RESEARCH STATEMENT

CIGOLE THOMAS

The broad area of my research is Algebraic Geometry, in which we use algebraic tools to study geometric objects called *varieties*. Varieties are mathematical objects that arise as simultaneous zero sets of a collection of polynomials. For example, the circle defined by the polynomial $f(x) = x^2 + y^2 - 1$ is a variety. The fundamental group, Γ of a manifold encodes its topology and mapping it into an algebraic group, G, provides a way to relate the geometric structure of G to the topology from Γ. This motivates the study of equivalence classes of homomorphisms from Γ to G, called *character varieties*. The dynamics of the action of the outer automorphism group of Γ on the G-character varieties of Γ over fields of characteristic zero has been extensively explored and studied in various cases. The goal of my research is to draw a parallel picture for the dynamics of this action on character varieties over fields of positive characteristic. This is particularly interesting since it connects different mathematical areas such as Algebraic Geometry, Number Theory, Algebraic Topology, and other related fields. Algebraic Geometry has exciting applications in industry, including approximating data sets which led me to explore possible ways of using varieties in Data Learning. My research is motivated by how different disciplines of mathematics come together by combining their tools to solve questions that could otherwise prove challenging to address.

Past. During my undergraduate studies at Indian Institue of Science Education and Research Mohali (IISER) India, I worked on different summer projects such as *Symmetry Groups of Platonic Solids*, *Flight Dynamics on Mimetic Butterflies*, *Automorphism Group of Free Groups, and Train Tracks* and *Hyperbolic geometry and Knot Theory*. This experience helped me familiarize myself with different mathematics disciplines before deciding that Geometry and Topology piques my interest the most. This led me to work on Hyperbolic Geometry for my master's thesis, where I studied decompositions of complex hyperbolic geometry. Together with my advisor Prof. Krishnendu Gongopadhyay, we proved the following in [5].

Theorem 1. *Let T be a holomorphic isometry of $H^n_{\mathbb{C}}$, that is, $T \in PU(n, 1)$. Then T is a product of at most four involutions and a complex k-reflection, where $k \leq 2$; $k = 0$ if T is elliptic; $k = 1$ if T is ellipto-translation or hyperbolic; $k = 2$ if T is ellipto-parabolic and $n > 2$.*

Present. My PhD research is carried out under the supervision of Prof. Sean Lawton at George Mason University (GMU). The primary focus of my research involves exploring the theoretical aspect of the dynamics of group actions on character varieties.

Informal Idea of Research. In simple terms, the theoretical part of my research involves checking the "symmetry" or "shape" of a "data set" as the size of the set gets increasingly bigger by applying a set of transformations to the set. For example, if the data set is in the form of a circle, then applying a rotation will not change the "shape" of the original data set. There are three main components to my research. The first one is to see if the specific data sets I am working with have an "almost round" geometry. In other words, the goal is to check if all the points or almost all the points in the data set are connected through this collection of transformations when the size of the data set goes to infinity. The second component involves looking at the "range of mobility of the points" on applying transformations to the data set. The third part focuses on counting the size of specific "data sets".

Detailed Explanation. The data sets in my research are algebraic varieties that are subsets of points in an affine space over a field, k, that are simultaneous zero sets of a collection of polynomials equipped with Zariski topology. When the polynomials that define the variety have integer coefficients, it is possible to look at the finite field points of the variety. Denote the finite field with q elements by $\mathbb{F}_q$. A character variety is constructed using two components: (1) an algebraic group G, a group that is a variety where group operations are regular maps and (2) a finitely generated group Γ. Then the *G-character variety* of Γ is the space of equivalence classes of group homomorphisms from Γ to G where two homomorphisms are equivalent if and only if their orbit closures intersect.

Using this terminology, my research is focused on understanding different aspects of the dynamics of the action of outer automorphism group, $\mathrm{Out}(\Gamma)$ on the G-character variety of Γ over finite fields, $\mathbb{F}_q$. The dynamics of mapping class group action on character varieties has been studied extensively for different classes of Γ and G. Such actions on finite field points of character variety have not been explored partially because of the absence of a natural geometric invariant measure. This makes common properties in a dynamic system like ergodicity challenging to use. I attempt to address this issue by looking at an analogy of such topological dynamics in an arithmetic setting. In this setting, comparable problems have been studied by Bourgain, Gamburd and Sarnak [1] using number theoretic techniques. Related problems have also been addressed in [3], [4].

My first approach to studying the dynamics is to look at the transitivity properties of the group action. In particular, I proved that the action is not transitive in certain cases. If the action is not transitive, the next step is to look for 'the biggest orbit' or a large orbit. I considered the set of epimorphisms as a potential candidate for a large orbit and showed that the $\mathrm{Out}(\Gamma)$ action is transitive on the set of epimorphisms when Γ has desirable properties. The next goal was to find a way to gauge how big is the large orbit or how well the action "mixes" the points in the variety. The character varieties over fields of characteristic zero have a well-defined geometric invariant measure. In this scenario, the concept of ergodicity determines how well a group action mixes the points in the variety. The action of a group, G on a variety, X is *ergodic* if $A \subset X$ is G-invariant implies A has measure zero or $X - A$ has measure zero. Since there is no well-defined notion of such a 'nice' measure in the finite field points of character variety, I defined asymptotic transitivity to understand the extent of "mixing" under a group action. Let G be an algebraic group and X a variety defined over integers such that G acts on the variety. Now consider the action of G on the finite field, $(\mathbb{Z}/p\mathbb{Z})$ points of the variety denoted by $X(\mathbb{Z}/p\mathbb{Z})$. Then we say that the action is *asymptotically transitive* if

$$\lim_{p\to\infty} \frac{\left| \max\limits_{v\in X(\mathbb{Z}/p\mathbb{Z})} \mathrm{Orb}(v) \right|}{|X(\mathbb{Z}/p\mathbb{Z})|} = 1.$$

I am interested in exploring how this group action unfolds in specific character varieties and verifying asymptotic transitivity. One such particular class of variety is the $\mathrm{SL}_n(\mathbb{C})$-character variety of free groups. This led me to calculate the E-polynomial of $\mathrm{SL}_n(\mathbb{C})$ for specific values of n which gives the number of $\mathbb{F}_q$ points as a function of q. This was accomplished by stratifying the variety based on its stabilizer type and exploring the subgroups in each stratum with the intention of understanding asymptotic transitivity. I proved that the action is not asymptotically transitive on the finite field points of the $\mathrm{SL}_n(\mathbb{F}_q)$-character variety of free group of rank r, F_r, when $n = 2, 3$.

Results. The first of the results shows that if a group G has a proper subgroup H such that there exist homomorphisms from Γ mapping onto both H and its complement $G - H$, then the action of G cannot be transitive on the set of homomorphisms from Γ to G and consequently on the character variety.

Theorem 2 (Non-Transitivity Theorem). *Let Γ be a group generated by $\{\gamma_1, \cdots, \gamma_r\}$ and G a reductive affine algebaric group over $\mathbb{Z}$. Suppose there exists $\rho \in \mathrm{Hom}(\Gamma, G)$ such that the image of the evaluation map, $ev(\rho) \in G^r$ and $\rho(\gamma_i) \in H \subset G$ for all i where H is a subgroup of G with the property that there exists $Y \in G - H$ and $\mu \in \mathrm{Hom}(\Gamma, G)$ such that $\mu(\gamma_i) = Y$ for some i. Then the action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Hom}(\Gamma, G)$ is not transitive. Moreover if ρ and μ are polystable, then the action of $\mathrm{Out}(\Gamma)$ on the character variety, $\mathfrak{X}_\Gamma(G)$ is not transitive.*

The next step is to look for a 'large orbit'. The following theorem proves that the action of $\mathrm{Out}(\Gamma)$ is transitive on the set of epimorphisms when the automorphism group of Γ has certain properties.
Define a *group of free type* to be a finitely generated group with an automorphism group that allows permutations, inversions, left multiplication and right multiplication of coordinates.

Theorem 3 (Large Orbit Theorem)**.** *Let G be a finite group and Γ be a group of free type with $n \geq 2r$ generators where r denotes the minimal number of generators for G. Then* $\mathrm{Aut}(\Gamma)$ *acts transitively on the set of epimorphisms from Γ to G.*

The next part of my research involved exploring asymptotic transitivity on certain character varieties. I primarily worked on the finite field points of $\mathrm{SL}_n(\mathbb{F}_q)$-character varieties of F_r, the free group on r generators when $n = 2, 3$. In each case, I stratified the space of homomorphisms by the stabilizer type of conjugation action. I calculated the size of each stratum and the E-polynomial of the character variety while providing a different proof for the same when $n = 3$. This led to the following theorem on asymptotic transitivity in these cases.

Theorem 4 (Asymptotic Transitivity)**.** *Let $\mathfrak{X}_{F_r}(\mathrm{SL}_n(\mathbb{F}_q))$ denote the finite field points of the* $\mathrm{SL}_3(\mathbb{C})$*-character variety of F_r for $n = 2, 3$. Then the action of* $\mathrm{Out}(F_r)$ *on $\mathfrak{X}_{F_r}(\mathrm{SL}_n(\mathbb{F}_q))$ is not asymptotically transitive. As $q \to \infty$, the ratio of the size of the maximal orbit to that of the variety is bounded above by $\frac{1}{2}$, that is,*

$$\lim_{q\to\infty} \frac{\left|\max\limits_{v\in\mathfrak{X}_{F_r}(\mathrm{SL}_n(\mathbb{F}_q))} \mathrm{Orb}(v)\right|}{|\mathfrak{X}_{F_r}(\mathrm{SL}_n(\mathbb{F}_q))|} \leq \frac{1}{2} \quad \textit{for } n = 2, 3.$$

Varieties in Data Learning. Exploring different ways of applying the theory of varieties to potentially understanding or approximating data sets allows us to see 'varieties in action'. This has been an interesting counterpart to 'action on varieties'. As part of the Industrial Immersion Program (IIP) at GMU, I have been working on studying how to use varieties for data learning. Given a data set, I want to know if it will be possible to approximate the data points to a variety. My objective is to use known results regarding varieties for training and data prediction. One of the existing works [2] focuses on using algebraic varieties for approximating data by learning the variety from a given sample of points. Another approach [7] uses supervised learning and network design for learning membership on algebraic varieties via deep neural networks.
Motivated by this, I am working on the following directions. An *algebraic hypersurface* is an algebraic variety defined by a single irreducible polynomial. Hypersurfaces are interesting candidates for approximating varieties since any variety, X of dimension r is birationally equivalent to a hypersurface in the projective affine space, $\mathbb{P}^{r+1}$ [6]. For a fixed dimension there are only fintiely many hypersurfaces. If the data points sit inside an ambient space of fixed dimension given by the number of variables/coordinates, then there are two paths that can be considered. The first one involves fixing a hypersurface and using the distance as a loss function. Then the number of data points that lies within a fixed tolerance level of the hypersurface is used to find the best variety that approximates the data set.
The second approach involves designing a neural network for multi-class classification with a finite subcollection of hypersurfaces of the fixed dimension. I am currently working on developing an algorithm that implements this approach. The strategy is to start with known examples of varieties and extend the algorithm to test and train using more complex ones.

Visualization. Finally, I am also interested in exploring visualizations of these orbits, especially for small primes p, as part of an effort to make abstract mathematics more accessible to a general audience. The following depicts the action of outer automorphism on certain abelian character varieties.

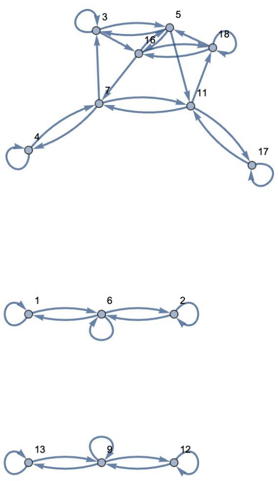

FIGURE 1. Orbits under the action of $\mathrm{Aut}(\mathbb{Z}^2)$ on $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{SL}_2(\mathbb{F}_2))$

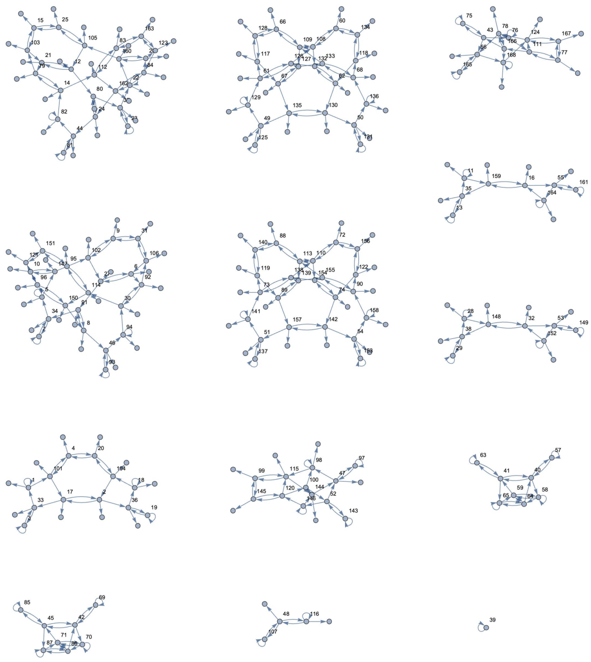

FIGURE 2. Orbits under the action of a set of generators of $\mathrm{Aut}(\mathbb{Z}^2)$ on $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{SL}_2(\mathbb{F}_3))$.

Future Directions. Since the non-transitivity theorem proves that the action of $\mathrm{Out}(F_r)$ on $\mathfrak{X}_{F_r}(\mathrm{SL}_n(\mathbb{F}_q))$ is not transitive, the natural subsequent step is to check if the action is asymptotically transitive.

Problem 1. *Prove or disprove that the action of* $\mathrm{Out}(F_r)$ *is not asympotically transitive on* $\mathfrak{X}_{F_r}(\mathrm{SL}_n(\mathbb{F}_q))$.

In the cases where asymptotic transitivity is disproved, the next path to pursue is to study the dynamics by estimating the orbit size. Since an upper bound for maximal orbits is proved (Theorem 4) in these cases, I want to compute a non-zero lower bound if it exists.

Problem 2. *Let* $\mathfrak{X}_{F_r}(\mathrm{SL}_3(\mathbb{F}_q))$ *be finite field points of the* $\mathrm{SL}_3(\mathbb{C})$ *character variety over* $\mathbb{F}_q$. *Compute a non-zero lower bound for the ratio of the size of the biggest orbit to that of the variety is bounded as* $q \to \infty$.

More broadly, one of the questions that I want to pursue is if this result can be generalized to SL_n-varieties of F_r.

Problem 3. *Let $\mathfrak{X}_{F_r}(\mathrm{SL}_n(\mathbb{F}_q))$ be finite field points of the $\mathrm{SL}_n(\mathbb{C})$ character variety over $\mathbb{F}_q$. Compute a non-zero lower bound and upper bound for the ratio of the size of the maximal orbit to that of the variety as $q \to \infty$.*

My long-term goal is to develop the theory of asymptotic transitivity in a formal manner. The techniques used in the proof of the theorems stated in this document provide some tools that could be used to check asymptotic transitivity. I want to build a toolbox for exploring asymptotic transitivity for general group actions on varieties. Another direction that I am interested in is investigating similar problems on relative character varieties defined by fixing the boundary components of the surface. I want to explore transitivity and asymptotic transitivity properties of the dynamics on relative character varieties.

Varieties in Data Learning. Moving forward, I am also interested in continuing with my work on data learning using varieties. The immediate goal is to complete the algorithm I am currently working on to the generality that would be possible. The logical subsequent step after that is to find and adapt it to data sets from real-life applications.
A parallel avenue that I want to explore is the possibility of using group actions on the data sets, which would enable the use of orbit structures to aid in approximating data to a variety.

Even though my expertise lies primarily in algebraic geometry, I am excited by the prospects at the interface of different mathematics disciplines. Therefore, I am interested in prospective collaborations within algebraic geometry and other areas where my research could be applied.

References

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