Statistical Field Theory

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1 Lecture: Introduction

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Motivation

This course concerns itself with **universality**: the idea that different physical systems can exhibit the same behaviour.

For example, a liquid gas system has a critical temperature, below which a first order phase transition can be found between liquid and gas. Above it there is no such distinction and one can only move continuously.

Experiments suggest that as a function of the temperature, the density is given by $|\rho^{\pm} - \rho_c| \propto |T - T_C|^{\beta}$ with $\beta \approx 0.327$ for $T \approx T_C$.

Another example is ferromagnets for which one also has a **critical temperature** called the Curie temperature T_C.

For $T > T_C$, $M = 0$, and for $T \leq T_C$ $M \propto (T_C - T)^{\beta}$ where the critical exponent β is as observed for gases.

In this course we will be studying the **classical statistical mechanics of fields**.

2 From Spins to Fields

2.1 The Ising Model

The Ising model is a simple model for a magnet. In d spatial dimensions, consider a lattice with N sites.

On the ith site we have a "spin" $S_i \in \{-1, 1\}$. The configuration $\{s_i\}$ has energy

$$E = -B \sum_{i}^{s_i} -J \sum_{\langle ij \rangle} s_i s_j, \tag{1}$$

where B represents an external magnetic field and J represents an interaction strength. Naturally, one should ask how the physics depends on B, J and the temperature T.

For:

- $J > 0$, the spins prefer to be aligned: $\uparrow\uparrow$ or $\downarrow\downarrow$. This is a **ferromagnet**.
- $J < 0$, the spins prefer to be anti-aligned: $\uparrow\downarrow$ or $\downarrow\uparrow$. This is an **anti-ferromagnet**.

  We will assume and fix $J > 0$.
- For $B > 0$ spins prefer $\uparrow$
- For $B < 0$ spins prefer $\downarrow$
- At low temperature, the system wants to minimise energy. This leads to an ordered state where all spins will align due to the lower energy preferenced state.
- At high temperature, entropy dominates which leads to a disordered state.

Recall that in the **canonical ensemble**, the probability of a given configuration $\{s_i\}$ is

$$p\left[s_i\right] = \frac{1}{Z} e^{-\beta E[s_i]}, \tag{2}$$

where $\beta = \frac{1}{T}$, $k_B = 1$ and Z is the **partition function**.

Definition 2.1: The **partition function** is given by

$$Z\left(T, B\right) = \sum_{\{s_i\}} e^{-\beta E[s_i]}. \tag{3}$$

Definition 2.2: The **thermodynamic free energy** $F_{\text{thermo}}\left(T, B\right) = \langle E \rangle - TS = -T \log Z$

Definition 2.3: The **magnetisation** is

$$m = \frac{1}{N} \left\langle \sum_{n=1}^{N} s_i \right\rangle \in \left[-1, 1\right]. \tag{4}$$

Note. The average $\langle X \rangle$ is an average over the probability distribution $p\left[s_i\right]$ at a fixed T.

Magnetisation is an **order parameter** as it can distinguish between ordered phases ($m \neq 0$) and disordered phases where $m \approx 0$. We can expand the expectation value in m such that

$$m = \sum_{\{s_i\}} \frac{1}{Z} e^{-\beta[E_i]} \frac{1}{N} \sum_i s_i = \frac{1}{N\beta} \frac{\partial}{\partial \beta} \log Z. \tag{5}$$

Therefore we want to compute the partition function as it will allow us to derive these observables.

This is easy in $d = 1$. This is hard in $d = 2$ and there is no general solution for generic lattices. For a square lattice with $B = 0$ it is possible and was famously solved by Onsager (winning him some famous prize). $d \geq 3$ are intractable analytically.

Our aim is to approximate in a way that correctly captures the long-distance behaviour.

We can define m for any $\{s_i\}$ by $m = \frac{1}{N}\sum s_i$, now no longer taking a statistical average. We can then write the partition function as

$$Z = \sum_m \sum_{\{s_i\}|m} e^{-\beta E[s_i]} =: \sum_m e^{-\beta F(m)}. \tag{6}$$

The spacing in allowed values of m is $\frac{2}{N}$. For $N \gg 1$ we can approximate m as continuous however and thus write the partition function as

$$Z \approx \frac{N}{2}\int_{-1}^{1} \mathrm{d}m\, e^{-\beta F(m)}. \tag{7}$$

We call $F(m)$ the **effective free energy** which depends on T, B and critically, m. This contains more information than F_{thermo}.

Let $f(m) = \frac{1}{N}F(m)$ and thus

$$Z \propto \int_{-1}^{1} \mathrm{d}m\, e^{-\beta N f(m)}. \tag{8}$$

For N large, $\beta f(m) \sim \mathcal{O}(1)$ as it is an **intensive** property (doesn't scale with the system). Performing a saddle point approximation, we can replace $f(m)$ by it's minimum where

$$\left.\frac{\partial f}{\partial m}\right|_{m=m_{\min}} = 0, \tag{9}$$

which gives us a partition function of

$$Z \propto e^{-\beta N f(M_{\min})}, \tag{10}$$

which gives us a thermodynamic free energy of

$$F_{\text{thermo}}(T, B) \approx F(M_{\min}(T, B), T, B). \tag{11}$$

Even with the saddle point approximation, computing $F(m)$ is hard. One such approach is to use the **mean field approximation** where we replace each spin by the average of the field, $s_i \to m$. This gives

$$E = -\beta\sum_i m - J\sum_{\langle ij\rangle} m^2 = -BNm - \frac{1}{2}NJqm^2, \tag{12}$$

where q is the number of nearest neighbours. $q = 2d + 2$ for a square lattice in d dimensions.

Therefore in the mean field approximation, the partition function becomes

$$Z \approx \sum_m \Omega(m)\, e^{-\beta E[n]}, \tag{13}$$

where $\Omega(m)$ is the number of configurations with $\frac{1}{N}\sum s_i = m$.