COMPUTATIONAL MATHS SAMPLE PAPER

MATH-BASED QUESTIONS AND SOLUTIONS

Question 3.

The power generated by a windmill varies with the wind speed. In an experiment, the following five measurements were obtained:

Wind Speed (Kmph)	14	22	30	38	46
Electric Power (W)	320	490	540	500	480

Determine the fourth-order polynomial in the Lagrange form that passes through the points.
Use the polynomial to calculate the power at a wind speed of 26 Kmph.

[25 Marks]

This question takes a lot of time on the calculator.

The Lagrange Formula:

$$f(x) = \sum_{i=1}^{n} y_i L_i(x) = \sum_{i=1}^{n} y_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

I think you will have to know this equation off by heart.

This is the table above written in another form that makes it easier to see which values should go where:

x_1	14	x_2	22	x_3	30	x_4	38	x_5	46
y_1	320	y_2	490	y_3	540	y_4	500	y_5	480

For a fourth-order Lagrange polynomial, expand the equation like so:

$$f(x) = y_1\left(\frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}\right) + y_2\left(\frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}\right)$$
$$+\ y_3\left(\frac{(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}\right)$$
$$+\ y_4\left(\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}\right)$$
$$+\ y_5\left(\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}\right)$$

Filling in the values from the table you get:

$$f(x) = 320\left(\frac{(x-22)(x-30)(x-38)(x-46)}{(14-22)(14-30)(14-38)(14-46)}\right) + 490\left(\frac{(x-14)(x-30)(x-38)(x-46)}{(22-14)(22-30)(22-38)(22-46)}\right)$$
$$+\ 540\left(\frac{(x-14)(x-22)(x-38)(x-46)}{(30-14)(30-22)(30-38)(30-46)}\right)$$
$$+\ 500\left(\frac{(x-14)(x-22)(x-30)(x-46)}{(38-14)(38-22)(38-30)(38-46)}\right)$$
$$+\ 480\left(\frac{(x-14)(x-22)(x-30)(x-38)}{(46-14)(46-22)(46-30)(46-38)}\right)$$

I don't recommend writing out the equation beyond this point because it will be too time consuming, especially when there is a second part to the question.

Next, you have to find the power at a wind speed of 26 kmph. In other words find $f(26)$. All you have to do is put in 26 for x:

$$f(26) = 320\left(\frac{(26-22)(26-30)(26-38)(26-46)}{(14-22)(14-30)(14-38)(14-46)}\right) + 490\left(\frac{(26-14)(26-30)(26-38)(26-46)}{(26-14)(26-30)(26-38)(26-46)}\right)$$
$$+\ 540\left(\frac{(26-14)(26-22)(26-38)(26-46)}{(30-14)(30-22)(30-38)(30-46)}\right)$$
$$+\ 500\left(\frac{(26-14)(26-22)(26-30)(26-46)}{(38-14)(38-22)(38-30)(38-46)}\right)$$
$$+\ 480\left(\frac{(26-14)(26-22)(26-30)(26-38)}{(46-14)(46-22)(46-30)(46-38)}\right)$$

$$= -12.5 + 229.6875 + 379.6875 - 78.125 + 17.1565$$
$$= 535.9065W$$

The answer is 535.9065 is a good estimate because it lies in between 490 and 540 in the table.

Question 4.

Using a four-term Taylor series expansion, derive a four-point backward difference formula for evaluating the first derivative of a function given by a set of unequally spaced points. The formula should give the derivative at point $x = x_i$, in terms of $x_i, x_{i-1}, x_{i-2}, x_{i-3}, f(x_i), f(x_{i-1}), f(x_{i-2})$ and $f(x_{i-3})$.

[25 Marks]

Taylor Series expansion:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(\xi)}{3!}h^3$$

$\frac{f'''(\xi)}{3!}h^3$ just represents the error of the estimation.

h is the step size (i.e. the distance between any two points' x values or $h = x_{i+1} - x_i$.

We have to use this formula to derive another formula that will have x_i, x_{i-1}, x_{i-2}, x_{i-3}, $f(x_i)$, $f(x_{i-1})$, $f(x_{i-2})$ and $f(x_{i-3})$ in it. The first step is to write out the equations for $f(x_{i-1})$, $f(x_{i-2})$ and $f(x_{i-3})$ in terms of $f(x_i)$ and in a similar layout to the one above:

The first thing we have to remember is that we are going backwards, not forwards. And so, the distance between the points x_i and x_{i-1} is $-h$ instead of just h.

$$f(x_{i-1}) = f(x_i) + f'(x_i)(-h) + \frac{f''(x_i)}{2!}(-h)^2 + \frac{f'''(\xi)}{3!}(-h)^3$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f'''(\xi)}{3!}h^3$$

That's the first equation we need. Now we do the same for $f(x_{i-2})$ and $f(x_{i-3})$. We have to consider that the distance between x_{i-2} and x_i is twice the distance than that between x_i and x_{i-1}. So, now we have to put in $-2h$ instead of h in the top equation:

$$f(x_{i-2}) = f(x_i) + f'(x_i)(-2h) + \frac{f''(x_i)}{2!}(-2h)^2 + \frac{f'''(\xi)}{3!}(-2h)^3$$

$$f(x_{i-2}) = f(x_i) - f'(x_i)2h + \frac{f''(x_i)}{2!}4h^2 - \frac{f'''(\xi)}{3!}8h^3$$

Similarly for $f(x_{i-3})$ (remember now that we are going backwards and the distance is three times that of the original equation):

$$f(x_{i-3}) = f(x_i) + f'(x_i)(-3h) + \frac{f''(x_i)}{2!}(-3h)^2 + \frac{f'''(\xi)}{3!}(-3h)^3$$

$$f(x_{i-3}) = f(x_i) - f'(x_i)3h + \frac{f''(x_i)}{2!}9h^2 - \frac{f'''(\xi)}{3!}27h^3$$

These are the three equations we need to combine to obtain the desired equation. The trick here is to remove the double differentiation parts from all of the equations (i.e. the $f''(x_i)$ parts). The way we do this is by adding and subtracting the equations in some way that the $f''(x_i)$ parts cancel out. The way I do it is too multiply the $f(x_{i-1})$ equation by 5, add the $f(x_{i-2})$ equation and finally subtract the $f(x_{i-3})$ equation, like so:

$$
\begin{aligned}
&5f(x_{i-1}) + f(x_{i-2}) - f(x_{i-3}) = \\
&5\,f(x_i) - f'(x_i)5h + \frac{f''(x_i)}{2!}5h^2 - \frac{f'''(\xi)}{3!}5h^3 + \\
&f(x_i) - f'(x_i)2h + \frac{f''(x_i)}{2!}4h^2 - \frac{f'''(\xi)}{3!}8h^3 - \\
&f(x_i) + f'(x_i)3h - \frac{f''(x_i)}{2!}9h^2 + \frac{f'''(\xi)}{3!}27h^3
\end{aligned}
$$

The part of the equation marked in red will become zero:

$$\frac{f''(x_i)}{2!}5h^2 + \frac{f''(x_i)}{2!}4h^2 - \frac{f''(x_i)}{2!}9h^2 = \frac{f''(x_i)}{2!}9h^2 - \frac{f''(x_i)}{2!}9h^2 = \frac{f''(x_i)}{2!}0h^2 = 0$$

So now we have:

$$
\begin{aligned}
&5f(x_{i-1}) + f(x_{i-2}) - f(x_{i-3}) \\
&\quad = 5\,f(x_i) - f'(x_i)5h - \frac{f'''(\xi)}{3!}5h^3 + f(x_i) - f'(x_i)2h - \frac{f'''(\xi)}{3!}8h^3 - f(x_i) + f'(x_i)3h \\
&\quad + \frac{f'''(\xi)}{3!}27h^3
\end{aligned}
$$

$$5f(x_{i-1}) + f(x_{i-2}) - f(x_{i-3}) = 5\,f(x_i) - f'(x_i)4h - \frac{f'''(\xi)}{3!}5h^3 - \frac{f'''(\xi)}{3!}8h^3 + \frac{f'''(\xi)}{3!}27h^3$$

$$f'(x_i)4h = 5\,f(x_i) - 5f(x_{i-1}) - f(x_{i-2}) + f(x_{i-3}) - \frac{f'''(\xi)}{3!}5h^3 - \frac{f'''(\xi)}{3!}8h^3 + \frac{f'''(\xi)}{3!}27h^3$$

$$f'(x_i) = \frac{5\,f(x_i) - 5f(x_{i-1}) - f(x_{i-2}) + f(x_{i-3})}{4h} - \frac{f'''(\xi)}{24}5h^2 - \frac{f'''(\xi)}{24}8h^2 + \frac{f'''(\xi)}{24}27h^2$$

$$f'(x_i) = \frac{5\,f(x_i) - 5f(x_{i-1}) - f(x_{i-2}) + f(x_{i-3})}{4h} + O(h^2)$$

$O(h^2)$ is the truncation error.

Question 5.

The central span of the Golden Gate bridge is 1260m long and the towers' height from the roadway is 150m. The shape of the main suspension cables can be approximately modeled (as a catenary) by the equation:

$$f(x) = C\left(\frac{e^{x/C} + e^{-x/C}}{2} - 1\right) \quad for \ -630 \le x \le 630m$$

where C=1347.

By using the equation $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$, determine the length (L) of the main suspension cables using three-point Gaussian Quadrature.
The given Gauss points and coefficients for the interval [-1, 1] are:

$$x_1 = -0.77459667, \qquad x_2 = 0, \qquad x_3 = 0.77459667$$

$$C_1 = 0.5555556, \qquad C_2 = 0.8888889, \qquad C_3 = 0.5555556$$

[25 Marks]

First we have do find $f'(x)$ from $f(x)$ to put in to the equation for L:

$$f(x) = C\left(\frac{e^{x/C} + e^{-x/C}}{2} - 1\right)$$

$$f(x) = \frac{C}{2}e^{x/C} + \frac{C}{2}e^{-x/C} - C$$

Differentiate with respect to x:

$$f'(x) = \frac{C}{2}\left(\frac{1}{C}\right)\left(e^{x/C}\right) - \frac{C}{2}\left(\frac{1}{C}\right)\left(e^{-x/C}\right)$$

$$f'(x) = \frac{1}{2}e^{x/C} - \frac{1}{2}e^{-x/C}$$

Now substitute the value of $f'(x)$ into the equation for L:

$$L = \int_{-630}^{630} \sqrt{1 + \left(\frac{1}{2}e^{x/C} - \frac{1}{2}e^{-x/C}\right)^2}\, dx$$

To use a three-point Gaussian Quatrature, you have to transform the integral into the following form:

$$L = \int_{-1}^{1} f(t)dt$$

To do this, you use the following equations to find x and dx in terms of t and dt respectively. The values for a and b are the limits of the original integral, in this case $a = -630$ and $b = 630$.

$$x = \frac{1}{2}[t(b-a) + a + b]$$
$$x = \frac{1}{2}[t(630 - -630) \ + (-630) + 630]$$
$$x = \frac{1}{2}[t(1260)]$$
$$x = 630t$$

$$dx = \frac{1}{2}(b-a)dt$$
$$dx = \frac{1}{2}(630 - (-630)dt$$
$$dx = \frac{1}{2}(1260)dt$$
$$dx = 630\ dt$$

So, we have to substitute the values for x and dx back into the equation for L:

$$L = \int_{-1}^{1} f(t)dt = \ 630\int_{-1}^{1}\sqrt{1+\left(\frac{1}{2}e^{630t/C} - \frac{1}{2}e^{-630t/C}\right)^2}\ dt$$

And finally we can apply the three-point Gaussian Quatrature to find the integral:

$$I = \ C_1 f(x_1) + C_2 f(x_2) + \ C_3 f(x_3)$$

The values for C_1, C_2, C_3, x_1, x_2 and x_3 are mentioned below the question.

$$I = \ 0.5555556 f(-0.77459667) + 0.8888889 f(0) + \ 0.5555556 f(0.77459667)$$

$f(-0.77459667)$ means that you put -0.77459667 in for t in the equation for L:

$$I = 0.5555556\ \sqrt{1+\left(\frac{1}{2}e^{630(-0.77459667)/C} - \frac{1}{2}e^{-630(-0.77459667)/C}\right)^2}$$
$$+ 0.8888889\ \sqrt{1+\left(\frac{1}{2}e^{630(0)/C} - \frac{1}{2}e^{-630(0)/C}\right)^2}$$
$$+ 0.5555556\ \sqrt{1+\left(\frac{1}{2}e^{630(0.77459667)/C} - \frac{1}{2}e^{-630(0.77459667)/C}\right)^2}$$

$$I = 2.073717433$$

If we look back at the equation:

$$L = 630\int_{-1}^{1}\sqrt{1+\left(\frac{1}{2}e^{630t/C} - \frac{1}{2}e^{-630t/C}\right)^2}\ dt$$

We see that the length is the integral multiplied by 630, or:

$$L = 630(I)$$

And so:

$$L = 630(2.073717433) = \ 1306.442m$$