Fermiones

$$\begin{cases} \dim\left(\Lambda^n(\mathcal{H})\right) = \binom{g}{n} \checkmark, \quad g := \dim \mathcal{H}. \\ \text{Bosones} \\ \dim\left(\odot^n(\mathcal{H})\right) = \binom{g+n-1}{n} \checkmark \\ \text{clásicos} \\ \dim\left(\otimes^n(\mathcal{H})\right) = g^n. \end{cases}$$

$\mathcal{H}$ Hilbert space of finite dimension.

We're going to have: $\mathcal{H}_1, \dots, \mathcal{H}_M$.

(Eigenspaces of the $\hat{H}$ operator)

$\hat{H}\psi = E_i \psi, \quad i = 1, \dots, M.$

The vector space of all ψ satisfying this equation is $\mathcal{H}_i$.

- $X_{N,M} = \{(n_1, \dots, n_M) \; ; \; n_1 + \dots + n_M = N\} \leftarrow$ $\boxed{\text{Measure space of } N \text{ Q.P. over } M \text{ energy levels}}$

n_i = occupation number of the E_i'th energy level.

Recall that we want to assign a measure to X_N.

$$\mu(n_1, \dots, n_M) = \prod_{i=1}^{n} \dim\left(\mathrm{Prod}^{n_i} \mathcal{H}_i\right). \rightarrow \text{Depends on the physical nature of the particle}$$

$$\boxed{X_{N,\infty} = \{(n_1, n_2, \dots) \in \mathbb{N} \; ; \; n_1 + n_2 + \dots = N\}}$$

$$\mu(n_1, \dots, n_M) := \dim\left((\mathrm{Prod}^{n_1}\mathcal{H}_1) \otimes \dots \otimes (\mathrm{Prod}^{n_M}\mathcal{H}_M)\right)$$

$n_1 + \dots + n_M = N < \infty.$

$$\boxed{\rho(n_1, \dots, n_M) = \frac{e^{-\beta E(n_1, \dots, n_M)}}{\int_X e^{-\beta E(n_1', \dots, n_M')} \, d\mu(n_1', \dots, n_M')}}.$$

$$\boxed{\sum_{k_1 + \dots + k_n = g} 1 = \binom{g+n-1}{n}}$$

Fermiones

$$\left.\begin{cases} \dim\left(\Lambda^n(\mathcal{H})\right) = \binom{g}{n} \checkmark, \quad g := \dim \mathcal{H}. \\ \text{Bosones} \\ \dim\left(\odot^n(\mathcal{H})\right) = \binom{g+n-1}{n} \checkmark \\ \text{clásicos} \\ \dim\left(\otimes^n(\mathcal{H})\right) = \frac{g^n}{n!}. \end{cases}\right\}$$

$\mathcal{H}$ Hilbert ~~space of~~ finite dimension.

M. S. C.

Bourbaki Algebra

$\leftarrow$ Added!

Clásicas: ① Distinguibles. ② No tienen ppio. de exclus.

① $\Leftrightarrow 1 \otimes 2 \neq 2 \otimes 1$ ✓
② $\Leftrightarrow 1 \otimes 1 \neq 0.$ ✓

} La construcción $\otimes$ recoge estas caracter...

Fermiones: ① Indistinguibles ② Tienen exclusión (en niveles de energía)

② $\iff$ $1 \wedge 1 = 0$ ✓

① $\iff$ $1 \wedge 2 = -2 \wedge 1$. ⤳ El $(-)$ no importa a nivel cuántico ✓

$\mathbb{P}(\Lambda^2 \mathcal{H})$

Projective space

Bosones: ① Indistinguibles ② No tienen exclusión (en n. de energ)

① $\iff$ $1 \odot 2 = 2 \odot 1$. ✓

② $\iff$ $1 \odot 1 \neq 0$. ✓

Según el profe: ($i \to \mathcal{E}$ levels) $\mu : \mathcal{S} \longrightarrow \mathbb{R}_+$, $\mathcal{S} \subseteq \mathcal{P}X$

$$\mu(n_1, n_2, \dots) := \prod_{i=1}^{\infty} \dim(\mathrm{Prod}^{n_i} \mathcal{H}_i). \checkmark$$

$\Omega := \mu$(Energy shell with value $\overline{E}$)

$= \mu(\{x \in X \,;\, E(x) = \overline{E}\})$. $\begin{cases} x = (n_1, \dots, n_M, \dots), \\ E(x) = \sum_i E_i n_i \end{cases}$

$$\boxed{\Omega = \sum_{\{x \,;\, E(x) = \overline{E}\}} \mu(x) = \sum_{\{x;\, E(x) = \overline{E}\}} \prod_{i=1}^{\infty} \dim(\mathrm{Prod}^{n_i} \mathcal{H}_i)}$$

(Ej) $\underbrace{\dim \mathcal{H}_1}_{g_1} = 1$, $\underbrace{\dim \mathcal{H}_2}_{g_2} = 2$, $\underbrace{\dim \mathcal{H}_3}_{g_3} = 3$.

$N = 3 \to$ Particles.

$E_1 = 0$, $E_2 = \varepsilon$, $E_3 = 2\varepsilon$.

→ Find the measure of the energy shell $\overline{E} = 2\varepsilon$.

Sol. $\{(n_1, n_2, n_3) \in X_3 \,;\, \underbrace{n_2 \varepsilon + 2 n_3 \varepsilon = 2\varepsilon}\}$

$\cancel{\varepsilon}(n_2 + 2n_3) = 2\cancel{\varepsilon} \iff \frac{n_2}{2} + n_3 = 1$.

$\Longrightarrow$ $n_1 + n_2 + n_3 = 3$ y $\frac{n_2}{2} + n_3 = 1$.

$n_1 = 3 \Rightarrow n_2 = n_3 = 0$ ✗ | $n_1 = 2 \Rightarrow \begin{cases} n_2 = 1 \text{ y } n_3 = 0 \text{ ✗} \\ n_2 = 0 \text{ y } n_3 = 1 \text{ ✓} \end{cases}$

$n_1 = 1 \Longrightarrow \begin{cases} n_2 = 2, & n_3 = 0 \checkmark \\ n_2 = 1, & n_3 = 1 \times \\ n_2 = 0, & n_3 = 2 \times \end{cases}$

$n_1 = 0 \Longrightarrow \begin{cases} n_2 = 3, & n_3 = 0 \times \\ n_2 = 2, & n_3 = 1 \times \\ n_2 = 1, & n_3 = 2 \times \\ n_2 = 0, & n_3 = 3 \times \end{cases}$

Conclusión: Solo hay 2: $\left.\begin{array}{l}(n_1 = 2, n_2 = 0, n_3 = 1) \\ (n_1 = 1, n_2 = 2, n_3 = 0)\end{array}\right\} s$

~~Clásicas~~ Maxwell!

$$\Omega_{M-B}(\bar{E} = 2\varepsilon) = \cancel{N!} \sum_s \prod_{i=1}^{3} \frac{g_i^{n_i}}{n_i!} = \cancel{6}\left(\frac{1^2 \cdot 2^0 \cdot 3^1}{2} + \frac{1^1 \cdot 2^2 \cdot 3^0}{2} \right) = 21. \checkmark\checkmark$$

Femiones: $\Omega_{F-D}(\bar{E} = 2\varepsilon) = \sum_s \prod_{i=1}^{3} \binom{g_i}{n_i} = \underbrace{\binom{1}{2}}_{0 \text{ (por convención)}}(\quad)(\quad) + \binom{1}{1}\binom{2}{2}\binom{3}{0} = 1 \checkmark\checkmark$

Algebraic Intermezzo $\dim V = n$; $\dim \Lambda^n V = 1$, $\dim \Lambda^{n+1} V = 0$.

$v_1 \wedge \cdots \wedge v_n \wedge v_i = 0$.

Bosones: $\Omega_{B-E}(\bar{E} = 2\varepsilon) = \sum_s \prod_{i=1}^{3} \binom{g_i + n_i - 1}{n_i}$

$$= \underbrace{\binom{1+2-1}{2}}_{1}\underbrace{\binom{2+0-1}{0}}_{1}\underbrace{\binom{3+1-1}{1}}_{\binom{3}{1}=3} + \underbrace{\binom{1+1-1}{1}}_{1}\underbrace{\binom{2+2-1}{2}}_{\frac{3!}{2!1!}=3}\underbrace{\binom{3+0-1}{0}}_{1}$$

$= 6.$ ◇

$t_R = \mu(n_1, \dots, n_M)$.

How are the n_i's statistically distributed when restricted to an energy shell $\bar{E}$?

$s = -k \int_{X_{\bar{E}}} \rho \log \rho \, d\mu$ In the context of $X_{\bar{E}}$ being an energy shell one uses the equiprobability postulate.

$= -k \log \Omega_{\bar{E}}$ ρ = uniform $= \dfrac{1}{\#\ \mu\text{-states in the energy shell}}$

Maximal entropy.

To find the answer to this issue, we have two ideas:

1) Use conditions $\sum_i n_i = N$, $\sum_i n_i E_i = E$, to maximize the entropy $-k \log \Omega_E$.

2) (Scheck). Consider only one energy level and use the usual expressions ($\mathcal{H} = 1$).

Maxwell $\mu(n) = \dfrac{(\dim \mathcal{H})^n}{n!}$, $\dim \mathcal{H} = 1$ (non-degenerate energies)

Fermiones $\mu(n) = \binom{\dim \mathcal{H}}{n}$, recall if $n > 1$, then $\mu(n) = 0$.

$= \binom{1}{n}$

Bosons. $\mu(n) = \binom{\cancel{\dim \mathcal{H}}^{\,1} + n - 1}{n} = \binom{n}{n} = 1$.

Next step: Understand density of states: $\sum_{\_} \longrightarrow \int_{\_} \rho(k)\, dk$.

Next: Examples!!