Introduction to control systems
TI0118

Homework I

Student Name

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Exercise 1

Find the inverse Laplace transform by hand calculations of the following:

1. $F_1(s) = \dfrac{s-10}{(s+2)(s+5)}$

2. $F_2(s) = \dfrac{100}{(s+1)(s^2+4s+13)}$

3. $F_3(s) = \dfrac{s+18}{s(s+3)^2}$

Verify your results using the Symbolic toolbox in Matlab/Octave.

Solution 1

1. $F_1(s) = \dfrac{s-10}{(s+2)(s+5)} \rightarrow F_1(s) = \dfrac{A}{s+2} + \dfrac{B}{s+5}$

   - $A = \lim\limits_{s\to -2} \dfrac{s-10}{s+5} = -4$
   - $B = \lim\limits_{s\to -5} \dfrac{s-10}{s+2} = 5$

   $$F_1(s) = \frac{-4}{s+2} + \frac{5}{s+5} \xrightarrow{\mathcal{L}^{-1}} f_1(t) = 5e^{-5t} - 4e^{-2t}$$

```
syms s
F(s) = (s - 10)/((s+2)*(s+5))
ilaplace(F)

ans =

5*exp(-5*t) - 4*exp(-2*t)
```

2. $F_2(s) = \dfrac{100}{(s+1)(s^2+4s+13)} \rightarrow F_2(s) = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+4s+13}$

- $A = \lim\limits_{s \to -1} \dfrac{100}{s^2+4s+13} = 10$
- $F_2(s) = \dfrac{100}{(s+1)(s^2+4s+13)} = \dfrac{10}{s+1} + \dfrac{Bs+C}{s^2+4s+3}$

$$\frac{100}{\cancel{(s+1)(s^2+4s+13)}} = \frac{(10+B)s^2+(B+C+40)s+130+C}{\cancel{(s+1)(s^2+4s+13)}}$$

$10 + B = 0 \rightarrow B = -10$
$130 + C = 100 \rightarrow C = -30$

$$F_2(s) = \frac{10}{s+1} - \frac{10s+30}{s^2+4s+13}$$

- Sabendo que:

$\mathcal{L}(Ae^{-at}\cos(wt)) = \dfrac{A(s+a)}{(s+a)^2+w^2}$ e
$\mathcal{L}(Be^{-at}\sin(wt)) = \dfrac{Bw}{(s+a)^2+w^2}$

Com isso, podemos deduzir que:
$\mathcal{L}(Ae^{-at}\cos(wt) + Be^{-at}\sin(wt)) = \dfrac{A(s+a)+Bw}{(s+a)^2+w^2}$

Logo, manipulando $F_2(s)$, temos:

$$F_2(s) = \frac{10}{s+1} - \frac{10s+30}{s^2+4s+13} = \frac{10}{s+1} - \frac{10(s+2) + \frac{10}{3} * 3}{(s+2)^2+3^2}$$

De modo que $a = 2$, $A = 10$, $w = 3$ e $B = \dfrac{10}{3}$

Por fim:

$$F_2(s) = \frac{10}{s+1} - \frac{10(s+2) + \frac{10}{3}3}{(s+2)^2+3^2} \xrightarrow{\mathcal{L}^{-1}}$$

$$f_2(t) = 10e^{-t} - 10e^{-2t}(\cos(3t) + \frac{1}{3}\sin(3t))$$

```
%Exercise 1.2

syms s
F2(s) = 100 /((s+1)*(s^2+ 4*s + 13))
ilaplace(F2)

ans =

10*exp(-t) - 10*exp(-2*t)*(cos(3*t) + sin(3*t)/3)
```

3. $F_3(s) = \dfrac{s+18}{s(s+3)^2} \rightarrow F_3(s) = \dfrac{A}{s} + \dfrac{B}{(s+3)^2} + \dfrac{C}{s+3}$

   - $A = \lim\limits_{s\to 0} \dfrac{s+18}{(s+3)^2} = \dfrac{18}{9} = 2$
   - $B = \lim\limits_{s\to -3} \dfrac{s+18}{s} = \dfrac{15}{-3} = -5$
   - $C = \lim\limits_{s\to -3} \left[\dfrac{d}{ds}\left(\dfrac{s+18}{s}\right)\right] = \lim\limits_{s\to -3} \dfrac{-18}{s^2} = \dfrac{-18}{9} = -2$

$$F_3(s) = \frac{2}{s} + \frac{-5}{(s+3)^2} + \frac{-2}{s+3} \xrightarrow{\mathcal{L}^{-1}} f_3(t) = 2 - 5te^{-3t} - 2e^-3t$$

```
%Exercise 1.3
syms s
F3(s) = (s+18) / (s*((s+3)^2))
ilaplace(F3)

ans =

2 - 5*t*exp(-3*t) - 2*exp(-3*t)
```

Exercise 2

For a system with the following transfer function:

$$G(s) = \frac{\alpha s + 10}{(s^2 + 12s + 32)}$$

1. For $\alpha = 1$, find and plot the unit step and impulse response.
2. For $\alpha = [-4,\ -2,\ -1,\ 0,\ 1,\ 2,\ 4]$, plot and compare the unit step and impulse response.
3. Discuss your results.

Solution 2

1. $G(s) = \dfrac{\alpha s + 10}{(s^2 + 12s + 32)} \rightarrow G(s) = \dfrac{s + 10}{(s+4)(s+8)}$

   - Para a função degrau, temos:

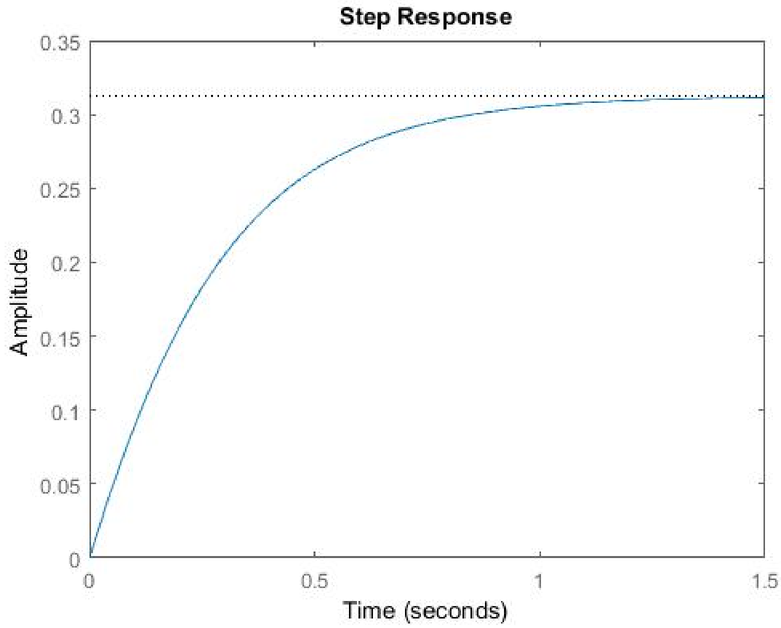
     $$Y(s) = \frac{s+10}{s(s+4)(s+8)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+8}$$

     o $A = \lim_{s \to 0} \dfrac{s+10}{(s+4)(s+8)} = \dfrac{10}{4*8} = \dfrac{5}{16}$

     o $B = \lim_{s \to -4} \dfrac{s+10}{s(s+8)} = \dfrac{6}{-16} = -\dfrac{3}{8}$

     o $C = \lim_{s \to -8} \dfrac{s+10}{s(s+4)} = \dfrac{2}{-8*-4} = \dfrac{1}{16}$

     $$Y(s) = \frac{\frac{5}{16}}{s} - \frac{\frac{3}{8}}{s+4} + \frac{\frac{1}{16}}{s+8} \xrightarrow{\mathcal{L}^{-1}} y(t) = \frac{5}{16} - \frac{3}{8}e^{-4t} + \frac{1}{16}e^{-8t}$$

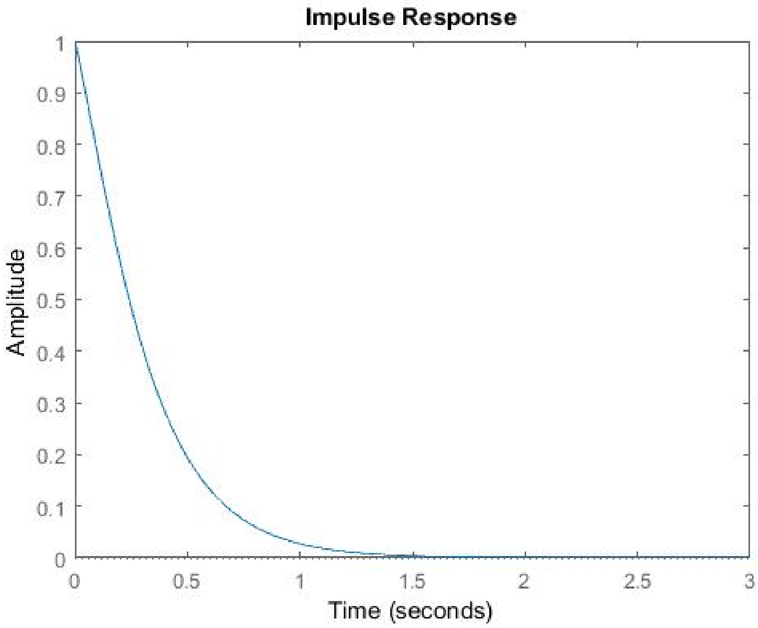


   - Para o impulso unitário, temos:

     $$Y(s) = \frac{s+10}{(s+4)(s+8)} = \frac{A}{s+4} + \frac{B}{s+8}$$

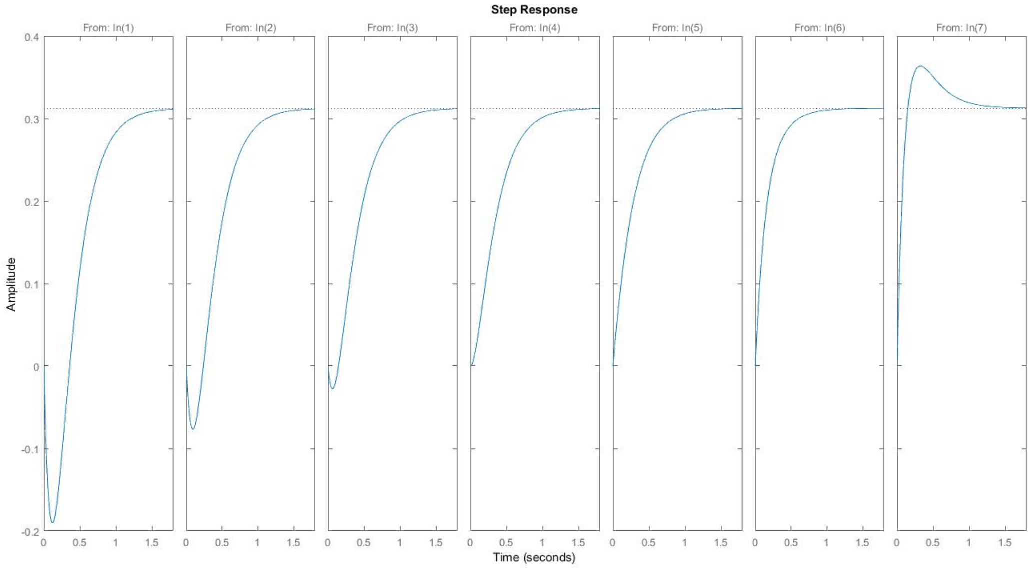
     o $A = \lim_{s \to -4} \dfrac{s+10}{s+8} = \dfrac{6}{4}$ $\qquad B = \lim_{s \to -8} \dfrac{s+10}{s+4} = \dfrac{2}{-4}$

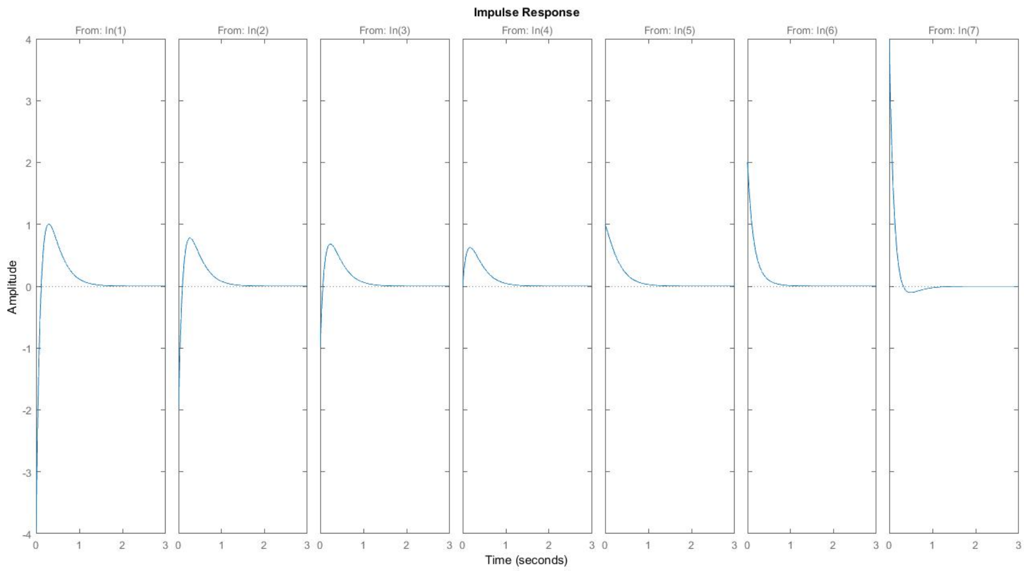
$$Y(s) = \frac{\frac{6}{4}}{s+4} - \frac{\frac{2}{4}}{s+8} \xrightarrow{\mathcal{L}^{-1}} y(t) = \frac{6}{4}e^{-4t} - \frac{2}{4}e^{-8t}$$




```
%Exercise 2.1
%Degrau
syms s
Y1(s)= (s+10)/(s*(s+4)*(s+8))
ilaplace(Y1)

num=[1 10];
den=[1 12 32];
G=tf(num,den);
step(G)

%Impulso
syms s
Y2(s)= (s+10)/((s+4)*(s+8))
ilaplace(Y2)

num=[1 10];
den=[1 12 32];
G=tf(num,den);
impulse(G)
```

2. Gráficos de $\alpha = [-4,\ -2,\ -1,\ 0,\ 1,\ 2,\ 4]$





```
%Exercise 2.2
%degrau
a = [-4 -2 -1 0 1 2 4]
for i=1:7
num=[a(i) 10];
den=[1 12 32];
G(i)=tf(num,den);
end
step(G)

%impulso
a = [-4 -2 -1 0 1 2 4]
for i=1:7
num=[a(i) 10];
den=[1 12 32];
G(i)=tf(num,den);
end
impulse(G)
```

3. Com os gráficos plotados, podemos perceber as situações de sobamortecimento, sobreamortecimento e amortecimento crítico. Quando $\alpha = 2$, podemos perceber o amortecimento crítico. Para $\alpha = 1$, temos sobreamortecimento. Enquanto para os outros α, temos sobamortecimento.

Exercise 3

Two tanks have cross-sectional areas S_1 and S_2 [m^2], respectively, and are arranged as shown in Figure 1.

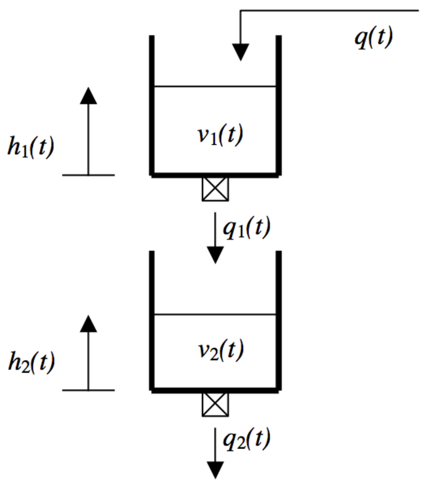


Figure 1: Two-tank system for Exercise 3.

The liquid heights on the two tanks are, respectively, $h_1(t)$ and $h_2(t)$ [m] and $v_1(t)$ and $v_2(t)$ are the liquid volumes in the tanks.

The first tank is fed by a flow-rate $q(t)$ [m^3/s] and has an output flow-rate given by $q_1(t) = K_1h_1(t)$ [m^3/s]. The first tank feeds the second one whose output flow-rate is given by $q_2(t) = K_2h_2(t)$ [m^3/s]. The mass conservation law for an incompressible fluid states that the derivatives of the liquid fluid $v(t)$ in the tank is given as

$$\frac{dv(t)}{dt} = q_{in}(t) - q_{out}(t)$$

1. Define a state space model for the system where $x_1(t) = v_1(t)$ and $x_2(t) = v_2(t)$ are the state variables, $u(t) = q(t)$ as input and $y(t) = h_2(t)$ as output.
2. Simulate the state space system, given the following values $S_1 = 2,\ S_2 = 4,\ K_1 = 3,\ K_2 = 4$ and the following inputs:
   (a) A step with amplitude 2 [m^3/s] (Hint: Use `stepDataOptions` for modifying the step amplitude).
   (b) A square signal with period 5 seconds, duration 30 seconds, and sampling every 0.1 second (Hint: Use `gensig` to generate the input signal).
3. Discuss your results.

Solution 3

1. Temos as seguintes equações pelo sistema:
   - $\dfrac{dv_1(t)}{dt} = q(t) - q_1(t) = q(t) - K_1h_1(t)$
   - $\dfrac{dv_2(t)}{dt} = q_1(t) - q_2(t) = K_1h_1(t) - K_2(t)h_2(t)$
   - $v_1(t) = h_1(t)S_1 \rightarrow h_1(t) = \dfrac{v_1(t)}{S_1}$
   - $v_2(t) = h_2(t)S_2 \rightarrow h_2(t) = \dfrac{v_2(t)}{S_2}$

   Substituindo pelas determinações da questão, temos:

   $\dot{x_1}(t) = u(t) - K_1h_1(t) = u(t) - \dfrac{K_1}{S_1}x_1(t)$

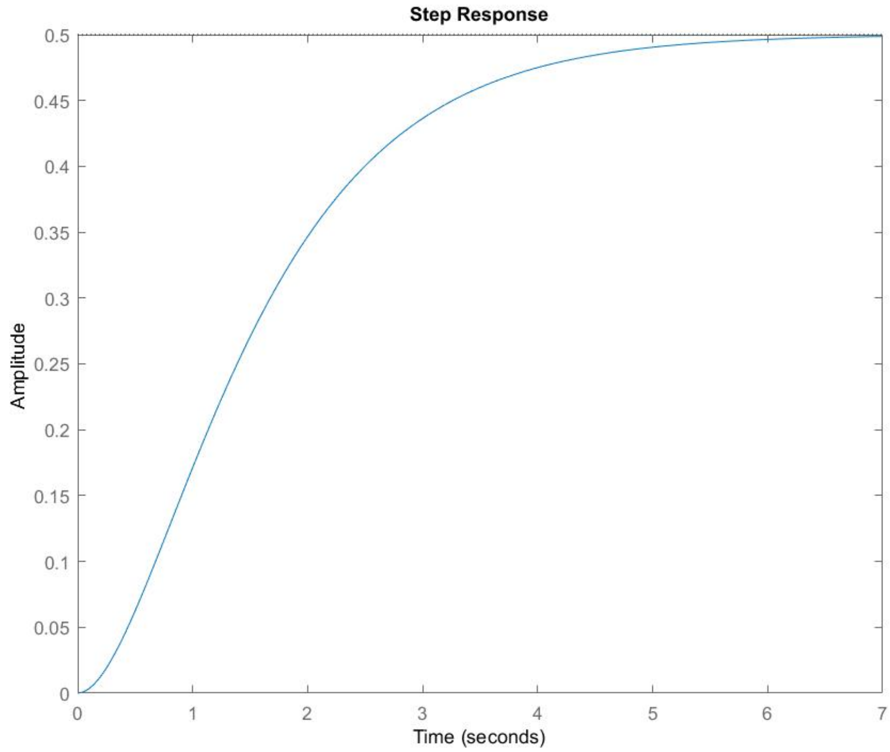
   $\dot{x_2}(t) = K_1h_1(t) - K_2h_2(t) = \dfrac{K_1}{S_1}x_1(t) - \dfrac{K_2}{S_2}x_2(t) \qquad y(t) = \dfrac{x_2(t)}{S_2}$

Com isso, achamos a representação em espaço de estado:

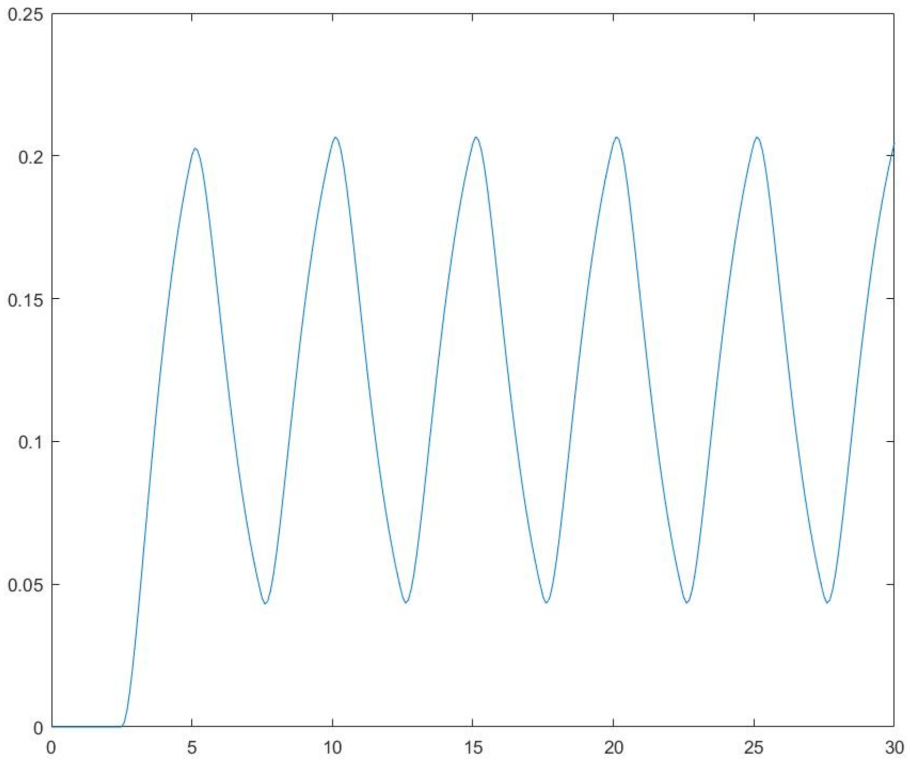
$$\dot{x} = Ax + Bu \rightarrow \dot{x} = \begin{bmatrix} -\dfrac{K_1}{S_1} & 0 \\ \dfrac{K_1}{S_1} & -\dfrac{K_2}{S_2} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = Cx + Du \rightarrow y = \begin{bmatrix} 0 & \dfrac{1}{S_2} \end{bmatrix} q + 0u$$

2. Para o degrau:




```
%Exercise3.2a
A = [(-3/2) 0;(3/2) (-4/4)];
B=[1; 0];
C=[0 (1/4)];
D=0;

[num,den] = ss2tf(A,B,C,D);
T = tf(num,den)
opt = stepDataOptions('StepAmplitude',2);
step(T,opt)
```




```
%Exercise3.2b
A = [(-3/2) 0;(3/2) (-4/4)];
B=[1; 0];
C=[0 (1/4)];
D=0;

[num,den] = ss2tf(A,B,C,D);
T = tf(num,den)
[u,t] = gensig('square',5,30,0.1)
[y,t]=lsim(T,u,t);
figure; plot(t,y)
```

Exercise 4

Given a system described by the IO model in Equation 1:

$$4\ddot{y}(t) + 7\dot{y}(t) + 3y(t) = \ddot{u}(t) + 4\dot{u}(t) + 4u(t) \tag{1}$$

1. Express the system as transfer function $G(s) = Y(s)/U(s)$.

2. Find a state space representation of the system.
3. Verify and comment your results on Matlab/Octave.

Solution 4

1. $4\ddot{y}(t) + 7\dot{y}(t) + 3y(t) = \ddot{u}(t) + 4\dot{u}(t) + 4u(t) \rightarrow$
$4s^2Y(s) + 7sY(s) + 3Y(s) = s^2U(s) + 4sU(s) + U(s) \rightarrow$
$Y(s)(4s^2 + 7s + 3) = U(s)(s^2 + 4s + 4) \rightarrow$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^2 + 4s + 4}{4s^2 + 7s + 3}$$

2. $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{s^2 + 4s + 4}{4s^2 + 7s + 3} * \dfrac{Z(s)}{Z(s)} \rightarrow$

$$Y(s) = (s^2 + 4s + 4)Z(s) \xrightarrow{\mathcal{L}^{-1}} y(t) = \ddot{z}(t) + 4\dot{z}(t) + 4z(t)$$
$$U(s) = (4s^2 + 7s + 3)Z(s) \xrightarrow{\mathcal{L}^{-1}} u(t) = 4\ddot{z}(t) + 7\dot{z}(t) + 3z(t)$$

Chamando $z(t) = q_1$ e $\dot{z}(t) = q_2$, temos que as variáveis de estado são:
$\dot{q_1} = q_2 \qquad \dot{q_2} = \frac{1}{4}u(t) - \frac{7}{4}q_2 - \frac{3}{4}q_1$

Achando a saída:
$y(t) = \ddot{z}(t) + 4\dot{z}(t) + 4z(t) \rightarrow y(t) = \frac{1}{4}u(t) - \frac{7}{4}q_2 - \frac{3}{4}q_1 + 4q_2 + 4q_1$
$y(t) = \frac{1}{4}u(t) + \frac{9}{4}q_2 + \frac{13}{4}q_1$

Com isso, chegamos a nossa representação em espaço de estado:
$$\dot{q} = Aq + Bu \rightarrow \dot{q} = \begin{bmatrix} 0 & 1 \\ -\frac{7}{4} & -\frac{3}{4} \end{bmatrix} q + \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} u$$

$$y = Cq + Du \rightarrow y = \begin{bmatrix} \frac{13}{4} & \frac{9}{4} \end{bmatrix} q + \frac{1}{4}u$$

3.
```
%Exercise4.3

num=[1 4 4];
den=[4 7 3];
T=tf(num,den);
Tss=ss(T)

Tss =

A =
x1      x2
x1   -1.75   -0.75
x2       1       0

B =
u1
x1    1
x2    0

C =
x1       x2
y1   0.5625   0.8125

D =
u1
y1   0.25

Continuous-time state-space model.
```