Problem 1
To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

Solution

A: known to have Hansen disease
B: known to have no Hansen disease
H: Positive Hansen disease result
$P(A) = 0.05$
$P(B) = 0.95$
$P(H|A) = 0.98$
$P(H|B) = 0.03$

So the probability that someone who tests positive for Hansen' disease on this new test actually has the disease is:

$$(A|H) = \frac{P(H|A) \times P(A)}{P(H)}$$

$$P(A|H) = \frac{P(H|A) \times P(A)}{P(H|A) \times P(A) + P(H|B) \times P(B)}$$

$$(A|H) = \frac{0.98 \times 0.05}{0.98 \times 0.05 + 0.03 \times 0.95}$$

$$P(A|H) = \frac{0.049}{0.0775} \approx 0.632$$

Problem 2
Proof the following distributions are normalized then calculate the mean and standard deviation of these distribution:

1. Univariate normal distribution.
2. (Optional) Multivariate normal distribution.

Solution

1. Univariate normal distribution.

The area under the normal distribution curve should be equal to 1. Next is the proof:

$$Put\ I = \int e^{\frac{-x^2}{2}} dx, then\ we\ have\ I^2 = \left(\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-y^2+x^2}{2}} dxdy$$

$$set\ x = rcos\theta,\ y = rsin\theta\ We\ have \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix} = \begin{bmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$Which\ J = [\frac{\partial(x,y)}{\partial(r,\theta)}]\ We\ have : dxdy = \begin{bmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{bmatrix} drd\theta = rdrd\theta$$

$$So : I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{x^2+y^2}{2}} dxdy = \int_0^{2\pi}\int_0^{\infty} e^{\frac{-r^2}{2}} rdrd\theta.$$

$$So : I^2 = \int_0^{2\pi}\int_0^{\infty} e^{\frac{-r^2}{2}} rdrd\theta = \int_0^{2\pi} [-e^{\frac{-r^2}{2}}]_0^{\infty} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi. \Rightarrow I = \sqrt{2\pi}$$

Calculatuing Mean:

First: We let

$$Z = \frac{(X - \mu)}{\sigma}$$

We have:

$$E[Z] = \int_{-\infty}^{+\infty} x f_Z(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{\frac{-x^2}{2}} dx = \frac{-1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} |_{-\infty}^{+\infty} = 0$$

Because :

$$X = \mu + \sigma Z$$

So:

$$E(X) = E(\mu) + E(\sigma Z) \Longleftrightarrow E(X) = \mu + E(Z)E(\sigma)$$

But:

$$E(Z) = 0 \rightarrow E(X) = \mu$$

Calculating Variance:

We have:

$$Var(Z) = E(Z^2) - ((E(Z))^2 = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{\frac{-x^2}{2}} dx$$

We let:

$$u = x, dv = xe^{\frac{-x^2}{2}}$$

$$Var(Z) = \frac{1}{\sqrt{2\pi}} (-xe^{\frac{-x^2}{2}} |_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx = 1$$

But:

$$X = \mu + \sigma Z$$

$$\Longleftrightarrow Var(X) = Var(\mu) + \sigma^2 Var(Z)$$

$$\rightarrow Var(X) = 0 + \sigma^2 1$$

$$\rightarrow Var(X) = \sigma^2$$