String Theory

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2025-02-08

Contents

1 Lecture: Introduction

23/01/2025

1.1 Motivation

String theory is a candidate theory of quantum gravity. Before we discuss string theory itself, it is worthwhile to discuss gravity, namely classical gravity. One has that the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{1}$$

which are the equations of motion for the action

$$S = S_{\text{E.H.}} + S_{\text{matter}}, \tag{2}$$

where

$$S_{EH} = \frac{1}{16\pi G_N}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}R, \tag{3}$$

where $g = \det g$. One can think of this as a field theory where the dynamical field is the metric, $g_{\mu\nu}$. One has stress energy tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{4}$$

When one translates to the quantum theory, the quantum fields and thus the stress energy tensor will too, $T_{\mu\nu} \to \hat{T}_{\mu\nu}$. It is difficult to imagine that the Einstein Hilbert action, and thus the left hand side of Einstein's equations are not quantum as well.

G_N appears as a coupling constant, and is a dimensional one with $[G_N] = M^{-1}L^3T^{-2}$. Then one can convert it to a mass scale (*planck*) in natural units with

$$8\pi G_N = \frac{\hbar c}{M_{\text{pl}}^2}. \tag{5}$$

Expanding around the "vacuum" of flat Minkowski space, then,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\text{pl}}}. \tag{6}$$

Then the action becomes

$$S_{\text{EH}} = \int_{\mathcal{M}} \mathrm{d}^4x\,\mathcal{L}_{\text{EH}}, \tag{7}$$

where

$$\mathcal{L}_{\text{EH}} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \tag{8}$$

where

$$\mathcal{L}_0 = -\frac{1}{4}\partial_\mu h^\rho{}_\rho \partial^\mu h^\sigma{}_\sigma + \frac{1}{2}\partial_\mu h^{\rho\sigma}\partial^\mu h_{\rho\sigma}, \tag{9}$$

which is the Lagrangian of a free massless spin 2 particle, one can call the *graviton.* We then have interaction Lagrangian

$$\mathcal{L}_{\text{int}} \sim \frac{1}{M_{\text{pl}}}h\left(\partial h\right)^2 + \frac{1}{M_{\text{pl}}^2}h^2\left(\partial h\right)^2. \tag{10}$$

However these interactions are irrelevant and thus lead to UV divergences at ≥ 2 loops. This is a non-renormalizable theory. The effective theory then goes $g_{\text{eff}} \sim \frac{E}{M_{\text{pl}}}$ becomes strong at $E \sim M_{\text{pl}}$. This is $M_{\text{pl}} \sim 10^{18}$ GeV which is so far above anything accessible to colliders today, string theory is often consider too academic a pursuit for those bound by phenomenological interests.

We need a UV completion of S_{EH} which will consist of

- a consistent quantum theory, namely, unitary, Lorentz invariant, UV finite (or renormalizable)
- a theory that reduces to GR at low energy $E \ll M_{\rm pl}$ such that

$$S_{\rm eff} \sim S_{\rm EH} + \cdots . \tag{11}$$

There is no other theory that does both. String theory is the unique candidate.

1.2 String Theory

A QFT is a theory of interacting relativistic particles. This necessarily involves unitarity, Lorentz invariance and locality. There are very few possibilities but still an infinite countable list.

String theory considers replacing point particles with strings. These can be closed or open one dimensional objects at fixed time. Propagating in time these generate two dimensional surfaces called *worldsheets*.

It is remarkable that this is unique and there are no free parameters. It is UV finite and closed strings give rise to gravitons and open strings to gauge fields and fermions. It also reduces to GR with matter at low energy.

It is not without baggage. It requires supersymmetry and extra dimensions to be well defined. For bosons this is $D = 26$ and for fermions $D = 10$.

While we have a unique theory, there are many ground states (like the Higgs boson) which make phenomenological predictions hard.

2 Lecture: Classical and Quantum String

25/01/2025

Work in $\mathbb{R}^{1,D-1}$ with metric

$$\eta_{\mu\nu} = \mathrm{diag}\left(-1, +1, \cdots, +1\right), \tag{12}$$

with line element

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu . \tag{13}$$

For timelike $\mathrm{d}x^\mu$, $\mathrm{d}s^2 < 0$.

We define a D-vector as X^μ with $\mu \in \{0, 1, \cdots, D-1\}$ and an inner product

$$X \cdot Y = X_\mu Y^\mu = \eta_{\mu\nu} X^\mu Y^\nu . \tag{14}$$

Sometimes we also think about a curved metric $G_{\mu\nu}\left(x\right)$.

2.1 Relativistic point particle

As a warm up, we will study a relativistic point particle of rest mass m moving in $\mathbb{R}^{1,D-1}$.

One should think of a worldline X as a map from a parameter $W \to \mathbb{R}^{1,D-1}$ that takes

$$\tau \mapsto X^\mu\left(\tau\right), \tag{15}$$

where τ is called the proper time.

The natural choice of an action is one proportional to the proper time and thus the length of the worldline. Namely, one takes

$$S = \mathcal{N} \int_W \sqrt{-\,\mathrm{d}s^2} = \mathcal{N} \int_W \mathrm{d}\tau \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}}. \tag{16}$$

Exercise 1: Fix $\mathcal{N}$ such that for $X^\mu(\tau) = (t(\tau), \vec{x}(\tau))$, the three velocity $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}$, $v = |\vec{v}|$, and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma(v) = \frac{1}{\sqrt{1-v^2}}, \tag{17}$$

one can set

$$S = \int_W \mathrm{d}t\, L(t), \tag{18}$$

with

$$L = \mathcal{N} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} \sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right)^2} = \mathcal{N}\sqrt{1-v^2}. \tag{19}$$

This Lagrangian has conjugate momenta

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = -\frac{\mathcal{N}\vec{v}}{\sqrt{1-v^2}} = m\gamma(v)\,\vec{v}, \tag{20}$$

where we have identified $\mathcal{N} = -m$.

By Legendre transform, we find a Hamiltonian

$$E = H = \vec{p}\cdot\vec{v} - L = mv^2\gamma(v) + m\gamma^{-1}(v) = m\gamma(v), \tag{21}$$

which is the correct relativistic energy of a point particle.

Therefore for a point particle, we can formally state the action

$$S = -m \int_W \mathrm{d}\tau = \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}}. \tag{22}$$

The form of this action makes the symmetries of it manifest (clear and explicit). Namely, one can see

- Poincare invariance, such that $X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + C^\mu$, where the Lorentz transformation is defined by

$$\Lambda^\mu{}_\rho \eta^{\rho\sigma} \Lambda^\nu{}_\sigma = \eta^{\mu\nu}. \tag{23}$$

- reparametrization invariance $\tau \to \widetilde{\tau}(\tau)$ given that $\frac{\mathrm{d}\widetilde{\tau}}{\mathrm{d}\tau} > 0$ as

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \to \frac{\mathrm{d}X^\mu}{\mathrm{d}\widetilde{\tau}} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\mathrm{d}\widetilde{\tau}}, \tag{24}$$

and as the measure transforms as $\mathrm{d}\tau = \frac{\mathrm{d}\widetilde{\tau}}{\mathrm{d}\tau}\,\mathrm{d}\widetilde{\tau}$, one has

$$S = -m \int_W \mathrm{d}\widetilde{\tau}\, \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\widetilde{\tau}}\frac{\mathrm{d}X^\nu}{\mathrm{d}\widetilde{\tau}}}. \tag{25}$$

This redundancy can be thought of as a gauge symmetry. If one gauge fixes $\widetilde{\tau} = t$, then we recover the previous action

$$S = -m \int \mathrm{d}t\, \sqrt{1 - v^2}, \tag{26}$$

however this form does not make the symmetries manifest, including the gauge one which is explicitly broken.

2.2 Relativistic String

Consider an open string and a closed string, with *worldsheets* that look like the plane and a cylinder.

These worldsheets Σ in $\mathbb{R}^{1,D-1}$ are the analogue of the world line for a 1D point particle.

We introduce worldsheet coordinates $\sigma^\alpha = (\tau, \sigma)$ with $\alpha = 0, 1$.

The spatial coordinate σ satisfies

- $\sigma \in [0, \pi]$ for the open string,
- $\sigma \sim \sigma + 2\pi$, for the closed string.

We need to describe how this worldsheet is embedded in spacetime. This is the *string configuration* and is a map

$$\begin{aligned} X : \Sigma &\to \mathbb{R}^{1,D-1} \\ (\tau, \sigma) &\mapsto X^\mu(\sigma, \tau) \end{aligned} \tag{27}$$

Definition 2.1: The **pullback** of the Minkowski metric $\eta_{\mu\nu}$ to Σ is given by

$$\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma^\alpha}\frac{\mathrm{d}X^\nu}{\mathrm{d}\sigma^\beta}. \tag{28}$$

This coincides with the induced metric on an embedded surface.

One can write it more explicitly with $\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma^0} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \dot{X}^\mu$ and $\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma^1} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma} = X'^\mu$ giving us

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}. \tag{29}$$

One can then define

$$S_{\text{Nambu-Goto}} = -T\int_\Sigma \mathrm{d}^2\sigma\, \sqrt{-\det\gamma} = -T\int_\Sigma \mathrm{d}^2\sigma\, \sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 \dot{X'}^2}, \tag{30}$$

which describes the natural notion of area on the string worldsheet, the extension of the notion of length of a worldline.

T has dimensions of M^2 and is the *tension* of the string as we will see.

2.3 Polyakov Action

We introduce a dynamical metric $g_{\alpha\beta}(\sigma,\tau)$ on Σ with determinant $g = \det g$ and signature $(-1,+1)$. Then we see

$$S_{\text{Polyakov}} = -\frac{T}{2}\int \mathrm{d}^2\sigma\,\sqrt{-g}\left(g^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right) = -\frac{T}{2}\int \mathrm{d}^2\sigma\,\sqrt{-g}\left(g^{\alpha\beta}\left(\partial_\alpha X\right)\cdot\left(\partial_\beta X\right)\right), \tag{31}$$

where one can also write

$$g^{\alpha\beta}\gamma_{\alpha\beta} = g^{\alpha\beta}\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}. \tag{32}$$

Note. The worldsheet is instrumental here. We are thinking of X as a map from the worldsheet to spacetime

$$X : \Sigma \to \mathbb{R}^{1,D-1}. \tag{33}$$

However one can think of the field theory itself as living on the worldsheet of the string. The worldsheet is then a 2D spacetime and the spacetime is a *field space* or a *target space*. Namely, one thinks of the spacetime coordinates as scalar fields on Σ and the string actions as defining 2D field theories.

The Polyakov action then, is a number of 2D scalar fields coupled to 2D gravity on Σ.

We then consider the equation of motion of the 2D metric $g_{\alpha\beta}$ which is

$$\frac{\delta S_{\text{Polyakov}}}{\delta g_{\alpha\beta}} = 0. \tag{34}$$

Equivalently, as the energy momentum tensor of the field theory is given by

$$\mathcal{T}^{\alpha\beta} = -\frac{1}{T}\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{Polyakov}}}{\delta g_{\alpha\beta}} = 0, \tag{35}$$

notice that one can integrate out $g_{\alpha\beta}$ and recover the Nambu-Goto action.

3 Lecture: Polyakov equivalence

28/01/2025

Recall that the two actions

$$S_{\text{NG}} = -T\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-\det\gamma} \qquad\qquad S_P = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-g}g^{\alpha\beta}\gamma_{\alpha\beta}, \tag{36}$$

for

$$\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}. \tag{37}$$

both describe a string on a worldsheet. The Nambu-Goto action is the area of the embedded worldsheet of the string in D dimensional spacetime. The Polyakov action defines a 2D scalar field theory coupled to 2D gravity. These two actions are dynamically equivalent. It remains to shown this.

Consider the variation of the Polyakov action,

$$\delta S_{\mathrm{P}} = -\frac{T}{2} \int \mathrm{d}^2\sigma \, \delta \left(\sqrt{-g} g^{\alpha\beta} \gamma_{\alpha\beta} \right), \tag{38}$$

where we see that

$$\delta \left(\sqrt{-g} g^{\alpha\beta} \gamma_{\alpha\beta} \right) = \sqrt{-g} \left(\delta g^{\alpha\beta} \gamma_{\alpha\beta} + \frac{\delta g}{2g} g^{\alpha\beta} \gamma_{\alpha\beta} \right). \tag{39}$$

We need to be able to compute this variation.

Lemma 3.1: For symmetric matrix $A_{\alpha\beta} \to A_{\alpha\beta} + (\delta A)_{\alpha\beta}$,

$$\delta (\det A) = \det A A^{\alpha\beta} \delta A_{\alpha\beta}. \tag{40}$$

Proof.

$$\begin{aligned} \delta (\det A) &= \delta (\exp (\log (\det A))) & (41) \\ &= \delta (\exp (\mathrm{Tr} (\log A))) & (42) \\ &= \delta (\mathrm{Tr} (\log A)) \det A & (43) \\ &= \mathrm{Tr} \left(A^{-1} \cdot \delta A \right) \det A. & (44) \end{aligned}$$

□

Therefore $\delta g = \delta (\det g) = g g^{\alpha\beta} \delta g_{\alpha\beta} = -g g_{\alpha\beta} \delta g^{\alpha\beta}$.

Thus, the variation of the Polyakov action becomes

$$\delta S_{\mathrm{P}} = \left(\gamma_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \left(g^{\rho\sigma} \gamma_{\rho\sigma} \right) \right) \delta g^{\alpha\beta} = 0, \tag{45}$$

which implies we have equation of motion $T^{\alpha\beta} = 0$ for $g_{\alpha\beta}$ equivalently written as

$$\gamma_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \left(g^{\rho\sigma} \gamma_{\rho\sigma} \right). \tag{46}$$

It suffices to take the determinant of this expression, giving

$$\det \gamma = \frac{1}{4} g \left(g^{\rho\sigma} \gamma_{\rho\sigma} \right)^2, \tag{47}$$

which implies

$$g = \frac{4 \det \gamma}{\left(g^{\rho\sigma} \gamma_{\rho\sigma} \right)^2}. \tag{48}$$

Substituting this into the Polyakov action, we see

$$S_{\mathrm{P}} = -\frac{T}{2} \int_{\Sigma} \mathrm{d}^2\sigma \, \frac{2\sqrt{-\det \gamma}}{\left(g^{\rho\sigma} \gamma_{\rho\sigma} \right)} \left(g^{\alpha\beta} \gamma_{\alpha\beta} \right) = S_{\mathrm{NG}}, \tag{49}$$

as desired. Namely, integrating out the metric recovers the Nambu-Goto action. Therefore they are classically equivalent descriptions of a string.

Recall that we are thinking of the Polyakov action as a two dimensional field theory defined on the string worldsheet,

$$S_{\mathrm{P}} = -\frac{T}{2} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-g} g^{\alpha\beta} \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}. \tag{50}$$

We are thinking of this theory as a number of scalar fields X^μ coupled to a curved background metric $g_{\alpha\beta}$ on the world sheet. Equivalently phrased, this is scalars coupled to 2D gravity.

Gravity has a large symmetry group, namely the group of diffeomorphisms of the manifold. In 4D, this is not enough to remove all degrees of freedom and one is left with the propagating components of the graviton.

In two dimensions, the symmetries of 2D GR is sufficient to render gravity to be purely gauge. There are no transverse directions for the graviton to be polarised in.

One may also naively want to add the Einstein Hilbert term to this action,

$$S_{\mathrm{EH}} = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \sqrt{-g} R\left[g\right], \tag{51}$$

however in 2D, this is a total derivative term which only contributes on the boundary and thus does not contribute. This is a manifestation that there are no dynamical degrees of freedom in 2D gravity.

3.1 Symmetries of the Polyakov action

The Polyakov action has a number of symmetries.

- The Polyakov action is invariant under the Poincare group with

  $$X^\mu \to \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + C^\mu, \tag{52}$$

  where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and $C^\mu \in \mathbb{R}^4$ is a translation.

  This is a global symmetry so one can compete Noether currents and conserved charges.

  > ***Proof.*** [(Ex 1.3)]
  >
  > □

- It is also invariant under reparametrization

  $$\sigma^\alpha \to \widetilde{\sigma}^\alpha\left(\sigma, \tau\right). \tag{53}$$

  These are 2D diffeomorphisms on the worldsheet. This is a gauge symmetry (redundancy in our description) and we know from GR that our metric tensor transforms as

  $$g_{\alpha\beta}\left(\sigma, \tau\right) \to \widetilde{g}_{\alpha\beta}\left(\widetilde{\sigma}, \widetilde{\tau}\right) \tag{54}$$

  $$= \frac{\partial \sigma^\delta}{\partial \widetilde{\sigma}^\alpha} \frac{\partial \sigma^\gamma}{\partial \widetilde{\sigma}^\beta} g_{\delta\gamma}\left(\sigma, \tau\right), \tag{55}$$

and as X^μ is a scalar field on the worldsheet,

$$X^\mu(\sigma,\tau) \to \widetilde{X}^\mu(\widetilde{\sigma},\widetilde{\tau}) = X^\mu(\sigma,\tau)\,. \tag{56}$$

Under infinitesimal transformation $\sigma^\alpha \to \widetilde{\sigma}^\alpha = \sigma^\alpha - \xi^\alpha$, one sees

$$\delta g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \qquad\qquad \delta X^\mu = \xi^\alpha \partial_\alpha X^\mu, \tag{57}$$

where ∇ is a (torsion-free) worldsheet covariant derivative

$$\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta + \Gamma_\alpha{}^\sigma{}_\beta \xi_\sigma, \tag{58}$$

where $\Gamma_\alpha{}^\sigma{}_\beta$ is the Christoffel symbol for $g_{\alpha\beta}$.

- The Polyakov action almost uniquely has another symmetry called *Weyl symmetry* such that

$$g_{\alpha\beta} \to \widetilde{g}_{\alpha\beta} = \Omega^2(\sigma,\tau)\, g_{\alpha\beta}, \tag{59}$$

where $X^\mu \to \widetilde{X}^\mu = X^\mu$. We treat this as a gauge symmetry.

> ***Proof.***
>
> □

Note. See Sheet 1 for a Polyakov-type action for point particle.

One can generalize the notion of strings to higher dimensional surfaces called *p-branes*. However the Polyakov action does not have Weyl symmetry in any other case but for strings ($p = 1$). This makes string theory much more tenable and branes very very hard.

3.2 Gauge fixing

As 2D gravity is (locally) *pure gauge*, and $g_{\alpha\beta} = g_{\beta\alpha}$ has 3 independent components, one can fix the 3 independent components using diffeomorphisms (providing 2 d.o.f.) and Weyl symmetry (providing 1 d.o.f.).

Namely, diffeomorphisms allow us to put a generic metric into the form

$$g_{\alpha\beta} \to e^{2\Phi}\eta_{\alpha\beta}, \tag{60}$$

from which a Weyl rescaling allows us to set

$$g_{\alpha\beta} \to \eta_{\alpha\beta} = \mathrm{diag}(-1,+1)\,, \tag{61}$$

generically. This is called *conformal gauge*.

The Polyakov action then becomes

$$S_{\text{conformal}} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-g}\eta^{\alpha\beta}(\partial_\alpha X)\cdot(\partial_\beta X)\,, \tag{62}$$

where we have abbreviated $a\cdot b = \eta_{\mu\nu}a^\mu b^\nu$.

This is a renormalizable 2D field theory. Even if we make the target spacetime curved $\eta_{\mu\nu} \to G_{\mu\nu}(X)$,

$$S_{\mathrm{NLSM}} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\, \eta^{\alpha\beta} G_{\mu\nu}(X)\,\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{63}$$

is a 2D *nonlinear sigma model.* It is on a generic curved manifold and is a renormalizable field theory. In higher spacetime dimensions this is not the case. Therefore strings are very unique in this sense when compared to higher dimensional branes.

However, we still need to impose the equation of motion for $g_{\alpha\beta}$ even though we have fixed it through symmetry. Recall that its equation of motion comes from

$$\mathcal{T}_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{P}}}{\delta g^{\alpha\beta}} = 0. \tag{64}$$

With this normalization, the stress energy tensor (with $g_{\alpha\beta} = \eta_{\alpha\beta}$) is of the form

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\left(\eta^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X\right). \tag{65}$$

> ***Proof.*** Taking the variation of the Polyakov action from Eq. (38) and setting $g_{\alpha\beta} = \eta_{\alpha\beta}$, we see that □

Therefore imposing this equation of motion for $g_{\alpha\beta}$ in the form $\mathcal{T}_{\alpha\beta} = 0$ becomes component wise

$$T_{10} = \dot{X}\cdot X' = 0, \qquad T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right). \tag{66}$$

These are the *Virasoro constraints* that must be imposed when we are looking at the Polyakov action in conformal gauge.

4 Lecture: Virasoro Constraints

30/01/2025

Recall that the conformal gauge action

$$S_{\mathrm{conformal}} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X, \tag{67}$$

requires the Virasoro constraint which can be stated as $\mathcal{T}_{\alpha\beta} = 0$ to be imposed. This action has equation of motion $\partial_\alpha \partial^\alpha X^\mu = 0$.

For open strings, we need to consider the boundary conditions at $\sigma = 0, \pi$.

Consider $X^\mu \to X^\mu + \delta X^\mu$ with $\delta X^\mu(\sigma, \tau_i) = \delta X^\mu(\sigma, \tau_f) = 0$ such that the variation vanishes at the boundaries. We then have

$$\delta S_{\mathrm{conformal}} = -T\int_{\tau_i}^{\tau_f}\mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\, \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta(\delta X), \tag{68}$$

which can be integrated by parts to give

$$\delta S_{\mathrm{conformal}} = T\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_0^\pi \mathrm{d}\sigma\left(\eta^{\alpha\beta}\partial_\alpha\partial_\beta X\right)\cdot \delta X - T\int_{\tau_i}^{\tau_f}\left(X'\cdot\delta X\Big|_{\sigma=\pi} - X'\cdot\delta X\Big|_{\sigma=0}\right). \tag{69}$$

For an arbitrary variation δX, $\delta S_{\text{conformal}} = 0$ vanishing requires the bulk piece to vanish,

$$\partial_\alpha \partial^\alpha X^\mu = 0, \tag{70}$$

but also the boundary terms must cumulatively vanish such that

$$X' \cdot \delta X \bigg|_{\sigma=0,\pi} = 0. \tag{71}$$

This is satisfied by two different boundary conditions for $X^\mu(\sigma, \tau)$.

- *Neumann* boundary conditions set $\frac{\partial X^\mu}{\partial \sigma} = 0$ at $\sigma = 0, \pi$.
- *Dirichlet* boundary conditions set $X^\mu \bigg|_{\sigma=0,\pi} = C^\mu$ for constant C^μ. This implies that $\delta X^\mu = 0$. This physically means that the string endpoint does not move in the X^μ direction. It is fixed.

Suppose we choose Dirichlet boundary conditions for $\mu = p+1, \cdots, D-1$ and Neumann for $\mu = 0, \cdots, p$. This is the statement that the $D-p$-dimensional hyperplane fixes the string's boundaries. It is only able to evolve in the transverse directions.

The conformal gauge action

$$S_{\text{conformal}} = -\frac{T}{2} \int_\Sigma \mathrm{d}^2\sigma \, \eta^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X, \tag{72}$$

has residual gauge invariance $\sigma^\alpha \to \widetilde{\sigma}^\alpha(\sigma, \tau)$ and $\eta_{\alpha\beta} \to \widetilde{\eta}_{\alpha\beta} = \Omega(\sigma)\, \eta_{\alpha\beta}$. Namely, some diffeomorphisms can be undone by Weyl transformations and thus remain in this action.

We can fix this residual gauge invariance by choosing *static gauge* such that

$$X^0(\sigma, \tau) = R\tau. \tag{73}$$

We set

$$X^\mu(\sigma, \tau) = \left(t(\sigma, \tau), \vec{x}(\sigma, \tau)\right), \tag{74}$$

with $\vec{x} \in \mathbb{R}^{D-1}$.

The Virasoro constraints $\dot{X} \cdot X' = 0$ and $\frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0$ becomes

$$\dot{\vec{x}} \cdot \vec{x}' = 0, \qquad \left|\dot{\vec{x}}\right|^2 + \left|\vec{x}'\right|^2 = R^2, \tag{75}$$

where

$$\dot{\vec{x}} = \frac{\partial \vec{x}}{\partial \tau}, \qquad \vec{x}' = \frac{\partial \vec{x}}{\partial \sigma}. \tag{76}$$

Noether currents of spacetime translations here $X^\mu \to X^\mu + C^\mu$ give charges $P^\mu_\alpha T \partial_\alpha X^\mu$.

The energy density of the string is given by

$$\varepsilon(\sigma, \tau) \equiv P^{\mu=0}_{\sigma=0} = T\dot{X}^0 = TR, \tag{77}$$

where the last equality holds in the static gauge.

The momentum density is given by

$$\rho_i(\sigma,\tau) = P^{\mu=i}_{\sigma=0} = \dot{\vec{x}}(\sigma,\tau)\,. \tag{78}$$

The first Virasoro constraint can thus be phrased as $\dot{\vec{x}} \cdot \vec{x}' = 0 \Leftrightarrow \vec{\rho} \cdot \vec{x}' = 0$ tells us that no momentum is carried along the string in the transverse direction.

We consider a simple solution. At $t = \tau = 0$, consider

$$x(\sigma,0) = R\cos\sigma, \qquad y(\sigma,0) = R\sin\sigma, \tag{79}$$

and $\dot{x}(\sigma,0) = \dot{y}(\sigma,0) = 0$. The string can't rotate.

Consider the time evolution in static gauge with $t = R\tau$ and thus

$$x(\sigma,\tau) = A(\tau)\cos\sigma, \qquad y(\sigma,\tau)\,A(\tau)\sin\sigma. \tag{80}$$

Plugging this into the string equation of motion we see

$$\partial_\alpha\partial^\alpha x = 0 \Rightarrow \ddot{A}(\tau) = -A(\tau) \Rightarrow A(\tau) = R_0\cos(\tau)\,, \tag{81}$$

where $A(0) = R_0$ and $\dot{A}(0) = 0$. This works identically for y. The first Virasoro constraint $\vec{x}\cdot\vec{x}' = 0$ is satisfied immediately as our motion is purely radial. Similarly, the second $\left|\dot{\vec{x}}\right|^2 + |\vec{x}'|^2 = R^2$ provides $R = R_0$.

The total energy of the string is then given by

$$E = \int_0^{2\pi} \varepsilon(\sigma,\tau)\,\mathrm{d}\sigma = 2\pi TR = 2\pi R_0 T, \tag{82}$$

where we use $\sigma \in [0, 2\pi]$ as it is closed.

Thus the energy of the string is the length of the string times the tension. This confirms that T has units of energy per unit length and is therefore a tension.

For the open string with Neumann boundary conditions $X'^\mu\Big|_{\sigma=0,\pi} = 0$. One can interpret this as the fact that the flow of energy momentum along the string vanishes, $P^\mu_\sigma\Big|_{\sigma=0,\pi}$. There is no flux of energy momentum out of the ends of the string.

Looking at the second Virasoro constraint at the endpoints only we see that

$$\left(\left|\dot{\vec{x}}\right|^2 + |\vec{x}'|^2\right)\Big|_{\sigma=0,\pi} = R^2 \Rightarrow \left(\left|\frac{\partial\vec{x}}{\partial\tau}\right|^2 + 0\right)\Big|_{\sigma=0,\pi} = R^2, \tag{83}$$

where $t = R\tau$, gives us

$$\left|\frac{\partial\vec{x}}{\partial t}\right|^2\Big|_{\sigma=0,\pi} = 1. \tag{84}$$

Therefore string endpoints move at the speed of light.

4.1 General solutions

We introduce *lightcone coordinates* given by

$$\sigma^{\pm} = \tau \pm \sigma, \tag{85}$$

which have derivatives

$$\partial_{\pm} = \frac{\partial}{\partial \sigma^{\pm}} = \frac{1}{2}\left(\frac{\partial}{\partial \tau} \pm \frac{\partial}{\partial \sigma}\right). \tag{86}$$

Then,

$$\partial_\alpha \partial^\alpha X^\mu = 0 \Rightarrow \partial_+ \partial_- X^\mu \left(\sigma +, \sigma^-\right) = 0. \tag{87}$$

We impose $X^\mu(\sigma^+, \sigma^-) = X^\mu(\sigma^-) + X^\mu(\sigma^+)$.

5 Lecture: Beginning String Quantization

01/02/2025

Recall that we introduced lightcone coordinates on the worldsheet Σ to be

$$\sigma^{\pm} = \tau \pm \sigma, \tag{88}$$

which give derivatives

$$\partial_{\pm} = \frac{\partial}{\partial \sigma^{\pm}} = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right). \tag{89}$$

The benefit of doing this is that the equation of motion in conformal gauge separates into left and right moving modes. This is a special feature of the massless wave equation in two dimensions.

Namely, $\partial_\alpha \partial^\alpha X^\mu = 0$ becomes $\partial_+ \partial_- X^\mu = 0$ and

$$X^\mu(\sigma, \tau) = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right). \tag{90}$$

One needs to solve this equation and consider its boundary conditions.

For a closed string, one has $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$. We know how to solve a periodic function, and thus have a general solution

$$X_L^\mu\left(\sigma^+\right) = x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \exp\left(-in\sigma^+\right), \tag{91}$$

where this is a constant term, linear term and 2π periodic functions. Note that the linear term is not periodic but only the sum of $X_L^\mu + X_R^\mu$ needs to be periodic, not them individually. Namely,

$$X_R^\mu\left(\sigma^-\right) = x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^\mu \exp\left(-in\sigma^-\right). \tag{92}$$

Note. $\alpha' = \frac{1}{2\pi T}$.

Further, reality of X^μ imposes that

$$\alpha_n^\mu = \left(\alpha_{-n}^\mu\right)^*, \qquad \widetilde{\alpha}_n^\mu = \left(\widetilde{\alpha}_{-n}^\mu\right)^*. \tag{93}$$

Exercise 2: Find the corresponding general solution for the open string with $\sigma \in [0, \pi]$ with Neumann boundary conditions such that $\partial_\sigma X^\mu \Big|_{\sigma=0,\pi} = 0$.

As any function on $[0, \pi]$ can be extended to a periodic function on $[0, 2\pi]$, the solution is near identical. The boundary condition is nontrivial to implement and ties together the left and right handed oscillators giving $\alpha_n^\mu = \widetilde{\alpha}_n^\mu$.

x^μ is naturally thought of as the center of the string. p^μ describes the motion of the string through spacetime. It can be identified with the spacetime momentum of the string.

5.1 Virasoro constraints

Recall it remains to impose the Virasoro constraints. In the original form, they are expressible as $\dot{X} \cdot X' = 0$ and $\frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0$. As $T_{\alpha\beta} = 0$, in lightcone coordinates, one has that the constraints take the form

$$T_{++} = \partial_+ X \cdot \partial_+ X = 0 \qquad T_{+-} = 0 \qquad T_{--} = \partial_- X \cdot \partial_- X = 0. \tag{94}$$

These pick out the left and right moving parts respectively.

Considering the mode expansion, in lightcone coordinates,

$$T_{++} = \alpha' \sum_{n\in\mathbb{Z}} L_N e^{-in\sigma^+} = 0 \qquad T_{--} = \alpha' \sum_{n\in\mathbb{Z}} \widetilde{L}_n e^{-in\sigma^-} = 0. \tag{95}$$

This holds for all σ and τ if $L_n = \widetilde{L}_n = 0$. One then finds that

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m} \cdot \alpha_m \qquad \widetilde{L}_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \widetilde{\alpha}_{n-m}\widetilde{\alpha}_m, \tag{96}$$

where one has defined $\alpha_0^\mu = \widetilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$.

Therefore, the Virasoro constraints are

$$\boxed{L_n = \widetilde{L}_n = 0, \quad \forall n \in \mathbb{Z}}. \tag{97}$$

These constraints are telling us that we have to pick a subset of the solution set for which the parameters obey these constraints.

Notice that for $n = 0$,

$$L_0 = \frac{\alpha'}{4} p_\mu p^\mu + \sum_{m>0} \alpha_m \cdot \alpha_{-m} \qquad \widetilde{L}_0 = \frac{\alpha'}{4} p_\mu p^\mu + \sum_{m>0} \widetilde{\alpha}_m \cdot \alpha_{-m}. \tag{98}$$

Recall that $X_L + X_R = X \sim X^\mu + P^\mu \tau + \cdots$ and that $P^\mu_\alpha = T\partial_\alpha X^\mu$, where the total momentum on the string is

$$P^\mu = \int_0^{2\pi} P^\mu_0 \, \mathrm{d}\sigma \, , \tag{99}$$

which is exactly equal to p^μ in the string expansion.

We can then identify $p_\mu p^\mu = -M^2$ to be the mass shell condition of the string. Namely, we see

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \widetilde{\alpha_n} \cdot \widetilde{\alpha}_{-n}. \tag{100}$$

When we quantize the string, we will see that these oscillator coordinates become harmonic oscillator creation and annihilation operators. Then these combinations will become number operators for the excitations of the string. Thus we arrive at the mass of the string in terms of its internal degrees of freedom. Excitations of the oscillator degrees of freedom of the string create heavier particles.

The second noteworthy thing here is that the left and right moving excitations implied by this equation is called *level matching*. Namely, the total occupation number of the left and right moving particles must match.

Both of these are consequences of Virasoro. We can now think about quantizing the string seriously.

5.2 Quantization

Quantizing the string looks promising as it is a two dimensional scalar field theory however there are complications. In particular, we need to impose constraints. There is also the question of residual gauge invariance.

We have two possible routes.

- Starting from the conformal gauge action, we can quantize it first and then impose the constraints and fix the gauge in the quantum theory. This is similar to Gupta-Bluer for QED. This gives rise to ghosts (states with negative norm) which are difficult to deal with. However when one imposes the constraints and gauge in this theory the ghosts decouple from physical states and one can define a valid Hilbert space. This uses Faddeev-Popov gauge fixing in the path integral formalism.
- If one imposes the gauge fixing and constraints in the classical theory first, we would expect that this gives a sensible quantum theory upon quantization. Generically, this approach is costly as symmetries like Lorentz invariance are not generally preservable when fixing a gauge. This breaking is facetious as the breaking is a result of a choice and we must recover a Lorentz invariant theory. However in the mean time, it depends on the choice of gauge orbit and thus is not manifestly Lorentz invariant.

We have to get the same quantum theory from both routes if it is consistent. Such symmetries can generically also be broken when attempting to transfer to the quantum theory by the appearance of *anomalies*.

5.3 Covariant Quantization

We proceed with the first method. We start with the conformal gauge action

$$S_{\text{conformal}} = -\frac{T}{2}\int_{\Sigma} \mathrm{d}^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X, \tag{101}$$

interpreting $X^\mu(\sigma,\tau)$ as massless scalar fields. We identify the conjugate momenta

$$\Pi_\mu(\sigma,\tau) = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T\dot{X}_\mu. \tag{102}$$

Quantization proceeds by promoting $X^\mu \to \hat{X}^\mu$ and $\Pi_\mu \to \hat{\Pi}_\mu$ and imposing the equal time commutation relation

$$\left[\hat{X}^\mu(\sigma,\tau), \hat{\Pi}_\nu(\sigma',\tau)\right] = i\delta^\mu_\nu \delta(\sigma - \sigma'). \tag{103}$$

In terms of the mode expansion, we then want to promote $x^\mu, p_\mu, \alpha^\mu_n$ and $\widetilde{\alpha}^\mu_n$ to operators identically. We then see that the reality constraint gives

$$\hat{\alpha}^\mu_n = \left(\hat{\alpha}^\mu_{-n}\right)^\dagger \qquad \hat{\widetilde{\alpha}}^\mu_n = \left(\hat{\widetilde{\alpha}}^\mu_{-n}\right)^\dagger. \tag{104}$$

We find that in particular that

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu, \tag{105}$$

and for the oscillator operators,

$$[\hat{\alpha}^\mu_n, \hat{\alpha}^\nu_m] = \left[\hat{\widetilde{\alpha}}_n, \hat{\widetilde{\alpha}}^\nu_m\right] = n\eta^{\mu\nu}\delta_{m+n,0}. \tag{106}$$

To make this more conventional we can define

$$\hat{a}^\mu_n = \frac{\hat{\alpha}^\mu_n}{\sqrt{n}} \qquad (\hat{a}^\mu_n)^\dagger = \frac{\hat{\alpha}^\mu_{-n}}{\sqrt{n}}, \tag{107}$$

which are the conventional harmonic oscillator operators with $\left[\hat{a}^\mu_n, (\hat{a}^\nu_m)^\dagger\right] = \delta_{n,m}\eta^{\mu\nu}$.

Therefore, we can as before, define a Fock space vacuum $|0\rangle$ such that $\hat{a}^\mu_n|0\rangle = 0$ and thus excited states by

$$\prod_{i=1}^{n}\left(\left(\hat{a}^{\mu_i}_{n_i}\right)^\dagger\right)^{n_i}|0\rangle. \tag{108}$$

Notice that $\left[\hat{a}^0_n, \left(\hat{a}^0_m\right)^\dagger\right] = -\delta_{nm}$ gives

$$\langle 0| a^0_1 {a^0_1}^\dagger |0\rangle = -1, \tag{109}$$

which is a problematic state of negative norm that we need to address.

6 Lecture: Continuing String Quantization

04/02/2025

Recall that $\hat{x}^\mu$ and $\hat{p}^\mu$ denote the string center of motion and our oscillator operators give a tower of simple harmonic oscillators.

Observe that a generic string ground state takes the form $|0, p^\mu\rangle$ defined by $\hat{P}^\mu |0, p\rangle = p^\mu |0, p\rangle$ and still $\hat{a}_n^\mu |0, p\rangle = 0$, with excited states as before. Recall that we still have negative norm states from the time-like oscillator creating ghosts with $\left[\hat{a}_n^0, \left(\hat{a}_m^0\right)^\dagger\right] = -\delta_{n,m}$ giving

$$\langle 0, p'| \, \hat{a}_1^0 \left(\hat{a}_1^0\right)^\dagger |0, p\rangle = -\delta^{(D-1)} \left(p - p'\right). \tag{110}$$

6.1 Light-cone quantization

We proceed with the light-cone quantization of the string. Taking the conformal gauge action

$$S_{\text{conformal}} = -\frac{T}{2} \int_\Sigma \mathrm{d}^2\sigma \, \eta^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X. \tag{111}$$

Recall that this has the residual gauge transformations of

$$\sigma \to \widetilde{\sigma}\left(\sigma, \tau\right) \qquad \tau \to \widetilde{\tau}\left(\sigma, \tau\right) \qquad \eta_{\alpha\beta} \to \widetilde{\eta}_{\alpha\beta} = \Omega^2\left(\sigma, \tau\right) \eta_{\alpha\beta}. \tag{112}$$

In worldsheet lightcone coordinates $\sigma^\pm = \tau \pm \sigma$ which have metric components $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ and $\eta_{++} = \eta_{--} = 0$, one has

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\alpha\beta} \, \mathrm{d}\sigma^\alpha \, \mathrm{d}\sigma^\beta && (113)\\ &= -\, \mathrm{d}\tau^2 + \mathrm{d}\sigma^2 && (114)\\ &= -\, \mathrm{d}\sigma^+ \, \mathrm{d}\sigma^- \,. && (115)\end{aligned}$$

Under $\sigma^+ \to \widetilde{\sigma}^+\left(\sigma^+\right)$ and $\sigma^- \to \widetilde{\sigma}^-\left(\sigma\right)$, it is clear the metric will transform as

$$\eta \to \widetilde{\eta} = \frac{\mathrm{d}\widetilde{\sigma}^+}{\mathrm{d}\sigma^+} \frac{\mathrm{d}\widetilde{\sigma}^-}{\mathrm{d}\sigma^-} \eta. \tag{116}$$

This can be absorbed by a Weyl transformation. This explains what the residual gauge symmetry is: namely, to independently redefine the two light cone coordinates describing the left and right movers. This is a small (measure zero) fraction of our previous gauge symmetry and is preserved by the conformal gauge condition.

We can use this residual gauge symmetry to eliminate one of the spacetime coordinates, say $X^0\left(\sigma, \tau\right) = X_L^0\left(\sigma^+\right) + X_R^0\left(\sigma^+\right)$. Namely, we can set

$$\widetilde{\sigma}^+ = \frac{2}{R} X^0\left(\sigma^+\right) \text{ and } \widetilde{\sigma}^- = \frac{2}{R} X_R^0\left(\sigma^-\right). \tag{117}$$

Then we see that $X^0 = X_L^0 + X_R^0 = \frac{R}{2}\left(\widetilde{\sigma}^+ + \widetilde{\sigma}^-\right) = R\widetilde{\tau}$.

This condition is exactly what we called the *static gauge condition* before. However this is not the most convenient gauge to take for quantization.

Definition 6.1: We define *spacetime lightcone coordinates* to be

$$X^{\pm} \equiv \sqrt{\frac{1}{2}}\left(X^0 \pm X^1\right), \tag{118}$$

and X^i for $i = 2, \cdots D-1$. Namely, $X^i \in \mathbb{R}^{D-2}$ lives in the transverse directions to $X^{\pm}$.

The spacetime metric then has components $\eta^{(D)}_{+-} = \eta^{(D)}_{-+} = -1$, $\eta^{(D)}_{\pm\pm} = 0$ and $\eta_{ij} = \delta_{ij}$, namely, in this basis,

$$\eta^D = \begin{pmatrix} & -1 & & & \\ -1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}. \tag{119}$$

Then we have an inner product given by

$$X_\mu Y^\mu = \eta^{(D)}_{\mu\nu} X^\mu Y^\mu = -X^+ Y^- - X^- Y^+ + \vec{X} \cdot \vec{Y}, \tag{120}$$

where $\vec{X} \cdot \vec{Y} = \delta_{ij} X^i Y^i$.

We can then define the *lightcone gauge condition*

$$X^+(\sigma, \tau) = x^+ + \alpha' p^+ \tau. \tag{121}$$

In this gauge, (and with lightcone coordinates on the worldsheet as well) the Virasoro constraints take the form

$$T_{++} = \eta_{\mu\nu} \partial_+ X^\mu \partial_+ X^\nu = -2\partial_+ X^+ \partial_+ X^- + \partial_+ \vec{X} \cdot \partial_+ \vec{X}, \tag{122}$$

where recall that ∂_+ is a worldsheet lightcone index and X^+ is a spacetime lightcone index.

The lightcone gauge condition implies that

$$\partial_+ X^+ = \frac{1}{2}\alpha' p^+. \tag{123}$$

Hence $T_{++} = T_{--} = 0$ implies

$$\partial_+ X^- = \frac{1}{\alpha' p^+} \partial_+ \vec{X} \cdot \partial_+ \vec{X} \qquad\qquad \partial_- X^- = \frac{1}{\alpha' p^+} \partial_- \vec{X} \cdot \partial_- \vec{X}. \tag{124}$$

We can use these constraints to determine X^- as well. Observe that as $X^-(\sigma, \tau) = X^-_L(\sigma^+) + X^-_R(\sigma^-)$ which have mode expansion

$$X^-_L\left(\sigma^+\right) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n=0} \frac{1}{n} \widetilde{\alpha}^-_n e^{-in\sigma^+} \tag{125}$$

$$X^-_R\left(\sigma^-\right) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n=0} \frac{1}{n} \alpha^-_n e^{-in\sigma^-}. \tag{126}$$

We can then solve for α_n^- in terms of the other oscillator variables. Plugging this into Eq. (124), we see

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m\in\mathbb{Z}}\vec{\alpha}_{n-m}\cdot\vec{\alpha}_m \tag{127}$$

$$\widetilde{\alpha}_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m\in\mathbb{Z}}\widetilde{\alpha}_{n-m}\cdot\widetilde{\alpha}_m, \tag{128}$$

where $\alpha_0^- = \widetilde{\alpha}_0^- = \sqrt{\frac{\alpha'}{2}}p^-$.

For $n = 0$, we see

$$M^2 = -p_\mu p^\mu = 2p^+p^- - |\vec{p_T}|^2 \tag{129}$$

$$= \frac{4}{\alpha'}\sum_{n>0}\vec{\alpha}_{-n}\cdot\vec{\alpha}_n \tag{130}$$

$$= \frac{4}{\alpha'}\sum_{n>0}\vec{\widetilde{\alpha}}_{-n}\cdot\vec{\widetilde{\alpha}}_n. \tag{131}$$

The only difference to before is that we have eliminated the oscillators in the lightcone directions. They are eliminated by the gauge constraints and imposing the Virasoro constraints. These constraints tell us that only the transverse oscillators of the string contribute.

This is analogous to the fixing of the gauge in QED in Gupta-Bleuer, removing the unphysical polarizations of the photon. The light cone components of the oscillators can be thought of as giving unphysical polarisations of the string.