Symmetries, Fields and Particles

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Lecture 1
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1 Introduction

Symmetries are hidden throughout undergraduate physics. Lagrangian mechanics relies on the principle of least action, where the action S is given by

$$S = \int_{t_1}^{t_2} \mathrm{d}t \, L\left(q\left(t\right), \dot{q}\left(t\right); t\right). \tag{1}$$

Classical trajectories minimise S which gives us the Euler Lagrange equation,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{2}$$

> **Theorem 1.1 (Noether's Theorem):** Invariance of L under some transformation implies an associated conserved quantity.

Example. Take a particle in a 3-dimensional potential which has Lagrangian

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(x, y, z\right). \tag{3}$$

There are a few notable symmetries here

1. L is independent of time t, i.e. under $t \mapsto t + \delta t$.

   Claim. The Hamiltonian $H = T + U$ is conserved.

   In general $H\left(x_i, p_i\right)$ is a function of $x_i = (x, y, z)$ and the conjugate momenta $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$ and is written in terms of the Lagrangian through Legendre transform as

   $$H\left(x_i, p_i; t\right) = \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L. \tag{4}$$

   Therefore, if L does not depend on time one has

   $$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 - \frac{\partial L}{\partial t} = 0, \tag{5}$$

   where we have used the Euler Lagrange equations to make the first term vanish.

2. If L is invariant under $x \mapsto x + \delta x$,

   $$\frac{\partial L}{\partial x} = 0 \overset{\text{EL}}{\Rightarrow} \frac{\partial L}{\partial \dot{x}} = p_x \text{ is constant.} \tag{6}$$

3. If L is invariant under rotations about the z axis then the z-component of angular momentum $L_z = xp_y - yp_x$ is constant.

   Similarly, in cylindrical coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$ and the Lagrangian becomes

   $$L = \frac{1}{2}\left(m\dot{\rho}^2 + \rho^2\dot{\theta} + \dot{z}^2\right) - U\left(\rho, z\right). \tag{7}$$

   Therefore, $\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = m\rho^2\dot{\theta} = xp_y - yp_x = \text{constant}.$

1.1 Symmetry in Quantum Mechanics

Given a system whose states are elements of a Hilbert space $\mathcal{H}$. Here, symmetry implies there exists some invertible operator $U : \mathcal{H} \to \mathcal{H}$ which preserves inner products, up to an overall phase $e^{i\phi}$ (e.g. expectation values, transition amplitudes).

Definition 1.1: Let $|\Phi\rangle, |\Psi\rangle$ be any normalised vectors in $\mathcal{H}$. Denote $|U\Psi\rangle = U|\Psi\rangle$. U is a **symmetry transformation operator** if

$$|\langle U\Phi|U\Psi\rangle| = |\langle \Phi|\Psi\rangle| . \tag{8}$$

Proposition 1.1(Wigner's theorem): Symmetry transformation operators are either

a) linear and unitary, or

b) anti-linear and anti-unitary, meaning for $\alpha, \beta \in \mathbb{C}$,

$$U\left(\alpha|\Psi\rangle + \beta|\Phi\rangle\right) = \alpha^* U|\Psi\rangle + \beta^*|\Phi\rangle , \tag{9}$$

and

$$\langle U\Phi|U\Psi\rangle = \langle \Phi|\Psi\rangle^* , \tag{10}$$

respectively.

Most symmetries fall into the former category, but a notable exception is time-reversal symmetry, falling into the latter.

Suppose we have a system with time independent Hamiltonian H. We can write down the time evolution of operators in the Schrödinger picture (where the states depend on time and the operators are static) as

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle . \tag{11}$$

Let's look at applying a symmetry operator U in each of the cases above.

a)

$$\begin{aligned} \langle U\Phi(t)|U\Psi(t)\rangle &= \langle \Phi(t)|\Psi(t)\rangle & (12)\\ &= \langle \Phi(t)|\, e^{-iHt}|\Psi(0)\rangle . & (13) \end{aligned}$$

We should find the same result by transforming $|\Psi(0)\rangle$ before the evolution

$$|U\Psi(t)\rangle = e^{-iHt}|U\Psi(0)\rangle , \tag{14}$$

which implies

$$\begin{aligned} \langle U\Phi(t)|U\Psi(t)\rangle &= \langle U\Phi(t)|\, e^{-iHt}|U\Psi(0)\rangle & (15)\\ &= \langle \Phi(t)|\, U^\dagger e^{-iHt}U|\Psi(0)\rangle . & (16) \end{aligned}$$

By comparing this to Eq. (13) we find that

$$U^\dagger e^{-iHt}U = e^{-iHt}. \tag{17}$$

Therefore U commutes with the Hamiltonian, $[U, H] = 0$.

Examples.

1) If H commutes with p, H cannot depend on x as $[x_i, p_j] = i\delta_{ij} \neq 0$. Therefore H is invariant under translations $x \to x + a$. One can construct a unitary operator that generates translations with $U = \exp(i\mathbf{p} \cdot \mathbf{a})$.
2) If H is rotationally symmetric the angular momentum operator commutes with H.

2 Lie Groups and algebras

2.1 Lie Groups

Definition 2.1: A **group** is a set G together with a binary operation $\circ$ such that the following properties hold

i) Closure: $g_2 \circ g_1 \in G$, $\forall g_1, g_2 \in G$,

ii) Associativity: $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$, $\forall g_1, g_2, g_3 \in G$,

iii) Identity: $\exists e \in G$ such that $g \circ e = e \circ g = g$, $\forall g \in G$,

iv) Inverse: $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \circ g^{-1} = e = g^{-1} \circ g$.

The identity e and inverse of g are unique.

Proof. Assume there exists e_1, e_2 which are both identities. Then we have that $e_1 \circ e_2 = e_1$ but also $e_1 \circ e_2 = e_2$ thus $e_1 = e_2$ and we have uniqueness.

For inverses, suppose g has two inverses h and j. One has that

$$g \circ h = e \text{ and } g \circ j = e. \tag{18}$$

Left multiplying by j and h respectively we see that

$$j \circ g \circ h = j \circ e \text{ and } h \circ g \circ j = h \circ e, \tag{19}$$

where simplifying (as we can by associativity) the left operation, we see

$$e \circ h = j \text{ and } e \circ j = h, \tag{20}$$

both of which imply $h = j$ and thus we have uniqueness. $\square$

Definition 2.2: A group $(G, \circ)$ is **commutative** (**abelian**) if

$$g_1 \circ g_2 = g_2 \circ g_1, \tag{21}$$

$\forall g_1, g_2 \in G$. Otherwise G is **non-commutative** (**non-abelian**).

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Definition 2.3: A **manifold** is a space which looks like Euclidean space ($\mathbb{R}^n$) locally. A **differentiable manifold** is one which satisfies certain smoothness conditions.

Definition 2.4: A **Lie group** consists of a differentiable manifold G along with a binary operation $\bullet$ such that the group axioms hold and that the operations $\left(\bullet, \cdot^{-1}\right)$ are smooth operations.

2.2 Matrix Lie Groups

The general linear group $GL\left(,\mathbb{F}\right)$ is the group of invertible $n \times n$ matrices over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Namely,

$$GL\left(n, \mathbb{F}\right) = \{M \in \mathrm{Mat}_n\left(\mathbb{F}\right) \mid \det M \neq 0\}. \tag{22}$$

The group operation is matrix multiplication and inverses are defined as $\det M \neq 0$.

The dimension of $GL\left(n, \mathbb{R}\right)$ is n^2, and thus we have n^2 free parameters.

For $GL\left(n, \mathbb{C}\right)$, the real dimension is $2n^2$ and the complex dimension is n^2.

There are a number of important subgroups of $GL\left(n, \mathbb{F}\right)$.

1. The *special linear group*, denoted $SL\left(n, \mathbb{F}\right) = \{M \in GL\left(n, \mathbb{F}\right) \mid \det M = 1\}$, where the constraint leaves us with a dimension of $n^2 - 1$.
2. The *orthogonal group*, denoted $O\left(n\right) = \{M \in GL\left(n, \mathbb{R}\right) \mid M^T M = I\}$. Notice that
$$M^T M = I \Rightarrow \det M = \pm 1. \tag{23}$$
3. The *special orthogonal group*, denoted $SO\left(n\right) = \{M \in O\left(n\right) \mid \det M = 1\}$
4. The *pseudo-orthogonal group*, where we define an $(n+m) \times (n+m)$ (metric) matrix by
$$\eta \equiv \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}. \tag{24}$$
This group is denoted
$$O\left(n, m\right) = \{M \in GL\left(n+m, \mathbb{R}\right) \mid M^T \eta M = \eta\}. \tag{25}$$
Similarly, there is a *special* subset of this group denoted $SO\left(n, m\right) \Rightarrow \det M = 1$.
5. The *unitary* matrices, which are denoted
$$U\left(n\right) = \{M \in GL\left(n, \mathbb{C}\right) \mid M^T M = I\}. \tag{26}$$
As before, we also have $SU\left(n\right)$ which restricts to matrices with $\det M = 1$.
6. The *pseudo-unitary* group, given by
$$U\left(n, m\right) = \{M \in GL\left(n, \mathbb{C}\right) \mid M^T \eta M = \eta\}. \tag{27}$$
7. The *symplectic group*, for which we define a fixed, antisymmetric $2n \times 2n$ matrix, such as
$$\Omega \equiv \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{28}$$
The symplectic group is then
$$\mathrm{Sp}\left(2n, \mathbb{R}\right) = \{M = GL\left(2n, \mathbb{R}\right) \mid M^T \Omega M = \Omega\}. \tag{29}$$
One can show that $M \in \mathrm{Sp}\left(2n, \mathbb{R}\right)$ satisfies $\det M = 1$.

Definition 2.5: Given a $2n \times 2n$ antisymmetric matrix A, its **Pfaffian** is given by

$$\mathrm{Pf}A \equiv \frac{1}{2^n n!} \varepsilon_{i_1 i_2 \cdots i_{2n}} A^{i_1 i_2} A^{i_3 i_4} \cdots A^{i_{2n-1} i_{2n}}, \tag{30}$$

where $\varepsilon_{i_1 i_2 \cdots i_n}$ is the totally antisymmetric symbol $\varepsilon_{i_1 i_2 \cdots i_n} = -\varepsilon_{i_2 i_1 \cdots i_n}$.

2.3 Group elements as transformations

We can define actions of group elements $g \in G$ on a set X. X might be G itself, but could also be a vector space (i.e. rotation matrices acting on vectors in $\mathbb{R}^3$).

Definition 2.6: The **left action** of G on X is a map $L : G \times X \to X$ such that for $x \in X$

- $L(e, x) = x$, for e, the identity of G,
- $L(g_2, L(g_1 x)) = L(g_2 g_1, x)$, $\forall x \in X$, $\forall g_1, g_2 \in G$.

The more usual notation is that $\forall g \in G$, we associate a map $g : X \to X$ such that $g(x) = gx$, however this is slightly less clear.

Definition 2.7: The **right action** of G on X is defined by $gX \to X$ such that $g(x) = xg^{-1}$, $\forall x \in X$ and $g \in G$.

The inverse preserves group composition. Namely,

$$g_2(g_1(x)) = x \underbrace{g_1^{-1} g_2^{-1}}_{(g_2 g_1)^{-1}} = (g_2 g_1)(x). \tag{31}$$

Definition 2.8: **Conjugation** by G on X is the action defined by

$$g(x) = gxg^{-1}, \tag{32}$$

$\forall g \in G_1$, $x \in X$.

Another definition worth making, even if it won't see immediate use is that of an *orbit*.

Definition 2.9: Given a group G and set X, an **orbit** of an element $x \in X$ is the set of elements of X which are in the image of an action of G on x.

Example. If the action is left, the orbit of $x \in X$ is written $Gx = \{gx \mid g \in G\}$.

It can be shown that the set of orbits under G 'partition' X as we will see.

2.4 Orthogonal groups

The orthogonal group, $O(n)$ in particular, represent rotations and reflections on $\mathbb{R}^n$. This preserves inner products such that

$$\langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \mathbf{v}_2^T \mathbf{v}_1, \tag{33}$$

given $R \in O(n)$,

$$\langle R\mathbf{v}_2, R\mathbf{v}_1\rangle = \mathbf{v}_2^T \underbrace{\left(R^T R\right)}_{I} \mathbf{v}_1 = \langle \mathbf{v}_2, \mathbf{v}_1\rangle . \tag{34}$$

This is similar for $U(n)$.

Consider

$$SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in (0, 2\pi) \right\}. \tag{35}$$

As cos and sin are smooth functions, this is a differentiable manifold. One can also show that $R(\theta_2) R(\theta_1) = R(\theta_1 + \theta_2)$.

Similarly, $SO(3)$ can represent rotations of vectors in $\mathbb{R}^3$ where the axis of the rotation is given by a unit vector $\mathbf{n} \in S^2$ and we rotate by an angle θ. Note that rotation by $\theta \in [-\pi, 0]$ about $\mathbf{n}$ is equivalent to a rotation by $-\theta$ about $-\mathbf{n}$ so we confine to $\theta \in [0, \pi]$.

Therefore we can depict the manifold of $SO(3)$ as a ball of radius π in $\mathbb{R}^3$, where the direction is specified by $\mathbf{n}$ and the distance from the origin is specified by $\theta \in [0, \pi]$. Antipodal points are identified such that $\pi\mathbf{n} = -\pi\mathbf{n}$.

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3 Lie Algebras

3.1 Pseudo orthogonal group

$SO(n, m)$ act on vectors in $\mathbb{R}^{n+m}$ and preserve the scalar product

$$v_2^T \eta v_1, \tag{36}$$

for $v_1, v_2 \in \mathbb{R}^{n+m}$. For example, $SO(1, 1)$ parametrise Lorentz boosts in one dimension and can be written in terms of the *rapidity* η as

$$SO(1,1) = \left\{ \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \middle| \eta \in \mathbb{R} \right\}. \tag{37}$$

As η is unbounded, $SO(1, 1)$ is clearly noncompact.

3.2 Parametrization of Lie Groups

At least in small neighbourhoods, we can assign coordinates on an n-dimensional manifold to be

$$x := \left(x^1, \cdots, x^n\right) \in \mathbb{R}^n. \tag{38}$$

This allows us to label elements $g(x) \in G$. Closure provides

$$g(y)\, g(x) = g(z). \tag{39}$$

Smoothness gives us that the components of z are continuously differentiable functions of x and y such that for $i \in 1, \cdots, n$,

$$z^i = \phi^i(x, y). \tag{40}$$

We choose the coordinate origin such that $g(0) = e$. Identity gives us that

$$g(0)\,g(x) = g(x) \Rightarrow \phi^i(x,0) = x^r \text{ and } \phi^i(0,y) = y^i. \tag{41}$$

Similarly, for inverses, we have that there exists some $\widetilde{x}$ such that $g(\widetilde{x}) = g(x)^{-1}$ and thus

$$\phi^i(\widetilde{x},x) = 0 = \phi^i(x,\widetilde{x}). \tag{42}$$

Lastly, associativity gives us

$$g(z)\,(g(y)\,g(x)) = (g(z)\,g(y))\,g(x) \Rightarrow \phi^i(\phi(x,y),z) = \phi^i(x,\phi(y,z)). \tag{43}$$

This appears like a Leibniz rule/Jacobi identity as we will see.

3.3 Lie Algebras

A Lie group is homogeneous. Any neighbourhood 'looks like' (or in a more formal sense, can be mapped to) any other neighbourhood.

For example, for $\varepsilon \in G$ close to g_1, $g_2 g^{-1}\varepsilon$ is close to g_2.

Thus no neighbourhood in particular is special. The natural choice of the representative neighbourhood to study is the one centered at the identity of G. We will linearize near the identity of G.

Definition 3.1: A Lie Algebra is a vector space V, which additionally has a vector product, the **Lie bracket**, $[\cdot,\cdot] : V \times V \to V$ satisfying the following properties for $X, Y, Z \in V$.

1) It is antisymmetric, $[X,Y] = -[Y,X]$,
2) It satisfies the Jacobi identity, $[X,[Y,Z]] + [Y,[X,Z]] + [Z,[X,Y]] = 0$,
3) It is linear such that for $\alpha, \beta \in \mathbb{F}$, $[X, \alpha Y + \beta Z] = \alpha [X,Y] + \beta [X,Z]$.

Note. Any vector space which has a vector product $\star : V \times V \to V$ can be made into a Lie Algebra with its Lie bracket given by

$$[X,Y] = X \star Y - Y \star X. \tag{44}$$

Definition 3.2: Let's choose a basis for V, given by $\{T_a\}$ for $a = 1, \cdots, n = \dim V$. We call these basis vectors **generators** of the Lie algebra, and we write their Lie brackets as

$$[T_a, T_b] = f^c{}_{abc} T_c, \tag{45}$$

where $f^c{}_{ab} \in \mathbb{F}$ are called **structure constants**.

Antisymmetry implies $f^c{}_{ba} = -f^c{}_{ab}$ and the Jacobi identity implies

$$f^e{}_{ad} f^d{}_{bc} + f^e{}_{cd} f^d{}_{ab} + f^e{}_{bd} f^d{}_{ca} = 0. \tag{46}$$

The general element of a Lie algebra can be written as a linear combination of $\{T_a\}$ as

$$X \in V \Rightarrow X = X^a T_a \text{ with } x^a \in \mathbb{F}, \tag{47}$$

which gives us the bracket of any two elements in terms of structure constants with

$$[X, Y] = X^a Y^b f^c{}_{abc} T_c. \tag{48}$$

3.4 Lie Groups and their Lie Algebras

Take $g(\theta) \in SO(2)$ to be

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{49}$$

where $e = I_2 = g(0)$. Points near the identity have $\theta \ll 1$ and thus Taylor expanding the components of $g(\theta)$ we see

$$\begin{aligned} g(\theta) &= I_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \theta_2^2 I_2 + \mathcal{O}\left(\theta^3\right) & (50) \\ &= e + \underbrace{\theta \left.\frac{\mathrm{d}g}{\mathrm{d}\theta}\right|_{g=0}}_{\text{tangent vector}} + \frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} + \mathcal{O}\left(\theta^2\right), & (51) \end{aligned}$$

where the linear term is tangent to the manifold. Here there is a one dimensional tangent space at e given by

$$T_e(SO(2)) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}. \tag{52}$$

This is the Lie algebra of $SO(2)$,

$$\mathfrak{so}(2) := L(SO(2)) := T_e(SO(2)). \tag{53}$$

It remains to show this.

Proof. Notice that

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} -ab & 0 \\ 0 & -ab \end{pmatrix} = -abI, \tag{54}$$

and thus for any two elements (matrices) of the Lie algebra, they commute (which is trivially antisymmetric and satisfying of Jacobi). Linearity similarly follows immediately by inspection. □

Similarly, one can show $\dim(SO(n)) = \frac{1}{2}n(n-1) \equiv d$, so we have coordinates $x_1 \cdots, x_d$. Consider a single-parameter family of $SO(n)$ elements,

$$M(t) := M(\mathbf{x}(t)) \in SO(n), \tag{55}$$

such that $M(0) = I_n$. Orthogonality ($M^T M = I$) implies

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(M^T(t) M(t)\right) \tag{56}$$

$$= \frac{\mathrm{d}M^T}{\mathrm{d}t} + M^T \frac{\mathrm{d}M}{\mathrm{d}t}, \tag{57}$$

where looking at $t = 0$, as $M(0) = I_n$ we see

$$\frac{\mathrm{d}M^T}{\mathrm{d}t} = -\frac{\mathrm{d}M}{\mathrm{d}t}, \tag{58}$$

which implies matrices in the tangent space of $SO(n)$ are antisymmetric (and thus traceless as well).

We have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \sum_i \frac{\partial M}{\partial x_i} \frac{\mathrm{d}x_i}{\mathrm{d}t}. \tag{59}$$

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4 The Exponential Map

Observe that

$$T_e(\mathcal{O}(n)) = T_e(SO(n)), \tag{60}$$

as $\det I = 1$, so all curves passing through I have $\det M = 1$.

4.1 Unitary Groups

Let $M(t)$ be a curve in $SU(n)$ with $M(0) = I$. For small t, write $M(t) = I + tX + \mathcal{O}(t^2)$, where $X = \frac{\mathrm{d}M}{\mathrm{d}t}\Big|_{t=0}$.

Unitarity of M provides that for all t,

$$I = M^\dagger M \tag{61}$$

$$U = I + t\left(X + X^\dagger\right) + \mathcal{O}\left(t^2\right), \tag{62}$$

which implies $X^\dagger = -X$, namely, elements of the tangent space are *anti-Hermitian.*

Claim. $\operatorname{tr} X = 0$ for $X \in L(SU(n))$ or $M \in SU(n)$

Proof. Look at

$$M(t) = \begin{pmatrix} 1 + tX_{11} & tX_{12} & \cdots & tX_{1n} \\ tX_{21} & t + tX_{22} & \cdots & tX_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ tX_{n1} & tX_{n2} & \cdots & 1 + tX_{nn} \end{pmatrix}. \tag{63}$$

Notice that

$$1 = \det M = 1 + \underbrace{t \operatorname{tr} X}_{0} + \mathcal{O}\left(t^2\right), \tag{64}$$

where the underbraced term (and higher order ones) must vanish. □

For $U(n)$, X can have non-zero trace.

4.2 Lie algebra of a matrix Lie group

Consider two curves $g_1(x(t))$ and $g_2(x(t))$ through the identity e of some Lie group G. We define

$$X_1 := \dot{g}_1\Big|_{t=0}, \qquad X_2 := \dot{g}_2\Big|_{t=0}. \tag{65}$$

One can define a product

$$g_3(z(t)) = g_2(y(t))\, g_1(x(t)) \in G, \tag{66}$$

satisfying

$$\dot{g}_3\Big|_{t=0} = (\dot{g}_2 g_1 + g_2 \dot{g}_1)\Big|_{t=0} \tag{67}$$

$$= X_2 + X_1 \in T_e(G), \tag{68}$$

another vector in the tangent space.

The Lie bracket arises from the *group commutator*.

Definition 4.1: The **group commutator** of $g_1, g_2 \in G$, is

$$[g_1, g_2]_G := g_1^{-1} g_2^{-1} g_1 g_2 := h \in G. \tag{69}$$

Returning to our two curves through the identity e, $g_i(t)$ for $i \in \{1, 2\}$, we can expand

$$g_i(t) = e + tX_i + t^2 W_i + \mathcal{O}\left(t^3\right). \tag{70}$$

We have that

$$g_1(t)\, g_2(t) = e + t(X_1 + X_2) + t^2 (X_1 X_2 + W_1 + W_2) + \mathcal{O}\left(t^3\right), \tag{71}$$

and

$$g_2(t)\, g_1(t) = e + t(X_1 + X_2) + t^2 (X_2 X_1 + W_1 + W_2) + \mathcal{O}\left(t^3\right). \tag{72}$$

If we then look at

$$h(t) = [g_2(t)\, g_1(t)]^{-1} g_1(t)\, g_2(t) = e + t^2 \underbrace{(X_1 X_2 - X_2 X_1)}_{[X_1, X_2]} + \cdots, \tag{73}$$

and thus the group commutator induces the Lie bracket in the algebra. As $h(t) \in G$, the tangent to $h(t)$ at e is $[X_1, X_2] \in L(G)$, and thus we have closure under the Lie bracket.

- We write the tangent space to a matrix Lie group $G \overset{\text{subgroup}}{<} GL(n, \mathbb{F})$ at a general element p as $T_p(G)$. Let $g(t)$ be a curve in the manifold through p with $g(t_0) = p$, and thus

$$g(t+\varepsilon) = g(t_0) + \dot{\varepsilon}(t_0) + \mathcal{O}\left(\varepsilon^2\right). \tag{74}$$

As both $g(t_0), g(t_0+\varepsilon) \in G$, there exists $h_p(\varepsilon) \in G$ such that

$$g(t_0+\varepsilon) = g(t_0)\, h_p(\varepsilon), \tag{75}$$

and as $\varepsilon \to 0$, $h_p(\varepsilon) \to e$. For small ε,

$$h_p(\varepsilon) = e + \varepsilon X_p + \mathcal{O}\left(\varepsilon^2\right), \tag{76}$$

for some $X_p \in L(G) = T_e(G)$. Neglecting $\mathcal{O}\left(\varepsilon^2\right)$,

$$\begin{aligned} e + \varepsilon X_p = h_p(\varepsilon) &= g^{-1}(t_0)\, g(t_0+\varepsilon) && (77)\\ &= g^{-1}(t_0)\left[g(t_0) + \varepsilon \dot{g}(t_0)\right] && (78)\\ &= e + \varepsilon \underbrace{g^{-1}(t_0)\, \dot{g}(t_0)}_{X_p}. && (79) \end{aligned}$$

Claim. Conversely, for any $X \in L(G)$, there exists a unique curve $g(t)$ with $g^{-1}(t)\,\dot{g}(t) = X$ and $g(0) = g_0$.

Proof. This is a consequence of existence and uniqueness of solutions of ODEs. The solution of this ODE is

$$g(t) = g_0 \exp(tX), \tag{80}$$

where

$$\exp tX := \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}. \tag{81}$$

□

4.3 One parameter subgroups

Given an $X \in L(G)$, the curve

$$g_X(t) = \exp tX, \tag{82}$$

forms an *abelian* subgroup of G, *generated* by X.

Notice that $g_X(t)$ is isomorphic to the group of real numbers under addition $(\mathbb{R}, +)$ if only $g_X(0) = e$. If there exist other $t_0 \neq 0$ such that $g_X(t_0) = 0$, then we have periodic structure and then $g_X(t)$ is isomorphic to the circle S^1. Lecture 5 22/10/2024

4.4 Lie Groups from Lie Algebras

Definition 4.2: Given a Lie algebra $L(G)$ of a Lie group G, we can define the **exponential map**:

$$\exp : L(G) \to G, \tag{83}$$

which for matrix Lie groups, is

$$X \mapsto \exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!}. \tag{84}$$

Locally, the map is bijective (one to one). For the proof see Hall Section 2.7. Globally, the map is generally, not.

Example. For example, $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$ and we have

$$L(U(1)) = \{ix \mid x \in \mathbb{R}\}, \tag{85}$$

where clearly $\exp(ix)$ is not one to one globally since $e^{2\pi n i} = 1$, $\forall n \in \mathbb{Z}$.

Example. $G = O(n)$. Let $X \in L(O(n)) \subset \mathrm{Skew}_n(\mathbb{R})$. Let $M = \exp tX$, and observe that as X is antisymmetric, $M^T = [\exp X]^T = \exp(-tX)$. Therefore,

$$MM^T = I = M^T M, \tag{86}$$

and thus we recover $M \in O(n)$.

Note. $\operatorname{tr} X = 0$. Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of X, and observe that

$$\begin{aligned} \det M &= \det(\exp tX) &(87)\\ &= \exp(\operatorname{tr} tX) &(88)\\ &= \exp(0) &(89)\\ &= 1. &(90) \end{aligned}$$

and thus $M \in SO(n)$. Thus elements of $O(n)$ with determinant -1 are not in the image of the exponential map.

Therefore, $O(n)$ is a disconnected manifold. One can think of $O(n)$ as two disconnected islands, one with $\det M = 1$ containing the identity called *proper rotations*, and another containing elements with $\det M = -1$ called *improper rotations* as they contain a reflection.

One can show that $A \in \mathrm{Skew}_n(\mathbb{R})$ implies $A \in L(SO(n))$ or $L(O(n))$.

Define $\gamma(t) := \exp tA$ to be a curve of matrices on some manifold. By above, we see that

$$(\gamma(t))^T (\gamma(t)) = I, \tag{91}$$

and thus $\det \gamma(t) = 1$ which implies $\gamma(t) \in SO(n)$. By construction, $A = \dot{\gamma}(t)\Big|_{t=0}$ and thus is tangent to the curve at the identity of $SO(n)$ suggesting $A \in L(SO(n))$. Therefore

$$\dim SO(n) = \dim L(SO(n)) = \dim(\mathrm{Skew}_n(\mathbb{R})) = \frac{n(n-1)}{2}. \tag{92}$$

4.5 Group product from Lie bracket

Recall the Baker-Campbell-Haussdorff (BCH) formula, namely that for $X, Y \in L(G)$, we have

$$\exp(tX)\exp(tY) = \exp(tZ), \tag{93}$$

where

$$Z = X + Y + \frac{t}{2}[X, Y] + \frac{t^2}{12}([X, [X, Y]] + [Y, [X, Y]]) + \mathcal{O}\left(t^3\right). \tag{94}$$

One can show this order by order in t. As $L(G)$ is closed under the Lie bracket, $Z \in L(G)$ and thus $\exp tZ \in G$.

5 Representation Theory

Groups and their elements represent transformations under which a system or object is invariant. Representations of groups tell us how the action of the group transforms vectors in a vector space.

We saw $GL(n, \mathbb{F})$ as a group of invertible matrices. These matrices are equivalently linear maps (automorphisms) on the vector space $\mathbb{F}^n$ with

$$GL(n, \mathbb{F}) : \mathbb{F}^n \to \mathbb{F}^n. \tag{95}$$

We generalize this notation to act on any vector space V such that

$$GL(V) : V \to V. \tag{96}$$

If V is finite dimensional, we can choose a basis and recover the original definition.

5.1 Lie group representations

Definition 5.1: A **representation** D of a group G is a smooth group homomorphism

$$D : G \to GL(V), \tag{97}$$

from G to the group of automorphisms on some vector space V called the **representation space**, associated with D.

That is, $\forall g \in G$, $D(g) : V \to V$ is an invertible, linear map such that for a vector $v \in V$,

$$v \mapsto D(g)\, v. \tag{98}$$

This map is linear such that

$$D(g)(\alpha v_1 + \beta v_2) = \alpha D(g)\, v_1 + \beta D(g)\, v_2, \tag{99}$$

$\forall \alpha, \beta \in \mathbb{F}$, $v_1, v_2 \in V$. Further, the group homomorphism holds such that we have

$$D(g_2 g_1) = D(g_2)\, D(g_1), \tag{100}$$

$\forall g_1, g_2 \in G$. This group homomorphism property implies that

$$D(e) = \mathrm{id}_V, \tag{101}$$

and by an identical argument,

$$D(g)^{-1} = D\left(g^{-1}\right). \tag{102}$$

Definition 5.2: The **dimension** of a representation D is the dimension of the representation space V on which it acts.

If V is finite dimensional, say $\dim V = N$, then $GL(V)$ is isomorphic to $GL(N, \mathbb{F})$. Lecture 6 24/10/2024

6 The Adjoint Representation

Definition 6.1: The **kernel** of a map $D : G \to GL(V)$ consists of all elements of G which map to the identity, $\mathrm{id}_V = I$.

Definition 6.2: A representation D is said to be **faithful** if $D(g) = \mathrm{id}_V$ only for $g = e$. Namely, if $\ker D = \{e\}$.

Faithfulness implies that D is injective, i.e. $D(g_1) = D(g_2) \Rightarrow g_1 = g_2$.

Proof. Assume D is faithful and that $D(g_1) = D(g_2)$. Then,

$$D\left(g_1^{-1}\right) D(g_1) = D\left(g_1^{-1}\right) D(g_2) \tag{103}$$

$$D\left(g_1^{-1} g_1\right) = D\left(g_1^{-1} g_2\right) \tag{104}$$

$$D(e) = D\left(g_1^{-1} g_2\right) \tag{105}$$

$$\mathrm{id}_V = D\left(g_1^{-1} g_2\right), \tag{106}$$

where as D is faithful, $g_1^{-1} g_2 = e \Rightarrow g_1 = g_2$. □

Examples. We look at $G = (\mathbb{R}, +)$.

1) For some fixed, $k \in \mathbb{R}$, $D(\alpha) = e^{k\alpha}$, $\forall \alpha \in G$ is a one-dimensional representation.

   One can check that this is a representation, namely, that it respects the group multiplication through a homomorphism

   $$D(\alpha) D(\beta) = e^{k\alpha} e^{k\beta} = e^{k(\alpha+\beta)} = D(\alpha + \beta). \tag{107}$$

   For $k \neq 0$, this is a faithful representation as $D(\alpha) = 1 \Rightarrow \alpha = 0$ and thus $\ker D = \{0 \equiv \mathrm{id}_G\}$.

2) For $k = 0$, $D(\alpha) = 1 \forall \alpha$, and thus $\ker D = G$. This is not faithful and is called the *trivial representation*.

3) We can similarly define $D(\alpha) = e^{ik\alpha}$, for $k \in \mathbb{R}$. This is not faithful as $\ker D = \{\frac{2\pi n}{k} \mid n \in \mathbb{Z}\}$. Here $V = \mathbb{C}$.

4) A two dimensional representation can also be defined with

   $$D(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{108}$$

   where $V = \mathbb{R}^2$.

5) Lastly, one can define an infinite dimensional representation. Let

$$V = \{\text{space of all real functions } f(x)\}, \tag{109}$$

and let

$$D(\alpha) f(x) = f(x-\alpha). \tag{110}$$

We see that $D(\alpha) f = f,\ \forall f \in V \Rightarrow \alpha = 0$, and thus the representation is faithful.

Definition 6.3: The **trivial representation** D_0 is where

$$D_0(g) = 1, \tag{111}$$

$\forall g \in G$. This is not faithful as $\ker D = G$ and the dimension of D_0 is 1.

Quantities which are invariant under group transformations, transform in the trivial representation. In physics, we call these **singlets**.

Note. One can form a trivial representation of any dimension M such that $D(g) = I_m$, $\forall g \in G$. This representation is *reducible* (as we will define) and can be thought of as m copies of the dimension one trivial representation.

Definition 6.4: If G is a matrix Lie group, then the **fundamental** or **defining representation** D_f is given by

$$D_f(g) = g, \tag{112}$$

$\forall g \in G$.

Only $D_f(e) = e$ thus it is faithful. If $G \subset GL(n, \mathbb{F})$, then $\dim D_f = n$.

Let G be a matrix Lie group and consider its Lie algebra as a vector space $V = L(G)$.

Definition 6.5: The **adjoint representation** $D^{\text{adj}} \equiv \text{Ad}$ is the map

$$\text{Ad} : G \to GL(L(G)), \tag{113}$$

such that $\forall g \in G$,

$$\text{Ad}_g : L(G) \to L(G), \tag{114}$$

with

$$\text{Ad}_g X = gXg^{-1}, \tag{115}$$

$\forall X \in L(G)$. This is action by conjugation.

Let's check that this is a representation.

- *Closure*: For $X \in L(G)$, there is a curve in G such that

$$g(t) = e + tX + \cdots. \tag{116}$$

For any $h \in G$, we have another curve

$$\widetilde{g}(t) = hg(t)h^{-1} \tag{117}$$
$$= e + t\underbrace{hXh^{-1}}_{\in L(G)} + \cdots . \tag{118}$$

Therefore $\mathrm{Ad}_h X = hXh^{-1} \in L(G)$ and thus we have closure.

- *Group homomorphism:* The group operation is preserved as

$$(\mathrm{Ad}_{g_2 g_1}) X = (g_2 g_1) X (g_2 g_1)^{-1} \tag{119}$$
$$= g_2 \left(g_1 X g_1^{-1}\right) g_2^{-1} \tag{120}$$
$$= \mathrm{Ad}_{g_2}(\mathrm{Ad}_{g_1} X) \tag{121}$$
$$= (\mathrm{Ad}_{g_2})(\mathrm{Ad}_{g_1})(X) . \tag{122}$$

- *The Lie bracket:* The Lie bracket is preserved as well as

$$\mathrm{Ad}_g([X,Y]) = g[X,Y]g^{-1} \tag{123}$$
$$= \left[gXg^{-1}, gYg^{-1}\right] \tag{124}$$
$$= [\mathrm{Ad}_g X, \mathrm{Ad}_g Y] . \tag{125}$$

6.1 Lie algebra representations

Definition 6.6: A **representation**, d, of a Lie algebra $L(G)$ is a map from $L(G)$ to a set of linear maps with $\mathfrak{gl}(V) = L(GL(V))$, where the Lie bracket is preserved (instead of the group operation).

That is, for each $X \in L(G)$, we have a map $d(X): V \to V$, a linear map (not necessarily invertible) such that

$$v \mapsto d(X)v, \tag{126}$$

$\forall v \in V$.

Linearity implies that for $X, Y \in L(G)$, we have $d(\alpha X + \beta Y) = \alpha d(X) + \beta d(Y)$. As we also want to preserve the bracket, we need

$$d([X,Y]) = [d(X), d(Y)], \tag{127}$$

$\forall X, Y \in L(G)$.

Definition 6.7: The **dimension** of $d = \dim V$.

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The Lie algebra also admits a trivial representation,

$$d_0(X) = 0 \in V, \tag{128}$$

$\forall X \in L(G)$.

The fundamental representation also follows identically and we have

$$d_f(X) = X \in V, \tag{129}$$

$\forall X \in L(G)$.

Lastly we rewrite the adjoint representation. Recall that it can be thought of as the action of the Lie algebra on itself.

Definition 6.8: The **adjoint representation** of a Lie algebra can be written

$$\mathrm{ad} : L(G) \to \mathfrak{gl}(L(G)) . \tag{130}$$

Then, for $X \in L(G)$,

$$\mathrm{ad}_X : L(G) \to L(G) , \tag{131}$$

such that

$$\mathrm{ad}_X Y = [X, Y] , \tag{132}$$

$\forall Y \in L(G)$.

6.2 From The Lie Group Reps to the Lie Algebra Reps

As before, consider tangent curves in G

$$g(t) = e + tX + \cdots . \tag{133}$$

We expand the corresponding elements of the representation D of G as

$$D(g(t)) = \mathrm{id}_V + td(X) + \cdots . \tag{134}$$

We use this expansion to define d from D and we can check that the Lie bracket is preserved. Namely,

$$D\left(g_1^{-1} g_2^{-1} g_1 g_2\right) = D(g_1)^{-1} D(g_1) D(g_2) , \tag{135}$$

where expanding the left hand side we see

$$D\left(g_1^{-1} g_2^{-1} g_1 g_2\right) = D\left(e + t^2 [X_1, X_2] + \cdots\right) \tag{136}$$

$$= \mathrm{id}_V + t^2 d([X_1, X_2]) . \tag{137}$$

Expanding $g_i(t) = e + tX_i + \cdots$, we see that the right hand side of Eq. (135) then becomes

$$D(g_1)^{-1} D(g_1) D(g_2) = \mathrm{id}_V + t^2 [d(X_1), d(X_2)] , \tag{138}$$

and thus equating the two sides, we arrive at

$$d([X_1, X_2]) = [d(X_1), d(X_2)] , \tag{139}$$

is a Lie algebra homomorphism.

Example. The adjoint representation ad_X can be obtained from Ad_g. Namely, given $Y \in L(G)$,

$$\mathrm{Ad}_g Y = gYg^{-1} \tag{140}$$

$$= (I + tX) Y (I - tX) \tag{141}$$

$$= Y + t[X, Y] \tag{142}$$

$$= (I + t\mathrm{ad}_X) Y, \tag{143}$$

and thus $\mathrm{ad}_X Y = [X, Y]$ as expected.

6.3 Useful concepts

Definition 6.9: Representations D_1 and D_2 of G (or d_1 and d_2 of $L(G)$) are **equivalent** if there exists an invertible linear maps R, such that

$$D_2(g) = RD_1(g)R^{-1}, \tag{144}$$

$\forall g \in G$ (or $X \in L(G)$).

Definition 6.10: A representation d of $L(G)$ with representation space V has an **invariant subspace** $W \subseteq V$ if $\forall w \in W$ and $X \in L(G)$,

$$d(X)w \in W. \tag{145}$$

Example. If all $d(X)$ are all upper triangular matrices, $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, then there is an invariant subspace

$$W = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \right\}. \tag{146}$$

Definition 6.11: An **irreducible representation** ("*irrep*") is a representation with no nontrivial invariant subspaces.

Otherwise, the representation is **reducible**.

Definition 6.12: A **direct sum** of vector spaces U and V is written

$$U \oplus W = \{(u, w) \mid u \in U, w \in W\}, \tag{147}$$

where $(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2)$ and $\alpha(u, w) = (\alpha u, \alpha w)$. Note that

$$\dim U \oplus W = \dim U + \dim W. \tag{148}$$

Definition 6.13: A **totally reducible** representation d of $L(G)$ (or $D(G)$) can be decomposed into irreducible pieces. Namely, it's representation spaces can be written as a direct sum of irreducible representation spaces,

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k, \tag{149}$$

such that $d(X)w_i \in W_i$ for all $X \in L(G)$ and $w_i \in W_i$. Then, there exists some basis where $d(X)$ becomes block diagonal such that

$$d(X) = \begin{pmatrix} d_1(X) & 0 & \cdots & 0 \\ 0 & d_2(X) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_k(X) \end{pmatrix}. \tag{150}$$

We often write $d = \widetilde{d}_1 \oplus \cdots \oplus \widetilde{d}_k$.

Definition 6.14: An N-dimensional representation (for N finite) D is **unitary** if $D(g) = U(N)$, $\forall g \in G$.

Identically d is unitary if $d(X)$ if $d(X) \in L(U(N))$, $\forall X \in L(G)$.

If all $D(g)$ are real, then $D(g) \in O(N)$ then D is said to be orthogonal. Most of these claims rely on d being finite dimensional.

Theorem 6.1 (Maschke): A finite-dimensional unitary representation is either irreducible or totally reducible.

Proof. (Sketch) For each invariant subspace W, the orthogonal component $W_\perp$ is also invariant. This implies we can separate the representation space into

$$V = W \oplus W_\perp. \tag{151}$$

Then similarly we can decompose W and $W_\perp$ into any further invariant spaces if they exist (and repeat until there are no more invariant subspaces). If V is finite dimensional then this process must terminate. □

Note. There are a few things of note after this definition storm. Maschke's theorem can be extended to

- all finite representations of discrete groups
- all finite representations of compact Lie groups

Example. Take $V = \{$ all 2 π periodic functions $f : \mathbb{R} \to \mathbb{R}$, $f(x + 2\pi) = f(x)\}$. Take the representation to be

$$(D(\alpha) f)(x = f(x - a)). \tag{152}$$

Recall that this is not faithful. We have invariant subspaces given by

$$W_n = \{f(x) = a_n \cos nx + b_n \sin nx \mid a_n, b_n \in \mathbb{R}\}, \tag{153}$$

which are one dimensional. One can then write

$$V = W_0 \oplus W_1 \oplus W_2 \oplus \cdots = \bigoplus_{n=0}^{\infty} W_n, \tag{154}$$

which is a direct sum of invariant subspaces, each occurring once. Lecture 8 29/10/2024

W_n is invariant as

$$a_n \cos n(x - \alpha) + b_n \sin n(x - \alpha) = a_n' \cos(nx) + b_n' \sin nx, \tag{155}$$

for some $a_n', b_n' \in \mathbb{R}$. Recall that the Fourier decomposition of any 2π periodic function can be written

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{156}$$

Definition 6.15: Let V and W be vector spaces. The **tensor product space** $V \otimes W$ is spanned by elements, **product vectors**, $v \otimes w$ with $v \in V$ and $w \in W$ satisfying

- linearity, such that $v \otimes (\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 v \otimes w_1 + \lambda_2 v \otimes w_2$, and identically in the first component.
- $\dim (V \otimes W) = (\dim V)(\dim W)$

With a product state $\Phi = v \otimes w$, we write

$$\Phi_A = \Phi_{\alpha a} = v_\alpha w_a, \tag{157}$$

where $\alpha = 1, \cdots, \dim V$, $a = 1, \cdots, \dim W$ and $A = 1, \cdots, \dim V \otimes W$.

Not all elements of $V \otimes W$ are product states (as they can be linear combinations).

Definition 6.16: Let $D^{(1)}$ and $D^{(2)}$ be representations of a group G with representation spaces V and W. These satisfy

$$D^{(1)}(g) : v_\alpha \mapsto D^{(1)}(g)_{\alpha\beta} v_\beta, \ v \in V, \tag{158}$$

$$D^{(2)}(g) : w_a \mapsto D^{(2)}(g)_{ab} w_b, \ w \in W. \tag{159}$$

The **tensor product representation** $D^{(1)} \otimes D^{(2)}$ is

$$\left(D^{(1)} \otimes D^{(2)}\right)(g)(v \otimes w) = \left(D^{(1)}(g)\, v\right) \otimes \left(D^{(2)}(g)\, w\right). \tag{160}$$

Let $g_t \in G$ be a curve in the Lie group G with $g_0 = e$ and $\dot{g}_0 = X \in L(G)$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(D^{(1)} \otimes D^{(2)}\right)(g_t)(v \otimes w)\right] = \left[\frac{\mathrm{d}}{\mathrm{d}t} D^{(1)}(g_t)\, v\right]_{t=0} \otimes D^{(2)}(g_0)\, w + D^{(1)}(g_0)\, v \otimes \left[\frac{\mathrm{d}}{\mathrm{d}t} D(g_t)\, w\right]_{t=0}. \tag{161}$$

Let $d^{(1)}$ and $d^{(2)}$ be Lie algebra representations corresponding to $D^{(1)}$ and $D^{(2)}$. Their tensor product is given by

$$\left(d^{(1)} \otimes d^{(2)}\right)(X) = d^{(1)}(X) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d^{(2)}(X). \tag{162}$$

There is an important corollary to Maschke's theorem.

Corollary 6.1: Representations of $d^{(1)} \otimes d^{(2)}$ can be, if finite, be written as the direct sum of irreducible representations of $L(G)$, $\widetilde{d}_i$ such that

$$d^{(1)} \otimes d^{(2)} = \widetilde{d}_1 \oplus \cdots \oplus \widetilde{d}_k = \bigoplus_{i=1}^{k} \widetilde{d}_i. \tag{163}$$

The is the desired decomposition into irreducible representations.

6.4 Angular momentum: $SO(3)$ and $SU(2)$

$SO(3)$ describes rotations in 3 dimensions and appears when studying the quantization of angular momentum in quantum mechanics. When studying spin angular momentum, we find half integer quantum numbers which lead to $SU(2)$ representations.

The Lie algebra of $SU(2)$ is given by

$$\begin{aligned}\mathfrak{su}(2) &= L(SU(2)) &(164)\\ &= \{\ 2\times 2 \text{ traceless, anti-hermitian matrices }\} &(165)\\ &= \left\{X\in \mathrm{Mat}_2(\mathbb{C}) \mid X^\dagger = -X, \mathrm{tr}\, X = 0\right\}. &(166)\end{aligned}$$

We can choose as a basis $t_a = -\frac{i}{2}\sigma_a$, where $a = 1,2,3$ and σ_a are the Pauli matrices. Recall that

$$\sigma_a\sigma_b = I\delta_{ab} + i\varepsilon_{abc}\sigma_c, \tag{167}$$

which implies

$$[T_a, T_b] = \varepsilon_{abc}T_c, \tag{168}$$

and thus the structure constants of $SU(2)$ are $f^c{}_{ab} = \varepsilon_{abc}$.

Similarly, for $SO(3)$, we see that

$$\mathfrak{so}(3) = L(SO(3)) = \mathrm{Skew}_3. \tag{169}$$

We have a basis of the form

$$\widetilde{T}_1 = \begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \qquad \widetilde{T}_2 = \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}, \qquad \widetilde{T}_3 = \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}, \tag{170}$$

namely, such that

$$\left(\widetilde{T}_a\right)_{bc} = -\varepsilon_{abc}\widetilde{T}_c, \tag{171}$$

and thus

$$\left[\widetilde{T}_a, \widetilde{T}_b\right] = \varepsilon_{abc}\widetilde{T}_c, \tag{172}$$

and thus $SO(3)$ has the same structure constants as $SU(2)$.

To show that these algebras are isomorphic, we would need an isomorphism

$$\phi : \mathfrak{g} \to \mathfrak{h}, \tag{173}$$

such that

$$\phi([X,Y]) = [\phi(X), \phi(Y)], \tag{174}$$

$\forall X, Y \in \mathfrak{g}$.

While, $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are (as their structure constants are the same, $SU(2)$ and $SO(3)$ are in fact not isomorphic, as we will see.

When we discussed $SO(3)$ earlier, we were picturing it as a 3-ball of radius π spanned by a unit vector $\mathbf{n}$ and an angle $0 \leq \theta \leq \pi$ with antipodes identified.

For $SU(2)$, take $U \in SU(2)$ we can write it as

$$U = a_0 I + i\mathbf{a} \cdot \boldsymbol{\sigma}, \tag{175}$$

with $(a_0, \mathbf{a}) \in \mathbb{R}^4$ and $a_0^2 + |\mathbf{a}|^2 = 1$. Therefore $SU(2)$ as a manifold is a unit sphere in $\mathbb{R}^4$, S^3.

Definition 6.17: Let H be a subgroup of G. For any $g \in G$, we can form a **left coset** of H as

$$gH = \{gh \mid h \in H\}, \tag{176}$$

and a right coset given by

$$Hg = \{hg \mid h \in H\}. \tag{177}$$

Definition 6.18: If $H \overset{\text{subgroup}}{<} G$ is a **normal subgroup** of G, $H \lhd G$ if $gH = Hg$, $\forall g \in G$.

Definition 6.19: Define a set G/H to be

$$G/H = \{gH \mid g \in G\}. \tag{178}$$

We define coset multiplication by

$$(g_2 H)(g_1 H) = (g_2 g_1) H. \tag{179}$$

Theorem 6.2: For $H \lhd G$, G/H is a group under coset multiplication, with $H = eH$ as the identity element.

Definition 6.20: Such a group G/H is called a **quotient group** or **factor group**.

Next, we will show that

$$SO(3) \simeq SU(2)/\mathbb{Z}_2, \tag{180}$$

with $\mathbb{Z}_2 = (I_2, -I_2)$. Lecture 9 31/10/2024

Definition 6.21: The center of a group is the set of all $x \in G$ which satisfy $xg = gx$, $\forall g \in G$.

Theorem 6.3: The center $Z(G) \trianglelefteq G$ is a normal subgroup of G.

Proof.

□

$SU(2)$ has centre $Z(SU(2)) = \{I_2, -I_2\} \cong \mathbb{Z}_2 = \{1, -1\}$.

We then look at cosets of the form $UZ(SU(2))$ for $U \in SU(2)$ and see

$$UZ(SU(2)) = \{U, -U\}. \tag{181}$$

The set of all such cosets forms the quotient group $SU(2)/\mathbb{Z}_2$ whose manifold is S^3 with antipodes identified, or equivalently just the upper half of S^3 ($a_0 \geq 0$) with opposite points on the equator identified.

One can see that this is just a curved picture of the $SO(3)$ manifold, as we claim

$$SO(3) \cong SU(2)/\mathbb{Z}_2. \tag{182}$$

We desire an explicit map to show this isomorphism.

One can define the map $\rho : SU(2) \to SO(3)$. For $A \in SU(2)$, $\rho(A) = R$ with components

$$R_{ij} = \frac{1}{2} \operatorname{tr}\left(\sigma_i A \sigma_j A^\dagger\right), \tag{183}$$

for $i = 1, 2, 3$. This is a 2 to 1 map as both $A, -A \mapsto \rho(A) = \rho(-A)$. This is called a **double covering** of $SO(3)$.

One also says that $SU(2)$ is the *double cover* of $SO(3)$.

Proposition 6.1: Every Lie algebra is the Lie algebra of exactly one **simply-connected** Lie group.

Definition 6.22: A manifold is **simply connected** if it is path connected and any closed loop can be smoothly contracted to a point.

6.5 Representations of $\mathfrak{su}(2)$

Observe that $T_a = -i\frac{\sigma_a}{2}$ are generators of the algebra. It is convenient to enlarge this real vector space to the field $\mathbb{C}$. Given a real vector space V,

$$V := \{\lambda^a T_a \mid \lambda^a \in \mathbb{R}\} = \operatorname{span}_{\mathbb{R}}\{T_a\}, \tag{184}$$

the *complexification* of V is

$$V_{\mathbb{C}} = \operatorname{span}_{\mathbb{C}}\{T_a\}. \tag{185}$$

For example, we have

$$\mathfrak{su}(n) = \{X \in \operatorname{Mat}_n(\mathbb{C}) \mid X^\dagger = -X, \operatorname{Tr} X = 0\}, \tag{186}$$

becomes

$$\mathfrak{su}_{\mathbb{C}}(n) = \{X \in \operatorname{Mat}_n(\mathbb{C}) \mid \operatorname{tr} X = 0\} \cong \mathfrak{sl}(n, \mathbb{C}). \tag{187}$$

Let $\mathfrak{g} = L(G)$ be a real Lie algebra and denote its complexification by $\mathfrak{g}_{\mathbb{C}} = L(G)_{\mathbb{C}}$. A representation d of $L(G)$ can be extended to $L(G)_{\mathbb{C}}$ by imposing

$$d(X + iY) = d(X) + id(Y), \tag{188}$$

where $X, Y \in L(G)$ and $X + iY \in L(G)_{\mathbb{C}}$.

Conversely, if we have a representation $d_{\mathbb{C}}$ of $L(G)_{\mathbb{C}}$ we can restrict it to the representation d of $L(G)$ by writing

$$d(X) = d_{\mathbb{C}}(X), \tag{189}$$

for $X \in L(G) \subset L(G)_{\mathbb{C}}$.

Definition 6.23: A **real form** of a complex Lie algebra $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ whose complexification is $\mathfrak{h}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}$.

In general a complex Lie algebra can have multiple non-isomorphic real forms.

Now moving to $\mathfrak{su}(2)$, we see

$$\mathfrak{su}(2)_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{\sigma_a \mid a = 1, 2, 3\}. \tag{190}$$

There exists a more convenient basis (Cartan-Weyl basis), with

$$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{191}$$

$$E_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{192}$$

$$E_0 = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{193}$$

Observe that we have

$$[H, E_\pm] = \pm 2E_\pm, \qquad [E_+, E_-] = H. \tag{194}$$

Recall that $\mathrm{ad}_X Y = [X, Y]$ and thus

$$[H, E_\pm] = \mathrm{ad}_H E_\pm = \pm 2E_\pm. \tag{195}$$

We also have

$$[H, H] = \mathrm{ad}_H H = 0. \tag{196}$$

We see that E_-, H and E_+ are eigenvectors of ad_H with eigenvalues of $-2, 0, 2$. These eigenvalues are called the **roots** of $\mathfrak{su}(2)$.

Let d be a finite dimensional irreducible representation ("irrep") of $\mathfrak{su}(2)$ with representation space V. We write an eigenvector of $d(H) = v_\lambda$ where

$$d(H)\, v_\lambda = \lambda v_\lambda. \tag{197}$$

Definition 6.24: The eigenvalues of $d(H)$ are called the **weights** of the representation d.

Note. Roots are the weights of the adjoint representation.

The operators $d(E_\pm)$ are called **ladder** operators as

$$d(H)(d(E_\pm)v_\lambda) = \left\{ d(E_\pm)d(H) + \underbrace{[d(H), d(E_\pm)]}_{d([H,E_\pm])} \right\} v_\lambda \tag{198}$$

$$= (\lambda \pm 2)(d(E_\pm)v_\lambda), \tag{199}$$

and thus $d(E_\pm)v_\lambda$ is also an eigenvector of $d(H)$ with eigenvalue $\lambda \pm 2$, *or*, $d(E_\pm)v_\lambda = 0$. Lecture 10 02/11/2024

If d is a finite $((n)$ $-$dimensional) representation, there must be a finite number of eigenvalues. We take d to be irreducible here. There must be some Λ such that

$$d(H)v_\Lambda = \Lambda v_\Lambda \text{ and } d(E_+)v_\Lambda = 0. \tag{200}$$

Such a Λ is called a **highest weight**.

Applying $d(E_-)$, n times, we see

$$v_{\Lambda-2n} = (d(E_-))^n v_\Lambda. \tag{201}$$

This process must terminate for some integer N as d is finite dimensional. This implies that we have a basis of eigenvectors for this representation, $\{v_\Lambda, v_{\Lambda-2}, \cdots, v_{\Lambda-2N}\}$.

We have that for $1 \leq n \leq N$,

$$d(H)d(E_+)v_{\Lambda-2n} = (\Lambda - 2n + 2)d(E_+)v_{\Lambda-2n}. \tag{202}$$

Seeking to show that this is the set of all possible eigenvectors, we check if $d(E_+)v_{\Lambda-2n} \propto V_{\Lambda-2n+2}$. Observe that

$$d(E_+)v_{\Lambda-2n} = d(E_+)d(E_-)v_{\Lambda-2n+2} \tag{203}$$

$$= \left(d(E_-)d(E_+) + \underbrace{[d(E_+), d(E_-)]}_{d(H)} \right) v_{\Lambda-2n+2} \tag{204}$$

$$= d(E_-)d(E_+)v_{\Lambda-2n+2} + (\Lambda - 2n + 2)v_{\Lambda-2n+2}. \tag{205}$$

This is a recursion relation. Consider $n = 1$, for which we would have $d(E_-)d(E_+)v_\Lambda = d(E_-)(0) = 0$ and thus

$$d(E_+)v_{\Lambda-2} = 0 + \Lambda v_\Lambda. \tag{206}$$

For $n = 2$, observe that

$$d(E_+)v_{\Lambda-4} = d(E_-)\underbrace{d(E_+)v_{\Lambda-2}}_{\Lambda v_\Lambda} + (\Lambda - 2)v_{\Lambda-2} \tag{207}$$

$$= \Lambda d(E_-) v_\Lambda + (\Lambda - 2) v_{\Lambda-2} \tag{208}$$
$$= (2\Lambda - 2) v_{\Lambda-2}. \tag{209}$$

In general, we have

$$d(E_+) v_{\Lambda-2n} = r_n v_{\Lambda-2n+2}. \tag{210}$$

Plugging this into Eq. (205), we find

$$r_n = r_{n-1} + \Lambda - 2n + 2, \tag{211}$$

with $r_1 = \Lambda$ from Eq. (206). This has solution

$$r_n = (\Lambda + 1 - n) n. \tag{212}$$

As established, a finite number of eigenvalues implies that for $n = N$, $d(E_-) v_{\Lambda-2N} = 0$. This implies that

$$r_{N+1} \overset{!}{=} 0 = [(\Lambda + 1) - (N+1)(N+1)] = (\Lambda - N)(N+1) = 0 \tag{213}$$
$$\Rightarrow \Lambda = N. \tag{214}$$

Note. From this we can infer that the highest weights Λ have to be non-negative integers N. We will use these highest weights to classify/label the irreducible representations.

Namely, the finite-dimensional irreducible representations of $L(SU(2)) = \mathfrak{su}(2)$ are labelled by $\Lambda \in \mathbb{Z}_{\geq 0}$, d_Λ with weights

$$S_\Lambda = \{-\Lambda, -\Lambda + 2, \cdots, \Lambda - 2, \Lambda\}. \tag{215}$$

S_Λ is called the **weight set** of d_Λ. The weights are non-degenerate and thus $\dim d_\Lambda = \Lambda + 1$.

- d_0 is the trivial representation with $\dim d_0 = 1$
- d_1 is the fundamental/defining representation with $\dim d_1 = 2$.
- d_2 is the adjoint representation and has $\dim d_2 = 3$.

This discussion appears in quantum mechanics when discussing angular momentum. In that context, the angular momentum operators $\mathbf{J} = (J_1, J_2, J_3)$ have eigenstates

$$\mathbf{J} \cdot \mathbf{J} |j, m\rangle = j(j+1) |jm\rangle \tag{216}$$
$$J_3 |j, m\rangle = m |jm\rangle, \tag{217}$$

with $2j \in \mathbb{Z}_{\geq 0}$, $2m \in \mathbb{Z}$ with $-j \leq m \leq j$.

Then we can translate between these domains with

$$d(H) = 2H_3 \tag{218}$$
$$\Lambda = 2j \tag{219}$$
$$d(E_\pm) = J_1 \pm iJ_2, \tag{220}$$

and the eigenvalues are $\lambda = 2m$.

6.6 Representations of $SU(2)$ and $SU(3)$

$SU(2)$ is simply connected (while $SO(3)$ is not), so a representation d_Λ of $\mathfrak{su}(2)$ gives a representation D_Λ of $SU(2)$ via the exponential map.

For $SO(3)$ recall that $SO(3) \cong SU(2)/\mathbb{Z}_2$. Namely, an element in $SO(3)$ corresponds to a pair of elements in $SU(2)$. $\{-A, A\}$, $A \in SU(2)$. The representation has to respect this.

D_Λ is a representation of $SO(3)$ iff it respects the identification of A with $-A$, namely,

$$D_\Lambda(-A) = D_\Lambda(A). \tag{221}$$

It is sufficient to check whether $D_\Lambda(-I) = D_\Lambda(I)$.

For $H = \sigma_3$, we have $-I = \exp(i\pi H) \in SU(2)$ and thus

$$D_\Lambda(-I) = \exp\left(i\pi d_{\Lambda(H)}\right). \tag{222}$$

As we established that $d_\Lambda(H)$ has eigenvalues $\lambda \in \{-\Lambda, -\Lambda+2, \cdots, \Lambda-2, \Lambda\}$, the eigenvalues of $D_\Lambda(-I)$ are

$$e^{i\pi\lambda} = (-1)^\lambda = (-1)^\Lambda, \tag{223}$$

as λ all have the same parity as Λ. Therefore

- for Λ even, we get suitable irreducible representations of both $SU(2)$ and $SO(3)$,
- for Λ odd, they are suitable only for $SU(2)$. These are *spinor representations*.

6.7 Tensor products of $\mathfrak{su}(2)$ irreducible representations

Given an arbitrary tensor product of representations, we want to decompose it into the direct sum of irreducible representations. Take irreps d_Λ and $d_{\Lambda'}$ with $\Lambda, \Lambda' \in \mathbb{Z}_{\geq 0}$ and the spaces V_Λ and V'_Λ (decomposed from $V_\Lambda \otimes V_{\Lambda'}$).

For $X \in \mathfrak{su}(2)$,

$$(d_\Lambda \otimes d_{\Lambda'})(X)(v \otimes v') = (d_\Lambda(X)v) \otimes v' + v \otimes (d_{\Lambda'}(X)v'), \tag{224}$$

where $\dim(d_\Lambda \otimes d_{\Lambda'}) = (\Lambda+1)(\Lambda'+1)$.

Such a decomposition implies we can write

$$d_\Lambda \otimes d_{\Lambda'} = \bigoplus_{\Lambda'' \in \mathbb{Z}_{>0}} \mathscr{L}^{\Lambda''}_{\Lambda,\Lambda'} d_{\Lambda''}, \tag{225}$$

where $\mathscr{L}$ are called *Littlewood-Richardson coefficients* (or multiplicities).

We have bases for $V_\Lambda, V_{\Lambda'}$ given by $\{v_\lambda\}$ with $\lambda \in S_\Lambda = \{-\Lambda, \cdots, \Lambda\}$ and identically $\{v_{\lambda'}\}$ with $\lambda' \in S_{\Lambda'}$. The basis for $V_\Lambda \otimes V_{\Lambda'}$ is

$$\{v_\lambda \otimes v_{\lambda'} \mid \lambda \in S_\Lambda,\ \lambda' \in S_{\Lambda'}\}. \tag{226}$$

As a result, lets look at the action on the diagonal element H. We have

$$(d_\Lambda \otimes d_{\Lambda'})(H)(v_\lambda \otimes v_{\lambda'}) = \lambda v_\lambda \otimes v_{\lambda'} + v_\lambda \otimes (\lambda' v_{\lambda'}) \tag{227}$$

$$= (\lambda + \lambda')\, v_\lambda \otimes v_{\lambda'}, \tag{228}$$

and thus the weights add. The weight set for the tensor product rep is then

$$S_{\Lambda,\Lambda'} = \{\lambda + \lambda' \mid \lambda \in S_\Lambda,\ \lambda' \in S_{\Lambda'}\}, \tag{229}$$

noting the multiplicities.