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CSCI 739 HW3

1.

a)

$$X(0) = \begin{pmatrix} \sin(0) & \cos(0) \\ \cos(0) & -\sin(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

Hermitian:

$$X(\varepsilon)^\dagger = X(\varepsilon)^T = \begin{pmatrix} \sin(\varepsilon) & \cos(\varepsilon) \\ \cos(\varepsilon) & -\sin(\varepsilon) \end{pmatrix}^T = \begin{pmatrix} \sin(\varepsilon) & \cos(\varepsilon) \\ \cos(\varepsilon) & -\sin(\varepsilon) \end{pmatrix} = X(\varepsilon)$$

Unitary:

$$X(\varepsilon) = X(\varepsilon)^\dagger \therefore X(\varepsilon)^\dagger X(\varepsilon) = X(\varepsilon)^2$$

$$X(\epsilon)^2 = \begin{pmatrix} \sin(\varepsilon) & \cos(\varepsilon) \\ \cos(\varepsilon) & -\sin(\varepsilon) \end{pmatrix} \begin{pmatrix} \sin(\varepsilon) & \cos(\varepsilon) \\ \cos(\varepsilon) & -\sin(\varepsilon) \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2(\varepsilon) + \cos^2(\varepsilon) & \sin(\varepsilon)\cos(\varepsilon) - \cos(\varepsilon)\sin(\varepsilon) \\ \cos(\varepsilon)\sin(\varepsilon) - \sin(\varepsilon)\cos(\varepsilon) & \cos^2(\varepsilon) + \sin^2(\varepsilon) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

b)

$$X(\varepsilon)|0\rangle = \begin{pmatrix} \sin(\varepsilon) & \cos(\varepsilon) \\ \cos(\varepsilon) & -\sin(\varepsilon) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin(\varepsilon) \\ \cos(\varepsilon) \end{pmatrix} = \sin(\varepsilon)\,|0\rangle + \cos(\varepsilon)\,|1\rangle$$

$$P_\varepsilon^{(1)}(0) = |\sin(\varepsilon)|^2 \ , \qquad P_\varepsilon^{(1)}(1) = |\cos(\varepsilon)|^2$$

$$P_\varepsilon^{(1)}(0) + P_\varepsilon^{(1)}(1) = \sin^2\varepsilon + \cos^2\varepsilon = 1$$

Ideal X: $\varepsilon = 0$

$$P_0^{(1)}(0) = \sin^2 0 = 0 \ , \qquad P_0^{(1)}(1) = \cos^2 0 = 1$$

Non-Ideal X: $\varepsilon \neq 0$

$$P_\varepsilon^{(1)}(0) = \sin^2\varepsilon \ , \qquad P_\varepsilon^{(1)}(1) = \cos^2\varepsilon$$

Because cosine produces any value between -1 and 1, cosine squared will be able to produce any value between 0 and 1 therefore, any value within the range for p is reproducible when changing epsilon $P_{\varepsilon}^{(1)}(1)$.

c)

When n is even:

$$X(\varepsilon)^n = X(\varepsilon)^{2k} = I^k , \qquad k \geq 0$$

$$|\psi_n(\varepsilon)\rangle = X(\varepsilon)^n|0\rangle = I|0\rangle = |0\rangle$$

When n is odd:

$$n = 2k + 1 , k \geq 0$$

$$\therefore X(\varepsilon)^n = X(\varepsilon)^{2k}X(\varepsilon)^1 = IX(\varepsilon) = X(\varepsilon)$$

$$|\psi_n(\varepsilon)\rangle = X(\varepsilon)^n|0\rangle = X(\varepsilon)|0\rangle = \sin(\varepsilon)\,|0\rangle + \cos(\varepsilon)\,|1\rangle$$

Because the X gate is Unitary regardless of epsilon being non-zero, this behavior will continue to occur regardless of epsilon. This also means that the value is determined by the parity of n and not the value of epsilon so we can no longer use epsilon tuning to generate all values of p given any value of n.

d) Because X gates remain unitary even when error prone, it may be much harder to detect errors because the state will remain the ideal or close too ideal (if odd number of gates). This means that we cannot as reliably use the probability/(gate outcome) to detect whether an error is present. This means many errors may end up going undetected.

2.

a) Hermitian proof:

$$H^\dagger = H^T = \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & -\frac{1}{2} \end{pmatrix}^T = \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & -\frac{1}{2} \end{pmatrix} = H$$

Finding eigenvalues:

$$det(H-\lambda I) = det\begin{pmatrix} \frac{1}{2}-\lambda & -1 \\ -1 & -\frac{1}{2}-\lambda \end{pmatrix} = 0$$

$$\left(\frac{1}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right) - (-1)(-1) = 0$$

$$-\frac{1}{4}+\lambda^2-1=0$$

$$\lambda = \pm\frac{\sqrt{5}}{2}$$

Proving orthogonal:

For $\lambda_- = -\frac{\sqrt{5}}{2}$:

$$\begin{pmatrix} \frac{1}{2}-\left(-\frac{\sqrt{5}}{2}\right) & -1 \\ -1 & -\frac{1}{2}-\left(-\frac{\sqrt{5}}{2}\right) \end{pmatrix}|\psi\rangle = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{-1+\sqrt{5}}{2} \end{pmatrix}\binom{x}{y} = \begin{pmatrix} \frac{1+\sqrt{5}}{2}x-y \\ -x+y\frac{-1+\sqrt{5}}{2} \end{pmatrix} = \binom{0}{0}$$

$$\frac{1+\sqrt{5}}{2}x-y=0\,, \qquad y=\frac{1+\sqrt{5}}{2}x$$

$$N_- = \sqrt{1+\left(\frac{1+\sqrt{5}}{2}\right)^2} = \sqrt{1+\frac{1+2\sqrt{5}+5}{4}} = \sqrt{\frac{5+\sqrt{5}}{2}}$$

$$|\psi_-\rangle = \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}}\begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}$$

For $\lambda_+ = \frac{\sqrt{5}}{2}$:

$$\begin{pmatrix} \frac{1}{2}-\left(\frac{\sqrt{5}}{2}\right) & -1 \\ -1 & -\frac{1}{2}-\left(\frac{\sqrt{5}}{2}\right) \end{pmatrix}|\psi\rangle = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{2}x - y \\ -x + y\frac{-1-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{1-\sqrt{5}}{2}x - y = 0\,, \qquad y = \frac{1-\sqrt{5}}{2}x$$

$$N_+ = \sqrt{\frac{5-\sqrt{5}}{2}}$$

$$|\psi_+\rangle = \frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}}\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

$$\langle\psi_-|\psi_+\rangle\ is\ 1*1+\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = 1-1 = 0$$

b)

$$\mathrm{E}_0 = -\frac{\sqrt{5}}{2}$$

$$|\psi_0\rangle = |\psi_-\rangle = \sqrt{\frac{5+\sqrt{5}}{2}}\ \left(|0\rangle + \left(\frac{1+\sqrt{5}}{2}\right)|1\rangle\right)$$

c)

i)

$$|\psi(\theta)\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$E(\theta) = \left(cos\left(\frac{\theta}{2}\right)\ \ sin\left(\frac{\theta}{2}\right)\right)\begin{pmatrix} \frac{1}{2} & -1 \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} cos\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$E(\theta) = \cos\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right) + \sin\left(\frac{\theta}{2}\right)\left(-\cos\left(\frac{\theta}{2}\right) - \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\right)$$

$$= \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right) - 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$= \mathbf{\frac{1}{2}\cos(\theta) - \sin(\theta)}$$

$$\frac{dE}{d\theta} = -\frac{1}{2}sin\theta - cos(\theta) = 0, \qquad tan(\theta) = -2$$

$$\theta_* = \boldsymbol{arctan(-2) + \pi \approx 2.034}$$

ii)

$$E(\theta_*) = \frac{1}{2}\cos(\theta_*) - \sin(\theta_*) = \frac{1}{2}\left(-\frac{1}{\sqrt{5}}\right) - \left(\frac{2}{\sqrt{5}}\right) E(\theta_*) = -\frac{5}{2\sqrt{5}} = -\frac{\boldsymbol{\sqrt{5}}}{\mathbf{2}}$$

3.

a)

$$\text{Input } x = (0,0) \rightarrow |00\rangle \text{ (Target } y_0 = 0)$$

$$|00\rangle \xrightarrow{R_y} (c_0|0\rangle + s_0|1\rangle) \otimes ((c_1|0\rangle + s_1|1\rangle)$$

$$\xrightarrow{CNOT} c_0c_1|00\rangle + c_0s_1|01\rangle + s_0c_1|11\rangle + s_0s_1|10\rangle$$

$$p_0 = (\text{amp of } |01\rangle)^2 + (\text{amp of } |11\rangle)^2 = (c_0s_1)^2 + (s_0c_1)^2 = c_0^2s_1^2 + s_0^2c_1^2$$

$$\text{Input } x = (0,1) \rightarrow |01\rangle \text{ (Target } y_1 = 1)$$

$$|01\rangle \xrightarrow{R_y} (c_0|0\rangle + s_0|1\rangle) \otimes ((-s_1|0\rangle + s_1|1\rangle)$$

$$\xrightarrow{CNOT} -c_0s_1|00\rangle + c_0c_1|01\rangle - s_0s_1|11\rangle + s_0c_1|10\rangle$$

$$p_1 = (c_0c_1)^2 + (-s_0s_1)^2 = c_0^2c_1^2 + s_0^2s_1^2$$

$$\text{Input } x = (1,0) \rightarrow |10\rangle \text{ (Target } y_2 = 1)$$

$$|10\rangle \xrightarrow{R_y} (-s_0|0\rangle + c_0|1\rangle) \otimes ((c_1|0\rangle + s_1|1\rangle)$$

$$\xrightarrow{CNOT} -s_0c_1|00\rangle - s_0s_1|01\rangle + c_0c_1|11\rangle + s_0c_1|10\rangle$$

$$p_2 = (-s_0s_1)^2 + (c_0c_1)^2 = s_0^2s_1^2 + c_0^2c_1^2$$

$$\text{Input } x = (1,1) \rightarrow |11\rangle \text{ (Target } y_3 = 0)$$

$$|11\rangle \xrightarrow{R_y} (c_0|0\rangle + s_0|1\rangle)\otimes((-s_1|0\rangle + s_1|1\rangle)$$
$$\xrightarrow{CNOT} s_0s_1|00\rangle + s_0c_1|01\rangle - c_0s_1|11\rangle + c_0c_1|10\rangle$$
$$p_3 = (-s_0s_1)^2 + (-c_0c_1)^2 = s_0^2s_1^2 + c_0^2c_1^2$$

This allows for linear separability as long as we solve for θ_0 *and* θ_1 where the p values are equivalent to the ones above ($p_0 = p_3 = 1$ *and* $p_1 = p_2 = 0$).

$$p_0 = 0 = c_0^2s_1^2 + s_0^2c_1^2$$
$$c_0^2s_1^2 = -(s_0^2c_1^2)$$
$$p_1 = 1 = c_0^2c_1^2 + s_0^2s_1^2$$

Because $c_0^2c_1^2 = -(s_0^2s_1^2)$ is true, and both sides cannot be negative as they are the product of squared values, both sides must always equal 0 in that equation. This allows for combinations of $(\theta_0 = n\pi, \theta_1 = m\pi)$, $(\theta_0 = n\pi, \theta_1 = k)$, or $(\theta_0 = k, \theta_1 = n\pi)$, where n and m can be any integer and k is any real number. However, to fulfill $P_1 = 1$, only $(\theta_0 = n\pi, \theta_1 = m\pi)$ is valid.

b)

$$If\ x = 0,\ \phi = 0\text{: } R_y(0)|0\rangle = cos(0)|0\rangle + sin(0)|1\rangle = |0\rangle\ \ and\ \ x = 1,\ \phi = \pi\text{: } R_y(\pi)|0\rangle$$
$$= \cos\frac{\pi}{2}|0\rangle + \sin\frac{\pi}{2}|1\rangle = |1\rangle$$

$$(R_y(\pi\, x_1)|0\rangle) \otimes (R_y(\pi\, x_2)|0\rangle) = |x_1\rangle \otimes |x_2\rangle = |x_1\, x_2\rangle$$

$$\text{Input } x = (0,0) \rightarrow| 00\rangle \text{ (Target } y_0 = 0)$$
$$|00\rangle \xrightarrow{R_y} (c_0|0\rangle + s_0|1\rangle)\otimes((c_1|0\rangle + s_1|1\rangle)$$
$$\xrightarrow{CNOT} c_0c_1|00\rangle + c_0s_1|01\rangle + s_0c_1|11\rangle + s_0s_1|10\rangle$$
$$p_0 = (\text{amp of } |01\rangle)^2 + (\text{amp of } |11\rangle)^2 = (c_0s_1)^2 + (s_0c_1)^2 = c_0^2s_1^2 + s_0^2c_1^2$$

$$\text{Input } x = (0,1) \rightarrow |01\rangle \text{ (Target } y_1 = 1)$$
$$|01\rangle \xrightarrow{R_y} (c_0|0\rangle + s_0|1\rangle)\otimes((-s_1|0\rangle + s_1|1\rangle)$$
$$\xrightarrow{CNOT} -c_0s_1|00\rangle + c_0c_1|01\rangle - s_0s_1|11\rangle + s_0c_1|10\rangle$$
$$p_1 = (c_0c_1)^2 + (-s_0s_1)^2 = c_0^2c_1^2 + s_0^2s_1^2$$

$$\text{Input } x = (1,0) \rightarrow |10\rangle \text{ (Target } y_2 = 0)$$
$$|11\rangle \xrightarrow{R_y} (c_0|0\rangle + s_0|1\rangle)\otimes((-s_1|0\rangle + s_1|1\rangle)$$
$$\xrightarrow{CNOT} s_0s_1|00\rangle + s_0c_1|01\rangle - c_0s_1|11\rangle + c_0c_1|10\rangle$$
$$p_2 = (-s_0s_1)^2 + (-c_0c_1)^2 = s_0^2s_1^2 + c_0^2c_1^2$$

$$\text{Input } x = (1,1) \rightarrow| 10\rangle \text{ (Target } y_3 = 0)$$

$$|10\rangle \xrightarrow{R_y} (-s_0|0\rangle + c_0|1\rangle)\otimes((c_1|0\rangle + s_1|1\rangle)$$
$$\xrightarrow{CNOT} -s_0c_1|00\rangle - s_0s_1|01\rangle + c_0c_1|11\rangle + s_0c_1|10\rangle$$
$$p_3 = (-s_0s_1)^2 + (c_0c_1)^2 = s_0^2s_1^2 + c_0^2c_1^2$$

Xor requires 00 and 11's P(1) to equal 0 and the rest to 1 however the encoding says that 00 and 10 are equivalent. This is impossible for both to be true. This means that angle encoding with this VQC cannot be linearly separable.

c)

$$L(\theta) = \frac{1}{4}\left((p_0 - 0)^2 + (p_1 - 1)^2 + (p_2 - 1)^2 + (p_3 - 0)^2\right)$$

$$p_1 = p_2 \; and \; p_0 = p_3 \therefore \; L(\theta) = \frac{1}{4}\left(p_0^2 + (p_1 - 1)^2 + (p_1 - 1)^2 + p_0^2\right) = \frac{1}{2}\left(p_0^2 + (p_1 - 1)^2\right)$$

$$p_0 = 1 - p_1 = c_0^2s_1^2 + s_0^2c_1^2 = 1 - c_0^2c_1^2 - s_0^2s_1^2$$

$$L(\theta) = \frac{1}{2}\left(p_0^2 + \left((1 - p_0) - 1\right)^2\right) = \frac{1}{2}\left(p_0^2 + (-p_0)^2\right) = \frac{1}{2}(2p_0^2) = p_0^2$$

$$\boldsymbol{L(\theta)} = \left(\cos^2\left(\frac{\theta_0}{2}\right)\sin^2\left(\frac{\theta_1}{2}\right) + \sin^2\left(\frac{\theta_0}{2}\right)\cos^2\left(\frac{\theta_1}{2}\right)\right)^2$$

The minimum $L(\theta) = 0$ when $p_0 = 0$.
Possible theta values are listed in a) ($\theta_0 = n\pi, \theta_1 = m\pi$). Both Thetas can be any integer multiple of pi. As multiple perfect solutions exist, and the circuit is super simple the convergence seems to be insensitive to initialization.

d)

$$p = n1/S$$

$$P(0) = 1 - p\,, P(1) = p$$

$$\langle Z\rangle = (P(0)(1)) + (P(1) * (-1)) = (1 - p)(1) + (p)(-1) = 1 - p - p = 1 - 2p$$

$$f_{VQC} = \frac{1 - \langle Z\rangle}{2} = \frac{1 - (1 - 2p)}{2} = \frac{2p}{2} = p$$

e) Skipped

f)

Basis encoding was best for a shallow VQC as there was no errors. Nonlinear feature mapping allows nonlinearly separable data to map to a higher dimensional space which allows for linear separability. Gates that induce entanglement create non-linear operations which allow close datapoints to move apart hence separate.

4.

a)

$$w_1x_1 + w_2x_2 + b = 0$$
$$(0,0): \quad b < 0$$
$$(0,1): w2 + b > 0$$
$$(1,0): w1 + b > 0$$
$$(1,1): w1 + w2 + b < 0$$

$$(w2 + b) + (w1 + b) > 0 \rightarrow w1 + w2 + 2b > 0$$
$$however$$
$$if\ w1 + w2 + b < 0\ and\ w1 + w2 + 2b > 0$$

$$then\ (w1 + w2 + 2b) - (w1 + w2 + b) > 0$$

$$But\ (w1 + w2 + 2b) - (w1 + w2 + b) = b < 0$$

This contradiction proves no linear boundary can exist.

b) The $U\phi(x;\beta)$ circuit initializes two qubits of $|00\rangle$ state

Then an entangling gate $e^{-i\beta Z\otimes Z} is\ applied\ CNOT_{0\rightarrow 1}\big(I \otimes R_z(2\beta)\big)CNOT_{0\rightarrow 1}$. This applies the $R_z(\pi x_1)$ to qubit 0 and $R_z(\pi x_2)$ to qubit 1.

c) $K(xi, xj) = |\langle \phi(xi) \mid \phi(xj) \rangle|^2$

$$Z \otimes Z = (1)(1) = 1$$

$$\beta = \frac{\pi}{4}$$

$$e^{-i\beta Z\otimes Z}|00\rangle = e^{-i\beta}|00\rangle = e^{-\frac{i\pi}{4}}|00\rangle$$

$$R_z(\phi) \mid 0\rangle = e^{-\frac{i\phi}{2}}|0\rangle.$$

$$R_z(\pi x_1) \mid 0\rangle = e^{-\frac{i\pi x_1}{2}}|0\rangle.$$

$$R_z(\pi x_2) \mid 0\rangle = e^{-\frac{i\pi x_2}{2}} |0\rangle$$

$$|\phi(x)\rangle = (e^{-\frac{i\pi x_1}{2}} \otimes e^{-\frac{i\pi x_2}{2}}) \cdot (e^{-\frac{i\pi}{4}}|00\rangle) = e^{-i(\frac{\pi}{4}+\frac{\pi(x_1+x_2)}{2})}|00\rangle$$

$$K = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

All values in matrix equal to $|00\rangle$ therefore they are a matrix of all 1s. as the matrix is all 1s K is symmetric to K^T.

d)

i)

$$\mathrm{b} = \frac{1}{|\mathrm{S_{near}}|} \sum_{\mathrm{s} \in \mathrm{S_{near}}} (\mathrm{y_s} - \sum_{\mathrm{i=1}}^{\mathrm{N}} \alpha_{\mathrm{i}} \mathrm{y_i} \mathrm{K}(\mathrm{x_i}, \mathrm{x_s}))$$

$$b = \frac{1}{4}\sum_{s=1}^{4}(y\_s - 0) = \frac{1}{4}(y_1 + y_2 + y_3 + y_4)b = \frac{1}{4}(-1+1+1-1) = 0$$

ii)

When b is 0, the entanglement gate $e^{-i\beta Z\otimes Z}$ becomes an identity matrix.

$U\phi(x;0) = (Rz(\pi x_1) \otimes Rz(\pi x_2))$ Applying this to $|00\rangle$ gives $|\ \phi(x)\rangle = e^{-i\pi(x1+x2)/2}|00\rangle$. This is still just $|00\rangle$. The kernel K is still 1 for all entries and accuracy is the same. This shows entanglement to not be critical in this case as it has no effect.

iii) Skip

iv)

$$f(x) = sign(\sum_{i=1}^{n} \alpha_i y_i K(x_i, x) + b)$$

$$\alpha_i = C,\ K(x_i, x) = 1, and\ b = 0$$

$$f(x) = sign\left(\sum_{i=1}^{n} Cy_i(1) + 0\right) = sign\big(C(-1+1+1-1)\big) = sign(0) = 0$$

Because the model predicts 0s instead of +1 or -1 it is still has no accuracy so C had no effect in this case..