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Riccati equation

In mathematics, a **Riccati equation** in the narrowest sense is any first-order ordinary differential equation that is quadratic in the unknown function. In other words, it is an equation of the form

$$y'(x) = q_0(x) + q_1(x)\,y(x) + q_2(x)\,y^2(x)$$

where $q_0(x) \neq 0$ and $q_2(x) \neq 0$. If $q_0(x) = 0$ the equation reduces to a Bernoulli equation, while if $q_2(x) = 0$ the equation becomes a first order linear ordinary differential equation.

The equation is named after Jacopo Riccati (1676–1754).[1]

More generally, the term **Riccati equation** is used to refer to matrix equations with an analogous quadratic term, which occur in both continuous-time and discrete-time linear-quadratic-Gaussian control. The steady-state (non-dynamic) version of these is referred to as the algebraic Riccati equation.

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Conversion to a second order linear equation

The **non-linear** Riccati equation can always be converted to a second order linear ordinary differential equation (ODE):[2] If

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2$$

then, wherever q_2 is non-zero and differentiable, $v = yq_2$ satisfies a Riccati equation of the form

$$v' = v^2 + R(x)v + S(x),$$

where $S = q_2 q_0$ and $R = q_1 + \frac{q_2'}{q_2}$, because

$$v' = (yq_2)' = y'q_2 + yq_2' = (q_0 + q_1 y + q_2 y^2)q_2 + v\frac{q_2'}{q_2} = q_0 q_2 + \left(q_1 + \frac{q_2'}{q_2}\right)v + v^2.$$

Substituting $v = -u'/u$, it follows that u satisfies the linear 2nd order ODE

$$u'' - R(x)u' + S(x)u = 0$$

since

$$v' = -(u'/u)' = -(u''/u) + (u'/u)^2 = -(u''/u) + v^2$$

so that

$$u''/u = v^2 - v' = -S - Rv = -S + Ru'/u$$

and hence

$$u'' - Ru' + Su = 0.$$

A solution of this equation will lead to a solution $y = -u'/(q_2 u)$ of the original Riccati equation.

Application to the Schwarzian equation

An important application of the Riccati equation is to the 3rd order Schwarzian differential equation

$$S(w) := (w''/w')' - (w''/w')^2/2 = f$$

which occurs in the theory of conformal mapping and univalent functions. In this case the ODEs are in the complex domain and differentiation is with respect to a complex variable. (The Schwarzian derivative $S(w)$ has the remarkable property that it is invariant under Möbius transformations, i.e. $S((aw+b)/(cw+d)) = S(w)$ whenever $ad - bc$ is non-zero.) The function $y = w''/w'$ satisfies the Riccati equation

$$y' = y^2/2 + f.$$

By the above $y = -2u'/u$ where u is a solution of the linear ODE

$$u'' + (1/2)fu = 0.$$

Since $w''/w' = -2u'/u$, integration gives $w' = C/u^2$ for some constant C. On the other hand any other independent solution U of the linear ODE has constant non-zero Wronskian $U'u - Uu'$ which can be taken to be C after scaling. Thus

$$w' = (U'u - Uu')/u^2 = (U/u)'$$

so that the Schwarzian equation has solution $w = U/u$.

Obtaining solutions by quadrature

The correspondence between Riccati equations and second-order linear ODEs has other consequences. For example, if one solution of a 2nd order ODE is known, then it is known that another solution can be obtained by quadrature, i.e., a simple integration. The same holds true for the Riccati equation. In fact, if one particular solution y_1 can be found, the general solution is obtained as

$$y = y_1 + u$$

Substituting

$$y_1 + u$$

in the Riccati equation yields

$$y_1' + u' = q_0 + q_1 \cdot (y_1 + u) + q_2 \cdot (y_1 + u)^2,$$

and since

$$y_1' = q_0 + q_1\, y_1 + q_2\, y_1^2,$$

it follows that

$$u' = q_1\, u + 2\, q_2\, y_1\, u + q_2\, u^2$$

or

$$u' - (q_1 + 2\, q_2\, y_1)\, u = q_2\, u^2,$$

which is a Bernoulli equation. The substitution that is needed to solve this Bernoulli equation is

$$z = \frac{1}{u}$$

Substituting

$$y = y_1 + \frac{1}{z}$$

directly into the Riccati equation yields the linear equation

$$z' + (q_1 + 2\, q_2\, y_1)\, z = -q_2$$

A set of solutions to the Riccati equation is then given by

$$y = y_1 + \frac{1}{z}$$

where z is the general solution to the aforementioned linear equation.

See also

- Linear-quadratic regulator
- Algebraic Riccati equation
- Linear-quadratic-Gaussian control

References

1. Riccati, Jacopo (1724) "Animadversiones in aequationes differentiales secundi gradus" (https://books.google.com/books?id=UjTw1w7tZsEC&pg=PA66#v=onepage&q&f=false)

(Observations regarding differential equations of the second order), *Actorum Eruditorum, quae Lipsiae publicantur, Supplementa*, **8** : 66-73. Translation of the original Latin into English (http://www.17centurymaths.com/contents/euler/rictr.pdf) by Ian Bruce.

2. Ince, E. L. (1956) [1926], *Ordinary Differential Equations*, New York: Dover Publications, pp. 23–25

Further reading

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- Zelikin, Mikhail I. (2000), *Homogeneous Spaces and the Riccati Equation in the Calculus of Variations*, Berlin: Springer-Verlag
- Reid, William T. (1972), *Riccati Differential Equations*, London: Academic Press

External links

- "Riccati equation" (https://www.encyclopediaofmath.org/index.php?title=Riccati_equation), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- Riccati Equation (http://eqworld.ipmnet.ru/en/solutions/ode/ode0123.pdf) at EqWorld: The World of Mathematical Equations.
- Riccati Differential Equation (http://mathworld.wolfram.com/RiccatiDifferentialEquation.html) at Mathworld
- MATLAB function (http://www.mathworks.com/help/toolbox/control/ref/care.html) for solving continuous-time algebraic Riccati equation.
- SciPy has functions for solving the continuous algebraic Riccati equation (http://docs.scipy.org/doc/scipy/reference/generated/scipy.linalg.solve_continuous_are.html) and the discrete algebraic Riccati equation (http://docs.scipy.org/doc/scipy/reference/generated/scipy.linalg.solve_discrete_are.html).