The Fundamental Theorem of Algebra via Linear Algebra

FTA Proof

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Theorem 1

Any nonconstant polynomial with complex coefficients has a complex root.

Theorem 2

For each $n \geq 1$, every $n \times n$ square matrix over $\mathbb{C}$ has an eigenvector in $\mathbb{C}^n$. Equivalently, for each $n \leq 1$, every linear operator on an n-dimensional complex vector space V has an eigenvector in V .

Lemma 3

Fix

- An integer $m > 1$
- A field F

Assumption

Suppose that, for every F-VS V whose dim not divisible by m, every linear operator on V has an eigenvector in V.

Proposition

$\Longrightarrow$ For every F-VS V with dim not divisible by m, each pair of commuting linear operators on V has a common eigenvector in V.

Applicability

- Lemma 3 has not many application
- We use Lemma 3, when m is a power of 2

Proof

Start

- We induct on the dimension d of the vector space as d runs through integers not divisible by m.

Proof

$d = 1$

A nonzero vector in a one-dimensional space is an eigenvector of every linear operator on the space (and two such operators commute)

Proof

case: $d > 1$

- d is not dividable by m
- We have settled all dimensions $< d$, which are not divisible by m
- Let A_1 and A_2 be commuting linear operators on an F-VS V with $dim_V = d$

Hypothesis of lemma $\implies$ A_1 has an eigenvalue λ in F

Proof

Let $U = im(A_1 - \lambda I_V), W = ker(A_1 - \lambda I_V)$

A_1-stability of subspaces of V

- $u \in U \implies A_1(u) \in U$
- $w \in W \implies A_1(w) \in W$

$dim_F W \geq 1$

Because λ is an eigenvalue of A_1

A_2-stability of U and V

- If $u \in U$, write $u = A_1(v) - \lambda v$
- $A_2(u)$ is also in U

$$A_2(u) = A_2(A_1(v)) - A_2(\lambda v)$$
$$= \mathrm{A}_1(A_2(v)) - \lambda(A_2(v)) = (A_1 - \lambda I_V)(A_2(v))$$

Proof

Note

- $\dim_F U + \dim_F W4 = d$ is not divisible by $m \implies$ one of U or W has dimension not divisible by m
- If the subspace U or W with dim not divisible by m is a proper subspace of $V \implies \mathrm{A}_1$ and A_2 have a common eigenvector in that subspace (and thus in V) by induction.
- The other case is that U or W is all of V and the other subspace is 0 since the dimensions add up to d. In that case $W = V$ since we already noted that $\dim_F W$ is positive.
- When $W = V$ every vector in V is an eigenvector for A_1, and one of them is an eigenvector for A_2 since V has dimension not divisible by m.

Lean

```lean
import Mathlib.LinearAlgebra.Matrix.IsDiag
import Mathlib.Data.Matrix.Basic
import Mathlib.LinearAlgebra.FiniteDimensional
import Mathlib.LinearAlgebra.Eigenspace.IsAlgClosed
import Mathlib.LinearAlgebra.Eigenspace.Basic
import Mathlib.LinearAlgebra.Finrank
import Mathlib.Algebra.Module.LinearMap
import Mathlib.FieldTheory.IsAlgClosed.Spectrum
open Matrix
open Fintype
```

Lean

```
variable {m : ℕ} [Fintype (Fin m)] [Field ℂ]

def IsEigenvector (A : Matrix (Fin m) (Fin m) ℂ) (v : Fin
    m → ℂ) := (v ≠ 0) ∧ (∃ μ : ℂ, (mulVec A v) = μ • v)

theorem exists_eigenvector (A : Matrix (Fin m) (Fin m) ℂ)
    : (m ≥ 1) → (∃ v : Fin m → ℂ, IsEigenvector A v) :=
by sorry
```

Lean

```
variable {m : ℕ} [Fintype (Fin m)] [Field ℂ]
```

Lean

```
def IsEigenvector (A : Matrix (Fin m) (Fin m) ℂ) (v : Fin
    m → ℂ) := (v ≠ 0) ∧ (∃ μ : ℂ, (mulVec A v) = μ · v)
```

Lean

```
theorem exists_eigenvector (A : Matrix (Fin m) (Fin m) ℂ)
    : (m ≥ 1) → (∃ v : Fin m → ℂ, IsEigenvector A v) :=
by sorry
```