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CSCI 739 HW2

1)

a.

$$U^{\dagger}U = I, \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \qquad H^{\dagger} = H^{T} = H$$

$$H^{\dagger}H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$$

$$\therefore H\ is\ unitary$$

b.

$$U^{\dagger}U = I$$
$$U^{\dagger}UU^{-1} = IU^{-1}$$
$$U^{\dagger}I = IU^{-1}$$
$$U^{\dagger} = U^{-1}$$

c.

$$H^{1}|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H^{2}|0\rangle = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}|0\rangle = I|0\rangle = |0\rangle$$

$$H^{2n} = I^{n} = I$$

$$H^{3} = H^{2}H = IH = H$$

For any even number of H gates applied to the ground state will not change.
For any odd number of H gates applied to the ground state will result in the plus basis state superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

d.

$$\mathrm{U}(\epsilon) = He^{i\epsilon\sigma_z}$$
$$U(0) = He^{i0\sigma_z} = He^{0} = H$$

e.

$$U(\epsilon)^{n} = He^{i\epsilon\sigma_z{}^{\mathrm{n}}} = He^{i\epsilon\sigma_z n}, \qquad |\psi_0\rangle = |0\rangle$$

$$|\psi_1\rangle = U(\epsilon)|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\epsilon} & e^{-i\epsilon}\\ e^{i\epsilon} & -e^{-i\epsilon}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\epsilon}\\ e^{i\epsilon}\end{pmatrix} = \frac{e^{i\epsilon}}{2}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{e^{i\epsilon}}{2}(|0\rangle + |1\rangle)$$

$$|\psi_2\rangle = U(\epsilon)|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\epsilon} & e^{-i\epsilon} \\ e^{i\epsilon} & -e^{-i\epsilon} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\epsilon} \\ e^{i\epsilon} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{2i\epsilon}+1 \\ e^{2i\epsilon}-1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{i\epsilon}\left(e^{i\epsilon}+e^{-i\epsilon}\right) \\ e^{i\epsilon}\left(e^{i\epsilon}-e^{-i\epsilon}\right) \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} (\cos(\epsilon)+i\sin(\epsilon))(\cos(\epsilon)+i\sin(\epsilon)+\cos(-\epsilon)+i\sin(-\epsilon) \\ (\cos(\epsilon)+i\sin(\epsilon))(\cos(\epsilon)+i\sin(\epsilon)-(\cos(-\epsilon)+i\sin(-\epsilon)) \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} (\cos(\epsilon)+i\sin(\epsilon))(\cos(\epsilon)+i\sin(\epsilon)+(\cos(\epsilon)-i\sin(\epsilon)) \\ (\cos(\epsilon)+i\sin(\epsilon))(\cos(\epsilon)+i\sin(\epsilon)-(\cos(\epsilon)-i\sin(\epsilon)) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{i\epsilon}(2\cos(\epsilon)) \\ e^{i\epsilon}(i2\sin(\epsilon)) \end{pmatrix} =$$

$$\frac{e^{i\epsilon}}{2}\begin{pmatrix} 2\cos(\epsilon) \\ i2\sin(\epsilon) \end{pmatrix} = e^{i\epsilon}\begin{pmatrix} \cos\epsilon \\ i\sin\epsilon \end{pmatrix} = e^{i\epsilon}(\cos\epsilon\,|0\rangle + i\sin\epsilon\ \ |1\rangle)$$

$$|\psi_{n+1}\rangle = U(\epsilon)|\psi_n\rangle$$

$$\alpha_{n+1} = \frac{1}{\sqrt{2}}(e^{i\epsilon}\alpha_n + e^{-i\epsilon}\beta_n)$$

$$\beta_{n+1} = \frac{1}{\sqrt{2}}(e^{i\epsilon}\alpha_n - e^{-i\epsilon}\beta_n)$$

f. When the error is 0, we can use an ideal H gate. This results in no change to the ground state when an even number of gates are applied. When an odd number of gates are applied, it results in the superposition which has ½ probability of both ground and excited states.

   For the noisy case, when a gate is only applied once, there is no difference than from ideal. However, once it exceeds that application with 2 or more, the probabilities begin to differ from the ideal. Where $P(0) = \cos^2\epsilon$ for the n = 2 noisy gates instead of $p(0) = 1$ and $P(1) = \sin^2\epsilon$ instead of $P(1) = 0$. Beyond 2 noisy gates this error builds upon itself whereas in the ideal case any pairs of H gates will cancel each other out.

2)

a. S

$$\mathrm{p}_l \approx \mathrm{A}\left(\frac{\mathrm{p}}{\mathrm{p}_{th}}\right)^{\frac{\mathrm{d}+1}{2}}$$

$$as\ d \to \infty$$

$when\ p < p_{th},$

$p_{th}\ grows\ faster\ than\ p$

$\therefore\ the\ logical\ error\ rate\ cannot\ be\ made\ arbitrarily\ small$

$when\ p > p_{th},$

$p\ grows\ faster\ than\ p_{th}$

$\therefore\ logical\ error\ rate\ can\ be\ made\ arbitrarily\ small\ by\ increasing\ d$

b.

$$p = 10^{-3}, \qquad p_{th} = 10^{-2}$$

$$\mathrm{d_{QSVM}} = \frac{2\log_{10} p_l}{\log_{10}\frac{\mathrm{p}}{\mathrm{p}_{th}}} - 1 = \frac{2\log_{10} 10^{-9}}{\log_{10}\frac{10^{-3}}{10^{-2}}} - 1 = 17, \qquad N_Q \approx 2d^2 = 578$$

$$d_{QBM} = \frac{2\log_{10} 10^{-10}}{\log_{10}\frac{10^{-3}}{10^{-2}}} - 1 = 19, \qquad N_Q \approx 722$$

$$d_{QNN} = \frac{2\log_{10} 10^{-11}}{\log_{10}\frac{10^{-3}}{10^{-2}}} - 1 = 21, \qquad N_Q \approx 882$$

$$d_{VQA} = \frac{2\log_{10} 10^{-12}}{\log_{10}\frac{10^{-3}}{10^{-2}}} - 1 = 23, \qquad N_Q \approx 1058$$

$$p = 10^{-3}\ \&\ p_{th} = 10^{-3} \therefore p_l = 1\ and\ is\ always\ wrong$$

$$p = 10^{-4}, \qquad p_{th} = 10^{-2}$$

$$d_{QVSM} = \frac{2\log_{10} 10^{-9}}{\log_{10}\frac{10^{-4}}{10^{-2}}} - 1 = 8, \qquad N_Q \approx 128$$

$$d_{QBM} = \frac{2\log_{10} 10^{-10}}{\log_{10}\frac{10^{-4}}{10^{-2}}} - 1 = 9, \qquad N_Q \approx 162$$

$$d_{QNN} = \frac{2\log_{10} 10^{-11}}{\log_{10}\frac{10^{-4}}{10^{-2}}} - 1 = 10, \qquad N_Q \approx 200$$

$$d_{VQA} = \frac{2\log_{10} 10^{-12}}{\log_{10}\frac{10^{-4}}{10^{-2}}} - 1 = 11, \qquad N_Q \approx 242$$

$$p = 10^{-4}, \qquad p_{th} = 10^{-3}$$

$$d_{QVSM} = \frac{2\log_{10} 10^{-9}}{\log_{10}\frac{10^{-4}}{10^{-3}}} - 1 = 17, \qquad N_Q \approx 578$$

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$$d_{VQA} = \frac{2\log_{10} 10^{-12}}{\log_{10}\frac{10^{-4}}{10^{-3}}} - 1 = 23, \qquad N_Q \approx 1058$$

c. The current hardware supports 100 to 100 qubits which seems almost within our qubit estimate range of 122 to 1058. Meaning it is likely possible with current hardware. Near term the qubit count seems feasible, but the error rates seem unrealistic. From all I can find multi gate qubits error rates are too high at around half a percent, which is nowhere near the error rate necessary. Until those error rates drop, long and near-term feasibility doesn't look promising.

d. It's most likely that QSVMs will achieve fault tolerant feasibility first because the targe Pl is so much lower than the rest. This results in a smaller d value and N_Q value which means less qubits are necessary to form a logical qubit/ least resource intensive of the group.

3)

a.

$$[3, 1, 4, 1] \rightarrow |0011\rangle \otimes |0001\rangle \otimes |0100\rangle \otimes |0001\rangle = |0011000101000001\rangle$$

$$4(n+1) = 4(3+1) = 16 \; qubits$$
$$1 \; time \; state$$

b.

$$\tilde{x} = \left[\frac{S_\pi}{9}, \frac{d_1}{9}, \frac{d_2}{9}, \frac{d_3}{9}\right] = \left[\frac{3}{9}, \frac{1}{9}, \frac{4}{9}, \frac{1}{9}\right] = \left[\frac{1}{3}, \frac{1}{9}, \frac{4}{9}, \frac{1}{9}\right]$$

$$||x||_2 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(\frac{1}{9}\right)^2} = \sqrt{\frac{1}{9} + \frac{1}{81} + \frac{16}{81} + \frac{1}{81}} = \sqrt{\frac{27}{81}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$\tilde{x} = \frac{x}{||x||_2} = \left[\frac{1}{3},\frac{1}{9},\frac{4}{9},\frac{1}{9}\right] \div \left(\frac{1}{\sqrt{3}}\right) = \left[\frac{1}{\sqrt{3}},\frac{1}{3\sqrt{3}},\frac{4}{3\sqrt{3}},\frac{1}{3\sqrt{3}}\right]$$

$$\left|\psi_{amp}^3\right\rangle = \sum_{j=0}^{3} \tilde{x}_j \; |j\rangle = \left(\frac{1}{\sqrt{3}}\right)|00\rangle + \left(\frac{1}{3\sqrt{3}}\right)|01\rangle + \left(\frac{4}{3\sqrt{3}}\right)|10\rangle + \left(\frac{1}{3\sqrt{3}}\right)|11\rangle$$

$$\tilde{x} = \left[\frac{1}{\sqrt{3}},\frac{1}{3\sqrt{3}},\frac{4}{3\sqrt{3}},\frac{1}{3\sqrt{3}}\right]$$

$$q = \log_2 4 = 2$$

c. Single Bit Full Value:

$$C_\pi = 4 \;\; (since\; C_\pi > \; v^3 = 3.141)$$

$$\theta_3 = 2 * arcsin\left(\frac{v^3}{C_\pi}\right) = 2 * arcsin(\frac{3.141}{4}) = 2 * arcsin(0.78525)$$
$$\approx 2 * 0.9033 \approx 1.8066\, radians$$

$$\left|\psi_\theta^3\right\rangle = R_y(\theta_3)|0\rangle = \left[cos\left(\frac{\theta_3}{2}\right), sin\left(\frac{\theta_3}{2}\right)\right] = \left[\sqrt{1 - \left(\frac{3.141}{4}\right)^2}, \frac{3.141}{4}\right]$$
$$\approx [0.619;\; 0.785]$$

Multi Bit single value: $\alpha = \pi$

$$\theta_0 = \alpha\,\frac{S_\pi}{9} = \pi\frac{3}{9} = \frac{\pi}{3} \approx 1.0472\, radians$$
$$\theta_1 = \alpha\frac{d_1}{9} = \frac{\pi}{9} \approx 0.3491\, radians$$
$$\theta_2 = \alpha\frac{d_2}{9} = \pi\frac{4}{9} \approx 1.3963\, radians$$
$$\theta_3 = \alpha\frac{d_3}{9} = \frac{\pi}{9} \approx 0.3491\, radians$$

$$\left|\psi_{multi}^3\right\rangle = \bigotimes_{i=0}^{3} R_y(\theta_i)|0\rangle$$
$$= R_y\left(\frac{\pi}{3}\right)|0\rangle \otimes R_y\left(\frac{\pi}{9}\right)|0\rangle \otimes R_y\left(\frac{4\pi}{9}\right)|0\rangle \otimes R_y\left(\frac{\pi}{9}\right)|0\rangle$$

With a smaller alpha we can handle noise better but there is a smaller dynamic range. The opposite is true for the larger alpha value. I chose alpha of pi because it is in balance between max alpha and the smallest.

d.

Encoding Method	Qubit count	Depth
Basis	$4(n+1) = 16$	1
Amplitude	$log_2(n+1) = 2$	exponential
Single w/ Full value	1	1
Multi w/ Single value	$n+1=4$	1

Basis is vulnerable to bit flips which causes a single digit gets corrupted. Whereas amplitude is affected by errors that change the entire coding. Single angles is very susceptible errors as one error causes the whole encoding to fail. Whereas multi bit angle encoding is more robust as errors only affect a single digit. This makes multibit angle encoding the most robust as it is susceptible to less frequent errors than basis and it's errors are localized unlike single bit angle encoding and amplitude encoding. The most efficient for large n values would be amplitude encoding as log n is smaller than linear.

4)

a. Basis encoding:

$$2 \rightarrow 0010, \quad 7 \rightarrow 0111, \quad 1 \rightarrow 0001, \quad 8 \rightarrow 1000$$

$$|\psi_{basis}^3\rangle = |BCD(2)\rangle \otimes |BCD(7)\rangle \otimes |BCD(1)\rangle \otimes |BCD(8)\rangle$$
$$= |0010\rangle \otimes |0111\rangle \otimes |0001\rangle \otimes |1000\rangle = |0010011100011000\rangle$$

Qubit required: $4(n+1) = 1$
Initialization pattern is using x gates on the 1 positions.

Amplitude encoding:

$$x = \left[\frac{S_e}{9}, \frac{d^1}{9}, \frac{d^2}{9}, \frac{d^3}{9}\right] = \left[\frac{2}{9}, \frac{7}{9}, \frac{1}{9}, \frac{8}{9}\right]$$

$$\|x\|_2 = \sqrt{\left(\frac{2}{9}\right)^2 + \left(\frac{7}{9}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{8}{9}\right)^2} = \sqrt{\frac{4}{81} + \frac{49}{81} + \frac{1}{81} + \frac{64}{81}} = \frac{\sqrt{118}}{9}$$

$$\tilde{x} = x/\|x\|_2 = \left[\frac{2}{9}, \frac{7}{9}, \frac{1}{9}, \frac{8}{9}\right] \div (\sqrt{118}/9)$$
$$= \left[\frac{2}{\sqrt{118}}, \frac{7}{\sqrt{118}}, \frac{1}{\sqrt{118}}, \frac{8}{\sqrt{118}}\right]$$

Qubits required: q = $\lceil \log_2(4) \rceil$ = 2 qubits

$$|\psi_{amp}^3\rangle = \sum_{j=0}^{3} \tilde{x}_j \, |j\rangle = \left(\frac{2}{\sqrt{118}}\right)|00\rangle + \left(\frac{7}{\sqrt{118}}\right)|01\rangle + \left(\frac{1}{\sqrt{118}}\right)|10\rangle + \left(\frac{8}{\sqrt{118}}\right)|11\rangle$$

Angle encoding single qubit:

$$C_e = 3 \ (since\ C_e > v^3 = 2.718)$$

$$\theta_3 = 2arcsin\left(\frac{v^3}{C_e}\right) = 2arcsin\left(\frac{2.718}{3}\right) = 2arcsin(0.90667) \approx 2 * 1.130 \approx 2.260\, radians$$

$$\left|\psi_\theta^3\right\rangle = R_y(\theta_3)|0\rangle = \left[cos\left(\frac{\theta_3}{2}\right), sin\left(\frac{\theta_3}{2}\right)\right] = \left[\sqrt{1 - \left(\frac{2.718}{3}\right)^2}, \frac{2.718}{3}\right] \approx [0.422;\ 0.906]$$

Angle encoding multi qubit: $\alpha = \pi$

Chose this alpha for the same reason as in Q3. It was halfway from balance between max and 0

$$\theta_0 = \alpha\frac{S_e}{9} = \pi\frac{2}{9} \approx 0.6981\, radians$$
$$\theta_1 = \alpha\frac{d_1}{9} = \pi\frac{7}{9} \approx 2.4435\, radians$$
$$\theta_2 = \alpha\frac{d_2}{9} = \pi\frac{1}{9} \approx 0.3491\, radians$$
$$\theta_3 = \alpha\frac{d_3}{9} = \pi\frac{8}{9} \approx 2.7925\, radians$$

$$\left|\psi_{multi}^3\right\rangle = \bigotimes_{i=0}^{3} R_y(\theta_i)|0\rangle$$
$$= R_y(0.6981)|0\rangle \otimes R_y(2.4435)|0\rangle \otimes R_y(0.3491)|0\rangle \otimes R_y(2.7925)|0\rangle$$

Precision vs resources:

$$0 < e - \sum_{k=0}^{m}\frac{1}{k!} < \frac{1}{m \times m!}$$

$$if\ m = 4: \frac{1}{4 \times 4!} \approx 0.01 > 0.0005$$

$$if\ m = 5: \frac{1}{5 \times 5!} \approx 0.001667 > 0.0005$$

$$if\ m = 6: \frac{1}{6 * 6!} \approx 0.000231 < 0.0005$$

$$\therefore the\ minimum\ required\ to\ guarentee\ n = 3\ decimal\ places\ is\ 6$$

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