

1)

a.

$$\langle 0|\psi\rangle = \alpha, P(|0\rangle) = |\alpha|^2$$

$$\langle 1|\psi\rangle = \beta, P(|1\rangle) = |\beta|^2$$

b.

$$|\alpha|^2 + |\beta|^2 = 1$$

$$P(|0\rangle) = |\alpha|^2, \quad P(|1\rangle) = |\beta|^2$$

$$\therefore P(|0\rangle) + P(|1\rangle) = 1$$

c.

$$P(|0\rangle) = |\alpha|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$P(|1\rangle) = |\beta|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

2)

a.

Tensor Product Notation:

$$|00\rangle = |0\rangle \otimes |0\rangle$$

$$|01\rangle = |0\rangle \otimes |1\rangle$$

$$|10\rangle = |1\rangle \otimes |0\rangle$$

$$|11\rangle = |1\rangle \otimes |1\rangle$$

Vectors Representation:

$$|00\rangle: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|11\rangle: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

b.

$$\begin{aligned} P(\text{total}) &= P(|00\rangle) + P(|01\rangle) + P(|10\rangle) + P(|11\rangle) \\ &= \left| \sqrt{\frac{1}{10}} \right|^2 + \left| \sqrt{\frac{3}{10}} \right|^2 + \left| \sqrt{\frac{4}{10}} \right|^2 + \left| \sqrt{\frac{5}{10}} \right|^2 = 1 \end{aligned}$$

$$P(|00\rangle) = \frac{1}{10}$$

$$P(|01\rangle) = \frac{4}{10}$$

$$P(|10\rangle) = \frac{2}{10}$$

$$P(|11\rangle) = \frac{3}{10}$$

c.

$$CNOT|00\rangle = |00\rangle$$

$$CNOT|01\rangle = |01\rangle$$

$$CNOT|10\rangle = |11\rangle$$

$$CNOT|11\rangle = |10\rangle$$

$$P(CNOT|00\rangle) = P(|00\rangle) = \frac{1}{10}$$

$$P(CNOT|01\rangle) = P(|01\rangle) = \frac{4}{10}$$

$$P(CNOT|10\rangle) = P(|11\rangle) = \frac{3}{10}$$

$$P(CNOT|11\rangle) = P(|10\rangle) = \frac{2}{10}$$

d. Flipped CNOT:

$|00\rangle \rightarrow |00\rangle$ right bit control is 0 \rightarrow no change
 $|01\rangle \rightarrow |11\rangle$ right bit control is 1 \rightarrow flip left bit
 $|10\rangle \rightarrow |10\rangle$ right bit control is 0 \rightarrow no change
 $|11\rangle \rightarrow |01\rangle$ right bit control is 1 \rightarrow flip left bit

$$U_{\text{FlippedControlCNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

e. The dagger LaTeX doesn't work ☹

$$U^{\dagger} = U^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$U^{\dagger}U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

The Matrix times its transpose is the identity matrix therefore it is unitary.

3)

a.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$\alpha = \cos\left(\frac{\theta}{2}\right), \quad \beta = e^{i\phi}\sin\left(\frac{\theta}{2}\right)$$

$$\arccos \alpha = \frac{\theta}{2}, \quad \theta = 2 \arccos|\alpha|$$

$$i\phi = \ln\left(\frac{\beta}{\sin\frac{\theta}{2}}\right) = \ln\frac{\beta}{\sin\frac{2\arccos\alpha}{2}} = \ln\frac{\beta}{\sqrt{1-\alpha^2}}$$

$$\phi = \frac{\ln\frac{\beta}{\sqrt{1-\alpha^2}}}{i}$$

$$\begin{aligned}
 |0\rangle: \alpha &= 1, \beta = 0 \\
 \theta &= \arccos 1 = 0 \\
 \phi &= 0
 \end{aligned}$$

$$\begin{aligned} |1\rangle: \alpha = 0, \beta = 1 \\ \theta = \arccos 0 = \pi \\ \phi = 0 \end{aligned}$$

b.

$$\begin{aligned} X(\alpha, \beta) &= (\beta, \alpha), & X(\theta, \phi) &= (\pi - \theta, -\phi) \\ Y(\alpha, \beta) &= (-i\beta, i\alpha), & Y(\theta, \phi) &= (\pi - \theta, \pi - \phi) \\ Z(\alpha, \beta) &= (\alpha, -\beta), & Z(\theta, \phi) &= (\theta, \phi + \pi) \\ H(\alpha, \beta) &= \left(\frac{\alpha + \beta}{\sqrt{2}}, \frac{\alpha - \beta}{\sqrt{2}}\right), & |0\rangle: H(\theta, \phi) &= (\pi/2, 0), \\ & & |1\rangle: H(\theta, \phi) &= (\pi/2, \pi) \\ P(\theta)(\alpha, \beta) &= (\alpha, e^{i\delta}\beta), & P(\theta_{input})(\theta, \phi) &= (\theta, \phi + \theta_{input}) \end{aligned}$$

c.

$$\begin{aligned} H|0\rangle &= \left(\frac{\alpha + \beta}{\sqrt{2}}, \frac{\alpha - \beta}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \end{aligned}$$

d.

$$H\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{2}H|0\rangle + \frac{1}{2}H|1\rangle = \frac{1}{2}(|0\rangle + |1\rangle) + \frac{1}{2}(|0\rangle - |1\rangle) = |0\rangle$$

The $|1\rangle$ cancel each other out (destructive interference) and we are left back with $|0\rangle$ meaning the inversion of an hgate is another hgate.

e. The final $|0\rangle$ state is a determined basis state as there is only the possibility of it being measured as a $|0\rangle$ so it cannot be a superposition.

4)

a.

$$\begin{aligned} (H \otimes H)|00\rangle &= H|0\rangle \otimes H|0\rangle \\ \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) |00\rangle &= H|0\rangle \otimes H|0\rangle \\ \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \otimes \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \end{aligned}$$

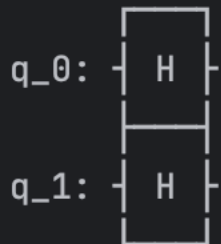
$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 1 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

When combining a sequence of gates to a sequence of states whether applying individual states to gates or all at once, either results in the same outcome. This is a property of tensor products that extends to all linear operators which is why it works with to n qubits.

- b. If we have multiple single qubit matrices their tensor product becomes a new matrix that can act on all states without interacting with each other. This allows us to apply operations simultaneously as if it were a multiple qubit system even though it is comprised of a combination of single qubits.

Circuit for $H \otimes H$ on $|00\rangle$:



Initial state: $|00\rangle$

Final state after $H \otimes H$:

$|00\rangle$: 0.500000+0.000000j

$|01\rangle$: 0.500000+0.000000j

$|10\rangle$: 0.500000+0.000000j

$|11\rangle$: 0.500000+0.000000j

c.

$|00\rangle$: 0.500000

$|01\rangle$: 0.500000

$|10\rangle$: 0.500000

$|11\rangle$: 0.500000

Verification:

Qiskit result: [0.5+0.j 0.5+0.j 0.5+0.j 0.5+0.j]

Theoretical: [0.5 0.5 0.5 0.5]

Error L2 norm: 0.0000000000

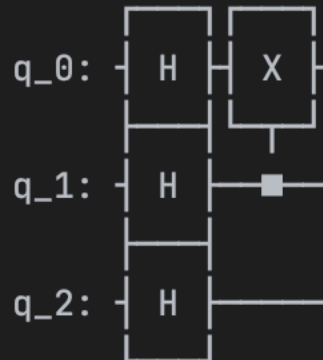
Alternative verification - $(H|0\rangle) \otimes (H|0\rangle)$:

Tensor product result: [0.5 0.5 0.5 0.5]

Matches Qiskit: True

=== CASE 2: H on all qubits, then CNOT(1→0) ===

Circuit for Case 2:



Final state amplitudes:

$|000\rangle$: 0.353553+0.000000j

$|001\rangle$: 0.353553+0.000000j

$|010\rangle$: 0.353553+0.000000j

$|011\rangle$: 0.353553+0.000000j

$|100\rangle$: 0.353553+0.000000j

$|101\rangle$: 0.353553+0.000000j

$|110\rangle$: 0.353553+0.000000j

$|111\rangle$: 0.353553+0.000000j

*After $H \otimes H \otimes H|000\rangle =$ equal superposition of all 8 states
CNOT(0 → 1) action:*

$|000\rangle \rightarrow |000\rangle$ (control = 0, no change)

$|001\rangle \rightarrow |001\rangle$ (control = 0, no change)

$|010\rangle \rightarrow |010\rangle$ (control = 0, no change)

$|011\rangle \rightarrow |011\rangle$ (control = 0, no change)

$|100\rangle \rightarrow |110\rangle$ (control = 1, flip)

$|101\rangle \rightarrow |111\rangle$ (control = 1, flip)

$|110\rangle \rightarrow |100\rangle$ (control = 1, flip)

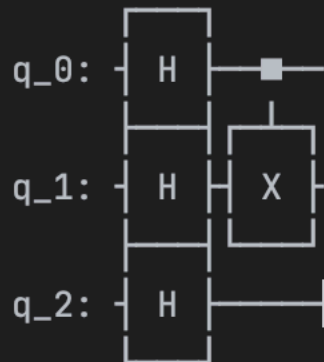
$|111\rangle \rightarrow |101\rangle$ (control = 1, flip)

Expected amplitudes: all $1/\sqrt{8} = 0.353553$

=== PROBLEM 4(d) ===

=== CASE 1: H on all qubits, then CNOT(0→1) ===

Circuit for Case 1:



Final state amplitudes:

$|000\rangle$: $0.353553+0.000000j$
 $|001\rangle$: $0.353553+0.000000j$
 $|010\rangle$: $0.353553+0.000000j$
 $|011\rangle$: $0.353553+0.000000j$
 $|100\rangle$: $0.353553+0.000000j$
 $|101\rangle$: $0.353553+0.000000j$
 $|110\rangle$: $0.353553+0.000000j$
 $|111\rangle$: $0.353553+0.000000j$

$H \otimes H \otimes H|000\rangle = \text{equal superposition of all 8 states}$

$CNOT(1 \rightarrow 0)$ action ($\text{control} = 1, \text{target} = 0$):

$|000\rangle \rightarrow |000\rangle$ ($\text{control} = 0, \text{no change}$)
 $|001\rangle \rightarrow |001\rangle$ ($\text{control} = 0, \text{no change}$)
 $|010\rangle \rightarrow |110\rangle$ ($\text{control} = 1, \text{flip}$)
 $|011\rangle \rightarrow |111\rangle$ ($\text{control} = 1, \text{flip}$)
 $|100\rangle \rightarrow |100\rangle$ ($\text{control} = 0, \text{no change}$)
 $|101\rangle \rightarrow |101\rangle$ ($\text{control} = 0, \text{no change}$)
 $|110\rangle \rightarrow |010\rangle$ ($\text{control} = 1, \text{flip}$)
 $|111\rangle \rightarrow |011\rangle$ ($\text{control} = 1, \text{flip}$)

Expected amplitudes: all $1/\sqrt{8} = 0.353553$

5)

a. Half adder $|110\rangle$

$$\begin{aligned}q_0 &= \text{input } A = 1 \\q_1 &= \text{input } B = 1 \\q_2 &= \text{Carry}(0) = 0\end{aligned}$$

$$\begin{aligned}\text{SUM} &= A \text{ XOR } B = 1 \text{ XOR } 1 = 0 \\ \text{CARRY} &= A \text{ AND } B = 1 \text{ AND } 1 = 1 \\ q_0 &\rightarrow 1 \\ q_1 &\rightarrow 0 \\ q_2 &\rightarrow 1\end{aligned}$$

Full adder $|1100\rangle$

$$\begin{aligned}q_0 &= \text{input } A = 1 \\q_1 &= \text{input } B = 1 \\q_3 &= \text{input } C = 0 \\q_2 &= \text{Carry} = 0\end{aligned}$$

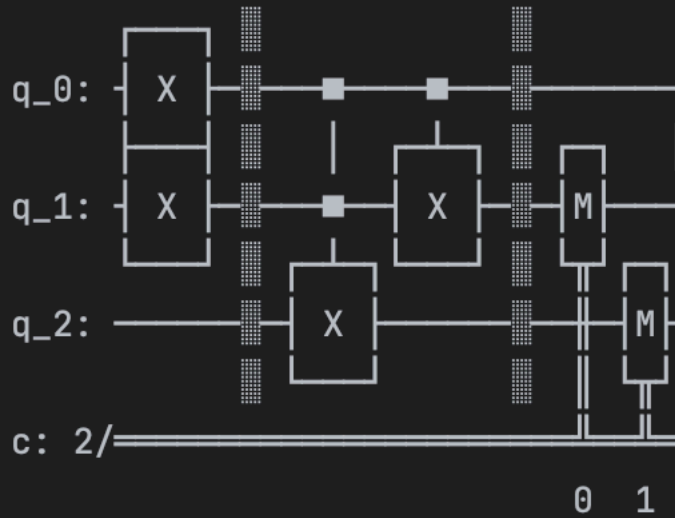
$$\begin{aligned}\text{SUM} &= A \text{ XOR } B \text{ XOR } C = 1 \text{ XOR } 1 \text{ XOR } 0 = 0 \\ \text{CARRY} &= (A \text{ AND } B) \text{ OR } (C \text{ AND } (A \text{ XOR } B)) = (1 \text{ AND } 1) \text{ OR } (0 \text{ AND } (1 \text{ XOR } 1)) = 1 \text{ OR } 0 \\ &= 1\end{aligned}$$

$$\begin{aligned}q_0 &\rightarrow 1 \\ q_1 &\rightarrow 1 \\ q_2 &\rightarrow 0 \\ q_3 &\rightarrow 1\end{aligned}$$

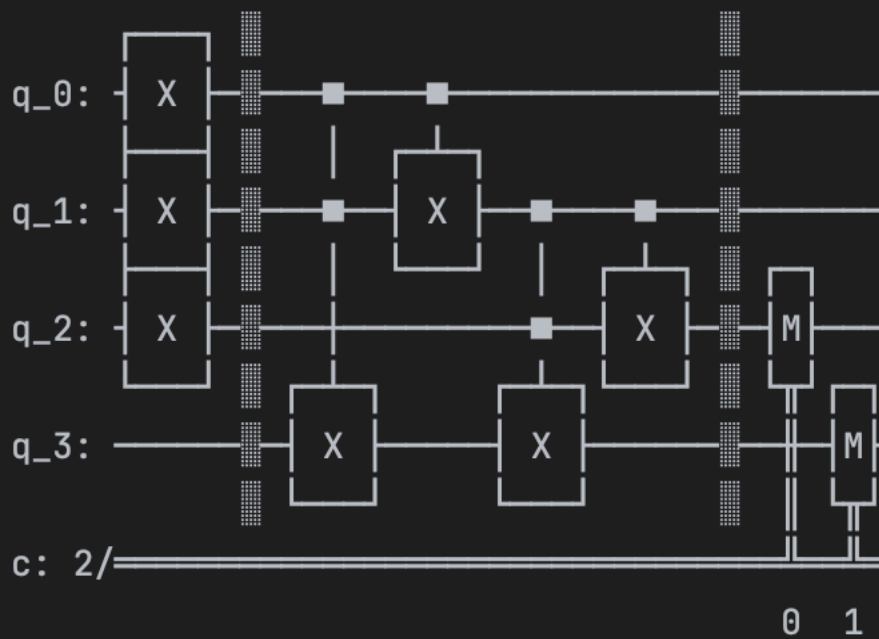
b - d.

=== Problem 5: Quantum Adders with Aer Simulator ===

--- Example Half Adder Circuit (for A=1, B=1) ---



--- Example Full Adder Circuit (for A=1, B=1, C_in=1)



b.

Half Adder:

Input A=0, B=0 -> Measured (Carry,Sum): '00'

Input A=0, B=1 -> Measured (Carry,Sum): '01'

Input A=1, B=0 -> Measured (Carry,Sum): '01'

Input A=1, B=1 -> Measured (Carry,Sum): '10'

c.

Half Adder:

Input A=0, B=0 -> Measured (Carry,Sum): '00'

Input A=0, B=1 -> Measured (Carry,Sum): '01'

Input A=1, B=0 -> Measured (Carry,Sum): '01'

Input A=1, B=1 -> Measured (Carry,Sum): '10'

Full Adder (with C_in=0):

Input A=0, B=0, C_in=0 -> Measured (Carry,Sum): '00'

Input A=0, B=1, C_in=0 -> Measured (Carry,Sum): '01'

Input A=1, B=0, C_in=0 -> Measured (Carry,Sum): '01'

Input A=1, B=1, C_in=0 -> Measured (Carry,Sum): '10'

d.

Half Adder:

Input A=0, B=0 -> Measured (Carry,Sum): '00'

Input A=0, B=1 -> Measured (Carry,Sum): '01'

Input A=1, B=0 -> Measured (Carry,Sum): '01'

Input A=1, B=1 -> Measured (Carry,Sum): '10'

Full Adder (with C_in=0):

Input A=0, B=0, C_in=0 -> Measured (Carry,Sum): '00'

Input A=0, B=1, C_in=0 -> Measured (Carry,Sum): '01'

Input A=1, B=0, C_in=0 -> Measured (Carry,Sum): '01'

Input A=1, B=1, C_in=0 -> Measured (Carry,Sum): '10'

The simulation results perfectly match classical binary addition. However real hardware would exhibit differences in outcomes due to noise.