} m \cdot \boldsymbol{\nabla} \delta m + \cdots\right), \tag{51}$$

where we use integration by parts to provide

$$\delta F = \int \mathrm{d}^d x \underbrace{\left(\alpha_2 m + \alpha_4 m^3 - \gamma \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} m \cdot + \cdots \right)}_{\frac{\delta F}{\delta m(\mathbf{x})}} \delta m. \tag{52}$$

If $m(\mathbf{x})$ minimises F, then $\delta F = 0$, $\forall \delta m$ and thus,

$$\frac{\delta F}{\delta m(\mathbf{x})} = 0 \Rightarrow \gamma \boldsymbol{\nabla}^2 m = \alpha_2 m + \alpha_4 m^3. \tag{53}$$

For $T > T_C$, $\alpha_2 > 0$ and thus $m = 0$.

For $T < T_C$, $\alpha_2 < 0$ and thus

$$m = \pm m_0 = \pm \sqrt{-\frac{\alpha_2}{\alpha_4}}. \tag{54}$$

Thus mean field theory arises as a saddle point approximation in Landau-Ginzburg theory.

4.1   Domain Walls

Take $T < T_C$ in which there are 2 ground states, $\pm m_0$. Thus, consider $m \to \pm m_0$ as $x \to \pm \infty$, namely we have two distinct domains partitioned by a plane at the origin.

Assume $m(\mathbf{x}) = m(x)$. Then, we have

$$\gamma \frac{\mathrm{d}^2 m}{\mathrm{d}x^2} = \alpha_2 m + \alpha_4 m^3. \tag{55}$$

This is solved by $m = m_0 \tanh\left(\frac{x-X}{W}\right)$ for $X \in \mathbb{R}$ and $W = \sqrt{-\frac{2\gamma}{\alpha_2}}$.

This describes the *domain wall* at $x = X$ of width W.

If our system has size L, then the free energy of $F[m_0] \propto L^d$. The cost of the domain wall is

$$\Delta F \equiv F[m(x)] - F(m_0) \tag{56}$$

$$\sim L^{d-1} \sqrt{\frac{-\gamma \alpha_2^3}{\alpha_4^2}}. \tag{57}$$

This scaling comes from the 'area' of the directions perpendicular to the domain wall, of which there are $d - 1$. Near the critical point $\alpha_2 \to 0 \Rightarrow W \to \infty$ and $\Delta F \to 0$.

Domain walls explain why the lower dimension is $d_\ell = 1$ for the Ising model. Let $d = 1$, and let $-\frac{1}{2}L < x < \frac{1}{2}L$, assuming $\alpha_2(T) < 0$. Assume boundary conditions

$$m\left(\pm \frac{1}{2}L\right) = m_0. \tag{58}$$

Recall the probability of a given configuration is $\frac{1}{Z} e^{-\beta F[m(\mathbf{x})]}$. Notice that we can also have two domain walls which define the boundaries of a region of spin $-m_0$.

This is an approximate saddle point. For well separated walls,

$$\Delta F_{2 \text{ walls}} \approx 2\Delta F. \tag{59}$$

Observe then that the relative probability of the two domain wall solution to the true ground state is

$$\frac{P\left(2 \text{ domain walls at } X, Y\right)}{P\left(m = m_0\right)} = e^{-2\beta\Delta F}, \tag{60}$$

where we integrate over all possible X and Y to get the relative probability of two domain walls anywhere such that

$$\frac{P\left(2 \text{ domain walls anywhere}\right)}{P\left(m = m_0\right)} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\mathrm{d}X}{W} \int_{X}^{\frac{L}{2}} \frac{\mathrm{d}Y}{W} e^{-2\beta\Delta F} \sim \left(\frac{L}{W}\right)^2 e^{-2\beta\Delta F}. \tag{61}$$

For $d > 1$, the exponential dominates as $L \to \infty$ and thus this tends to zero (energy beats entropy). In $d = 1$, however, this diverges as $L \to \infty$ as the 'entropy term' beats the energy term (which is independent of L in $d = 1$).

Therefore, in $d = 1$, it is much more probable to see two domain walls than $m = m_0$. Similarly, any region with $m \approx \pm m_0$ is unstable to formation of domain walls and thus the *ordered phase doesn't exist*!

5   Lecture: The Path Integral

We want to go beyond the saddle point approximation to calculate Z. To begin, we change our notation $m\left(\mathbf{x}\right) \to \phi\left(\mathbf{x}\right)$ and set $B = 0$. Then,

$$F\left[\phi\left(\mathbf{x}\right)\right] = \int \mathrm{d}^d x \left(\frac{1}{2}\alpha_2\left(T\right)\phi^2 + \frac{1}{4}\alpha_4\left(T\right)\phi^4 + \frac{1}{2}\gamma\left(T\right)\left(\boldsymbol{\nabla}\phi\right)^2 + \cdots\right). \tag{62}$$

Evaluating the path integral is easy if F is only quadratic in ϕ. It is still possible if higher order terms are small, and very hard if not impossible otherwise.

For $T > T_C$, let $\mu^2 = \alpha_2\left(T\right) > 0$. Consider a quadratic approximation to F. We have

$$F\left[\phi\left(\mathbf{x}\right)\right] = \frac{1}{2} \int \mathrm{d}^d x \left(\mu^2\phi^2 + \gamma\left(\boldsymbol{\nabla}\phi\right)^2\right). \tag{63}$$

For $T < T_C$, $\alpha_2\left(T\right) < 0$ which implies $\langle\phi\rangle = \pm m_0$. Let $\widetilde{\phi} = \phi - \langle\phi\rangle$. This implies

$$F = \underbrace{F\left[m_0\right]}_{\text{const}} + \frac{1}{2} \int \mathrm{d}^d x \left(\alpha_2'\left(T\right)\widetilde{\phi}^2 + \gamma\left(T\right)\left(\boldsymbol{\nabla}\widetilde{\phi}\right)^2 + \cdots\right), \tag{64}$$

where $\alpha_2' + 3m_0^2\alpha_4 = -2\alpha_2 > 0$. Therefore the quadratic approximation gives that the system acts as if it is above the critical temperature under $\phi \to \widetilde{\phi}$ and $\mu^2 = \alpha_2' = 2\left|\alpha_2\right| > 0$.

5.1 Thermodynamic free energy

We aim to compute corrections to F_{thermo} from fluctuations in $\phi(\mathbf{x})$. We ignore $F[m_0]$ and $F_0(T)$ here as they do not contribute with any physically significance).

The Fourier transform of the field is

$$\phi_{\mathbf{k}} = \int \mathrm{d}^d x \, e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x}), \tag{65}$$

where ϕ is real thus $\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$. We often call the *wavevector* $\mathbf{k}$ a *momentum* (as in qft one has $p = \hbar k$).

Recall that we assumed ϕ doesn't vary on scales less than a. Expressing this more formally, we can state that $\phi_{\mathbf{k}} = 0$, $\forall |k| > \Lambda$ where $\Lambda = \frac{\pi}{a}$ is called the *UV cutoff*.

Identically, the inverse Fourier transform is given by

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}. \tag{66}$$

If our system occupies a finite volume cubic region with $V = L^d$, then $\mathbf{k}$ now takes discrete values $\mathbf{k} = \frac{2\pi}{L}n$ where $\mathbf{n} \in \mathbb{Z}^d$. Therefore our Fourier transform becomes discrete with

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^d} = \left(\frac{1}{L}\right)^d \sum_{\mathbf{n}}, \tag{67}$$

and identically,

$$\phi(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot x} \phi_{\mathbf{k}}. \tag{68}$$

Substituting this expression into the free energy, we see

$$F[\phi_{\mathbf{k}}] = \frac{1}{2} \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{\mathrm{d}^d k_2}{(2\pi)^d} \left(\mu^2 - \gamma \mathbf{k}_1 \cdot \mathbf{k}_2\right) \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{x}}, \tag{69}$$

where we see that as the Fourier transform of the delta function is

$$\int \mathrm{d}^d x \, e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{x}} = (2\pi)^d (\mathbf{k}_1 + \mathbf{k}_2), \tag{70}$$

we have with $\mathbf{k} = \mathbf{k}_1$,

$$F = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\mu^2 + \gamma \mathbf{k}^2\right) \phi_{\mathbf{k}} \phi_{-\mathbf{k}}. \tag{71}$$

Using the reality condition of $\phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^*$, we can write this as

$$F = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\mu^2 + \gamma \mathbf{k}^2\right) |\phi_{\mathbf{k}}|^2, \tag{72}$$

where as this is an even function of each component, we can integrate only over the region in which $k_x > 0$ to give

$$F = \int_{k_x > 0} \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\mu^2 + \gamma \mathbf{k}^2 \right) |\phi_{\mathbf{k}}|^2 \tag{73}$$

$$= \frac{1}{V} \sum_{\mathbf{k}|k_x > 0} \left(\mu^2 + \gamma t^2 \right) |\phi_{\mathbf{k}}|^2 . \tag{74}$$

We define the measure

$$\int \mathcal{D}\phi\left(\mathbf{x}\right) = N \prod_{\mathbf{k}|k_x > 0} \int \mathrm{d}\operatorname{Re}\phi_{\mathbf{k}}\,\mathrm{d}\operatorname{Im}\phi_{\mathbf{k}} , \tag{75}$$

where N is some normalization constant.

Thus, the partition function can be written

$$Z = N \left(\Pi_{\mathbf{k}|k_x > 0} \right) \left(\prod_{\mathbf{k}|k_x > 0} \int \mathrm{d}\operatorname{Re}\phi_{\mathbf{k}}\,\mathrm{d}\operatorname{Im}\phi_{\mathbf{k}} \right) \exp\left[-\frac{\beta}{V} \sum_{\mathbf{k}|k_x > 0} \left(\mu^2 + \gamma \mathbf{k}^2 \right) |\phi_{\mathbf{k}}|^2 \right], \tag{76}$$

where pulling the sum out of the exponential we see

$$Z = N \left(\Pi_{\mathbf{k}|k_x > 0} \right) \left(\prod_{\mathbf{k}|k_x > 0} \int \mathrm{d}\operatorname{Re}\phi_{\mathbf{k}}\,\mathrm{d}\operatorname{Im}\phi_{\mathbf{k}} \exp\left[-\frac{\beta}{V} \left(\mu^2 + \gamma \mathbf{k}^2 \right) \left((\operatorname{Re}\phi_{\mathbf{k}})^2 + (\operatorname{Im}\phi_{\mathbf{k}})^2 \right) \right] \right). \tag{77}$$

Recall that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{x^2}{a}} = \sqrt{\pi a}. \tag{78}$$

Therefore we see that

$$Z = N \prod_{\mathbf{k}|k_x > 0} \left[\sqrt{\frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2}} \right] = N \prod_{\mathbf{k}|k_x > 0} \frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2}. \tag{79}$$

Therefore,

$$\frac{F_{\text{thermo}}}{V} = -\frac{T}{V} \log Z \tag{80}$$

$$= -\frac{T}{V} \sum_{\mathbf{k}|k_x > 0} \log\left(\frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2} \right) + \frac{T}{V} \log N. \tag{81}$$

Recall that this is a correction to the previous results. Therefore to compute the contribution of these fluctuations to the heat capacity, recall that

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} \tag{82}$$

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$$= \frac{\partial \left(\beta F_{\text{thermo}} \right)}{\partial \beta}, \tag{83}$$

and

$$C = \frac{\partial \langle E \rangle}{\partial T} \tag{84}$$

$$= -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} \tag{85}$$

$$= -\beta^2 \frac{\partial^2 \left(\beta F_{\text{thermo}} \right)}{\partial \beta^2}, \tag{86}$$

and thus

$$\frac{C}{V} = -\beta^2 \frac{\partial^2}{\partial \beta^2} \left[-\frac{1}{V} \sum_{\mathbf{k} | k_x > 0} \log \left(\frac{\pi V T}{\mu^2 + \gamma \mathbf{k}^2} \right) \right]. \tag{87}$$

Take $\mu^2 = T - T_C$ and $\gamma \in \mathbb{R}$ for simplicity and assume $T > T_C$. Then,

$$\frac{C}{V} = \frac{1}{V} \sum_{\mathbf{k} | k_x > 0} \left[1 + \frac{2T}{\mu^2 + \gamma \mathbf{k}^2} + \frac{T^2}{(\mu^2 + \gamma \mathbf{k}^2)^2} \right], \tag{88}$$

where converting back to the integral notation, we see that

$$\frac{C}{V} = \int_{k_x > 0} \frac{\mathrm{d}^d k}{(2\pi)^d} \left[1 + \frac{2T}{\mu^2 + \gamma \mathbf{k}^2} + \frac{T^2}{(\mu^2 + \gamma \mathbf{k}^2)^2} \right] \tag{89}$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[1 + \frac{2T}{\mu^2 + \gamma \mathbf{k}^2} + \frac{T^2}{(\mu^2 + \gamma \mathbf{k}^2)^2} \right], \tag{90}$$

the first term gives us $\frac{1}{2} k_0$ for every degree of freedom (as the equipartition theorem predicts). The other terms may diverge as $T \to T_C$ (i.e. as $\mu^2 \to 0$). The integral may not converge, and $\mathbf{k} = 0$ is an example of an *IR divergence* corresponding to a long wavelength/low energy. The final term in particular we see is proportional to

$$\int_0^\Lambda \frac{\mathrm{d}k \, k^{d-1}}{(\mu^2 + \gamma k^2)}, \tag{91}$$

where setting $k = \sqrt{\frac{\mu^2}{\gamma}} x$, as $T \to T_C$

$$\int_0^\Lambda \frac{\mathrm{d}k \, k^{d-1}}{(\mu^2 + \gamma k^2)} = \frac{\mu^{d-1}}{\gamma^{\frac{d}{2}}} \int_0^{\Lambda \sqrt{\frac{x}{\mu^2}}} \frac{\mathrm{d}x \, x^{d-1}}{(1 + x^2)^2} = \begin{cases} \Lambda^{d-4}, & d > 4, \\ \log \Lambda, & d = 4, \\ \mu^{d-4} & d < 4. \end{cases} \tag{92}$$

Similarly, the second term is proportional to

$$\int_0^\Lambda \frac{\mathrm{d}k \, k^{d-1}}{\mu^2 + \gamma k^2} \sim \begin{cases} \Lambda^{d-2}, & d > 2, \\ \log \Lambda, & d = 2, \\ \mu^{-1} d = 1. \end{cases} \tag{93}$$

Therefore the contributions of fluctuations is finite as $T \to T_C$ in $d \geq 4$, however for $d < 4$, these contributions diverge as here

$$c \sim \mu^{d-2} \sim |T - T_C|^{-\alpha}, \tag{94}$$

where $\alpha = 2 - \frac{d}{2}$. These fluctuations explain why the MFT value of $\alpha = 0$ is wrong, but this value is also wrong! The reason for this is that we can't neglect the ϕ^4 term.

6 Lecture

6.1 Correlation Functions

We know

$$\langle \phi\left(\mathbf{x}\right) \rangle = \begin{cases} 0, & T > T_C, \\ \pm m_0, & T < T_C \end{cases}. \tag{95}$$

Fluctuations about this mean are captured by correlation functions. Namely,

$$\langle \left(\phi\left(\mathbf{x}\right) - \langle \phi\left(\mathbf{x}\right)\rangle\right)\left(\phi\left(\mathbf{y}\right) - \langle \phi\left(\mathbf{y}\right)\rangle\right) \rangle = \langle \phi\left(\mathbf{x}\right) \phi\left(\mathbf{y}\right)\rangle - \langle \phi\left(\mathbf{x}\right)\rangle \langle \phi\left(\mathbf{y}\right)\rangle. \tag{96}$$

We can compute this by including a spatially varying magnetic field $B\left(\mathbf{x}\right)$ such that the free energy becomes $F\left[\phi, B\right]$, Namely,

$$F\left[\phi, B\right] = \int \mathrm{d}^d x \left(\frac{1}{2}\mu^2 \phi^2 + \frac{1}{2}\gamma \left(\boldsymbol{\nabla}\phi\right)^2 - B\left(\mathbf{x}\right)\phi\left(\mathbf{x}\right)\right). \tag{97}$$

Then, we see

$$Z\left[B\left(\mathbf{x}\right)\right] = \int \mathcal{D}\phi e^{-\beta F[\phi, B]}. \tag{98}$$

This functional contains lots of information. For example,

$$\frac{1}{\beta}\frac{\delta \log Z}{\delta B\left(\mathbf{x}\right)} = \frac{1}{\beta Z}\frac{\delta Z}{\delta B\left(\mathbf{x}\right)} = \frac{1}{2}\int \mathcal{D}\phi \phi\left(\mathbf{x}\right) e^{-\beta F[\phi, B]} = \langle \phi\left(\mathbf{x}\right)\rangle_B. \tag{99}$$

The second derivative similarly provides

$$\frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B\left(\mathbf{x}\right)\delta B\left(\mathbf{y}\right)} = \frac{1}{\beta^2 Z}\frac{\delta^2 Z}{\delta B\left(\mathbf{x}\right)\delta B\left(\mathbf{y}\right)} - \frac{1}{\beta^2 Z^2}\frac{\delta Z}{\delta B\left(\mathbf{x}\right)}\frac{\delta Z}{\delta B\left(\mathbf{y}\right)} \tag{100}$$

$$= \langle \phi\left(\mathbf{x}\right)\phi\left(\mathbf{y}\right)\rangle_B - \langle \phi\left(\mathbf{x}\right)\rangle_B \langle \phi\left(\mathbf{y}\right)\rangle_B. \tag{101}$$

Then if we set $B = 0$ after evaluating the functional derivatives, we see

$$\frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B\left(\mathbf{x}\right)\delta B\left(\mathbf{y}\right)}\bigg|_{B=0} = \langle \phi\left(\mathbf{x}\right)\phi\left(\mathbf{y}\right)\rangle - \langle \phi\rangle^2, \tag{102}$$

which is the correlation function we desired.

Now we move to calculate $Z[B(\mathbf{x})]$ in Fourier space with

$$F[\phi, B] = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{1}{2} \left(\mu^2 + \gamma \mathbf{k}^2 \right) |\phi_k|^2 - B_{-\mathbf{k}} \phi_{\mathbf{k}} \right]. \tag{103}$$

We complete the square by writing $\hat{\phi}_{\mathbf{k}} = \phi_{\mathbf{k}} - \frac{B_{\mathbf{k}}}{\mu^2 + \gamma \mathbf{k}^2}$ which gives us

$$F = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{1}{2} \left(\mu^2 + \gamma \mathbf{k}^2 \right) \left| \hat{\phi}_{\mathbf{k}} \right|^2 - \frac{1}{2} \frac{|B_{\mathbf{k}}|^2}{\mu^2 + \gamma \mathbf{k}^2} \right]. \tag{104}$$

Shifting $\phi_{\mathbf{k}} \to \hat{\phi}_{\mathbf{k}}$ in the path integral leaves the measure invariant as it still spans all possible functions. Thus, the partition function becomes

$$Z = N \int \prod_k \mathrm{d}\operatorname{Re}\hat{\phi}_{\mathbf{k}} \, \mathrm{d}\operatorname{Im}\hat{\phi}_{\mathbf{k}} \, e^{-\beta F} \tag{105}$$

$$= e^{-\beta F_{\text{thermo}}} \exp\left(\frac{\beta}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{|B_{\mathbf{k}}|^2}{\mu^2 + \gamma \mathbf{k}^2} \right) \tag{106}$$

$$= e^{-\beta F_{\text{thermo}}} \exp\left(\frac{\beta}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, B(\mathbf{x}) \, G(\mathbf{x} - \mathbf{y}) \, B(\mathbf{y}) \right), \tag{107}$$

where

$$G(\mathbf{x}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\mu^2 + \gamma \mathbf{k}^2}. \tag{108}$$

Thus, we arrive at the correlation function,

$$\langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle - \langle \phi \rangle^2 = \frac{1}{\beta} G(\mathbf{x} - \mathbf{y}). \tag{109}$$

Let $\xi^2 = \frac{\gamma}{\mu^2}$.

Claim. For $r = |x|$,

$$G(\mathbf{x}) \sim \begin{cases} \frac{1}{r^{d-2}}, & r \ll \xi, \\ \frac{e^{-\frac{r}{\xi}}}{\xi^{\frac{d-3}{2}} r^{\frac{d-1}{2}}}, & r \gg \xi. \end{cases} \tag{110}$$

> *Proof.*
>
> $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

This tells us that fluctuations occur over sizes $|x| \lesssim \xi$ and decay rapidly for $|x| > \xi$. In general, this property defines the **correlation length** ξ.

Claim. G is a **Green function** such that

$$\left(-\gamma \boldsymbol{\nabla}^2 + \mu^2 \right) G(\mathbf{x}) = \delta^{(d)}(\mathbf{x}). \tag{111}$$

Proof.

□

There is another critical exponent we can define. Namely, as

$$\xi \sim \frac{1}{|T - T_C|^{\nu}}, \tag{112}$$

which at the critical temperature diverges $\xi \to \infty$, we define η such that

$$\langle \phi(\mathbf{x}) \phi(0) \rangle \sim \frac{1}{r^{d-2+\eta}}. \tag{113}$$

Table 2: Predictions

MFT + Quadratic fluctuations	$d = 2$	$d = 3$		$d \geq 4$
η	0	$\frac{1}{4}$	0.0363	0
ν	$\frac{1}{2}$	1	0.63	$\frac{1}{2}$

6.2 Susceptibility

Recall that the susceptibility in MFT (without spatial variations) was how easily the mean field value is changed when the magnetic field is increased.

In more generality, we define

$$\chi(\mathbf{x}, \mathbf{y}) = \frac{\delta \langle \phi(\mathbf{x}) \rangle_B}{\delta B(\mathbf{y})} \tag{114}$$

$$= \frac{\delta}{\delta B(\mathbf{y})} \left[\frac{1}{\beta} \frac{\delta \log Z}{\delta B(\mathbf{x})} \right] \tag{115}$$

$$= \beta \left(\left\langle \phi(\mathbf{x}) \phi(\mathbf{y}) - \langle \phi \rangle^2 \right\rangle \right) \tag{116}$$

$$= G(\mathbf{x} - \mathbf{y}), \tag{117}$$

which motivates the intuition of interpreting χ as the variance in ϕ at $\mathbf{x}$ when we increase the magnetic field at $\mathbf{y}$.

If there is no spatial variation, then

$$\langle \phi \rangle = \frac{1}{V} \int d^d x \, \langle \phi(\mathbf{x}) \rangle \tag{118}$$

$$\delta \langle \phi \rangle = \frac{1}{V} \int d^d x \, d^d y \, \frac{\delta \langle \phi(\mathbf{x}) \rangle}{\delta B(\mathbf{y})} \delta B(\mathbf{y}) \tag{119}$$

$$= \frac{1}{V} \int d^d x \, d^d y \, \chi(\mathbf{x}, \mathbf{y}) \, \delta B \tag{120}$$

$$= \frac{1}{V} \int d^d x \, d^d y \, G(\mathbf{x} - \mathbf{y}) \tag{121}$$

$$= \int d^d x \, G(\mathbf{x}). \tag{122}$$

This diverges as $|x| \to \infty$ and $T \to T_C$. Taking

$$\xi \sim \int r^{d-1} \, \mathrm{d}r \, \frac{e^{-\frac{r}{\xi}}}{\xi^{\frac{d-3}{2}} r^{\frac{d-1}{2}}}, \tag{123}$$

where with $r = \xi y$, we see

$$\chi \sim \xi^2 \sim \frac{1}{|T - T_C|}. \tag{124}$$

Thus this predicts $\gamma = 1$ as before.

7   Lecture: The Renormalization Group

31/10/2024

The "Ginzburg criterion" states that mean field theory can't be trusted when fluctuations in ϕ are large compared to $\langle \phi \rangle \sim \pm m_0$.

Observe that the average fluctuation size can be approximated with

$$R = \frac{\displaystyle\int_{\substack{|y|<\xi \\ |x|<\xi}} \mathrm{d}^d x \, \mathrm{d}^d y \, \langle (\phi(\mathbf{x}) - \langle \phi(\mathbf{x}) \rangle)(\phi(\mathbf{y}) - \langle \phi(\mathbf{y}) \rangle) \rangle}{\displaystyle\int_{|x|<\xi} \mathrm{d}^d x \, \langle \phi(\mathbf{x}) \rangle \int_{|y|<\xi} \mathrm{d}^d y \, \langle \phi(\mathbf{y}) \rangle}, \tag{125}$$

which becomes

$$R \sim \frac{\displaystyle\int_{|x|<\xi,|y|<\xi} \frac{1}{\beta} G(\mathbf{x}') \, \mathrm{d}^d x' \, \mathrm{d}^d y'}{\left(\xi^d m_0\right)^2} \tag{126}$$

$$\sim \frac{\xi^d}{\xi^{2d} m_0^2} \int_0^\xi r^{d-1} \, \mathrm{d}r \, \frac{1}{r^{d-2}} \tag{127}$$

$$\sim \frac{\xi^{2-d}}{m_0^2} \tag{128}$$

$$\sim \frac{\left(|T - T_C|^{-\frac{1}{2}}\right)^{2-d}}{\left(|T - T_C|^{\frac{1}{2}}\right)^2} \tag{129}$$

$$\sim |T - T_C|^{\frac{d-1}{2}}, \tag{130}$$

thus we see that mean field theory will be bad as $T \to T_C$ for $d < 4 = d_c$, the upper critical dimension. However for $d \geq 4$, we see that mean field theory does work as the fluctuations are small.

8   The Renormalization Group

Now, we want to include non-quadratic terms in F such as ϕ^4.

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Recall that the free energy (if we rescale ϕ such that we can set $\gamma = 1$) is given by

$$F[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2} \left(\boldsymbol{\nabla} \phi \right)^2 + \frac{1}{2} \mu^2 \phi^2 + g\phi^4 + \cdots \right]. \tag{131}$$

We now allow all possible terms in F consistent with analyticity and symmetries. This essentially restricts us to terms of the form ϕ^{2n} or $\phi^5 \left(\boldsymbol{\nabla} \phi \right)^2 \boldsymbol{\nabla}^2 \phi$.

Terms like ϕ^{17} break the $\mathbb{Z}_2$ symmetry, and ϕ^{-2} ones break analyticity.

Each valid term has a coupling constant $\left(\mu^2, g, \cdots \right)$. We view these as coordinates in an infinite dimensional "theory space" where these coordinates parametrize all possible theories.

We further rescale ϕ such that it absorbs the leading $\frac{1}{\beta}$ in the path integral, so it now reads

$$Z = \int \mathcal{D}\phi e^{-F[\phi]}. \tag{132}$$

To make this well defined recall that we introduce a UV cut-off such that $\phi_{\mathbf{k}} = 0$, for $|\mathbf{k}| > \Lambda \sim \frac{1}{a}$ where a is the size of the boxes we course grain over.

Suppose we only care about the physics on some long distance scale L and thus don't care about ϕ_k for $|k| \gg L^{-1}$. We are then going to construct a new theory with a lower cutoff $\Lambda' = \frac{\Lambda}{\zeta}$ with $\zeta > 1$. This should work as long as $\Lambda' \gg L^{-1}$.

Thus, we write

$$\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^- + \phi_{\mathbf{k}}^+, \tag{133}$$

where

$$\phi_{\mathbf{k}}^- = \begin{cases} \phi_{\mathbf{k}}, & |\mathbf{k}| < \Lambda', \\ 0, & |\mathbf{k}| > \Lambda^-, \end{cases} \tag{134}$$

and similarly

$$\phi_{\mathbf{k}}^- = \begin{cases} \phi_{\mathbf{k}}, & \Lambda' < |\mathbf{k}| < \Lambda, \\ 0, & \text{otherwise}, \end{cases} \tag{135}$$

which are the short-wavelength (UV) modes we want to integrate out.

Notice that we can write

$$F[\phi_{\mathbf{k}}] = F_0 \left[\phi_{\mathbf{k}}^- \right] + F_0 \left[\phi_{\mathbf{k}}^+ \right] + F_I \left[\phi_{\mathbf{k}}^-, \phi_{\mathbf{k}}^+ \right], \tag{136}$$

where F_I is the interaction terms.

Then, the partition function/path integral becomes

$$Z = \int \prod_{|\mathbf{k}| < \Lambda} \mathrm{d}\phi_{\mathbf{k}}^- \, e^{-F[\phi_{\mathbf{k}}^-]} \int \prod_{\Lambda' < |\mathbf{k}| < \Lambda} \mathrm{d}\phi_{\mathbf{k}} \, e^{-F_0[\phi_{\mathbf{k}}^+] - F_I[\phi_{\mathbf{k}}^-, \phi_{\mathbf{k}}^+]} \tag{137}$$

$$= \int \prod_{|\mathbf{k}| < \Lambda'} \mathrm{d}\phi_{\mathbf{k}}^- \, e^{-F'[\phi_{\mathbf{k}}']}, \tag{138}$$

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where $F'\left[\phi_{\mathbf{k}}^-\right]$ is the *Wilsonian effective free energy* obtained by "integrating out" the UV modes to obtain a theory with a lower cutoff Λ'.

F' here must take the same general form as F since we included all possible terms,

$$F'\left[\phi\right] = \int \mathrm{d}^d x \left(\frac{1}{2}\gamma'\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}\mu'^2\phi^2 + g'\phi^4 + \cdots\right), \tag{139}$$

but the values of the coupling constants will differ. We want to compare this with the original theory, but they have different cut-offs. So we rescale $\mathbf{k}' = \zeta\mathbf{k}$, then $|\mathbf{k}'| < \xi\Lambda' = \Lambda$, or similarly, $\mathbf{x}' = \frac{\mathbf{x}}{\zeta}$ so $\mathbf{k}\cdot\mathbf{x} = \mathbf{k}'\cdot\mathbf{x}'$.

Therefore observe that

$$\int \mathrm{d}^d x \frac{1}{2}\gamma'\left(\boldsymbol{\nabla}\phi\right)^2 = \int \zeta^d \,\mathrm{d}^d x' \frac{1}{2}\gamma'\zeta^{-2}\left(\boldsymbol{\nabla}'\phi\right)^2. \tag{140}$$

We then rescale $\phi'\left(\mathbf{x}'\right) = \zeta^{\frac{d-2}{2}}\sqrt{\gamma'}\phi\left(\mathbf{x}\right)$, this gives us

$$F'\left[\phi'\right] = \int \mathrm{d}^d x' \left[\frac{1}{2}\left(\boldsymbol{\nabla}'\phi'\right)^2 + \frac{1}{2}\mu\left(\zeta\right)^2\phi'^2 + g\left(\zeta\right)\phi'^4\right], \tag{141}$$

which we can now compare to the original theory as it has the same cutoff and normalization of the first term ($\gamma = 1$). As ζ increases (i.e. as we decrease our coupling), we obtain a flow in theory space.

The map $R\left(\zeta\right): F \mapsto F'$ is a **renormalization group** (RG) transformation that obeys

$$R\left(\zeta_1\right)R\left(\zeta_2\right) = R\left(\zeta_1\zeta_2\right), \tag{142}$$

but $R\left(\zeta\right)$ is not invertible and thus is actually a *semigroup* not a group (it was given a bad name).

In summary there are three steps to the renormalization group:

1) Integrate out high momentum modes $\frac{\Lambda}{3} < |\mathbf{k}| < \Lambda$,

2) Rescale $\mathbf{k}' = \zeta\mathbf{k}$ and $\mathbf{x}' = \frac{x}{\zeta}$,

3) Rescale the fields to get canonically normalized gradient terms.

The natural question is how does RG flow behave as $\zeta \to \infty$? It is possible that the flow diverges in theory space, i.e. some or all coefficients diverge. It is also possible for the flow to approach a fixed point. These are of incredible importance.

One might also expect the flow to approach a limit cycle or wander aimlessly, but these do not happen.

Let's focus on a fixed point. After step 2, $\xi' = \frac{\xi}{\zeta}$. However, at a fixed point, $\xi' = \xi$. The only solution is that $\xi = 0$ or ∞.

In the disordered phase, $T > T_C$, $\xi = 0$ in the limit $T \to \infty$.

In the ordered phase, $T < T_C$, $\xi = 0$ in the limit $T \to 0$.

These are **trivial fixed points**. We are interested in the case for which $\xi = \infty$, which occurs at a critical point of a continuous phase transition.

9   Lecture: Scaling Dimension

05/11/2024

All points on RG trajectories have the same long-distance physics, controlled by the fixed point. Namely, we have different microscopic theories with the same long-distance behaviour. This is an example of universality.

As $\xi' = \xi/\zeta = \infty$ as $\zeta \to \infty$, we must have that $\zeta = \infty$ for all points on the trajectory approaching the fixed point.

At a fixed point, we want to classify couplings.

- If we turn on a (combination of) couplings and RG flow takes us back to the fixed point, this coupling is said to be **irrelevant**. These don't change the long distance physics.

- If we do the same and RG flow takes us away from the fixed point, this coupling is called **relevant**.

- If we had a line (or surface) of fixed points, then RG flowing along the line is called a **marginal** coupling. These are very rare.

Typically there are just a few relevant directions/couplings and infinitely many irrelevant directions. The basin of attraction of a fixed point is called its **critical surface**.

The **codimension** of this surface is the number of relevant deformations about this fixed point.

All theories on the critical surface flow to the fixed point and therefore have the same long-distance behaviour. This is why universality exists.

9.1   Scaling

With dimensional analysis, we can measure dimensions in units of inverse length. As $[x] = -1$, $\left[\frac{\partial}{\partial x}\right] = 1$, $[F] = 0$ such that e^{-F} makes sense and thus as $F[\phi] = \int \mathrm{d}^d x \left(\boldsymbol{\nabla}\phi\right)^2 + \cdots$, we have $[\phi] = \frac{d-2}{2}$.

At the critical point,

$$\langle \phi(\mathbf{x})\, \phi(\mathbf{0}) \rangle \sim \frac{1}{r^{d-2+\eta}}, \tag{143}$$

where dimensional analysis tells us $\eta = 0$, which experiments contradict. The reason for this is that there is another length scale present. Making this dimensionally constant, we see

$$\langle \phi(\mathbf{x})\, \phi(\mathbf{0}) \rangle \sim \frac{a^2}{r^{d-2+\eta}}. \tag{144}$$

$\eta \neq 0$ can come from rescaling the field (Step 3) in the RG procedure: $\phi'(\mathbf{x}') = \zeta^{\Delta_\phi} \phi(\mathbf{x})$.

Δ_ϕ is the **scaling dimension** of ϕ and $\frac{d-2}{2}$ is sometimes called the "naive dimension". Using the scaling dimension, we see that at the critical point

$$\langle \phi'(\mathbf{x}')\, \phi'(0) \rangle = \zeta^{2\Delta_\phi} \langle \phi(\mathbf{x})\, \phi(0) \rangle \sim \frac{\zeta^{2\Delta_\phi}}{r^{d-2+\eta}}. \tag{145}$$

But as the critical point is a fixed point, we have

$$\frac{1}{r'^{\,d-2+\eta}} = \frac{\zeta^{d-2+\eta}}{r^{d-2+\eta}}, \tag{146}$$

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and thus

$$\Delta_\phi = \frac{d-2}{2} + \frac{\eta}{2}, \tag{147}$$

where $\frac{\eta}{2}$ is called the **anomalous dimension**.

Near the fixed point, as $\xi' = \xi/\zeta \Rightarrow \Delta_\xi = -1$. We previously found $\xi \sim t^{-\nu}$ with t being the *reduced temperature*

$$t = \frac{|T - T_C|}{T_C}, \tag{148}$$

where $t' = \zeta^{\Delta_t} t$ changes under the RG flow. We can then deduce $\Delta_t = \frac{1}{\nu}$.

One might expect there to be relations between these critical exponents. Indeed, writing

$$F_{\text{thermo}}(t) = \int d^d x\, f(t), \tag{149}$$

notice that the RG flow does not change the value of $F_{\text{thermo}} = -\log Z$, namely, it does not change the physics.

Therefore

$$F'_{\text{thermo}}(t') = F_{\text{thermo}}(t). \tag{150}$$

At a fixed point, $f' \equiv f$, therefore near the fixed point,

$$\int d^d x'\, f'(t') = \int d^d x\, \zeta^{-d} f\left(\zeta^{\Delta_t} t\right). \tag{151}$$

This implies we have $f(t) \sim t^{\frac{d}{\Delta_t}} = t^{d\nu}$.

With spins correlated over "blocks" of size ξ, extensivity implies $F_{\text{thermo}} \sim$ num. of blocks $\sim \left(\frac{L}{\xi}\right)^d \sim t^{d\nu}$. One can think of this as the size of blocks over which spins are correlated.

Further as

$$c \sim \frac{\partial^2 f}{\partial t^2} \sim t^{d\nu-2}, \tag{152}$$

but $c \sim t^{-\alpha}$, and thus

$$\alpha = 2 - d\nu. \tag{153}$$

This is called the **hyperscaling relation**.

As a second example, consider $T < T_C$ but close to it. This implies $\phi \sim t^\beta$, then $\Delta_\phi = \beta \Delta_t = \frac{\beta}{\nu}$, and thus

$$\beta = \nu \Delta_\phi = \frac{1}{2}(d - 2 + \eta)\nu. \tag{154}$$

Further, with a magnetic field B, the free energy F includes a term $\int \mathrm{d}^d x \, B(\mathbf{x}) \, \phi(\mathbf{x})$. At a critical point F is invariant under a renormalization group flow and thus $\Delta_B \delta_\phi = d$ and thus

$$\Delta_B = \frac{d+2-\eta}{2}. \tag{155}$$

As

$$\chi = \left(\frac{\partial \phi}{\partial B}\right)_T \sim t^{-\gamma} \Rightarrow \Delta_\phi - \Delta_B = -\frac{\gamma}{\nu} \Rightarrow \gamma = \nu(2-\eta). \tag{156}$$

Lastly, near the critical point, $\phi \sim B^{\frac{1}{\delta}}$, and therefore

$$\delta = \frac{\Delta_B}{\Delta_\phi} = \frac{d+2-\eta}{d-2+\eta}. \tag{157}$$

Thus we have shown, if we know η, ν, we can compute α, β, γ and δ.

This works well for the Ising model in $d = 2, 3$ and for $d = 4$ MFT with quadratic fluctuations. It disagrees however with mean field theory for β and δ for $d > 4$.

This will be resolved when we consider a more accurate expansion of fields, (e.g. we should have taken $\phi \sim \sqrt{\frac{t}{g}}$).