Symmetries, Fields and Particles

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Lecture 1
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1 Introduction

Symmetries are hidden throughout undergraduate physics. Lagrangian mechanics relies on the principle of least action, where the action S is given by

$$S = \int_{t_1}^{t_2} \mathrm{d}t \, L\left(q\left(t\right), \dot{q}\left(t\right); t\right). \tag{1}$$

Classical trajectories minimise S which gives us the Euler Lagrange equation,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{2}$$

Theorem 1.1 (Noether's Theorem): Invariance of L under some transformation implies an associated conserved quantity.

Example. Take a particle in a 3-dimensional potential which has Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(x, y, z\right). \tag{3}$$

There are a few notable symmetries here

1. L is independent of time t, i.e. under $t \mapsto t + \delta t$.

   Claim. The Hamiltonian $H = T + U$ is conserved.

   In general $H\left(x_i, p_i\right)$ is a function of $x_i = (x, y, z)$ and the conjugate momenta $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$ and is written in terms of the Lagrangian through Legendre transform as

   $$H\left(x_i, p_i; t\right) = \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L. \tag{4}$$

   Therefore, if L does not depend on time one has

   $$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 - \frac{\partial L}{\partial t} = 0, \tag{5}$$

   where we have used the Euler Lagrange equations to make the first term vanish.

2. If L is invariant under $x \mapsto x + \delta x$,

   $$\frac{\partial L}{\partial x} = 0 \overset{\text{EL}}{\Rightarrow} \frac{\partial L}{\partial \dot{x}} = p_x \text{ is constant.} \tag{6}$$

3. If L is invariant under rotations about the z axis then the z-component of angular momentum $L_z = xp_y - yp_x$ is constant.

   Similarly, in cylindrical coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$ and the Lagrangian becomes

   $$L = \frac{1}{2}\left(m\dot{\rho}^2 + \rho^2\dot{\theta} + \dot{z}^2\right) - U\left(\rho, z\right). \tag{7}$$

   Therefore, $\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = m\rho^2\dot{\theta} = xp_y - yp_x = \text{constant}$.

1.1 Symmetry in Quantum Mechanics

Given a system whose states are elements of a Hilbert space $\mathcal{H}$. Here, symmetry implies there exists some invertible operator $U : \mathcal{H} \to \mathcal{H}$ which preserves inner products, up to an overall phase $e^{i\phi}$ (e.g. expectation values, transition amplitudes).

Definition 1.1: Let $|\Phi\rangle, |\Psi\rangle$ be any normalised vectors in $\mathcal{H}$. Denote $|U\Psi\rangle = U|\Psi\rangle$. U is a **symmetry transformation operator** if

$$|\langle U\Phi|U\Psi\rangle| = |\langle\Phi|\Psi\rangle|. \tag{8}$$

Proposition 1.1(Wigner's theorem): Symmetry transformation operators are either

a) linear and unitary, or

b) anti-linear and anti-unitary, meaning for $\alpha, \beta \in \mathbb{C}$,

$$U\left(\alpha\left|\Psi\right\rangle + \beta\left|\Phi\right\rangle\right) = \alpha^* U\left|\Psi\right\rangle + \beta^*\left|\Phi\right\rangle, \tag{9}$$

and

$$\langle U\Phi|U\Psi\rangle = \langle\Phi|\Psi\rangle^*, \tag{10}$$

respectively.

Most symmetries fall into the former category, but a notable exception is time-reversal symmetry, falling into the latter.

Suppose we have a system with time independent Hamiltonian H. We can write down the time evolution of operators in the Schrödinger picture (where the states depend on time and the operators are static) as

$$\left|\Psi\left(t\right)\right\rangle = e^{-iHt}\left|\Psi\left(0\right)\right\rangle. \tag{11}$$

Let's look at applying a symmetry operator U in each of the cases above.

a)

$$\begin{aligned}\langle U\Phi\left(t\right)|U\Psi\left(t\right)\rangle &= \langle\Phi\left(t\right)|\Psi\left(t\right)\rangle \qquad (12)\\ &= \langle\Phi\left(t\right)|\, e^{-iHt}\left|\Psi\left(0\right)\right\rangle. \qquad (13)\end{aligned}$$

We should find the same result by transforming $\left|\Psi\left(0\right)\right\rangle$ before the evolution

$$\left|U\Psi\left(t\right)\right\rangle = e^{-iHt}\left|U\Psi\left(0\right)\right\rangle, \tag{14}$$

which implies

$$\begin{aligned}\langle U\Phi\left(t\right)|U\Psi\left(t\right)\rangle &= \langle U\Phi\left(t\right)|\, e^{-iHt}\left|U\Psi\left(0\right)\right\rangle \qquad (15)\\ &= \langle\Phi\left(t\right)|\, U^\dagger e^{-iHt}U\left|\Psi\left(0\right)\right\rangle. \qquad (16)\end{aligned}$$

By comparing this to Eq. (13) we find that

$$U^\dagger e^{-iHt}U = e^{-iHt}. \tag{17}$$

Therefore U commutes with the Hamiltonian, $[U, H] = 0$.

Examples.

1) If H commutes with p, H cannot depend on x as $[x_i, p_j] = i\delta_{ij} \neq 0$. Therefore H is invariant under translations $x \to x + a$. One can construct a unitary operator that generates translations with $U = \exp\left(i\mathbf{p}\cdot\mathbf{a}\right)$.
2) If H is rotationally symmetric the angular momentum operator commutes with H.

2 Lie Groups and algebras

2.1 Lie Groups

Definition 2.1: A **group** is a set G together with a binary operation $\circ$ such that the following properties hold

i) Closure: $g_2 \circ g_1 \in G$, $\forall g_1, g_2 \in G$,

ii) Associativity: $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$, $\forall g_1, g_2, g_3 \in G$,

iii) Identity: $\exists e \in G$ such that $g \circ e = e \circ g = g$, $\forall g \in G$,

iv) Inverse: $\forall g \in G$, $\exists g^{-1} \in G$ such that $g \circ g^{-1} = e = g^{-1} \circ g$.

The identity e and inverse of g are unique.

Proof. Assume there exists e_1, e_2 which are both identities. Then we have that $e_1 \circ e_2 = e_1$ but also $e_1 \circ e_2 = e_2$ thus $e_1 = e_2$ and we have uniqueness.

For inverses, suppose g has two inverses h and j. One has that

$$g \circ h = e \text{ and } g \circ j = e. \tag{18}$$

Left multiplying by j and h respectively we see that

$$j \circ g \circ h = j \circ e \text{ and } h \circ g \circ j = h \circ e, \tag{19}$$

where simplifying (as we can by associativity) the left operation, we see

$$e \circ h = j \text{ and } e \circ j = h, \tag{20}$$

both of which imply $h = j$ and thus we have uniqueness. □

Definition 2.2: A group $(G, \circ)$ is **commutative** (**abelian**) if

$$g_1 \circ g_2 = g_2 \circ g_1, \tag{21}$$

$\forall g_1, g_2 \in G$. Otherwise G is **non-commutative** (**non-abelian**).

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Definition 2.3: A **manifold** is a space which looks like Euclidean space ($\mathbb{R}^n$) locally. A **differentiable manifold** is one which satisfies certain smoothness conditions.

Definition 2.4: A **Lie group** consists of a differentiable manifold G along with a binary operation $\bullet$ such that the group axioms hold and that the operations $\left(\bullet, \cdot^{-1}\right)$ are smooth operations.

2.2 Matrix Lie Groups

The general linear group $GL(,\mathbb{F})$ is the group of invertible $n \times n$ matrices over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Namely,

$$GL(n,\mathbb{F}) = \{M \in \mathrm{Mat}_n(\mathbb{F}) \mid \det M \neq 0\}. \tag{22}$$

The group operation is matrix multiplication and inverses are defined as $\det M \neq 0$.

The dimension of $GL(n,\mathbb{R})$ is n^2, and thus we have n^2 free parameters.

For $GL(n,\mathbb{C})$, the real dimension is $2n^2$ and the complex dimension is n^2.

There are a number of important subgroups of $GL(n,\mathbb{F})$.

1. The *special linear group*, denoted $SL(n,\mathbb{F}) = \{M \in GL(n,\mathbb{F}) \mid \det M = 1\}$, where the constraint leaves us with a dimension of $n^2 - 1$.
2. The *orthogonal group*, denoted $O(n) = \{M \in GL(n,\mathbb{R}) \mid M^T M = I\}$. Notice that
$$M^T M = I \Rightarrow \det M = \pm 1. \tag{23}$$
3. The *special orthogonal group*, denoted $SO(n) = \{M \in O(n) \mid \det M = 1\}$
4. The *pseudo-orthogonal group*, where we define an $(n+m) \times (n+m)$ (metric) matrix by
$$\eta \equiv \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}. \tag{24}$$
This group is denoted
$$O(n,m) = \{M \in GL(n+m,\mathbb{R}) \mid M^T \eta M = \eta\}. \tag{25}$$
Similarly, there is a *special* subset of this group denoted $SO(n,m) \Rightarrow \det M = 1$.
5. The *unitary* matrices, which are denoted
$$U(n) = \{M \in GL(n,\mathbb{C}) \mid M^T M = I\}. \tag{26}$$
As before, we also have $SU(n)$ which restricts to matrices with $\det M = 1$.
6. The *pseudo-unitary* group, given by
$$U(n,m) = \{M \in GL(n,\mathbb{C}) \mid M^T \eta M = \eta\}. \tag{27}$$
7. The *symplectic group*, for which we define a fixed, antisymmetric $2n \times 2n$ matrix, such as
$$\Omega \equiv \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{28}$$
The symplectic group is then
$$\mathrm{Sp}(2n,\mathbb{R}) = \{M = GL(2n,\mathbb{R}) \mid M^T \Omega M = \Omega\}. \tag{29}$$
One can show that $M \in \mathrm{Sp}(2n,\mathbb{R})$ satisfies $\det M = 1$.

Definition 2.5: Given a $2n \times 2n$ antisymmetric matrix A, its **Pfaffian** is given by

$$\mathrm{Pf}A \equiv \frac{1}{2^n n!} \varepsilon_{i_1 i_2 \cdots i_{2n}} A^{i_1 i_2} A^{i_3 i_4} \cdots A^{i_{2n-1} i_{2n}}, \tag{30}$$

where $\varepsilon_{i_1 i_2 \cdots i_n}$ is the totally antisymmetric symbol $\varepsilon_{i_1 i_2 \cdots i_n} = -\varepsilon_{i_2 i_1 \cdots i_n}$.

2.3 Group elements as transformations

We can define actions of group elements $g \in G$ on a set X. X might be G itself, but could also be a vector space (i.e. rotation matrices acting on vectors in $\mathbb{R}^3$).

Definition 2.6: The **left action** of G on X is a map $L : G \times X \to X$ such that for $x \in X$

- $L(e, x) = x$, for e, the identity of G,
- $L(g_2, L(g_1 x)) = L(g_2 g_1, x)$, $\forall x \in X$, $\forall g_1, g_2 \in G$.

The more usual notation is that $\forall g \in G$, we associate a map $g : X \to X$ such that $g(x) = gx$, however this is slightly less clear.

Definition 2.7: The **right action** of G on X is defined by $gX \to X$ such that $g(x) = xg^{-1}$, $\forall x \in X$ and $g \in G$.

The inverse preserves group composition. Namely,

$$g_2(g_1(x)) = x \underbrace{g_1^{-1} g_2^{-1}}_{(g_2 g_1)^{-1}} = (g_2 g_1)(x). \tag{31}$$

Definition 2.8: Conjugation by G on X is the action defined by

$$g(x) = gxg^{-1}, \tag{32}$$

$\forall g \in G_1$, $x \in X$.

Another definition worth making, even if it won't see immediate use is that of an *orbit*.

Definition 2.9: Given a group G and set X, an **orbit** of an element $x \in X$ is the set of elements of X which are in the image of an action of G on x.

Example. If the action is left, the orbit of $x \in X$ is written $Gx = \{gx \mid g \in G\}$.

It can be shown that the set of orbits under G 'partition' X as we will see.

2.4 Orthogonal groups

The orthogonal group, $O(n)$ in particular, represent rotations and reflections on $\mathbb{R}^n$. This preserves inner products such that

$$\langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \mathbf{v}_2^T \mathbf{v}_1, \tag{33}$$

given $R \in O(n)$,

$$\langle R\mathbf{v}_2, R\mathbf{v}_1 \rangle = \mathbf{v}_2^T \underbrace{\left(R^T R\right)}_{I} \mathbf{v}_1 = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle . \tag{34}$$

This is similar for $U(n)$.

Consider

$$SO(2) = \left\{ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in (0, 2\pi) \right\}. \tag{35}$$

As cos and sin are smooth functions, this is a differentiable manifold. One can also show that $R(\theta_2) R(\theta_1) = R(\theta_1 + \theta_2)$.

Similarly, $SO(3)$ can represent rotations of vectors in $\mathbb{R}^3$ where the axis of the rotation is given by a unit vector $\mathbf{n} \in S^2$ and we rotate by an angle θ. Note that rotation by $\theta \in [-\pi, 0]$ about $\mathbf{n}$ is equivalent to a rotation by $-\theta$ about $-\mathbf{n}$ so we confine to $\theta \in [0, \pi]$.

Therefore we can depict the manifold of $SO(3)$ as a ball of radius π in $\mathbb{R}^3$, where the direction is specified by $\mathbf{n}$ and the distance from the origin is specified by $\theta \in [0, \pi]$. Antipodal points are identified such that $\pi\mathbf{n} = -\pi\mathbf{n}$.

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3 Lie Algebras

3.1 Pseudo orthogonal group

$SO(n, m)$ act on vectors in $\mathbb{R}^{n+m}$ and preserve the scalar product

$$v_2^T \eta v_1, \tag{36}$$

for $v_1, v_2 \in \mathbb{R}^{n+m}$. For example, $SO(1,1)$ parametrise Lorentz boosts in one dimension and can be written in terms of the *rapidity* η as

$$SO(1,1) = \left\{ \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \middle| \eta \in \mathbb{R} \right\}. \tag{37}$$

As η is unbounded, $SO(1,1)$ is clearly noncompact.

3.2 Parametrization of Lie Groups

At least in small neighbourhoods, we can assign coordinates on an n-dimensional manifold to be

$$x := \left(x^1, \cdots, x^n\right) \in \mathbb{R}^n. \tag{38}$$

This allows us to label elements $g(x) \in G$. Closure provides

$$g(y)\, g(x) = g(z). \tag{39}$$

Smoothness gives us that the components of z are continuously differentiable functions of x and y such that for $i \in 1, \cdots, n$,

$$z^i = \phi^i(x, y). \tag{40}$$

We choose the coordinate origin such that $g(0) = e$. Identity gives us that

$$g(0)\,g(x) = g(x) \Rightarrow \phi^i(x,0) = x^r \text{ and } \phi^i(0,y) = y^i. \tag{41}$$

Similarly, for inverses, we have that there exists some $\widetilde{x}$ such that $g(\widetilde{x}) = g(x)^{-1}$ and thus

$$\phi^i(\widetilde{x},x) = 0 = \phi^i(x,\widetilde{x}). \tag{42}$$

Lastly, associativity gives us

$$g(z)\,(g(y)\,g(x)) = (g(z)\,g(y))\,g(x) \Rightarrow \phi^i(\phi(x,y),z) = \phi^i(x,\phi(y,z)). \tag{43}$$

This appears like a Leibniz rule/Jacobi identity as we will see.

3.3 Lie Algebras

A Lie group is homogeneous. Any neighbourhood 'looks like' (or in a more formal sense, can be mapped to) any other neighbourhood.

For example, for $\varepsilon \in G$ close to g_1, $g_2 g^{-1}\varepsilon$ is close to g_2.

Thus no neighbourhood in particular is special. The natural choice of the representative neighbourhood to study is the one centered at the identity of G. We will linearize near the identity of G.

Definition 3.1: A Lie Algebra is a vector space V, which additionally has a vector product, the **Lie bracket**, $[\cdot,\cdot] : V \times V \to V$ satisfying the following properties for $X, Y, Z \in V$.

1) It is antisymmetric, $[X,Y] = -[Y,X]$,
2) It satisfies the Jacobi identity, $[X,[Y,Z]] + [Y,[X,Z]] + [Z,[X,Y]] = 0$,
3) It is linear such that for $\alpha, \beta \in \mathbb{F}$, $[X, \alpha Y + \beta Z] = \alpha [X,Y] + \beta [X,Z]$.

Note. Any vector space which has a vector product $\star : V \times V \to V$ can be made into a Lie Algebra with its Lie bracket given by

$$[X,Y] = X \star Y - Y \star X. \tag{44}$$

Definition 3.2: Let's choose a basis for V, given by $\{T_a\}$ for $a = 1, \cdots, n = \dim V$. We call these basis vectors **generators** of the Lie algebra, and we write their Lie brackets as

$$[T_a, T_b] = f^c{}_{abc} T_c, \tag{45}$$

where $f^c{}_{ab} \in \mathbb{F}$ are called **structure constants**.

Antisymmetry implies $f^c{}_{ba} = -f^c{}_{ab}$ and the Jacobi identity implies

$$f^e{}_{ad} f^d{}_{bc} + f^e{}_{cd} f^d{}_{ab} + f^e{}_{bd} f^d{}_{ca} = 0. \tag{46}$$

The general element of a Lie algebra can be written as a linear combination of $\{T_a\}$ as

$$X \in V \Rightarrow X = X^a T_a \text{ with } x^a \in \mathbb{F}, \tag{47}$$

which gives us the bracket of any two elements in terms of structure constants with

$$[X,Y]=X^aY^bf^c{}_{abc}T_c. \tag{48}$$

3.4 Lie Groups and their Lie Algebras

Take $g(\theta)\in SO(2)$ to be

$$g(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \tag{49}$$

where $e=I_2=g(0)$. Points near the identity have $\theta\ll 1$ and thus Taylor expanding the components of $g(\theta)$ we see

$$g(\theta)=I_2+\theta\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}-\theta_2^2I_2+\mathcal{O}\left(\theta^3\right) \tag{50}$$

$$=e+\underbrace{\theta\frac{\mathrm{d}g}{\mathrm{d}\theta}\bigg|_{g=0}}_{\text{tangent vector}}+\frac{\mathrm{d}^2g}{\mathrm{d}\theta^2}+\mathcal{O}\left(\theta^2\right), \tag{51}$$

where the linear term is tangent to the manifold. Here there is a one dimensional tangent space at e given by

$$T_e\left(SO(2)\right)=\left\{\begin{pmatrix}0 & -a\\ a & 0\end{pmatrix}\bigg|a\in\mathbb{R}\right\}. \tag{52}$$

This is the Lie algebra of $SO(2)$,

$$\mathfrak{so}(2):=L\left(SO(2)\right):=T_e\left(SO(2)\right). \tag{53}$$

It remains to show this.

Proof. Notice that

$$\begin{pmatrix}0 & -a\\ a & 0\end{pmatrix}\begin{pmatrix}0 & -b\\ b & 0\end{pmatrix}=\begin{pmatrix}-ab & 0\\ 0 & -ab\end{pmatrix}=-abI, \tag{54}$$

and thus for any two elements (matrices) of the Lie algebra, they commute (which is trivially antisymmetric and satisfying of Jacobi). Linearity similarly follows immediately by inspection. □

Similarly, one can show $\dim\left(SO(n)\right)=\frac{1}{2}n(n-1)\equiv d$, so we have coordinates $x_1\cdots,x_d$. Consider a single-parameter family of $SO(n)$ elements,

$$M(t):=M\left(\mathbf{x}(t)\right)\in SO(n), \tag{55}$$

such that $M(0)=I_n$. Orthogonality ($M^TM=I$) implies

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\left(M^T(t)\,M(t)\right) \tag{56}$$

$$= \frac{\mathrm{d}M^T}{\mathrm{d}t} + M^T \frac{\mathrm{d}M}{\mathrm{d}t}, \tag{57}$$

where looking at $t = 0$, as $M(0) = I_n$ we see

$$\frac{\mathrm{d}M^T}{\mathrm{d}t} = -\frac{\mathrm{d}M}{\mathrm{d}t}, \tag{58}$$

which implies matrices in the tangent space of $SO(n)$ are antisymmetric (and thus traceless as well).

We have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \sum_i \frac{\partial M}{\partial x_i} \frac{\mathrm{d}x_i}{\mathrm{d}t}. \tag{59}$$

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Observe that

$$T_e(\mathcal{O}(n)) = T_e(SO(n)), \tag{60}$$

as $\det I = 1$, so all curves passing through I have $\det M = 1$.

3.5 Unitary Groups

Let $M(t)$ be a curve in $SU(n)$ with $M(0) = I$. For small t, write $M(t) = I + tX + \mathcal{O}(t^2)$, where $X = \left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=0}$.

Unitarity of M provides that for all t,

$$I = M^\dagger M \tag{61}$$

$$U = I + t\left(X + X^\dagger\right) + \mathcal{O}\left(t^2\right), \tag{62}$$

which implies $X^\dagger = -X$, namely, elements of the tangent space are *anti-Hermitian.*

Claim. $\operatorname{tr} X = 0$ for $X \in L(SU(n))$ or $M \in SU(n)$

Proof. Look at

$$M(t) = \begin{pmatrix} 1 + tX_{11} & tX_{12} & \cdots & tX_{1n} \\ tX_{21} & t + tX_{22} & \cdots & tX_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ tX_{n1} & tX_{n2} & \cdots & 1 + tX_{nn} \end{pmatrix}. \tag{63}$$

Notice that

$$1 = \det M = 1 + \underbrace{t \operatorname{tr} X}_{0} + \mathcal{O}\left(t^2\right), \tag{64}$$

where the underbraced term (and higher order ones) must vanish. ☐

For $U(n)$, X can have non-zero trace.

3.6 Lie algebra of a matrix Lie group

Consider two curves $g_1(x(t))$ and $g_2(x(t))$ through the identity e of some Lie group G. We define

$$X_1 := \dot{g}_1\Big|_{t=0}, \qquad X_2 := \dot{g}_2\Big|_{t=0}. \tag{65}$$

One can define a product

$$g_3(z(t)) = g_2(y(t))\, g_1(x(t)) \in G, \tag{66}$$

satisfying

$$\dot{g}_3\Big|_{t=0} = (\dot{g}_2 g_1 + g_2 \dot{g}_1)\Big|_{t=0} \tag{67}$$

$$= X_2 + X_1 \in T_e(G), \tag{68}$$

another vector in the tangent space.

The Lie bracket arises from the *group commutator*.

Definition 3.3: The **group commutator** of $g_1, g_2 \in G$, is

$$[g_1, g_2]_G := g_1^{-1} g_2^{-1} g_1 g_2 := h \in G. \tag{69}$$

Returning to our two curves through the identity e, $g_i(t)$ for $i \in \{1, 2\}$, we can expand

$$g_i(t) = e + tX_i + t^2 W_i + \mathcal{O}\left(t^3\right). \tag{70}$$

We have that

$$g_1(t)\, g_2(t) = e + t(X_1 + X_2) + t^2 (X_1 X_2 + W_1 + W_2) + \mathcal{O}\left(t^3\right), \tag{71}$$

and

$$g_2(t)\, g_1(t) = e + t(X_1 + X_2) + t^2 (X_2 X_1 + W_1 + W_2) + \mathcal{O}\left(t^3\right). \tag{72}$$

If we then look at

$$h(t) = [g_2(t)\, g_1(t)]^{-1} g_1(t)\, g_2(t) = e + t^2 \underbrace{(X_1 X_2 - X_2 X_1)}_{[X_1, X_2]} + \cdots, \tag{73}$$

and thus the group commutator induces the Lie bracket in the algebra. As $h(t) \in G$, the tangent to $h(t)$ at e is $[X_1, X_2] \in L(G)$, and thus we have closure under the Lie bracket.

- We write the tangent space to a matrix Lie group $G \overset{\text{subgroup}}{<} GL(n, \mathbb{F})$ at a general element p as $T_p(G)$. Let $g(t)$ be a curve in the manifold through p with $g(t_0) = p$, and thus

$$g(t + \varepsilon) = g(t_0) + \dot{\varepsilon}(t_0) + \mathcal{O}\left(\varepsilon^2\right). \tag{74}$$

As both $g(t_0), g(t_0 + \varepsilon) \in G$, there exists $h_p(\varepsilon) \in G$ such that

$$g(t_0 + \varepsilon) = g(t_0)\, h_p(\varepsilon), \tag{75}$$

and as $\varepsilon \to 0$, $h_p(\varepsilon) \to e$. For small ε,

$$h_p(\varepsilon) = e + \varepsilon X_p + \mathcal{O}\left(\varepsilon^2\right), \tag{76}$$

for some $X_p \in L(G) = T_e(G)$. Neglecting $\mathcal{O}\left(\varepsilon^2\right)$,

$$\begin{aligned} e + \varepsilon X_p = h_p(\varepsilon) &= g^{-1}(t_0)\, g(t_0+\varepsilon) && (77)\\ &= g^{-1}(t_0)\left[g(t_0) + \varepsilon \dot{g}(t_0)\right] && (78)\\ &= e + \varepsilon \underbrace{g^{-1}(t_0)\, \dot{g}(t_0)}_{X_p}. && (79)\end{aligned}$$

Claim. Conversely, for any $X \in L(G)$, there exists a unique curve $g(t)$ with $g^{-1}(t)\,\dot{g}(t) = X$ and $g(0) = g_0$.

Proof. This is a consequence of existence and uniqueness of solutions of ODEs. The solution of this ODE is

$$g(t) = g_0 \exp(tX), \tag{80}$$

where

$$\exp tX := \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}. \tag{81}$$

□

3.7 One parameter subgroups

Given an $X \in L(G)$, the curve

$$g_X(t) = \exp tX, \tag{82}$$

forms an *abelian* subgroup of G, *generated* by X.

Notice that $g_X(t)$ is isomorphic to the group of real numbers under addition $(\mathbb{R}, +)$ if only $g_X(0) = e$. If there exist other $t_0 \neq 0$ such that $g_X(t_0) = 0$, then we have periodic structure and then $g_X(t)$ is isomorphic to the circle S^1.