Problem 1
Logistic regression, likelihood, maximize likelihood, negative log likelihood.

Solution

$$p(C1 \mid \phi) = y(\phi) = \sigma(\mathbf{w}^T \phi)$$
$$p(C2 \mid \phi) = 1 - p(C1 \mid \phi)$$

For a dataset ϕ_n, t_n, where $t_n \in 0, 1$ and $\phi_n = \phi(x_n)$ with $n = 1, ..., N$ the likelihood function can be written as:

$$p(t \mid \mathbf{w}) = \prod_{n=1}^{N} y_n^{t_n}(1 - y_n)^{1-t_n}$$

where:

$$t_n = (t_1, ...t_N)^T, y_n = p(C1 \mid \phi_n)$$

Taking negative logarithm we have:

$$L = -logp(t \mid \mathbf{w}) = -\Sigma_{n=1}^{N} t_n log y_n + ((1 - t_n) log(1 - y_n))$$

Where

$$y_n = \sigma(a_n), a_n = \mathbf{w}^T \phi_n$$

We have:

$$\frac{\partial L}{\partial w} = \Sigma_{n=1}^{N} (\frac{\partial L}{\partial y_n} \times \frac{\partial yn}{\partial a_n} \times \frac{\partial a_n}{\partial w})$$
$$\frac{\partial L}{\partial w_n} = -\frac{t_n}{y_n} + \frac{(1 - t_n)}{(1 - y_n)} = \frac{y_n - t_n}{y_n(1 - y_n)}$$
$$\frac{\partial y}{\partial a_n} = \sigma(a_n)(1 - \sigma(a_n)) = y_n(1 - y_n)$$
$$\frac{\partial a_n}{\partial w} = \phi_n$$

So :

$$\frac{\partial L}{\partial w} = \Sigma_{n=1}^{N} (\frac{\partial L}{\partial y_n} \times \frac{\partial yn}{\partial a_n} \times \frac{\partial a_n}{\partial w}) = \Sigma_{n=1}^{N} \frac{y_n - t_n}{y_n(1 - y_n)} \times y_n(1 - y_n) \times \phi_n = \Sigma_{n=1}^{N} (y_n - t_n)\phi_n$$

Problem 2
Find the function f(x) with f'(x) = f(x)(1-f(x))

Solution

$$f'(x) = f(x)(1 - f(x))$$

$$\Longleftrightarrow \frac{d(f(x))}{dx} = f(x)(1 - f(x))$$

$$\longrightarrow \frac{d(f(x))}{f(x)(1 - f(x)} = dx$$

$$\longrightarrow \int \frac{d(f(x))}{f(x)(1 - f(x)} = \int dx$$

$$\longrightarrow \int \frac{1}{f(x)(1 - f(x))} d(f(x)) = \int dx$$

$$\longrightarrow \int \frac{1}{f(x)} + \frac{1}{1-f(x)} d(f(x)) = \int dx$$

$$\Longleftrightarrow ln(f(x)) - ln(1-f(x)) = x$$

$$\Longleftrightarrow \frac{f(x)}{1-f(x)} = e^x$$

$$\longrightarrow f(x) = e^x(1-f(x))$$

$$\Longleftrightarrow f(x) = e^x - e^x f(x)$$

$$\Longleftrightarrow 1 = \frac{e^x}{f(x)} - e^x$$

$$\Longleftrightarrow f(x) = \frac{e^x}{1+e^x} = \sigma(x)$$