

# Symmetries, Fields and Particles

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Lecture 1  
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## 1 Introduction

Symmetries are hidden throughout undergraduate physics. Lagrangian mechanics relies on the principle of least action, where the action  $S$  is given by

$$S = \int_{t_1}^{t_2} dt L(q(t), \dot{q}(t); t). \quad (1)$$

Classical trajectories minimise  $S$  which gives us the Euler Lagrange equation,

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0. \quad (2)$$

**Theorem 1.1 (Noether's Theorem):** Invariance of  $L$  under some transformation implies an associated conserved quantity.

**Example.** Take a particle in a 3-dimensional potential which has Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z). \quad (3)$$

There are a few notable symmetries here

1.  $L$  is independent of time  $t$ , i.e. under  $t \mapsto t + \delta t$ .

**Claim.** The Hamiltonian  $H = T + U$  is conserved.

In general  $H(x_i, p_i)$  is a function of  $x_i = (x, y, z)$  and the conjugate momenta  $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$  and is written in terms of the Lagrangian through Legendre transform as

$$H(x_i, p_i; t) = \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L. \quad (4)$$

Therefore, if  $L$  does not depend on time one has

$$\frac{dH}{dt} = 0 - \frac{\partial L}{\partial t} = 0, \quad (5)$$

where we have used the Euler Lagrange equations to make the first term vanish.

2. If  $L$  is invariant under  $x \mapsto x + \delta x$ ,

$$\frac{\partial L}{\partial x} = 0 \stackrel{\text{EL}}{\Rightarrow} \frac{\partial L}{\partial \dot{x}} = p_x \text{ is constant.} \quad (6)$$

3. If  $L$  is invariant under rotations about the  $z$  axis then the  $z$ -component of angular momentum  $L_z = xp_y - yp_x$  is constant.

Similarly, in cylindrical coordinates  $x = \rho \cos \theta$ ,  $y = \rho \sin \theta$  and the Lagrangian becomes

$$L = \frac{1}{2} \left( m\dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \dot{z}^2 \right) - U(\rho, z). \quad (7)$$

Therefore,  $\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = m\rho^2 \dot{\theta} = xp_y - yp_x = \text{constant}$ .

## 1.1 Symmetry in Quantum Mechanics

Given a system whose states are elements of a Hilbert space  $\mathcal{H}$ . Here, symmetry implies there exists some invertible operator  $U : \mathcal{H} \rightarrow \mathcal{H}$  which preserves inner products, up to an overall phase  $e^{i\phi}$  (e.g. expectation values, transition amplitudes).

**Definition 1.1:** Let  $|\Phi\rangle, |\Psi\rangle$  be any normalised vectors in  $\mathcal{H}$ . Denote  $|U\Psi\rangle = U|\Psi\rangle$ .  $U$  is a **symmetry transformation operator** if

$$|\langle U\Phi | U\Psi \rangle| = |\langle \Phi | \Psi \rangle|. \quad (8)$$

**Proposition 1.1(Wigner's theorem):** Symmetry transformation operators are either

- a) linear and unitary, or
- b) anti-linear and anti-unitary, meaning for  $\alpha, \beta \in \mathbb{C}$ ,

$$U(\alpha|\Psi\rangle + \beta|\Phi\rangle) = \alpha^*U|\Psi\rangle + \beta^*U|\Phi\rangle, \quad (9)$$

and

$$\langle U\Phi|U\Psi\rangle = \langle\Phi|\Psi\rangle^*, \quad (10)$$

respectively.

Most symmetries fall into the former category, but a notable exception is time-reversal symmetry, falling into the latter.

Suppose we have a system with time independent Hamiltonian  $H$ . We can write down the time evolution of operators in the Schrödinger picture (where the states depend on time and the operators are static) as

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle. \quad (11)$$

Let's look at applying a symmetry operator  $U$  in each of the cases above.

a)

$$\langle U\Phi(t)|U\Psi(t)\rangle = \langle\Phi(t)|\Psi(t)\rangle \quad (12)$$

$$= \langle\Phi(t)|e^{-iHt}|\Psi(0)\rangle. \quad (13)$$

We should find the same result by transforming  $|\Psi(0)\rangle$  before the evolution

$$|U\Psi(t)\rangle = e^{-iHt}|U\Psi(0)\rangle, \quad (14)$$

which implies

$$\langle U\Phi(t)|U\Psi(t)\rangle = \langle U\Phi(t)|e^{-iHt}|U\Psi(0)\rangle \quad (15)$$

$$= \langle\Phi(t)|U^\dagger e^{-iHt}U|\Psi(0)\rangle. \quad (16)$$

By comparing this to Eq. (13) we find that

$$U^\dagger e^{-iHt}U = e^{-iHt}. \quad (17)$$

Therefore  $U$  commutes with the Hamiltonian,  $[U, H] = 0$ .

### Examples.

- 1) If  $H$  commutes with  $p$ ,  $H$  cannot depend on  $x$  as  $[x_i, p_j] = i\delta_{ij} \neq 0$ . Therefore  $H$  is invariant under translations  $x \rightarrow x + a$ . One can construct a unitary operator that generates translations with  $U = \exp(i\mathbf{p} \cdot \mathbf{a})$ .
- 2) If  $H$  is rotationally symmetric the angular momentum operator commutes with  $H$ .

## 2 Lie Groups and algebras

### 2.1 Lie Groups

**Definition 2.1:** A **group** is a set  $G$  together with a binary operation  $\circ$  such that the following properties hold

- i) Closure:  $g_2 \circ g_1 \in G, \forall g_1, g_2 \in G,$
- ii) Associativity:  $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1, \forall g_1, g_2, g_3 \in G,$
- iii) Identity:  $\exists e \in G$  such that  $g \circ e = e \circ g = g, \forall g \in G,$
- iv) Inverse:  $\forall g \in G, \exists g^{-1} \in G$  such that  $g \circ g^{-1} = e = g^{-1} \circ g.$

The identity  $e$  and inverse of  $g$  are unique.

**Proof.** Assume there exists  $e_1, e_2$  which are both identities. Then we have that  $e_1 \circ e_2 = e_1$  but also  $e_1 \circ e_2 = e_2$  thus  $e_1 = e_2$  and we have uniqueness.

For inverses, suppose  $g$  has two inverses  $h$  and  $j$ . One has that

$$g \circ h = e \text{ and } g \circ j = e. \quad (18)$$

Left multiplying by  $j$  and  $h$  respectively we see that

$$j \circ g \circ h = j \circ e \text{ and } h \circ g \circ j = h \circ e, \quad (19)$$

where simplifying (as we can by associativity) the left operation, we see

$$e \circ h = j \text{ and } e \circ j = h, \quad (20)$$

both of which imply  $h = j$  and thus we have uniqueness.  $\square$

**Definition 2.2:** A group  $(G, \circ)$  is **commutative (abelian)** if

$$g_1 \circ g_2 = g_2 \circ g_1, \quad (21)$$

$\forall g_1, g_2 \in G$ . Otherwise  $G$  is **non-commutative (non-abelian)**.

**Definition 2.3:** A **manifold** is a space which looks like Euclidean space  $(\mathbb{R}^n)$  locally. A **differentiable manifold** is one which satisfies certain smoothness conditions.

**Definition 2.4:** A **Lie group** consists of a differentiable manifold  $G$  along with a binary operation  $\bullet$  such that the group axioms hold and that the operations  $(\bullet, \cdot^{-1})$  are smooth operations.

Lecture 2  
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## 2.2 Matrix Lie Groups

The general linear group  $GL(n, \mathbb{F})$  is the group of invertible  $n \times n$  matrices over the field  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . Namely,

$$GL(n, \mathbb{F}) = \{M \in \text{Mat}_n(\mathbb{F}) \mid \det M \neq 0\}. \quad (22)$$

The group operation is matrix multiplication and inverses are defined as  $\det M \neq 0$ .

The dimension of  $GL(n, \mathbb{R})$  is  $n^2$ , and thus we have  $n^2$  free parameters.

For  $GL(n, \mathbb{C})$ , the real dimension is  $2n^2$  and the complex dimension is  $n^2$ .

There are a number of important subgroups of  $GL(n, \mathbb{F})$ .

1. The *special linear group*, denoted  $SL(n, \mathbb{F}) = \{M \in GL(n, \mathbb{F}) \mid \det M = 1\}$ , where the constraint leaves us with a dimension of  $n^2 - 1$ .
2. The *orthogonal group*, denoted  $O(n) = \{M \in GL(n, \mathbb{R}) \mid M^T M = I\}$ . Notice that

$$M^T M = I \Rightarrow \det M = \pm 1. \quad (23)$$

3. The *special orthogonal group*, denoted  $SO(n) = \{M \in O(n) \mid \det M = 1\}$
4. The *pseudo-orthogonal group*, where we define an  $(n+m) \times (n+m)$  (metric) matrix by

$$\eta \equiv \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}. \quad (24)$$

This group is denoted

$$O(n, m) = \{M \in GL(n+m, \mathbb{R}) \mid M^T \eta M = \eta\}. \quad (25)$$

Similarly, there is a *special* subset of this group denoted  $SO(n, m) \Rightarrow \det M = 1$ .

5. The *unitary* matrices, which are denoted

$$U(n) = \{M \in GL(n, \mathbb{C}) \mid M^T M = I\}. \quad (26)$$

As before, we also have  $SU(n)$  which restricts to matrices with  $\det M = 1$ .

6. The *pseudo-unitary* group, given by

$$U(n, m) = \{M \in GL(n, \mathbb{C}) \mid M^T \eta M = \eta\}. \quad (27)$$

7. The *symplectic group*, for which we define a fixed, antisymmetric  $2n \times 2n$  matrix, such as

$$\Omega \equiv \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad (28)$$

The symplectic group is then

$$\text{Sp}(2n, \mathbb{R}) = \{M \in GL(2n, \mathbb{R}) \mid M^T \Omega M = \Omega\}. \quad (29)$$

One can show that  $M \in \text{Sp}(2n, \mathbb{R})$  satisfies  $\det M = 1$ .

**Definition 2.5:** Given a  $2n \times 2n$  antisymmetric matrix  $A$ , its **Pfaffian** is given by

$$\text{Pf}A \equiv \frac{1}{2^n n!} \varepsilon_{i_1 i_2 \dots i_{2n}} A^{i_1 i_2} A^{i_3 i_4} \dots A^{i_{2n-1} i_{2n}}, \quad (30)$$

where  $\varepsilon_{i_1 i_2 \dots i_n}$  is the totally antisymmetric symbol  $\varepsilon_{i_1 i_2 \dots i_n} = -\varepsilon_{i_2 i_1 \dots i_n}$ .

### 2.3 Group elements as transformations

We can define actions of group elements  $g \in G$  on a set  $X$ .  $X$  might be  $G$  itself, but could also be a vector space (i.e. rotation matrices acting on vectors in  $\mathbb{R}^3$ ).

**Definition 2.6:** The **left action** of  $G$  on  $X$  is a map  $L : G \times X \rightarrow X$  such that for  $x \in X$

- $L(e, x) = x$ , for  $e$ , the identity of  $G$ ,
- $L(g_2, L(g_1, x)) = L(g_2 g_1, x)$ ,  $\forall x \in X, \forall g_1, g_2 \in G$ .

The more usual notation is that  $\forall g \in G$ , we associate a map  $g : X \rightarrow X$  such that  $g(x) = gx$ , however this is slightly less clear.

**Definition 2.7:** The **right action** of  $G$  on  $X$  is defined by  $gX \rightarrow X$  such that  $g(x) = xg^{-1}$ ,  $\forall x \in X$  and  $g \in G$ .

The inverse preserves group composition. Namely,

$$g_2(g_1(x)) = x \underbrace{g_1^{-1} g_2^{-1}}_{(g_2 g_1)^{-1}} = (g_2 g_1)(x). \quad (31)$$

**Definition 2.8: Conjugation** by  $G$  on  $X$  is the action defined by

$$g(x) = xg^{-1}, \quad (32)$$

$\forall g \in G_1, x \in X$ .

Another definition worth making, even if it won't see immediate use is that of an *orbit*.

**Definition 2.9:** Given a group  $G$  and set  $X$ , an **orbit** of an element  $x \in X$  is the set of elements of  $X$  which are in the image of an action of  $G$  on  $x$ .

**Example.** If the action is left, the orbit of  $x \in X$  is written  $Gx = \{gx \mid g \in G\}$ .

It can be shown that the set of orbits under  $G$  'partition'  $X$  as we will see.

### 2.4 Orthogonal groups

The orthogonal group,  $O(n)$  in particular, represent rotations and reflections on  $\mathbb{R}^n$ . This preserves inner products such that

$$\langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \mathbf{v}_2^T \mathbf{v}_1, \quad (33)$$

given  $R \in O(n)$ ,

$$\langle R\mathbf{v}_2, R\mathbf{v}_1 \rangle = \mathbf{v}_2^T \underbrace{(R^T R)}_I \mathbf{v}_1 = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle. \quad (34)$$

This is similar for  $U(n)$ .

Consider

$$SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \middle| \theta \in (0, 2\pi) \right\}. \quad (35)$$

As  $\cos$  and  $\sin$  are smooth functions, this is a differentiable manifold. One can also show that  $R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$ .

Similarly,  $SO(3)$  can represent rotations of vectors in  $\mathbb{R}^3$  where the axis of the rotation is given by a unit vector  $\mathbf{n} \in S^2$  and we rotate by an angle  $\theta$ . Note that rotation by  $\theta \in [-\pi, 0]$  about  $\mathbf{n}$  is equivalent to a rotation by  $-\theta$  about  $-\mathbf{n}$  so we confine to  $\theta \in [0, \pi]$ .

Therefore we can depict the manifold of  $SO(3)$  as a ball of radius  $\pi$  in  $\mathbb{R}^3$ , where the direction is specified by  $\mathbf{n}$  and the distance from the origin is specified by  $\theta \in [0, \pi]$ . Antipodal points are identified such that  $\pi\mathbf{n} = -\pi\mathbf{n}$ .

Lecture 3  
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## 3 Lie Algebras

### 3.1 Pseudo orthogonal group

$SO(n, m)$  act on vectors in  $\mathbb{R}^{n+m}$  and preserve the scalar product

$$v_2^T \eta v_1, \quad (36)$$

for  $v_1, v_2 \in \mathbb{R}^{n+m}$ . For example,  $SO(1, 1)$  parametrise Lorentz boosts in one dimension and can be written in terms of the *rapidity*  $\eta$  as

$$SO(1, 1) = \left\{ \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \middle| \eta \in \mathbb{R} \right\}. \quad (37)$$

As  $\eta$  is unbounded,  $SO(1, 1)$  is clearly noncompact.

### 3.2 Parametrization of Lie Groups

At least in small neighbourhoods, we can assign coordinates on an  $n$ -dimensional manifold to be

$$x := (x^1, \dots, x^n) \in \mathbb{R}^n. \quad (38)$$

This allows us to label elements  $g(x) \in G$ . Closure provides

$$g(y)g(x) = g(z). \quad (39)$$

Smoothness gives us that the components of  $z$  are continuously differentiable functions of  $x$  and  $y$  such that for  $i \in 1, \dots, n$ ,

$$z^i = \phi^i(x, y). \quad (40)$$

We choose the coordinate origin such that  $g(0) = e$ . Identity gives us that

$$g(0)g(x) = g(x) \Rightarrow \phi^i(x, 0) = x^i \text{ and } \phi^i(0, y) = y^i. \quad (41)$$

Similarly, for inverses, we have that there exists some  $\tilde{x}$  such that  $g(\tilde{x}) = g(x)^{-1}$  and thus

$$\phi^i(\tilde{x}, x) = 0 = \phi^i(x, \tilde{x}). \quad (42)$$

Lastly, associativity gives us

$$g(z)(g(y)g(x)) = (g(z)g(y))g(x) \Rightarrow \phi^i(\phi(x, y), z) = \phi^i(x, \phi(y, z)). \quad (43)$$

This appears like a Leibniz rule/Jacobi identity as we will see.

### 3.3 Lie Algebras

A Lie group is homogeneous. Any neighbourhood ‘looks like’ (or in a more formal sense, can be mapped to) any other neighbourhood.

For example, for  $\varepsilon \in G$  close to  $g_1$ ,  $g_2 g^{-1} \varepsilon$  is close to  $g_2$ .

Thus no neighbourhood in particular is special. The natural choice of the representative neighbourhood to study is the one centered at the identity of  $G$ . We will linearize near the identity of  $G$ .

**Definition 3.1:** A Lie Algebra is a vector space  $V$ , which additionally has a vector product, the **Lie bracket**,  $[\cdot, \cdot] : V \times V \rightarrow V$  satisfying the following properties for  $X, Y, Z \in V$ .

- 1) It is antisymmetric,  $[X, Y] = -[Y, X]$ ,
- 2) It satisfies the Jacobi identity,  $[X, [Y, Z]] + [Y, [X, Z]] + [Z, [X, Y]] = 0$ ,
- 3) It is linear such that for  $\alpha, \beta \in \mathbb{F}$ ,  $[X, \alpha Y + \beta Z] = \alpha [X, Y] + \beta [X, Z]$ .

**Note.** Any vector space which has a vector product  $\star : V \times V \rightarrow V$  can be made into a Lie Algebra with its Lie bracket given by

$$[X, Y] = X \star Y - Y \star X. \quad (44)$$

**Definition 3.2:** Let's choose a basis for  $V$ , given by  $\{T_a\}$  for  $a = 1, \dots, n = \dim V$ . We call these basis vectors **generators** of the Lie algebra, and we write their Lie brackets as

$$[T_a, T_b] = f_{abc}^c T_c, \quad (45)$$

where  $f_{ab}^c \in \mathbb{F}$  are called **structure constants**.

Antisymmetry implies  $f_{ba}^c = -f_{ab}^c$  and the Jacobi identity implies

$$f_{ad}^e f_{bc}^d + f_{cd}^e f_{ab}^d + f_{bd}^e f_{ca}^d = 0. \quad (46)$$

The general element of a Lie algebra can be written as a linear combination of  $\{T_a\}$  as

$$X \in V \Rightarrow X = X^a T_a \text{ with } x^a \in \mathbb{F}, \quad (47)$$



which gives us the bracket of any two elements in terms of structure constants with

$$[X, Y] = X^a Y^b f_{abc}^c T_c. \quad (48)$$

### 3.4 Lie Groups and their Lie Algebras

Take  $g(\theta) \in SO(2)$  to be

$$g(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (49)$$

where  $e = I_2 = g(0)$ . Points near the identity have  $\theta \ll 1$  and thus Taylor expanding the components of  $g(\theta)$  we see

$$g(\theta) = I_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \theta^2 I_2 + \mathcal{O}(\theta^3) \quad (50)$$

$$= e + \underbrace{\theta \frac{dg}{d\theta} \Big|_{g=0}}_{\text{tangent vector}} + \frac{d^2 g}{d\theta^2} + \mathcal{O}(\theta^2), \quad (51)$$

where the linear term is tangent to the manifold. Here there is a one dimensional tangent space at  $e$  given by

$$T_e(SO(2)) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \Big| a \in \mathbb{R} \right\}. \quad (52)$$

This is the Lie algebra of  $SO(2)$ ,

$$\mathfrak{so}(2) := L(SO(2)) := T_e(SO(2)). \quad (53)$$

It remains to show this.

**Proof.** Notice that

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} -ab & 0 \\ 0 & -ab \end{pmatrix} = -abI, \quad (54)$$

and thus for any two elements (matrices) of the Lie algebra, they commute (which is trivially antisymmetric and satisfying of Jacobi). Linearity similarly follows immediately by inspection.  $\square$

Similarly, one can show  $\dim(SO(n)) = \frac{1}{2}n(n-1) \equiv d$ , so we have coordinates  $x_1 \cdots, x_d$ . Consider a single-parameter family of  $SO(n)$  elements,

$$M(t) := M(\mathbf{x}(t)) \in SO(n), \quad (55)$$

such that  $M(0) = I_n$ . Orthogonality ( $M^T M = I$ ) implies

$$0 = \frac{d}{dt} (M^T(t) M(t)) \quad (56)$$

$$= \frac{dM^T}{dt} + M^T \frac{dM}{dt}, \quad (57)$$

where looking at  $t = 0$ , as  $M(0) = I_n$  we see

$$\frac{dM^T}{dt} = -\frac{dM}{dt}, \quad (58)$$

which implies matrices in the tangent space of  $SO(n)$  are antisymmetric (and thus traceless as well).

We have

$$\frac{dM}{dt} = \sum_i \frac{\partial M}{\partial x_i} \frac{dx_i}{dt}. \quad (59)$$

Lecture 4  
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## 4 The Exponential Map

Observe that

$$T_e(\mathcal{O}(n)) = T_e(SO(n)), \quad (60)$$

as  $\det I = 1$ , so all curves passing through  $I$  have  $\det M = 1$ .

### 4.1 Unitary Groups

Let  $M(t)$  be a curve in  $SU(n)$  with  $M(0) = I$ . For small  $t$ , write  $M(t) = I + tX + \mathcal{O}(t^2)$ , where  $X = \left. \frac{dM}{dt} \right|_{t=0}$ .

Unitarity of  $M$  provides that for all  $t$ ,

$$I = M^\dagger M \quad (61)$$

$$U = I + t(X + X^\dagger) + \mathcal{O}(t^2), \quad (62)$$

which implies  $X^\dagger = -X$ , namely, elements of the tangent space are *anti-Hermitian*.

**Claim.**  $\text{tr } X = 0$  for  $X \in L(SU(n))$  or  $M \in SU(n)$

**Proof.** Look at

$$M(t) = \begin{pmatrix} 1 + tX_{11} & tX_{12} & \cdots & tX_{1n} \\ tX_{21} & 1 + tX_{22} & \cdots & tX_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ tX_{n1} & tX_{n2} & \cdots & 1 + tX_{nn} \end{pmatrix}. \quad (63)$$

Notice that

$$1 = \det M = 1 + \underbrace{t \text{tr } X}_0 + \mathcal{O}(t^2), \quad (64)$$

where the underbraced term (and higher order ones) must vanish. □

For  $U(n)$ ,  $X$  can have non-zero trace.

## 4.2 Lie algebra of a matrix Lie group

Consider two curves  $g_1(x(t))$  and  $g_2(x(t))$  through the identity  $e$  of some Lie group  $G$ . We define

$$X_1 := \dot{g}_1 \Big|_{t=0}, \quad X_2 := \dot{g}_2 \Big|_{t=0}. \quad (65)$$

One can define a product

$$g_3(z(t)) = g_2(y(t))g_1(x(t)) \in G, \quad (66)$$

satisfying

$$\dot{g}_3 \Big|_{t=0} = (\dot{g}_2 g_1 + g_2 \dot{g}_1) \Big|_{t=0} \quad (67)$$

$$= X_2 + X_1 \in T_e(G), \quad (68)$$

another vector in the tangent space.

The Lie bracket arises from the *group commutator*.

**Definition 4.1:** The **group commutator** of  $g_1, g_2 \in G$ , is

$$[g_1, g_2]_G := g_1^{-1} g_2^{-1} g_1 g_2 := h \in G. \quad (69)$$

Returning to our two curves through the identity  $e$ ,  $g_i(t)$  for  $i \in \{1, 2\}$ , we can expand

$$g_i(t) = e + tX_i + t^2W_i + \mathcal{O}(t^3). \quad (70)$$

We have that

$$g_1(t)g_2(t) = e + t(X_1 + X_2) + t^2(X_1X_2 + W_1 + W_2) + \mathcal{O}(t^3), \quad (71)$$

and

$$g_2(t)g_1(t) = e + t(X_1 + X_2) + t^2(X_2X_1 + W_1 + W_2) + \mathcal{O}(t^3). \quad (72)$$

If we then look at

$$h(t) = [g_2(t)g_1(t)]^{-1}g_1(t)g_2(t) = e + t^2 \underbrace{(X_1X_2 - X_2X_1)}_{[X_1, X_2]} + \cdots, \quad (73)$$

and thus the group commutator induces the Lie bracket in the algebra. As  $h(t) \in G$ , the tangent to  $h(t)$  at  $e$  is  $[X_1, X_2] \in L(G)$ , and thus we have closure under the Lie bracket.

- We write the tangent space to a matrix Lie group  $G \stackrel{\text{subgroup}}{<} GL(n, \mathbb{F})$  at a general element  $p$  as  $T_p(G)$ . Let  $g(t)$  be a curve in the manifold through  $p$  with  $g(t_0) = p$ , and thus

$$g(t + \varepsilon) = g(t_0) + \dot{g}(t_0)\varepsilon + \mathcal{O}(\varepsilon^2). \quad (74)$$

As both  $g(t_0), g(t_0 + \varepsilon) \in G$ , there exists  $h_p(\varepsilon) \in G$  such that

$$g(t_0 + \varepsilon) = g(t_0) h_p(\varepsilon), \quad (75)$$

and as  $\varepsilon \rightarrow 0$ ,  $h_p(\varepsilon) \rightarrow e$ . For small  $\varepsilon$ ,

$$h_p(\varepsilon) = e + \varepsilon X_p + \mathcal{O}(\varepsilon^2), \quad (76)$$

for some  $X_p \in L(G) = T_e(G)$ . Neglecting  $\mathcal{O}(\varepsilon^2)$ ,

$$e + \varepsilon X_p = h_p(\varepsilon) = g^{-1}(t_0) g(t_0 + \varepsilon) \quad (77)$$

$$= g^{-1}(t_0) [g(t_0) + \varepsilon \dot{g}(t_0)] \quad (78)$$

$$= e + \varepsilon \underbrace{g^{-1}(t_0) \dot{g}(t_0)}_{X_p}. \quad (79)$$

**Claim.** Conversely, for any  $X \in L(G)$ , there exists a unique curve  $g(t)$  with  $g^{-1}(t) \dot{g}(t) = X$  and  $g(0) = g_0$ .

**Proof.** This is a consequence of existence and uniqueness of solutions of ODEs. The solution of this ODE is

$$g(t) = g_0 \exp(tX), \quad (80)$$

where

$$\exp tX := \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}. \quad (81)$$

□

### 4.3 One parameter subgroups

Given an  $X \in L(G)$ , the curve

$$g_X(t) = \exp tX, \quad (82)$$

forms an *abelian* subgroup of  $G$ , *generated* by  $X$ .

Notice that  $g_X(t)$  is isomorphic to the group of real numbers under addition  $(\mathbb{R}, +)$  if only  $g_X(0) = e$ . If there exist other  $t_0 \neq 0$  such that  $g_X(t_0) = e$ , then we have periodic structure and then  $g_X(t)$  is isomorphic to the circle  $S^1$ .

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### 4.4 Lie Groups from Lie Algebras

**Definition 4.2:** Given a Lie algebra  $L(G)$  of a Lie group  $G$ , we can define the **exponential map**:

$$\exp : L(G) \rightarrow G, \quad (83)$$

which for matrix Lie groups, is

$$X \mapsto \exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!}. \quad (84)$$

Locally, the map is bijective (one to one). For the proof see Hall Section 2.7. Globally, the map is generally, not.

**Example.** For example,  $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$  and we have

$$L(U(1)) = \{ix \mid x \in \mathbb{R}\}, \quad (85)$$

where clearly  $\exp(ix)$  is not one to one globally since  $e^{2\pi ni} = 1, \forall n \in \mathbb{Z}$ .

**Example.**  $G = O(n)$ . Let  $X \in L(O(n)) \subset \text{Skew}_n(\mathbb{R})$ . Let  $M = \exp tX$ , and observe that as  $X$  is antisymmetric,  $M^T = [\exp X]^T = \exp(-tX)$ . Therefore,

$$MM^T = I = M^T M, \quad (86)$$

and thus we recover  $M \in O(n)$ .

**Note.**  $\text{tr } X = 0$ . Let  $\lambda_1, \dots, \lambda_n$  be eigenvalues of  $X$ , and observe that

$$\det M = \det(\exp tX) \quad (87)$$

$$= \exp(\text{tr } tX) \quad (88)$$

$$= \exp(0) \quad (89)$$

$$= 1. \quad (90)$$

and thus  $M \in SO(n)$ . Thus elements of  $O(n)$  with determinant  $-1$  are not in the image of the exponential map.

Therefore,  $O(n)$  is a disconnected manifold. One can think of  $O(n)$  as two disconnected islands, one with  $\det M = 1$  containing the identity called *proper rotations*, and another containing elements with  $\det M = -1$  called *improper rotations* as they contain a reflection.

One can show that  $A \in \text{Skew}_n(\mathbb{R})$  implies  $A \in L(SO(n))$  or  $L(O(n))$ .

Define  $\gamma(t) := \exp tA$  to be a curve of matrices on some manifold. By above, we see that

$$(\gamma(t))^T (\gamma(t)) = I, \quad (91)$$

and thus  $\det \gamma(t) = 1$  which implies  $\gamma(t) \in SO(n)$ . By construction,  $A = \dot{\gamma}(t) \Big|_{t=0}$  and thus is tangent to the curve at the identity of  $SO(n)$  suggesting  $A \in L(SO(n))$ . Therefore

$$\dim SO(n) = \dim L(SO(n)) = \dim(\text{Skew}_n(\mathbb{R})) = \frac{n(n-1)}{2}. \quad (92)$$

### 4.5 Group product from Lie bracket

Recall the Baker-Campbell-Hausdorff (BCH) formula, namely that for  $X, Y \in L(G)$ , we have

$$\exp(tX)\exp(tY) = \exp(tZ), \quad (93)$$

where

$$Z = X + Y + \frac{t}{2}[X, Y] + \frac{t^2}{12}([X, [X, Y]] + [Y, [X, Y]]) + \mathcal{O}(t^3). \quad (94)$$

One can show this order by order in  $t$ . As  $L(G)$  is closed under the Lie bracket,  $Z \in L(G)$  and thus  $\exp tZ \in G$ .

## 5 Representation Theory

Groups and their elements represent transformations under which a system or object is invariant. Representations of groups tell us how the action of the group transforms vectors in a vector space.

We saw  $GL(n, \mathbb{F})$  as a group of invertible matrices. These matrices are equivalently linear maps (automorphisms) on the vector space  $\mathbb{F}^n$  with

$$GL(n, \mathbb{F}) : \mathbb{F}^n \rightarrow \mathbb{F}^n. \quad (95)$$

We generalize this notation to act on any vector space  $V$  such that

$$GL(V) : V \rightarrow V. \quad (96)$$

If  $V$  is finite dimensional, we can choose a basis and recover the original definition.

### 5.1 Lie group representations

**Definition 5.1:** A **representation**  $D$  of a group  $G$  is a smooth group homomorphism

$$D : G \rightarrow GL(V), \quad (97)$$

from  $G$  to the group of automorphisms on some vector space  $V$  called the **representation space**, associated with  $D$ .

That is,  $\forall g \in G$ ,  $D(g) : V \rightarrow V$  is an invertible, linear map such that for a vector  $v \in V$ ,

$$v \mapsto D(g)v. \quad (98)$$

This map is linear such that

$$D(g)(\alpha v_1 + \beta v_2) = \alpha D(g)v_1 + \beta D(g)v_2, \quad (99)$$

$\forall \alpha, \beta \in \mathbb{F}$ ,  $v_1, v_2 \in V$ . Further, the group homomorphism holds such that we have

$$D(g_2 g_1) = D(g_2) D(g_1), \quad (100)$$

$\forall g_1, g_2 \in G$ . This group homomorphism property implies that

$$D(e) = \text{id}_V, \quad (101)$$

and by an identical argument,

$$D(g)^{-1} = D(g^{-1}). \quad (102)$$

**Definition 5.2:** The **dimension** of a representation  $D$  is the dimension of the representation space  $V$  on which it acts.

If  $V$  is finite dimensional, say  $\dim V = N$ , then  $GL(V)$  is isomorphic to  $GL(N, \mathbb{F})$ .