String Theory

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2025-01-26

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1 Lecture: Introduction

23/01/2025

1.1 Motivation

String theory is a candidate theory of quantum gravity. Before we discuss string theory itself, it is worthwhile to discuss gravity, namely classical gravity. One has that the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{1}$$

which are the equations of motion for the action

$$S = S_{\text{E.H.}} + S_{\text{matter}}, \tag{2}$$

where

$$S_{EH} = \frac{1}{16\pi G_N} \int_{\mathcal{M}} \mathrm{d}^4 x \, \sqrt{-g} R, \tag{3}$$

where $g = \det g$. One can think of this as a field theory where the dynamical field is the metric, $g_{\mu\nu}$. One has stress energy tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{4}$$

When one translates to the quantum theory, the quantum fields and thus the stress energy tensor will too, $T_{\mu\nu} \to \hat{T}_{\mu\nu}$. It is difficult to imagine that the Einstein Hilbert action, and thus the left hand side of Einstein's equations are not quantum as well.

G_N appears as a coupling constant, and is a dimensional one with $[G_N] = M^{-1}L^3T^{-2}$. Then one can convert it to a mass scale (*planck*) in natural units with

$$8\pi G_N = \frac{\hbar c}{M_{\text{pl}}^2}. \tag{5}$$

Expanding around the "vacuum" of flat Minkowski space, then,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\text{pl}}}. \tag{6}$$

Then the action becomes

$$S_{\text{EH}} = \int_{\mathcal{M}} \mathrm{d}^4x\, \mathcal{L}_{\text{EH}}, \tag{7}$$

where

$$\mathcal{L}_{\text{EH}} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \tag{8}$$

where

$$\mathcal{L}_0 = -\frac{1}{4}\partial_\mu h^\rho{}_\rho \partial^\mu h^\sigma{}_\sigma + \frac{1}{2}\partial_\mu h^{\rho\sigma}\partial^\mu h_{\rho\sigma}, \tag{9}$$

which is the Lagrangian of a free massless spin 2 particle, one can call the *graviton.* We then have interaction Lagrangian

$$\mathcal{L}_{\text{int}} \sim \frac{1}{M_{\text{pl}}} h\left(\partial h\right)^2 + \frac{1}{M_{\text{pl}}^2} h^2 \left(\partial h\right)^2. \tag{10}$$

However these interactions are irrelevant and thus lead to UV divergences at ≥ 2 loops. This is a non-renormalizable theory. The effective theory then goes $g_{\text{eff}} \sim \frac{E}{M_{\text{pl}}}$ becomes strong at $E \sim M_{\text{pl}}$. This is $M_{\text{pl}} \sim 10^{18}$ GeV which is so far above anything accessible to colliders today, string theory is often consider too academic a pursuit for those bound by phenomenological interests.

We need a UV completion of S_{EH} which will consist of

- a consistent quantum theory, namely, unitary, Lorentz invariant, UV finite (or renormalizable)
- a theory that reduces to GR at low energy $E \ll M_{\text{pl}}$ such that

$$S_{\text{eff}} \sim S_{\text{EH}} + \cdots. \tag{11}$$

There is no other theory that does both. String theory is the unique candidate.

1.2 String Theory

A QFT is a theory of interacting relativistic particles. This necessarily involves unitarity, Lorentz invariance and locality. There are very few possibilities but still an infinite countable list.

String theory considers replacing point particles with strings. These can be closed or open one dimensional objects at fixed time. Propagating in time these generate two dimensional surfaces called *worldsheets*.

It is remarkable that this is unique and there are no free parameters. It is UV finite and closed strings give rise to gravitons and open strings to gauge fields and fermions. It also reduces to GR with matter at low energy.

It is not without baggage. It requires supersymmetry and extra dimensions to be well defined. For bosons this is $D = 26$ and for fermions $D = 10$.

While we have a unique theory, there are many ground states (like the Higgs boson) which make phenomenological predictions hard.

2 Lecture: Classical and Quantum String

25/01/2025

Work in $\mathbb{R}^{1,D-1}$ with metric

$$\eta_{\mu\nu} = \mathrm{diag}\left(-1, +1, \cdots, +1\right), \tag{12}$$

with line element

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu\,. \tag{13}$$

For timelike $\mathrm{d}x^\mu$, $\mathrm{d}s^2 < 0$.

We define a D-vector as X^μ with $\mu \in \{0, 1, \cdots, D-1\}$ and an inner product

$$X \cdot Y = X_\mu Y^\mu = \eta_{\mu\nu} X^\mu Y^\nu. \tag{14}$$

Sometimes we also think about a curved metric $G_{\mu\nu}\left(x\right)$.

2.1 Relativistic point particle

As a warm up, we will study a relativistic point particle of rest mass m moving in $\mathbb{R}^{1,D-1}$.

One should think of a worldline X as a map from a parameter $W \to \mathbb{R}^{1,D-1}$ that takes

$$\tau \mapsto X^\mu\left(\tau\right), \tag{15}$$

where τ is called the proper time.

The natural choice of an action is one proportional to the proper time and thus the length of the worldline. Namely, one takes

$$S = \mathcal{N} \int_W \sqrt{-\,\mathrm{d}s^2} = \mathcal{N} \int_W \mathrm{d}\tau\, \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}}. \tag{16}$$

Exercise 1: Fix $\mathcal{N}$ such that for $X^\mu\left(\tau\right) = \left(t\left(\tau\right), \vec{x}\left(\tau\right)\right)$, the three velocity $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}$, $v = |\vec{v}|$, and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\left(v\right) = \frac{1}{\sqrt{1 - v^2}}, \tag{17}$$

one can set

$$S = \int_W \mathrm{d}t\, L\left(t\right), \tag{18}$$

with

$$L = \mathcal{N}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1}\sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right)^2} = \mathcal{N}\sqrt{1-v^2}. \tag{19}$$

This Lagrangian has conjugate momenta

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = -\frac{\mathcal{N}\vec{v}}{\sqrt{1-v^2}} = m\gamma\left(v\right)\vec{v}, \tag{20}$$

where we have identified $\mathcal{N} = -m$.

By Legendre transform, we find a Hamiltonian

$$E = H = \vec{p}\cdot\vec{v} - L = mv^2\gamma\left(v\right) + m\gamma^{-1}\left(v\right) = m\gamma\left(v\right), \tag{21}$$

which is the correct relativistic energy of a point particle.

Therefore for a point particle, we can formally state the action

$$S = -m\int_W \mathrm{d}\tau = \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}}. \tag{22}$$

The form of this action makes the symmetries of it manifest (clear and explicit). Namely, one can see

- Poincare invariance, such that $X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + C^\mu$, where the Lorentz transformation is defined by

$$\Lambda^\mu{}_\rho \eta^{\rho\sigma}\Lambda^\nu{}_\sigma = \eta^{\mu\nu}. \tag{23}$$

- reparametrization invariance $\tau \to \widetilde{\tau}\left(\tau\right)$ given that $\frac{\mathrm{d}\widetilde{\tau}}{\mathrm{d}\tau} > 0$ as

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \to \frac{\mathrm{d}X^\mu}{\mathrm{d}\widetilde{\tau}} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\widetilde{\tau}}, \tag{24}$$

and as the measure transforms as $\mathrm{d}\tau = \frac{\mathrm{d}\widetilde{\tau}}{\mathrm{d}\tau}\,\mathrm{d}\widetilde{\tau}$, one has

$$S = -m\int_W \mathrm{d}\widetilde{\tau}\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\widetilde{\tau}}\frac{\mathrm{d}X^\nu}{\mathrm{d}\widetilde{\tau}}}. \tag{25}$$

This redundancy can be thought of as a gauge symmetry. If one gauge fixes $\widetilde{\tau} = t$, then we recover the previous action

$$S = -m\int \mathrm{d}t\,\sqrt{1-v^2}, \tag{26}$$

however this form does not make the symmetries manifest, including the gauge one which is explicitly broken.

2.2 Relativistic String

Consider an open string and a closed string, with *worldsheets* that look like the plane and a cylinder.

These worldsheets Σ in $\mathbb{R}^{1,D-1}$ are the analogue of the world line for a 1D point particle.

We introduce worldsheet coordinates $\sigma^\alpha = (\tau, \sigma)$ with $\alpha = 0, 1$.

The spatial coordinate σ satisfies

- $\sigma \in [0, \pi]$ for the open string,
- $\sigma \sim \sigma + 2\pi$, for the closed string.

We need to describe how this worldsheet is embedded in spacetime. This is the *string configuration* and is a map

$$\begin{aligned} X : \Sigma &\to \mathbb{R}^{1,D-1} \\ (\tau, \sigma) &\mapsto X^\mu (\sigma, \tau) \end{aligned} \tag{27}$$

Definition 2.1: The **pullback** of the Minkowski metric $\eta_{\mu\nu}$ to Σ is given by

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma^\alpha} \frac{\mathrm{d}X^\nu}{\mathrm{d}\sigma^\beta}. \tag{28}$$

This coincides with the induced metric on an embedded surface.

One can write it more explicitly with $\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma^0} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \dot{X}^\mu$ and $\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma^1} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma} = X'^\mu$ giving us

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix}. \tag{29}$$

One can then define

$$S_{\text{Nambu-Goto}} = -T \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det \gamma} = -T \int_\Sigma \mathrm{d}^2\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 \dot{X'}^2}, \tag{30}$$

which describes the natural notion of area on the string worldsheet, the extension of the notion of length of a worldline.

T has dimensions of M^2 and is the *tension* of the string as we will see.

2.3 Polyakov Action

We introduce a dynamical metric $g_{\alpha\beta} (\sigma, \tau)$ on Σ with determinant $g = \det g$ and signature $(-1, +1)$. Then we see

$$S_{\text{Polyakov}} = -\frac{T}{2} \int \mathrm{d}^2\sigma \sqrt{-g} \left(g^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu\right) = -\frac{T}{2} \int \mathrm{d}^2\sigma \sqrt{-g} \left(g^{\alpha\beta} \left(\partial_\alpha X\right) \cdot \left(\partial_\beta X\right)\right), \tag{31}$$

where one can also write

$$g^{\alpha\beta} \gamma_{\alpha\beta} = g^{\alpha\beta} \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}. \tag{32}$$

Note. The worldsheet is instrumental here. We are thinking of X as a map from the worldsheet to spacetime

$$X : \Sigma \to \mathbb{R}^{1,D-1}. \tag{33}$$

However one can think of the field theory itself as living on the worldsheet of the string. The worldsheet is then a 2D spacetime and the spacetime is a *field space* or a *target space*. Namely, one thinks of the spacetime coordinates as scalar fields on Σ and the string actions as defining 2D field theories.

The Polyakov action then, is a number of 2D scalar fields coupled to 2D gravity on Σ.

We then consider the equation of motion of the 2D metric $g_{\alpha\beta}$ which is

$$\frac{\delta S_{\text{Polyakov}}}{\delta g_{\alpha\beta}} = 0. \tag{34}$$

Equivalently, as the energy momentum tensor of the field theory is given by

$$\mathcal{T}^{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{Polyakov}}}{\delta g_{\alpha\beta}} = 0, \tag{35}$$

notice that one can integrate out $g_{\alpha\beta}$ and recover the Nambu-Goto action