

HOCHSCHULE RHEINMAIN



PHYSICS LAB 3

Experiment P3-3

Torsional Pendulum

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1 Introduction

1.1 Terms and Definitions

Free Harmonic and Damped Oscillation

If a system capable of oscillation is deflected out of its equilibrium position and is experiencing a restoring force proportional to its deflection this system is called an *harmonic oscillator*. If a dampening force such as friction is introduced, the system no longer oscillates freely but rather damped.

Both, damped and harmonic oscillations are considered *free* if there is no continuous, the oscillation driving stimulus present.

Natural Angular Frequency of a Harmonic Oscillation

Depending of the very characteristics of the given system it will oscillate at a distinct frequency - the natural angular frequency ω_0 .

Differential Equation of the Damped Harmonic Oscillation

$$I\ddot{\varphi} = -D\varphi - \rho\dot{\varphi} + M \cos(\omega t) \quad (1.1)$$

Damping cases

Rotational Inertia

STEINER's Theorem

Eddy Current Brake

Constant Current Constant Voltage Operation of a PSU

Capacitance of a Parallel Plate Capacitor

Time-Constant of an RC-Circuit

1.2 Preparation

Deriving the Equation for Damped Free Oscillation

$$\vec{M}_{Inert} + \vec{M}_{Frict} + \vec{M}_{Rest} = 0 \quad \Leftrightarrow \quad J \cdot \ddot{\varphi}(t) - k \cdot \dot{\varphi}(t) - D^* \cdot \varphi(t) = 0 \quad (1.2)$$

can be written as

$$\ddot{\varphi}(t) + 2\delta \cdot \dot{\varphi}(t) + \omega_0^2 \cdot \varphi(t) = 0 \quad (1.3)$$

with

$$-\frac{k}{J} = 2\delta, \quad -\frac{D^*}{J} = \omega_0^2 \quad (1.4)$$

whereas eq. (1.3) is a second degree harmonic differential equation. The chosen approach is:

$$\varphi(t) = \hat{\varphi}e^{\lambda t}, \quad \dot{\varphi}(t) = \lambda \hat{\varphi}e^{\lambda t}, \quad \ddot{\varphi}(t) = \lambda^2 \hat{\varphi}e^{\lambda t} \quad (1.5)$$

Plugged into eq. (1.3) gives

$$\begin{aligned} (\lambda^2 + 2\delta\lambda + \omega_0^2) \hat{\varphi} e^{\lambda t} &= 0 \\ \lambda_{1,2} &= -\delta \pm \sqrt{\delta^2 - \omega_0^2} \end{aligned} \quad (1.6)$$

Here two possible cases are to be distinguished:

$$\lambda_{1,2} = \begin{cases} -\delta \pm i\omega_d & \text{for } \delta^2 < \omega_0^2 \quad (\text{a}) \\ -\delta \pm \omega_d & \text{for } \delta^2 \geq \omega_0^2 \quad (\text{b}) \end{cases} \quad (1.7)$$

In eq. (1.3):

$$\varphi_1(t) = \varphi_1 e^{-\delta + i\omega_d t}, \quad \varphi_2(t) = \varphi_2 e^{-\delta - i\omega_d t} \quad (1.8)$$

Linear combination of $\varphi_1(t)$ and $\varphi_2(t)$ lastly leads to

$$\varphi(t) = \varphi_1 e^{-\delta + i\omega_d t} + \varphi_2 e^{-\delta - i\omega_d t} = \hat{\varphi} e^{-\delta t} \cdot \cos(\omega_d t + \varphi_0) \quad (1.9)$$

Unknown Angular Inertia of the Pendulum

The angular inertia of the pendulum J_P can be determined by adding a known angular inertia J_R of a cylindrical rod. As ?? delivers

$$\omega_0 = \sqrt{\frac{D^*}{J}} \quad (1.10)$$

and $\omega = \frac{2\pi}{T}$, the following relation is given:

$$\omega_{0,P} = \sqrt{\frac{D^*}{J_P}} = \frac{2\pi}{T_P} \Leftrightarrow 2\pi = T_P \cdot \sqrt{\frac{D^*}{J_P}} \quad (1.11)$$

$$\omega_{0,P+R} = \sqrt{\frac{D^*}{J_P + J_R}} = \frac{2\pi}{T_{P+R}} \Leftrightarrow 2\pi = T_{P+R} \cdot \sqrt{\frac{D^*}{J_P + J_R}} \quad (1.12)$$

When eq. (1.11) and eq. (1.12) are equated:

$$\begin{aligned} T_P \cdot \sqrt{\frac{D^*}{J_P}} &= T_{P+R} \cdot \sqrt{\frac{D^*}{J_P + J_R}} \\ \Leftrightarrow \left(\frac{T_{P+R}}{T_P} \right)^2 &= \frac{J_P + J_R}{J_P} = 1 + \frac{J_R}{J_P} \\ \Leftrightarrow \frac{J_R}{J_P} &= \left(\frac{T_{P+R}}{T_P} \right)^2 - 1 \\ \Rightarrow J_P &= \frac{J_R}{\left(\frac{T_{P+R}}{T_P} \right)^2 - 1} \end{aligned} \quad (1.13)$$

The angular inertia of the pendulum can be calculated without knowing the torsion constant D^* .

Rotational Inertia of a Cylindrical Rod

Inertia of a rotating mass dimensionless mass is proportional to the square of the distance to its rotational axis. As the mass of a cylindrical body is distributed over its volume, it is necessary to integrate over all dm along the distance r from the center of rotation.

$$J_Z = \int r^2 dm \quad (1.14)$$

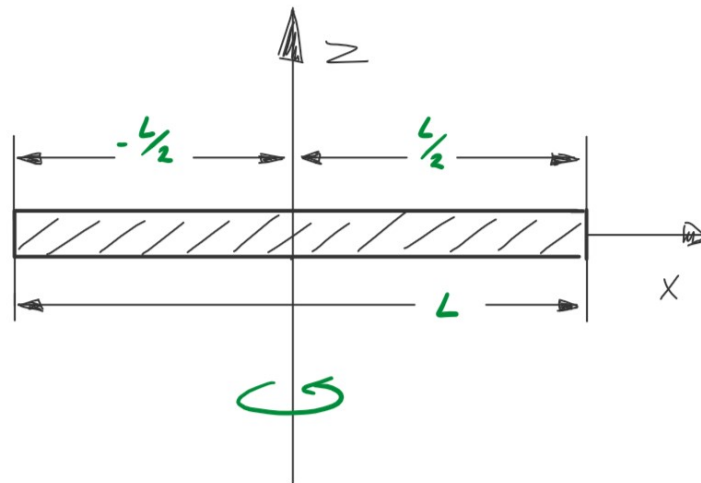


Figure 1.1: Scheme of a orthogonally to its center axis rotating rod.

With

$$\rho = \frac{dm}{dx} = \frac{M}{L} \quad \Leftrightarrow \quad dm = \frac{M}{L} dx \quad (1.15)$$

plugged into eq. (1.14) with respect to the integration limits as of fig. 1.1 gives

$$J_Z = \int_{-L/2}^{L/2} \frac{M}{L} x^2 dx = \frac{1}{12} ML^2 \quad (1.16)$$

Equations for the Sensor Capacitances

To derive:

$$C_1(\varphi) = \varepsilon_0 \frac{\pi D^2}{16d} \left(1 - \frac{2\varphi}{\pi} \right) \quad (1.17)$$

$$C_2(\varphi) = \varepsilon_0 \frac{\pi D^2}{16d} \left(1 + \frac{2\varphi}{\pi} \right) \quad (1.18)$$

with

$$A_{1,2}(\varphi) = \frac{1}{16} \pi D^2 \left(1 \pm \frac{2\varphi}{\pi} \right) \quad (1.19)$$

One half of the stator pairs together with the rotor plate forms two capacitors connected in series. With each capacitor having the same value at any time the total capacitance equates to

$$C_{1,2}(\varphi) = \varepsilon_0 \varepsilon_r \frac{A(\pm\varphi)}{2d} \quad (1.20)$$

Where $A(\varphi)$ can be expressed as

$$\begin{aligned} A(\varphi) &= \frac{1}{8} D^2 (\pi \pm \varphi) \\ A(\varphi) &= \frac{1}{8} D^2 \left(\frac{\pi^2}{\pi} \pm \frac{\pi\varphi}{\pi} \right) \\ A(\varphi) &= \frac{1}{8} \pi D^2 \left(1 \pm \frac{\varphi}{\pi} \right) \end{aligned} \quad (1.21)$$

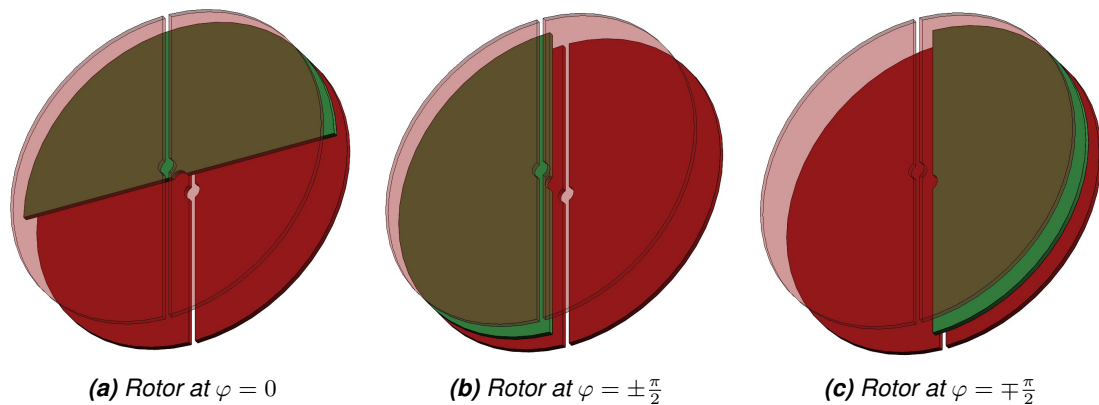


Figure 1.2: Schematic assembly of the angular sensor. The semi circular rotor plate (green) sandwiched between the two stators (red). The area of the rotor facing one of the vertical stator pairs varies with the angular displacement φ of the rotor.

Say the zero position is chosen such as the whole area of the rotor takes effect (see fig. 1.2c) eq. (1.21) maximizes. Thus the absolute capacitance of one of the capacitors is maximized. Stepping the rotor about $\varphi = \frac{\pi}{2}$ as seen in fig. 1.2a halves the effective area of the capacitor halving the total capacitance. At an angular displacement of $\varphi = \pi$ the capacitance equates to zero respectively. Combining eq. (1.20) and eq. (1.21) gives ¹

$$C_{1,2} = \varepsilon_0 \frac{\pi D^2}{16d} \left(1 \pm \frac{2\varphi}{\pi} \right) \quad (1.22)$$

$$\left(1 \pm \frac{2\varphi}{\pi} \right) \begin{cases} = 0 & \text{for } \varphi = \pm \frac{\pi}{2} \\ = 1 & \text{for } \varphi = 0 \\ = 2 & \text{for } \varphi = \mp \frac{\pi}{2} \end{cases}$$

Time to Reach the Threshold Voltage

The charging curve of a capacitor is given by eq. (1.23).

$$U_C(t) = U_0(1 - e^{-\frac{t}{\tau}}) \quad (1.23)$$

Being interested at the time t_{th} it takes to reach a certain threshold voltage U_{th} eq. (1.23) can be transformed as follows:

$$\begin{aligned} 1 - \frac{U_{th}}{U_0} &= e^{-\frac{t_{th}}{\tau}} \\ &\Leftrightarrow \\ t_{th} &= -\ln\left(1 - \frac{U_{th}}{U_0}\right) \cdot \tau \end{aligned} \quad (1.24)$$

with the time constant $\tau = R \cdot C$.

Determining the Angular Deflection by the Difference of Timer Ticks

The time to reach the threshold voltage as of eq. (1.24) is captured independently due to each capacitor being connected to individual GPIOs.

¹The solution is missing a factor of 2 in front of φ

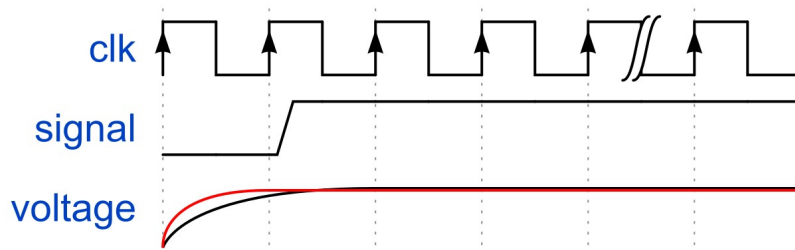


Figure 1.3: Timing diagram showing the signal transision.

Since the charging curve of the capacitors differs in an anti-proportional manner when an angular deflection takes place the absolute value of the time difference gives the the angle about zero while the sign gives the direction. Therefore, taken these considerations in account and merging eq. (1.20) and eq. (1.24) gives:

$$\begin{aligned}
 \Delta t_{th}(\varphi) &= t_{th,1} - t_{th,2} = -\ln\left(1 - \frac{U_{th}}{U_0}\right) R [C_1(\varphi) - C_2(\varphi)] \\
 &= -\varepsilon_0 R \frac{\pi D^2}{16d} \ln\left(1 - \frac{U_{th}}{U_0}\right) \left[\left(1 + \frac{2\varphi}{\pi}\right) - \left(1 - \frac{2\varphi}{\pi}\right)\right] \\
 &= -\varepsilon_0 R \frac{4D^2}{16d} \ln\left(1 - \frac{U_{th}}{U_0}\right) \cdot \varphi
 \end{aligned} \tag{1.25}$$

Here ε_0 , R , D , d , U_{th} and U_0 remain constant and can be gathered as a proportionality factor. This reduces eq. (1.25) to

$$\Delta t_{th}(\varphi) = \chi \cdot \varphi \tag{1.26}$$

The μC checks the state of the input pin once every cycle. To take that into account the difference in threshold time Δt_{th} has to be divided by the cycle time Δt of the μC which gives the number of cycles it took for the input pins to switch state from low to high. If a change takes place at a non integer multiple of Δt the μC will register a transision on the subsequent cycle, thus, for the cycle count n applies $n \in \mathbb{N}$. Furthermore, a non-integer value for n has to be rounded up to the next integer value.

Mathematically the above considerations yield

$$n(\varphi) = \left\lceil \frac{|\Delta t_{th}(\varphi)|}{\Delta t} \right\rceil = \lceil \chi' \cdot |\varphi| \rceil \quad \text{with} \quad n(\varphi) : n(\varphi) \in \mathbb{N} \tag{1.27}$$

which translates into the amount of deflection and

$$\frac{|n(\varphi)|}{n(\varphi)} = \pm 1 \tag{1.28}$$

to distinguish between a CW/CCW rotation.

Sensitivity of the Angular Sensor

As seen in eq. (1.26) the tick rate relates linearly with the angular displacement φ . Therefore, the maximum resolution of the angular sensor expressed as *ticks per radian* is χ' .

$$\frac{dn(\varphi)}{d\varphi} = \chi' \cdot \varphi \frac{d}{d\varphi} = \chi' \tag{1.29}$$

The clock frequency of the μC is $f = 16 \text{ MHz}$ which gives a cycle time of $\delta t = 62.5 \text{ ns}$.

To ready the capacitors for the next charging cycle they need to be discharged as quick as possible. Considering that a time to discharge the capacitors $< \Delta t$ makes no significant difference the unknown value R of the resistor can be approximated as

$$\begin{aligned} 3\tau &= \Delta t = 3RC \\ \Leftrightarrow \\ \frac{\Delta t}{3C_{max}} &= R \end{aligned} \quad (1.30)$$

In the equation above the assumptions are made that a discharge rate of 95% is sufficient and the circuit needs to be able to discharge the capacitor within the timeframe Δt while being at its maximum capacitance. Thus

$$\begin{aligned} C_{max} &= \varepsilon_0 \frac{\pi D^2}{8d} \\ &= 8,85 \cdot 10^{-12} \frac{\text{As}}{\text{Vm}} \frac{\pi \cdot 0.12^2 \text{m}^2}{8 \cdot 0.01 \text{m}} \\ &= 5.01 \text{ pF} \end{aligned} \quad (1.31)$$

in eq. (1.30) gives a value for the resistance as

$$\frac{62.5 \text{ ns}}{3 \cdot 5.01 \text{ pF}} \approx 4160.9 \text{ k}\Omega \quad (1.32)$$

This lies between the two more common E-Series values of $4.7 \text{ k}\Omega$ and $3.9 \text{ k}\Omega$. For further calculations the latter is chosen as a higher resistance would increase the discharge time.

Plugging in the given values of for ε_0 , D , d , U_{th} , U_0 and the calculated values for Δt and R equates eq. (1.26) to

$$\begin{aligned} \chi' &= -\varepsilon_0 R \frac{4D^2}{16d} \ln \left(1 - \frac{U_{th}}{U_0} \right) \Delta t^{-1} \\ &= -8.85 \cdot 10^{-12} \frac{\text{As}}{\text{Vm}} \cdot 3.9 \text{ k}\Omega \cdot \frac{4 \cdot 0.12^2 \text{m}^2}{16 \cdot 0.01 \text{m}} \ln \left(1 - \frac{2.5 \text{ V}}{5 \text{ V}} \right) \cdot \frac{1}{62.5 \text{ ns}} \\ &\approx 0.138 \text{ rad}^{-1} \end{aligned} \quad (1.33)$$

2 Set-Up of Experiment

3 Execution

4 Evaluation

5 Conclusion

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List of Symbols

ω_0 Angular frequency

A Appendix

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Table A.1: Handwritten notes corresponding each measurement.

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