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CSCI 739 HW1

1)

a.

$$\langle 0|\psi \rangle = \alpha, P(|0\rangle) = |\alpha|^2$$

$$\langle 1|\psi \rangle = \beta, P(|1\rangle) = |\beta|^2$$

b.

$$|\alpha|^2 + |\beta|^2 = 1$$

$$P(|0\rangle) = |\alpha|^2, \quad P(|1\rangle) = |\beta|^2$$

$$\therefore P(|0\rangle) + P(|1\rangle) = 1$$

c.

$$P(|0\rangle) = |\alpha|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$P(|1\rangle) = |\beta|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

2)

a.

Tensor Product Notation:

$$\begin{aligned} |00\rangle &= |0\rangle \otimes |0\rangle \\ |01\rangle &= |0\rangle \otimes |1\rangle \\ |10\rangle &= |1\rangle \otimes |0\rangle \\ |11\rangle &= |1\rangle \otimes |1\rangle \end{aligned}$$

Vectors Representation:

$$\begin{aligned} |00\rangle: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ |01\rangle: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$|10\rangle: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 & \cdot & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|11\rangle: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 & \cdot & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

b.

$$\begin{aligned} P(\text{total}) &= P(|00\rangle) + P(|01\rangle) + P(|10\rangle) + P(|11\rangle) \\ &= \left| \sqrt{\frac{1}{10}} \right|^2 + \left| \sqrt{\frac{3}{10}} \right|^2 + \left| \sqrt{\frac{4}{10}} \right|^2 + \left| \sqrt{\frac{5}{10}} \right|^2 = 1 \end{aligned}$$

$$P(|00\rangle) = \frac{1}{10}$$

$$P(|01\rangle) = \frac{4}{10}$$

$$P(|10\rangle) = \frac{2}{10}$$

$$P(|11\rangle) = \frac{3}{10}$$

c.

$$\begin{aligned} CNOT|00\rangle &= |00\rangle \\ CNOT|01\rangle &= |01\rangle \\ CNOT|10\rangle &= |11\rangle \\ CNOT|11\rangle &= |10\rangle \end{aligned}$$

$$P(CNOT|00\rangle) = P(|00\rangle) = \frac{1}{10}$$

$$P(CNOT|01\rangle) = P(|01\rangle) = \frac{4}{10}$$

$$P(CNOT|10\rangle) = P(|11\rangle) = \frac{3}{10}$$

$$P(CNOT|00\rangle) = P(|10\rangle) = \frac{2}{10}$$

d. Flipped CNOT:

- $|00\rangle \rightarrow |00\rangle$  right bit control is 0  $\rightarrow$  no change
- $|01\rangle \rightarrow |11\rangle$  right bit control is 1  $\rightarrow$  flip left bit
- $|10\rangle \rightarrow |10\rangle$  right bit control is 0  $\rightarrow$  no change
- $|11\rangle \rightarrow |01\rangle$  right bit control is 1  $\rightarrow$  flip left bit

$$U_{\text{FlippedControlCNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

e. The dagger LaTeX doesn't work  $\otimes$

$$U^{\text{dagger}} = U^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$U^{\text{dagger}}U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

The Matrix times its transpose is the identity matrix therefore it is unitary.

3)

a.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$\alpha = \cos\left(\frac{\theta}{2}\right), \quad \beta = e^{i\phi}\sin\left(\frac{\theta}{2}\right)$$

$$\arccos \alpha = \frac{\theta}{2}, \quad \theta = 2 \arccos |\alpha|$$

$$i\phi = \ln\left(\frac{\beta}{\sin\frac{\theta}{2}}\right) = \ln\frac{\beta}{\sin\frac{2\arccos\alpha}{2}} = \ln\frac{\beta}{\sqrt{1-\alpha^2}}$$

$$\phi = \frac{\ln\frac{\beta}{\sqrt{1-\alpha^2}}}{i}$$

$$|0\rangle: \alpha = 1, \beta = 0$$

$$\theta = \arccos 1 = 0$$

$$\phi = 0$$

$$\begin{aligned} |1\rangle: \alpha &= 0, \beta = 1 \\ \theta &= \arccos 0 = \pi \\ \phi &= 0 \end{aligned}$$

b.

$$X(\alpha, \beta) = (\beta, \alpha), \quad X(\theta, \phi) = (\pi - \theta, -\phi)$$

$$Y(\alpha, \beta) = (-i\beta, i\alpha), \quad Y(\theta, \phi) = (\pi - \theta, \pi - \phi)$$

$$Z(\alpha, \beta) = (\alpha, -\beta), \quad Z(\theta, \phi) = (\theta, \phi + \pi)$$

$$\begin{aligned} H(\alpha, \beta) &= \left( \frac{\alpha + \beta}{\sqrt{2}}, \frac{\alpha - \beta}{\sqrt{2}} \right), \quad |0\rangle: H(\theta, \phi) = (\pi/2, 0), \\ |1\rangle: H(\theta, \phi) &= (\pi/2, \pi) \end{aligned}$$

$$P(\theta)(\alpha, \beta) = (\alpha, e^{i\delta}\beta), \quad P(\theta_{input})(\theta, \phi) = (\theta, \phi + \theta_{input})$$

c.

$$\begin{aligned} H|0\rangle &= \left( \frac{\alpha + \beta}{\sqrt{2}}, \frac{\alpha - \beta}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \end{aligned}$$

d.

$$H \left( \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right) = \frac{1}{2} H|0\rangle + \frac{1}{2} H|1\rangle = \frac{1}{2} (|0\rangle + |1\rangle) + \frac{1}{2} (|0\rangle - |1\rangle) = |0\rangle$$

The  $|1\rangle$  cancel each other out (destructive interference) and we are left back with  $|0\rangle$  meaning the inversion of an hgate is another hgate.

- e. The final  $|0\rangle$  state is a determined basis state as there is only the possibility of it being measured as a  $|0\rangle$  so it cannot be a superposition.

4)

a.

$$\begin{aligned} (H \otimes H)|00\rangle &= H|0\rangle \otimes H|0\rangle \\ \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) |00\rangle &= H|0\rangle \otimes H|0\rangle \\ \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \otimes \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \end{aligned}$$

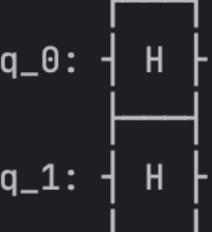
$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 1 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

When combining a sequence of gates to a sequence of states whether applying individual states to gates or all at once, either results in the same outcome. This is a property of tensor products that extends to all linear operators which is why it works with  $n$  qubits.

- b. If we have multiple single qubit matrices their tensor product becomes a new matrix that can act on all states without interacting with each other. This allows us to apply operations simultaneously as if it were a multiple qubit system even though it is comprised of a combination of single qubits.

**Circuit for  $H \otimes H$  on  $|00\rangle$ :**



Initial state:  $|00\rangle$

Final state after  $H \otimes H$ :

```

 $|00\rangle: 0.500000+0.000000j$ 
 $|01\rangle: 0.500000+0.000000j$ 
 $|10\rangle: 0.500000+0.000000j$ 
 $|11\rangle: 0.500000+0.000000j$ 

```

c.

```
|00>: 0.500000  
|01>: 0.500000  
|10>: 0.500000  
|11>: 0.500000
```

Verification:

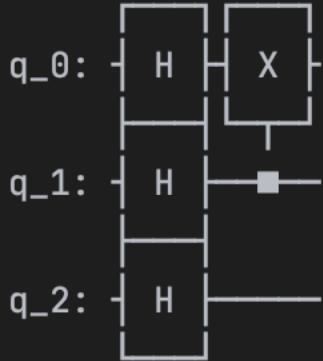
```
Qiskit result: [0.5+0.j 0.5+0.j 0.5+0.j 0.5+0.j]  
Theoretical: [0.5 0.5 0.5 0.5]  
Error L2 norm: 0.0000000000
```

Alternative verification -  $(H|0\rangle) \otimes (H|0\rangle)$ :

```
Tensor product result: [0.5 0.5 0.5 0.5]  
Matches Qiskit: True
```

```
==== CASE 2: H on all qubits, then CNOT(1→0) ===
```

Circuit for Case 2:



Final state amplitudes:

```
|000>: 0.353553+0.000000j
|001>: 0.353553+0.000000j
|010>: 0.353553+0.000000j
|011>: 0.353553+0.000000j
|100>: 0.353553+0.000000j
|101>: 0.353553+0.000000j
|110>: 0.353553+0.000000j
|111>: 0.353553+0.000000j
```

After  $H \otimes H \otimes H |000\rangle =$  equal superposition of all 8 states

$CNOT(0 \rightarrow 1)$  action:

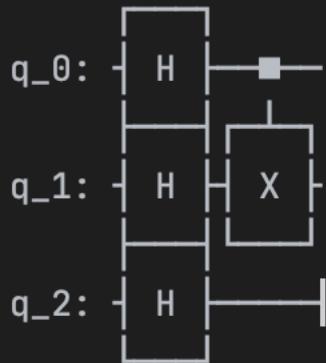
```
|000> → |000> (control = 0, no change)
|001> → |001> (control = 0, no change)
|010> → |010> (control = 0, no change)
|011> → |011> (control = 0, no change)
|100> → |110> (control = 1, flip)
|101> → |111> (control = 1, flip)
|110> → |100> (control = 1, flip)
|111> → |101> (control = 1, flip)
```

Expected amplitudes: all  $1/\sqrt{8} = 0.353553$

==== PROBLEM 4(d) ====

==== CASE 1: H on all qubits, then CNOT(0→1) ====

Circuit for Case 1:



Final state amplitudes:

$|000\rangle: 0.353553+0.000000j$   
 $|001\rangle: 0.353553+0.000000j$   
 $|010\rangle: 0.353553+0.000000j$   
 $|011\rangle: 0.353553+0.000000j$   
 $|100\rangle: 0.353553+0.000000j$   
 $|101\rangle: 0.353553+0.000000j$   
 $|110\rangle: 0.353553+0.000000j$   
 $|111\rangle: 0.353553+0.000000j$

$H \otimes H \otimes H|000\rangle = \text{equal superposition of all 8 states}$   
CNOT(1 → 0) action (control = 1, target = 0):

$|000\rangle \rightarrow |000\rangle$  (control = 0, no change)  
 $|001\rangle \rightarrow |001\rangle$  (control = 0, no change)  
 $|010\rangle \rightarrow |110\rangle$  (control = 1, flip)  
 $|011\rangle \rightarrow |111\rangle$  (control = 1, flip)  
 $|100\rangle \rightarrow |100\rangle$  (control = 0, no change)  
 $|101\rangle \rightarrow |101\rangle$  (control = 0, no change)  
 $|110\rangle \rightarrow |010\rangle$  (control = 1, flip)  
 $|111\rangle \rightarrow |011\rangle$  (control = 1, flip)

Expected amplitudes: all  $1/\sqrt{8} = 0.353553$

5)

a. Half adder  $|110\rangle$

$$\begin{aligned}q_0 &= \text{input } A = 1 \\q_1 &= \text{input } B = 1 \\q_2 &= \text{Carry}(0) = 0\end{aligned}$$

$$\begin{aligned}\text{SUM} &= A \text{ XOR } B = 1 \text{ XOR } 1 = 0 \\ \text{CARRY} &= A \text{ AND } B = 1 \text{ AND } 1 = 1 \\ q_0 &\rightarrow 1 \\ q_1 &\rightarrow 0 \\ q_2 &\rightarrow 1\end{aligned}$$

Full adder  $|1100\rangle$

$$\begin{aligned}q_0 &= \text{input } A = 1 \\q_1 &= \text{input } B = 1 \\q_3 &= \text{input } C = 0 \\q_2 &= \text{Carry} = 0\end{aligned}$$

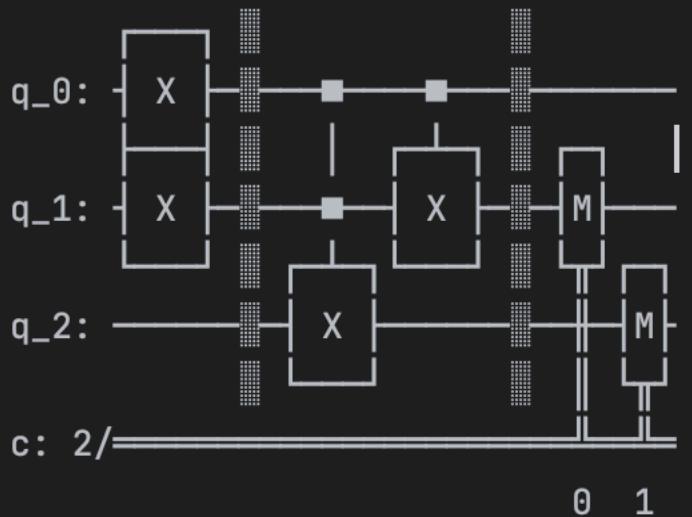
$$\begin{aligned}\text{SUM} &= A \text{ XOR } B \text{ XOR } C = 1 \text{ XOR } 1 \text{ XOR } 0 = 0 \\ \text{CARRY} &= (A \text{ AND } B) \text{ OR } (C \text{ AND } (A \text{ XOR } B)) = (1 \text{ AND } 1) \text{ OR } (0 \text{ AND } (1 \text{ XOR } 1)) = 1 \text{ OR } 0 \\ &= 1\end{aligned}$$

$$\begin{aligned}q_0 &\rightarrow 1 \\q_1 &\rightarrow 1 \\q_2 &\rightarrow 0 \\q_3 &\rightarrow 1\end{aligned}$$

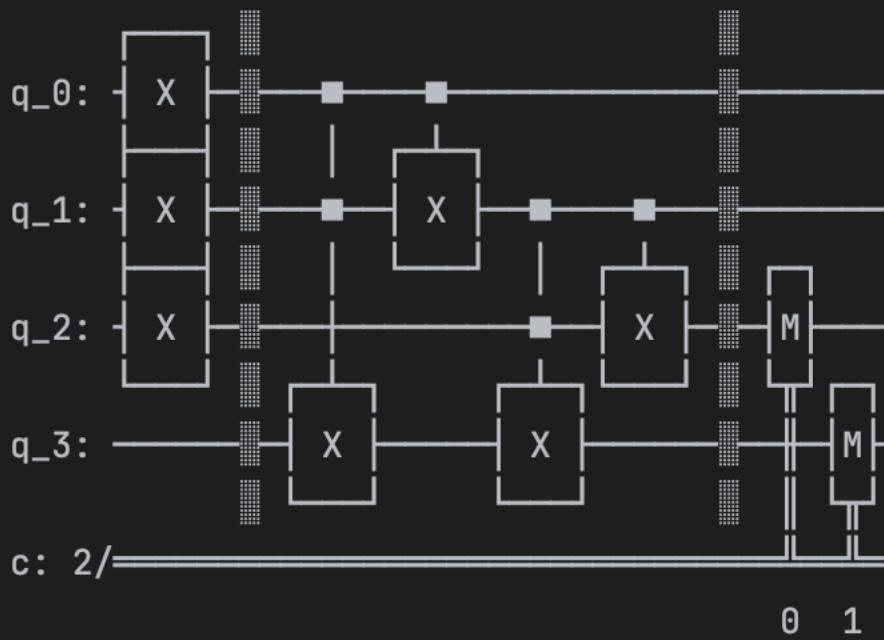
b - d.

==== Problem 5: Quantum Adders with Aer Simulator ====

--- Example Half Adder Circuit (for A=1, B=1) ---



--- Example Full Adder Circuit (for A=1, B=1, C\_in=1)



b.

**Half Adder:**

```
Input A=0, B=0 -> Measured (Carry,Sum): '00'  
Input A=0, B=1 -> Measured (Carry,Sum): '01'  
Input A=1, B=0 -> Measured (Carry,Sum): '01'  
Input A=1, B=1 -> Measured (Carry,Sum): '10'
```

c.

**Half Adder:**

```
Input A=0, B=0 -> Measured (Carry,Sum): '00'  
Input A=0, B=1 -> Measured (Carry,Sum): '01'  
Input A=1, B=0 -> Measured (Carry,Sum): '01'  
Input A=1, B=1 -> Measured (Carry,Sum): '10'
```

**Full Adder (with C\_in=0):**

```
Input A=0, B=0, C_in=0 -> Measured (Carry,Sum): '00'  
Input A=0, B=1, C_in=0 -> Measured (Carry,Sum): '01'  
Input A=1, B=0, C_in=0 -> Measured (Carry,Sum): '01'  
Input A=1, B=1, C_in=0 -> Measured (Carry,Sum): '10'
```

d.

**Half Adder:**

```
Input A=0, B=0 -> Measured (Carry,Sum): '00'  
Input A=0, B=1 -> Measured (Carry,Sum): '01'  
Input A=1, B=0 -> Measured (Carry,Sum): '01'  
Input A=1, B=1 -> Measured (Carry,Sum): '10'
```

**Full Adder (with C\_in=0):**

```
Input A=0, B=0, C_in=0 -> Measured (Carry,Sum): '00'  
Input A=0, B=1, C_in=0 -> Measured (Carry,Sum): '01'  
Input A=1, B=0, C_in=0 -> Measured (Carry,Sum): '01'  
Input A=1, B=1, C_in=0 -> Measured (Carry,Sum): '10'
```

The simulation results perfectly match classical binary addition. However real hardware would exhibit differences in outcomes due to noise.