Problem 1 (160 pts)**.**

Assume $G = (V, E)$ is an undirected graph with $u, v, w \in V$. For each of the following relations, prove or disprove that it is an equivalence relation:

(a) Define R_1 so that uR_1v when u is adjacent to v.

(b) Define R_2 so that uR_2v when u and v are both adjacent to w.

(c) Define R_3 so that uR_3v when $\deg(u) = \deg(v)$.

Problem 2 (160 pts)**.**

(a) Suppose $G = (V, E)$ is an undirected graph which is connected and has $n - 1$ edges. Show that G contains no cycles. What type of graph is G?

(b) Suppose $G = (V, E)$ is an undirected graph which is connected and has $n - 1$ edges. Prove or disprove the following: for any two nodes $u, v \in V$, there exists a unique simple path between u and v.

Problem 3 (160 pts)**.**

Using mathematical induction, show that the sum of the first n odd natural numbers is n^2.

Problem 4 (160 pts)**.**

Let Σ be an alphabet and $L_2 = \{w \in \Sigma^* : |w| = 2\}$, where Σ^* represents the set of all finite length strings with symbols draw from Σ.

(a) Suppose for some $n \geq 0$, you are given $L_{2n} = \{w \in \Sigma^* : |w| = 2n\}$. Define $L_{2(n+1)}$ similarly and provide an inductive definition which uses L_2 and L_{2n} to generate $L_{2(n+1)}$.

(b) Define $\star$ to align with the inductive definition in Part (a), so that for example, $L_4 \star L_2 = L_6$. Now provide an inductive definition which uses $\star$ to generate $L_{even} = \{L_k : k = 2n, n \geq 0\}$. Notice that this is a set of languages (a set of sets of strings), not a set of strings. Does order matter when invoking $\star$?

- Importantly, note that there is more than one way you can define $\star$ which will make sense in the context of the inductive definition. Also, how $\star$ is defined can impact the answers to questions posed about $\star$ in the following parts of this problem.

(c) Provide an inductive definition to generate $L_{odd} = \{L_k : k = 2n + 1, n \geq 0\}$. Can you also use $\star$ as you defined it for this purpose, or do you need to modify $\star$ or define something similar but different?

(d) Provide an inductive definition to generate $L_{square} = \{L_k : k = n^2, n \geq 0\}$. *Hint: consider applying the result of Problem 3 in addition to the other parts of this problem.*

Problem 5 (160 pts)**.**

Professor Blurphy has an interesting group of students this semester. They don't like to be crowded, but also don't wish to be isolated either. Assume that the seats in the classroom are arranged in a rectangular grid (i.e. as a 2D matrix), and define a neighbor of a seat to be one of the (up to) 8 seats immediately adjacent, including diagonally. When the students enter the classroom, they choose their seats as follows:

- The first student chooses a seat at random
- Each subsequent student sits so that there is exactly one neighboring seat occupied when they sit down (though they will not move if other students sit near them later).

Assume that there are always enough seats to accommodate this arrangement, and that the students stick to this plan for all seats (including the edges which have fewer than 8 neighbors).

(a) What is the maximum number of neighbors possible for any one student? Will there always necessarily be some student with this many neighbors?

(b) What is the minimum number of neighbors possible for any one student? Will there always necessarily be some student with this many neighbors?

(c) One day, Professor Blurphy decides to ask the students to work in groups of 2. Assuming the students will want to be neighbors with their groupmate, is this possible with the initial seating arrangement? Explain your reasoning. You may assume that their is an even number of students.

(d) In the following lecture, Professor Blurphy decides to ask the students to work in groups of 3. Assuming the students want to be neighbors with each of their groupmates, is this possible with the initial seating arrangement? Explain your reasoning. Will this always be true for any size of group? What limitations exist with respect to group size? You may assume that the size of the group divides the number of students.