Hochschule RheinMain

Physics Lab 3

Experiment P3-3
Torsional Pendulum

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1 Introduction

1.1 Terms and Definitions

Free Harmonic and Damped Oscillation

If a system capable of oscillation is deflected out of its equilibrium position and is experiencing a restoring force proportional to its deflection this system is called an *harmonic oscillator*. If a dampening force such as friction is introduced, the system no longer oscillates freely but rather damped.
Both, damped and harmonic oscillations are considered *free* if there is no continuous, the oscillation driving stimulus present.

Natural Angular Frequency of a Harmonic Oscillation

Depending of the very characteristics of the given system it will oscillate at a distinct frequency - the natural angular frequency ω_0.

Differential Equation of the Damped Harmonic Oscillation

$$I\ddot{\varphi} = -D\varphi - \rho\dot{\varphi} + M\cos(\omega t) \tag{1.1}$$

Damping cases

Overdamped: The system returns to equilibrium without oscillating.
Critically damped: The system returns to equilibrium as quickly as possible without oscillating.
Underdamped: The system oscillates (at reduced frequency compared to the undamped case) with the amplitude gradually decreasing to zero.

Rotational Inerta

A cylindrical rot has a rotational inertia about its center of: $I_S = \frac{1}{12}ml^2$

STEINER's Theorem

The STEINER's Theorem: $I = I_S + md^2$

Eddy Current Brake

The equivalent circuit of a conductor exposed to a changing magnetic field is composed of only a voltage source and a resistor. The voltage across the this resistor is induced as of the law of electromagnetic induction: $U_{ind} = -NA\frac{\partial B}{\partial t}$. The resulting current creates a magnetic field opposing the inducing field.

Constant Current Constant Voltage Operation of a PSU

As the name implies a power supply in CC/CV operation mode keeps the output current and output voltage constant indipendently from a load applied.

Capacitance of a Parallel Plate Capacitor

The capacitance of a parallel plate capacitor is: $C = \varepsilon_0\varepsilon_r \cdot \frac{A}{d}$

Time-Constant of an RC-Circuit

The time constant of an RC circuit: $\tau = RC$

1.2 Preparation

Deriving the Equation for Damped Free Oscillation

$$\vec{M}_{Inert} + \vec{M}_{Frict} + \vec{M}_{Rest} = 0 \quad \Leftrightarrow \quad J \cdot \ddot{\varphi}(t) - k \cdot \dot{\varphi}(t) - D^* \cdot \varphi(t) = 0 \tag{1.2}$$

can be written as

$$\ddot{\varphi}(t) + 2\delta \cdot \dot{\varphi}(t) + \omega_0^2 \cdot \varphi(t) = 0 \tag{1.3}$$

with

$$-\frac{k}{J} = 2\delta, \quad -\frac{D^*}{J} = \omega_0^2 \tag{1.4}$$

whereas eq. (1.3) is a second degree harmonic differential equation. The chosen approach is:

$$\varphi(t) = \hat{\varphi} e^{\lambda t}, \quad \dot{\varphi}(t) = \lambda \hat{\varphi} e^{\lambda t}, \quad \ddot{\varphi}(t) = \lambda^2 \hat{\varphi} e^{\lambda t} \tag{1.5}$$

Pluged into eq. (1.3) gives

$$\begin{aligned} \left(\lambda^2 + 2\delta\lambda + \omega_0^2\right) \hat{\varphi} e^{\lambda t} &= 0 \\ \lambda_{1,2} &= -\delta \pm \sqrt{\delta^2 - \omega_0^2} \end{aligned} \tag{1.6}$$

Here two possible cases are to be distinguished:

$$\lambda_{1,2} = \begin{cases} -\delta \pm i\omega_d & \text{for} \quad \delta^2 < \omega_0^2 \quad \text{(a)} \\ -\delta \pm \omega_d & \text{for} \quad \delta^2 \geq \omega_0^2 \quad \text{(b)} \end{cases} \tag{1.7}$$

In eq. (1.3):

$$\varphi_1(t) = \varphi_1 e^{-\delta + i\omega_d t}, \quad \varphi_2(t) = \varphi_2 e^{-\delta - i\omega_d t} \tag{1.8}$$

Linear combination of $\varphi_1(t)$ and $\varphi_2(t)$ lastly leads to

$$\varphi(t) = \varphi_1 e^{-\delta + i\omega_d t} + \varphi_2 e^{-\delta - i\omega_d t} = \hat{\varphi} e^{-\delta t} \cdot \cos(\omega_d t + \varphi_0) \tag{1.9}$$

Unknown Angular Inertia of the Pendulum

The angular inertia of the pendulum J_P can be determined by adding a known angular inertia J_R of a cylindrical rod. As **??** delivers

$$\omega_0 = \sqrt{\frac{D^*}{J}} \tag{1.10}$$

and $\omega = \frac{2\pi}{T}$, the following relation is given:

$$\omega_{0,P} = \sqrt{\frac{D^*}{J_P}} = \frac{2\pi}{T_P} \quad \Leftrightarrow \quad 2\pi = T_P \cdot \sqrt{\frac{D^*}{J_P}} \tag{1.11}$$

$$\omega_{0,P+R} = \sqrt{\frac{D^*}{J_P + J_R}} = \frac{2\pi}{T_{P+R}} \quad \Leftrightarrow \quad 2\pi = T_{P+R} \cdot \sqrt{\frac{D^*}{J_P + J_R}} \tag{1.12}$$

When eq. (1.11) and eq. (1.12) are equated:

$$\begin{aligned}
& T_P \cdot \sqrt{\frac{D^*}{J_P}} = T_{P+R} \cdot \sqrt{\frac{D^*}{J_P + J_R}} \\
\Leftrightarrow \quad & \left(\frac{T_{P+R}}{T_P}\right)^2 = \frac{J_P + J_R}{J_P} = 1 + \frac{J_R}{J_P} \\
\Leftrightarrow \quad & \frac{J_R}{J_P} = \left(\frac{T_{P+R}}{T_P}\right)^2 - 1 \\
\Rightarrow \quad & J_P = \frac{J_R}{\left(\frac{T_{P+R}}{T_P}\right)^2 - 1} \qquad (1.13)
\end{aligned}$$

The angular inertia of the pendulum can be calculated without knowing the torsion constant D^*.

Rotational Inertia of a Cylindrical Rod

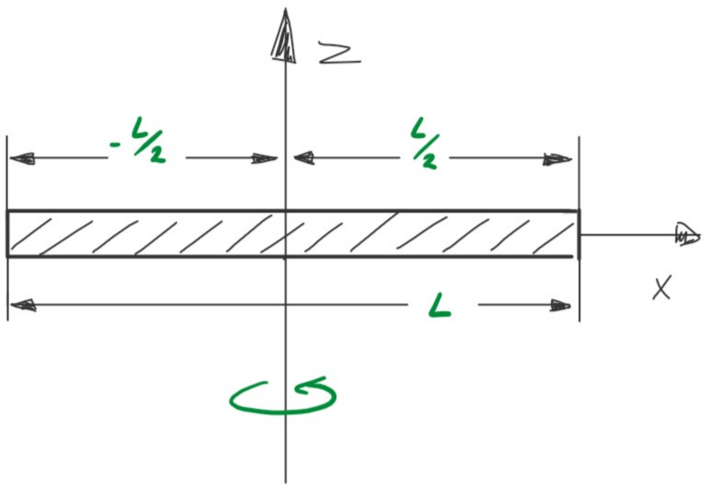


Figure 1.1: Scheme of a orthogonally to its center axis rotating rod.

Inertia of a rotating mass dimensionless mass is proportional to the square of the distance to its rotational axis. As the mass of a cylindrical body is distributed over its volume, it is necessary to integrate over all dm along the distance r from the center of rotation.

$$J_Z = \int r^2 dm \qquad (1.14)$$

With

$$\rho = \frac{dm}{dx} = \frac{M}{L} \qquad \Leftrightarrow \qquad dm = \frac{M}{L} dx \qquad (1.15)$$

plugged into eq. (1.14) with respect to the integration limits as of fig. 1.1 gives

$$J_Z = \int_{-L/2}^{L/2} \frac{M}{L} x^2 dx = \frac{1}{12} M L^2 \qquad (1.16)$$

Equations for the Sensor Capacitances

To derive:

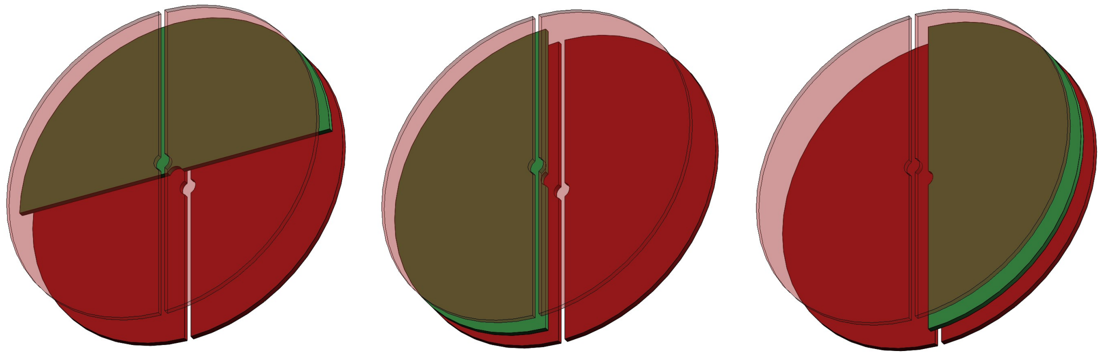
$$C_1(\varphi) = \varepsilon_0 \frac{\pi D^2}{16d}\left(1 - \frac{2\varphi}{\pi}\right) \tag{1.17}$$

$$C_2(\varphi) = \varepsilon_0 \frac{\pi D^2}{16d}\left(1 + \frac{2\varphi}{\pi}\right) \tag{1.18}$$

with

$$A_{1,2}(\varphi) = \frac{1}{16}\pi D^2 \left(1 \pm \frac{2\varphi}{\pi}\right) \tag{1.19}$$

One half of the stator pairs together with the rotor plate forms two capacitors connected in series. With



(a) Rotor at $\varphi = 0$ *(b) Rotor at* $\varphi = \pm\frac{\pi}{2}$ *(c) Rotor at* $\varphi = \mp\frac{\pi}{2}$

Figure 1.2: Schematical assembly of the angular sensor. The semi circular rotor plate (green) sandwiched between the two stators (red). The area of the rotor facing one of the vertical stator pairs varies with the angular displacement φ of the rotor.

each capacitor having the same value at any time the total capacitance equates to

$$C_{1,2}(\varphi) = \varepsilon_0 \varepsilon_r \frac{A(\pm\varphi)}{2d} \tag{1.20}$$

Where $A(\varphi)$ can be expressed as

$$A(\varphi) = \frac{1}{8}D^2 (\pi \pm \varphi)$$

$$A(\varphi) = \frac{1}{8}D^2 \left(\frac{\pi^2}{\pi} \pm \frac{\pi\varphi}{\pi}\right)$$

$$A(\varphi) = \frac{1}{8}\pi D^2 \left(1 \pm \frac{\varphi}{\pi}\right) \tag{1.21}$$

Say the zero position is chosen such as the whole area of the rotor takes effect (see fig. 1.2c) eq. (1.21) maximizes. Thus the absolute capacitance of one of the capacitors is maximized. Stepping the rotor about $\varphi = \frac{\pi}{2}$ as seen in fig. 1.2a halves the effective area of the capacitor halfing the total capacitance. At an angular displacement of $\varphi = \pi$ the capacitance equates to zero respectively.
Combining eq. (1.20) and eq. (1.21) gives [1]

$$C_{1,2} = \varepsilon_0 \frac{\pi D^2}{16d}\left(1 \pm \frac{2\varphi}{\pi}\right) \tag{1.22}$$

$$\left(1 \pm \frac{2\varphi}{\pi}\right) \begin{cases} = 0 & \text{for} \quad \varphi = \pm\frac{\pi}{2} \\ = 1 & \text{for} \quad \varphi = 0 \\ = 2 & \text{for} \quad \varphi = \mp\frac{\pi}{2} \end{cases}$$

[1] The solution is missing a factor of 2 in front of φ

Time to Reach the Threshold Voltage

The charging curve of a capacitor is given by eq. (1.23).

$$U_C(t) = U_0(1 - e^{-\frac{t}{\tau}}) \tag{1.23}$$

Being interested at the time t_{th} it takes to reach a certain threshold voltage U_{th} eq. (1.23) can be transformed as follows:

$$1 - \frac{U_{th}}{U_0} = e^{-\frac{t_{th}}{\tau}}$$
$$\Leftrightarrow$$
$$t_{th} = -\ln\left(1 - \frac{U_{th}}{U_0}\right) \cdot \tau \tag{1.24}$$

with the time constant $\tau = R \cdot C$.

Determining the Angular Deflection by the Difference of Timer Ticks

The time to reach the threshold voltage as of eq. (1.24) is captured indipendently due to each capacitor being connected to individual GPIOs.
Since the charging curve of the capacitors differs in an anti-proportional manner when an angular deflection takes place the absolute value of the time difference gives the the angle about zero while the sign gives the direction. Therefore, taken these considerations in account and merging eq. (1.20) and eq. (1.24) gives:

$$\begin{aligned}\Delta t_{th}(\varphi) = t_{th,1} - t_{th,2} &= -\ln\left(1 - \frac{U_{th}}{U_0}\right) R\left[C_1(\varphi) - C_2(\varphi)\right] \\ &= -\varepsilon_0 R \frac{\pi D^2}{16d} \ln\left(1 - \frac{U_{th}}{U_0}\right)\left[\left(1 + \frac{2\varphi}{\pi}\right) - \left(1 - \frac{2\varphi}{\pi}\right)\right] \\ &= -\varepsilon_0 R \frac{4D^2}{16d} \ln\left(1 - \frac{U_{th}}{U_0}\right) \cdot \varphi \end{aligned} \tag{1.25}$$

Here $\varepsilon_0, R, D, d, U_{th}$ and U_0 remain constant and can be gathered as a proportionality factor. This reduces eq. (1.25) to

$$\Delta t_{th}(\varphi) = \chi \cdot \varphi \tag{1.26}$$

The µC checks the state of the input pin once every cycle. To take that into account the difference in threshold time Δt_{th} has to be devided by the cycle time Δt of the µC which gives the number of cycles it took for the input pins to switch state from low to high. If a change takes place at a non integer multiple of Δt the µC will register a transision on the subsequent cycle, thus, for the cycle count n applies $n \in \mathbb{N}$. Furthermore, a non-integer value for n has to be rounded up to the next integer value.
Mathmatically the above considerations yield

$$n(\varphi) = \left\lceil \frac{|\Delta t_{th}(\varphi)|}{\Delta t} \right\rceil = \lceil \chi' \cdot |\varphi| \rceil \qquad \text{with} \qquad n(\varphi) : n(\varphi) \in \mathbb{N} \tag{1.27}$$

which translates into the amount of deflection and

$$\frac{|n(\varphi)|}{n(\varphi)} = \pm 1 \tag{1.28}$$

to distinguish between a CW/CCW rotation.

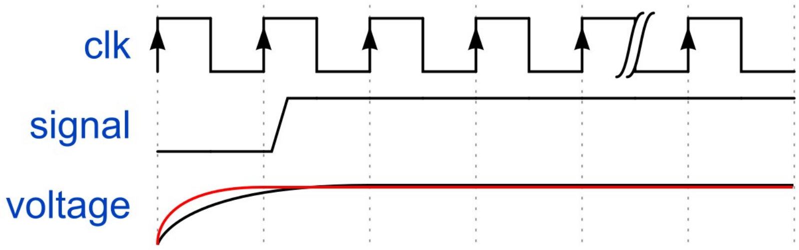


Figure 1.3: Timing diagram showing the signal transision.

Sensitivity of the Angular Sensor

As seen in eq. (1.26) the tick rate relates linearly with the angular displacement φ. Therefore, the maximum resolution of the angular sensor expressed as *ticks per radiant* **is** χ'.

$$\frac{dn(\varphi)}{d\varphi} = \chi' \cdot \varphi \frac{d}{d\varphi} = \chi' \tag{1.29}$$

The clock frequency of the µC is $f = 16\,\mathrm{MHz}$ which gives a cycle time of $\delta t = 62.5\,\mathrm{ns}$.
To ready the capacitors for the next charging cycle they need to be discharged as quick as possible. Considering that a time to discharge the capacitors $< \Delta t$ makes no significant difference the unknown value R of the resistor can be approximated as

$$\begin{aligned} 3\tau = \Delta t &= 3RC \\ &\Leftrightarrow \\ \frac{\Delta t}{3C_{max}} &= R \end{aligned} \tag{1.30}$$

In the equation above the assumptions are made that a discharge rate of 95% is sufficiant and the circuit needs to be able to discharge the capacitor within the timeframe Δt while being at its maximum capacitance. Thus

$$\begin{aligned} C_{max} &= \varepsilon_0 \frac{\pi D^2}{8d} \\ &= 8,85 \cdot 10^{-12}\,\frac{\mathrm{As}}{\mathrm{Vm}}\,\frac{\pi \cdot 0.12^2\,\mathrm{m}^2}{8 \cdot 0.01\,\mathrm{m}} \\ &= 5.01\,\mathrm{pF} \end{aligned} \tag{1.31}$$

in eq. (1.30) gives a value for the resistance as

$$\frac{62.5\,\mathrm{ns}}{3 \cdot 5.01\,\mathrm{pF}} \approx 4160.9\,\mathrm{k\Omega} \tag{1.32}$$

This lies between the two more common E-Series values of $4.7\,\mathrm{k\Omega}$ and $3.9\,\mathrm{k\Omega}$. For further calculations the latter is chosen as a higher resistance would increase the discharge time.

Plugging in the given values of for $\varepsilon_0, D, d, U_{th}, U_0$ and the calculated values for Δt and R equates eq. (1.26) to

$$\begin{aligned} \chi' &= -\varepsilon_0 R \frac{4D^2}{16d} \ln\left(1 - \frac{U_{th}}{U_0}\right) \Delta t^{-1} \\ &= -8.85 \cdot 10^{-12}\,\frac{\mathrm{As}}{\mathrm{Vm}} \cdot 3.9\,\mathrm{k\Omega} \cdot \frac{4 \cdot 0.12^2\,\mathrm{m}^2}{16 \cdot 0.01\,\mathrm{m}} \ln\left(1 - \frac{2.5\,\mathrm{V}}{5\,\mathrm{V}}\right) \cdot \frac{1}{62.5\,\mathrm{ns}} \\ &\approx 0.138\,\mathrm{rad}^{-1} \end{aligned} \tag{1.33}$$

2 Set-Up of Experiment

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ω_0 Angular frequency

A Appendix

Figure A.1: During the course of the experiment captured oscillograms.

Table A.1: Handwritten notes corresponding each measurement.

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