Symmetries, Fields and Particles

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Contents

Lecture 1
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1 Introduction

Symmetries are hidden throughout undergraduate physics. Lagrangian mechanics relies on the principle of least action, where the action S is given by

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L\left(q\left(t\right), \dot{q}\left(t\right); t\right). \tag{1}$$

Classical trajectories minimise S which gives us the Euler Lagrange equation,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{2}$$

Theorem 1.1 (Noether's Theorem): Invariance of L under some transformation implies an associated conserved quantity.

Example. Take a particle in a 3-dimensional potential which has Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(x, y, z\right). \tag{3}$$

There are a few notable symmetries here

1. L is independent of time t, i.e. under $t \mapsto t + \delta t$.

   Claim. The Hamiltonian $H = T + U$ is conserved.

In general $H(x_i, p_i)$ is a function of $x_i = (x, y, z)$ and the conjugate momenta $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$ and is written in terms of the Lagrangian through Legendre transform as

$$H(x_i, p_i; t) = \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L. \tag{4}$$

Therefore, if L does not depend on time one has

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 - \frac{\partial L}{\partial t} = 0, \tag{5}$$

where we have used the Euler Lagrange equations to make the first term vanish.

2. If L is invariant under $x \mapsto x + \delta x$,

$$\frac{\partial L}{\partial x} = 0 \overset{\mathrm{EL}}{\Rightarrow} \frac{\partial L}{\partial \dot{x}} = p_x \text{ is constant.} \tag{6}$$

3. If L is invariant under rotations about the z axis then the z-component of angular momentum $L_z = xp_y - yp_x$ is constant.

   Similarly, in cylindrical coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$ and the Lagrangian becomes

$$L = \frac{1}{2}\left(m\dot{\rho}^2 + \rho^2\dot{\theta} + \dot{z}^2\right) - U(\rho, z). \tag{7}$$

   Therefore, $\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = m\rho^2\dot{\theta} = xp_y - yp_x = \text{constant}$.

1.1 Symmetry in Quantum Mechanics

Given a system whose states are elements of a Hilbert space $\mathcal{H}$. Here, symmetry implies there exists some invertible operator $U : \mathcal{H} \to \mathcal{H}$ which preserves inner products, up to an overall phase $e^{i\phi}$ (e.g. expectation values, transition amplitudes).

Definition 1.1: Let $|\Phi\rangle, |\Psi\rangle$ be any normalised vectors in $\mathcal{H}$. Denote $|U\Psi\rangle = U|\Psi\rangle$. U is a **symmetry transformation operator** if

$$|\langle U\Phi|U\Psi\rangle| = |\langle\Phi|\Psi\rangle|. \tag{8}$$

Proposition 1.1(Wigner's theorem): Symmetry transformation operators are either

a) linear and unitary, or

b) anti-linear and anti-unitary, meaning for $\alpha, \beta \in \mathbb{C}$,

$$U(\alpha|\Psi\rangle + \beta|\Phi\rangle) = \alpha^* U|\Psi\rangle + \beta^*|\Phi\rangle, \tag{9}$$

and

$$\langle U\Phi|U\Psi\rangle = \langle\Phi|\Psi\rangle^*, \tag{10}$$

respectively.

Most symmetries fall into the former category, but a notable exception is time-reversal symmetry, falling into the latter.

Suppose we have a system with time independent Hamiltonian H. We can write down the time evolution of operators in the Schrödinger picture (where the states depend on time and the operators are static) as

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle. \tag{11}$$

Let's look at applying a symmetry operator U in each of the cases above.

a)

$$\begin{aligned}\langle U\Phi(t)|U\Psi(t)\rangle &= \langle\Phi(t)|\Psi(t)\rangle \quad (12)\\ &= \langle\Phi(t)|e^{-iHt}|\Psi(0)\rangle. \quad (13)\end{aligned}$$

We should find the same result by transforming $|\Psi(0)\rangle$ before the evolution

$$|U\Psi(t)\rangle = e^{-iHt}|U\Psi(0)\rangle, \tag{14}$$

which implies

$$\begin{aligned}\langle U\Phi(t)|U\Psi(t)\rangle &= \langle U\Phi(t)|e^{-iHt}|U\Psi(0)\rangle \quad (15)\\ &= \langle\Phi(t)|U^\dagger e^{-iHt}U|\Psi(0)\rangle. \quad (16)\end{aligned}$$

By comparing this to Eq. (13) we find that

$$U^\dagger e^{-iHt}U = e^{-iHt}. \tag{17}$$

Therefore U commutes with the Hamiltonian, $[U,H]=0$.

Examples.

1) If H commutes with p, H cannot depend on x as $[x_i, p_j] = i\delta_{ij} \neq 0$. Therefore H is invariant under translations $x \to x+a$. One can construct a unitary operator that generates translations with $U = \exp(i\mathbf{p}\cdot\mathbf{a})$.
2) If H is rotationally symmetric the angular momentum operator commutes with H.

2 Lie Groups and algebras

2.1 Lie Groups

Definition 2.1: A **group** is a set G together with a binary operation $\circ$ such that the following properties hold

i) Closure: $g_2 \circ g_1 \in G$, $\forall g_1, g_2 \in G$,

ii) Associativity: $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$, $\forall g_1, g_2, g_3 \in G$,

iii) Identity: $\exists e \in G$ such that $g \circ e = e \circ g = g$, $\forall g \in G$,

iv) Inverse: $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \circ g^{-1} = e = g^{-1} \circ g$.

The identity e and inverse of g are unique.

Proof. Assume there exists e_1, e_2 which are both identities. Then we have that $e_1 \circ e_2 = e_1$ but also $e_1 \circ e_2 = e_2$ thus $e_1 = e_2$ and we have uniqueness.

For inverses, suppose g has two inverses h and j. One has that

$$g \circ h = e \text{ and } g \circ j = e. \tag{18}$$

Left multiplying by j and h respectively we see that

$$j \circ g \circ h = j \circ e \text{ and } h \circ g \circ j = h \circ e, \tag{19}$$

where simplifying (as we can by associativity) the left operation, we see

$$e \circ h = j \text{ and } e \circ j = h, \tag{20}$$

both of which imply $h = j$ and thus we have uniqueness. □

Definition 2.2: A group $(G, \circ)$ is **commutative** (**abelian**) if

$$g_1 \circ g_2 = g_2 \circ g_1, \tag{21}$$

$\forall g_1, g_2 \in G$. Otherwise G is **non-commutative** (**non-abelian**).