

Statistical Field Theory

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1 Lecture: Introduction

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Motivation

This course concerns itself with **universality**: the idea that different physical systems can exhibit the same behaviour.

For example, a liquid gas system has a critical temperature, below which a first order phase transition can be found between liquid and gas. Above it there is no such distinction and one can only move continuously.

Experiments suggest that as a function of the temperature, the density is given by $|\rho^\pm - \rho_c| \propto |T - T_C|^\beta$ with $\beta \approx 0.327$ for $T \approx T_C$.

Another example is ferromagnets for which one also has a **critical temperature** called the Curie temperature T_C .

For $T > T_C$, $M = 0$, and for $T \leq T_C$ $M \propto (T_C - T)^\beta$ where the critical exponent β is as observed for gases.

In this course we will be studying the **classical statistical mechanics of fields**.

1.1 The Ising Model

The Ising model is a simple model for a magnet. In d spatial dimensions, consider a lattice with N sites.

On the i th site we have a “spin” $S_i \in \{-1, 1\}$. The configuration $\{s_i\}$ has energy

$$E = -B \sum_i s_i - J \sum_{\langle ij \rangle} s_i s_j, \quad (1)$$

where B represents an external magnetic field and J represents an interaction strength. Naturally, one should ask how the physics depends on B , J and the temperature T .

For:

- $J > 0$, the spins prefer to be aligned: $\uparrow\uparrow$ or $\downarrow\downarrow$. This is a **ferromagnet**.
- $J < 0$, the spins prefer to be anti-aligned: $\uparrow\downarrow$ or $\downarrow\uparrow$. This is an **anti-ferromagnet**.

We will assume and fix $J > 0$.

- For $B > 0$ spins prefer \uparrow
- For $B < 0$ spins prefer \downarrow
- At low temperature, the system wants to minimise energy. This leads to an ordered state where all spins will align due to the lower energy preferred state.
- At high temperature, entropy dominates which leads to a disordered state.

Recall that in the **canonical ensemble**, the probability of a given configuration $\{s_i\}$ is

$$p[s_i] = \frac{1}{Z} e^{-\beta E[s_i]}, \quad (2)$$

where $\beta = \frac{1}{T}$, $k_B = 1$ and Z is the **partition function**.

Definition 1.1: The **partition function** is given by

$$Z(T, B) = \sum_{\{s_i\}} e^{-\beta E[s_i]}. \quad (3)$$

Definition 1.2: The **thermodynamic free energy** $F_{\text{thermo}}(T, B) = \langle E \rangle - TS = -T \log Z$

Definition 1.3: The **magnetisation** is

$$m = \frac{1}{N} \left\langle \sum_{n=1}^N s_i \right\rangle \in [-1, 1]. \quad (4)$$

Note. The average $\langle X \rangle$ is an average over the probability distribution $p[s_i]$ at a fixed T .

Magnetisation is an **order parameter** as it can distinguish between ordered phases ($m \neq 0$) and disordered phases where $m \approx 0$. We can expand the expectation value in m such that

$$m = \sum_{\{s_i\}} \frac{1}{Z} e^{-\beta E[s_i]} \frac{1}{N} \sum_i s_i = \frac{1}{N\beta} \frac{\partial}{\partial B} \log Z. \quad (5)$$

Therefore we want to compute the partition function as it will allow us to derive these observables.

This is easy in $d = 1$. This is hard in $d = 2$ and there is no general solution for generic lattices. For a square lattice with $B = 0$ it is possible and was famously solved by Onsager (winning him some famous prize). $d \geq 3$ are intractable analytically.

Our aim is to approximate in a way that correctly captures the long-distance behaviour.

We can define m for any $\{s_i\}$ by $m = \frac{1}{N} \sum s_i$, now no longer taking a statistical average. We can then write the partition function as

$$Z = \sum_m \sum_{\{s_i\}|m} e^{-\beta E[s_i]} =: \sum_m e^{-\beta F(m)}. \quad (6)$$

The spacing in allowed values of m is $\frac{2}{N}$. For $N \gg 1$ we can approximate m as continuous however and thus write the partition function as

$$Z \approx \frac{N}{2} \int_{-1}^1 dm e^{-\beta F(m)}. \quad (7)$$

We call $F(m)$ the **effective free energy** which depends on T, B and critically, m . This contains more information than F_{thermo} .

Let $f(m) = \frac{1}{N} F(m)$ and thus

$$Z \propto \int_{-1}^1 dm e^{-\beta N f(m)}. \quad (8)$$

For N large, $\beta f(m) \sim \mathcal{O}(1)$ as it is an **intensive** property (doesn't scale with the system). Performing a saddle point approximation, we can replace $f(m)$ by it's minimum where

$$\left. \frac{\partial f}{\partial m} \right|_{m=m_{\min}} = 0, \quad (9)$$

which gives us a partition function of

$$Z \propto e^{-\beta N f(M_{\min})}, \quad (10)$$

which gives us a thermodynamic free energy of

$$F_{\text{thermo}}(T, B) \approx F(M_{\min}(T, B), T, B). \quad (11)$$

Even with the saddle point approximation, computing $F(m)$ is hard. One such approach is to use the **mean field approximation** where we replace each spin by the average of the field, $s_i \rightarrow m$. This gives

$$E = -\beta \sum_i m - J \sum_{\langle ij \rangle} m^2 = -BNm - \frac{1}{2} NJqm^2, \quad (12)$$

where q is the number of nearest neighbours. $q = 2d + 2$ for a square lattice in d dimensions.

Therefore in the mean field approximation, the partition function becomes

$$Z \approx \sum_m \Omega(m) e^{-\beta E[m]}, \quad (13)$$

where $\Omega(m)$ is the number of configurations with $\frac{1}{N} \sum s_i = m$.

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Let N_\uparrow be the number of up spins and thus $N_\downarrow = N - N_\uparrow$ down spins. The magnetisation is therefore

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N}. \quad (14)$$

The number of configurations with this magnetisation is

$$\Omega(m) = \frac{N!}{N_\uparrow!(N - N_\uparrow)!} = \frac{N}{N_\uparrow!N_\downarrow!}. \quad (15)$$

Using the Stirling approximation,

$$\log n! = n \log n - n, \quad (16)$$

we have

$$\log \Omega \approx N (\log N - 1) - N_\uparrow (\log N_\uparrow - 1) - N_\downarrow (\log N_\downarrow - 1), \quad (17)$$

and thus

$$\frac{\log \Omega}{N} \approx \log 2 - \frac{1}{2} (1 + m) \log (1 + m) - \frac{1}{2} (1 - m) \log (1 - m). \quad (18)$$

Therefore, we have in the mean field approximation that

$$e^{-\beta N f(m)} \approx \Omega(m) e^{-\beta E(m)} \quad (19)$$

$$\Rightarrow f(m) \approx -Bm - \frac{1}{2} J q m^2 - T \left[\log 2 - \frac{1}{2} (1 + m) \log (1 + m) - \frac{1}{2} (1 - m) \log (1 - m) \right]. \quad (20)$$

We now move to minimise f such that $\frac{\partial f}{\partial m} = 0$. This implies

$$\beta (B + J q m) = \frac{1}{2} \log \left(\frac{1 + m}{1 - m} \right) \quad (21)$$

$$\Rightarrow m = \tanh \left[\beta \underbrace{(B + J q m)}_{B_{\text{eff}}} \right]. \quad (22)$$

Each spin feels an *effective* B field $B_{\text{eff}}(m)$.

2.1 Landau Theory of phase transitions

At a phase transition some quantity, called the **order parameter**, is not smooth. For us this is m . For small m , we have

$$f(m) = -T \log 2 - Bm + \frac{1}{2} \left(T - \underbrace{Jq}_{T_C} \right) m^2 + \frac{1}{12} T m^4 + \dots \quad (23)$$

Notice we have introduced the critical temperature, $T_C = Jq$.

The natural question is: In equilibrium at $m = m_{\min}$, how does this behave as we vary the temperature T or the external magnetic field B .

For $B = 0$ this reduces to

$$f(m) = \frac{1}{2}(T - T_C)m^2 + \frac{1}{2}Tm^4 + \dots \quad (24)$$

For $T > T_C$, $m_{\min} = 0$, and for $T < T_C$, $m_{\min} = \pm m_0 = \pm \sqrt{\frac{3(T_C - T)}{T}}$. This is valid for $m_0 \ll 1$ i.e. T close to T_C .

Plotting the minima as a function of T , we see a branching at $T = T_C$ and while continuous the function is not smooth and thus we have a phase transition.

- For $T > T_C$: $m = 0$ is called the *disordered phase*,
- For $T < T_C$: $m \neq 0$ is called the *ordered phase*.

This is an example of a *continuous phase transition*, or in modern language *second order phase transition*.

Notes.

- Notice that F is invariant under \mathbb{Z}_2 symmetry with simultaneous $m \rightarrow -m$ and $B \rightarrow -B$ (inherited from the Ising model's symmetry of $s_i \rightarrow -s_i$ and $B \rightarrow -B$).
- For $T < T_C$, either $m = +m_0$ or $m = -m_0$ so the \mathbb{Z}_2 symmetry of the system does not preserve the ground state. This is called **spontaneous symmetry breaking**.
- At finite N , Z is analytic in T and B . Therefore, phase transitions can only occur in the limit $N \rightarrow \infty$.
- Spontaneous symmetry breaking also only occurs for $N \rightarrow \infty$. Namely,

$$m = \lim_{B \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \sum \langle s_i \rangle, \quad (25)$$

if the order of these limits is reversed we would get $m = 0$ for all temperatures T . This follows from the \mathbb{Z}_2 symmetry as at finite N , \mathbb{Z}_2 symmetry implies Z is an even function in B and thus F_{thermo} is as well. Then,

$$\langle m \rangle = -\frac{1}{N} \frac{\partial F_{\text{thermo}}}{\partial B} \Big|_{B=0} = 0, \quad (26)$$

as the derivative of an even function is an odd function which necessarily vanishes at zero.

Definition 2.1: The **heat capacity** is defined

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial}{\partial T} \left(-\frac{\partial \log Z}{\partial \beta} \right) = \beta^2 \frac{\partial^2 \log Z}{\partial \beta^2}, \quad (27)$$

This implies

$$\log Z = -\beta N f(m_{\min}) \quad (28)$$

$$= \begin{cases} \text{const}, & T > T_C, \\ \frac{3N}{4} \frac{(T_C - T)^2}{T^2} + \text{const}, & T < T_C. \end{cases} \quad (29)$$

It is more fruitful to work with the **specific heat capacity** given by

$$c = \frac{C}{N} = \begin{cases} 0, & T \rightarrow T_C \text{ from above}, \\ \frac{3}{2}, & T \rightarrow T_C \text{ from below}. \end{cases} \quad (30)$$

Therefore c is *discontinuous* at $T = T_C$.

So far our discussion has assumed $B = 0$ in the limit. We now work with finite $B > 0$. This gives us

$$f(m) \approx -Bm + \frac{1}{2}(T - T_C)m^2 + \frac{1}{12}Tm^4 + \dots \quad (31)$$

For $T > T_C$ we have a concave curve with a single positive minima $m_{\min} \approx \frac{B}{T}$ for $T \rightarrow \infty$.

For $T < T_C$, we still have a double well with two local minima, but now they are no longer degenerate and there is a true global minimum. The other is referred to as a *metastable state*, as it may be stable on short time scales, but given sufficient time and perturbations, would decay to the true minimum m_{\min} .

As a function of T the global minimum $m_{\min}(T)$ is now a smooth function of temperature which implies we see no phase transition if T is varied at fixed $B \neq 0$.

However, we can now instead vary B at fixed $T < T_C$. For $B > 0$ we see the weighted double well as we did before and for $B = 0$ the degenerate double well. For $B < 0$ we see the global minima discontinuously shifts (from m_0 to $-m_0$) as B decreases from $B > 0$ to $B < 0$.

This is an example of a **discontinuous phase transition** which is also called a **first order phase transition** as our order parameter is the first derivative of $m_{\text{equil}} = -\frac{\partial f_{\text{thermo}}}{\partial B}$. As this is discontinuous, we call it a first order phase transition.

Therefore, as a function of B and T for $B = 0$ and $T < T_C$ we see a line of first order phase transitions. For $B = 0$ and $T = T_C$, this becomes a second order phase transition.