Extreme Values in Financial Statistics

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par

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Mihi cura futuri
— Ovide, Métamorphoses, 13, 363

To my friends and loved ones…

Acknowledgements

TO BE FILLED

Lausanne, 14 Août 2015 K. M–H.

Preface

A preface is not mandatory. It would typically be written by some other person (eg your thesis director).
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Lausanne, 14 Août 2015 K. M–H.

Abstract

Key words:

Résumé

Mots clefs :

Contents

List of Figures

List of Tables

1 Introduction

1.1 A few words to set the scene

In real life, it is not uncommon to have at one's disposal data about a phenomenon occurring through time. It may be as simple as daily rainfall data in a cit for the past two years, or it could be the weekly opening prices of a stock for the past decade.

Most of the time, people would like to use the data at their disposal to make predictions to answer questions, from the prosaic ones such as 'Will it rain tomorrow ?' to more consequential ones such as 'Will I make a profit if I cling to my shares today and sell them only tomorrow ?'. Of course, those are only vaguely worded questions : it is impossible to answer them satisfactorily without knowing the context, the objectives etc. behind them.

Yet, what these questions have in common is that they focus on the normal 'behaviour' that is to be expected in the future. Depending on the specific issue that is considered, the 'average behaviour' may not be the most interesting thing. For instance, suppose that a government wants to build a network of dams[1]. The dams are meant to protect the country from future floods for the next one hundred years, therefore the question that needs to be answered is one of 'worst case event' : "Over the next century, how severe may be the worst flood ?".

Extreme events are the kind of events we will be interested in this master thesis project. Although Extreme Value Theory has applications in many fields[2], we will here apply it more specifically to financial data.

1.2 Formalising the settings

Let $(X_n)_{n\geq 0}$ be a sequence of independent identically distributed random variables with common cumulative distribution function F_X. The sequence of maxima is defined by M_0=X_0 and $\forall n \geq 1$, M_n=$\max_{0\leq i\leq n}(X_i)$. We would like to determine the limiting distribution of the

[1]As was done in The Netherlands beginning in the fifties
[2]including climate science, seismology, insurance etc.

sequence $(M_n)_{n\geq 0}$.[3] This is a matter that will keep us busy quite a long time but the first thing to do is to re-formulate it.

Indeed, let us do a quick and simple computation :

$$\begin{aligned} F_n(t) &= \Pr(\{M_n \leq t\}) \\ &= \Pr(\{\max_{0\leq i\leq n}(X_i) \leq t\}) \\ &= \Pr(\{X_1 \leq t\} \cap \cdots \cap \{X_n \leq t\}) \\ &= (F_X(t))^n \end{aligned} \tag{1.1}$$

Here we see that not much information will be drawn from this result by taking the limit n→ $+\infty$. The limiting distribution will be degenerate. Indeed, let us consider the upper end-point of F_X[4], z^+. Then,

$$\begin{aligned} &\forall z < z^+ \lim_{z\to\infty} F_n(z) = 0 \\ &\forall z \geq z^+ \lim_{z\to\infty} F_n(z) = 1 \end{aligned} \tag{1.2}$$

It turns out we cannot use the limiting distribution directly. A common approach[5] is to consider a sequence of the maxima, centred and normalised.

We will thus consider in all what follows the sequence defined by $(M_n^*)_{n\geq 0} = (\frac{M_n - b_n}{a_n})_{n\geq 0}$ where $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are a sequence of real numbers and positive real numbers respectively. Finding a result on whether such a sequence admits a limiting distributions, and the conditions under which the result holds, will be one of our goals.

[3] If we can determine the limiting distribution of the maxima from the data, then we will have a means to make predictions on the occurrence of future extreme events.

[4] that is the smallest z such that $F_X(z)$ be equal to one. For the Normal distribution, z will be $+\infty$, by contrast for a continuous Uniform Distribution $U([a,b])$ it will be b.

[5] adopted by the mathematicians that laid the grounds for EVT

2 Investigating results on the limiting distribution

2.1 Playing with the (original) sequence of maxima

Here, we will generate finite size sequences (N = 10000) of independent identically distributed random variables following respectively :

- a standard Normal Distribution $\mathcal{N}(0,1)$
- a Cauchy Distribution *Cauchy(0,1)*
- an Exponential Distribution *Exp(1)*

We will compute the sequence of maxima, neither centred nor normalised, and draw the scatter plot as well as the plot of the maxima M_n as a function of the time steps n. We will also draw the $\frac{1}{n}$-quantiles of the distributions (distributions of the sample, not of the maxima) as a function of the time steps n. This will lead us to make an interesting observation.

2.1.1 Sample following a Normal distribution

Computing the quantiles The Normal distribution is a particular case because, unlike in the cases of the Cauchy and the Exponential distribution, there is no explicit form to the cumulative distribution function. We will thus use a "well-known"[1] inequality, holding $\forall\, t > 0$:

$$(\frac{1}{t} - \frac{1}{t^3}) * \frac{\exp(-\frac{t^2}{2})}{\sqrt{2*\pi}} < 1 - \Phi(t) < \frac{1}{t} * \frac{\exp(-\frac{t^2}{2})}{\sqrt{2*\pi}} \tag{2.1}$$

[1] Many textbooks mention it, though it is not necessarily what springs to the mind when thinking about the properties of Gaussian RVs.

From there, it is easy to see that the following holds :

$$1 - \Phi(t) \sim_{t \to +\infty} \frac{1}{t} * \frac{\exp(-\frac{t^2}{2})}{\sqrt{2 * \pi}} \tag{2.2}$$

When n grows large, the $\frac{1}{n}$-quantile grows very large so it is valid to replace $1 - \Phi(t)$ by its equivalent in the equation satisfied by the quantiles :

$$\begin{aligned} & F_X(q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\ \iff \quad & \frac{1}{q_{\frac{1}{n}}} * \frac{\exp(-\frac{q_{\frac{1}{n}}^2}{2})}{\sqrt{2 * \pi}} = \frac{1}{n} \\ \iff \quad & \log(q_{\frac{1}{n}}) + \log(\exp(-\frac{q_{\frac{1}{n}}^2}{2})) + \log(\sqrt{2 * \pi}) = \log(n) \end{aligned} \tag{2.3}$$

This equation cannot be solved analytically, we will resolve it iteratively. Th starting point is $\log(n) = \frac{t_0^2}{2}$, which gives us $t_0 = \sqrt{(2 * \log(n))}$. If we then run the Newton-Raphson algorithm, we see that the corrections to t_0 from the next iterations are small enough that we can keep t_0 as solution.[2].

Figure 2.1 – Below, a realisation of the sequence of maxima for i.i.d. standard unit Gaussian RVs

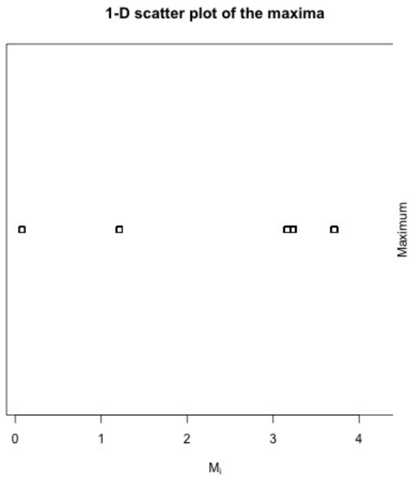

Figure 2.2 – Scatter Plot of the Maxima, n = 10000

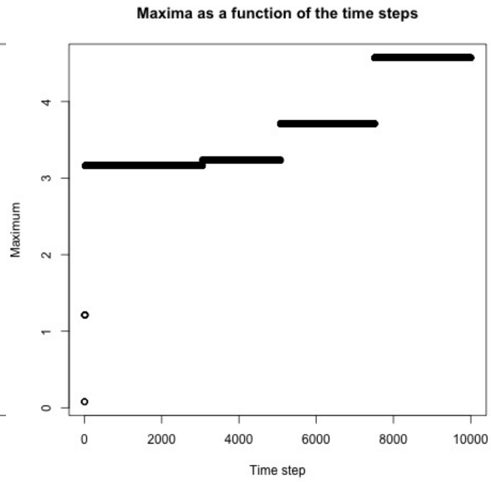

Figure 2.3 – Maxima against the time steps

[2] See the fourth of the figures below

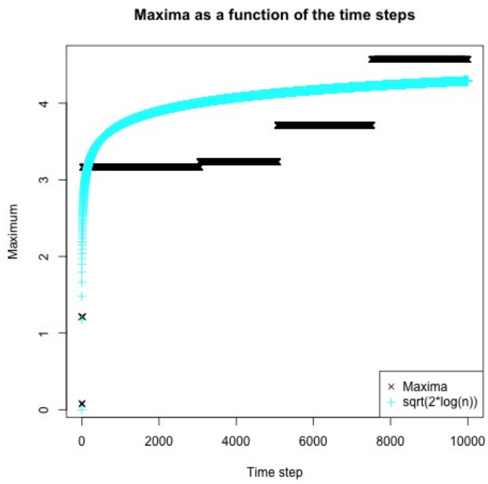

Figure 2.4 – Maxima against the time steps and function n→ $\sqrt{2*\log(n)}$

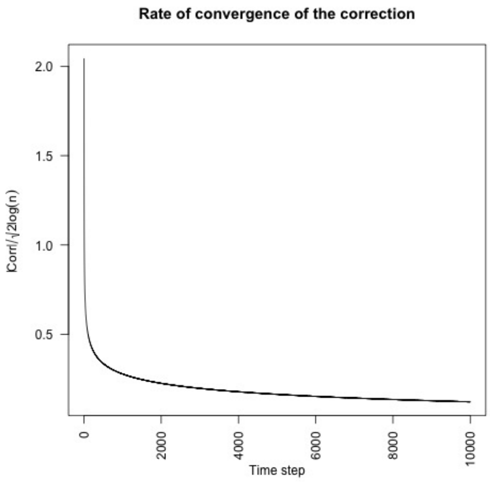

Figure 2.5 – The correction becomes negligible compared to the starting term as n grows large

2.1.2 Sample following a Cauchy distribution

Computing the quantiles Let $X_1, \cdots, X_n$ be i.i.d. RVs ~ *Cauchy(0,1)*. The distribution function is $F_X(t) = \frac{1}{\pi} * \arctan(x) - \frac{1}{2}$. The $\frac{1}{n}$-quantiles satisfy the equation :

$$
\begin{aligned}
& F_X(q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\
\iff \quad & \frac{\arctan(q_{\frac{1}{n}})}{\pi} + \frac{1}{2} = 1 - \frac{1}{n} \\
\iff \quad & \frac{\arctan(q_{\frac{1}{n}})}{\pi} = \frac{2-n}{n} \\
\iff \quad & q_{\frac{1}{n}} = \tan(\frac{\pi}{2} * \frac{2-n}{n})
\end{aligned}
\tag{2.4}
$$

Figure 2.6 – Below, a realisation of the sequence of maxima for i.i.d. *Cauchy(0,1)* RVs

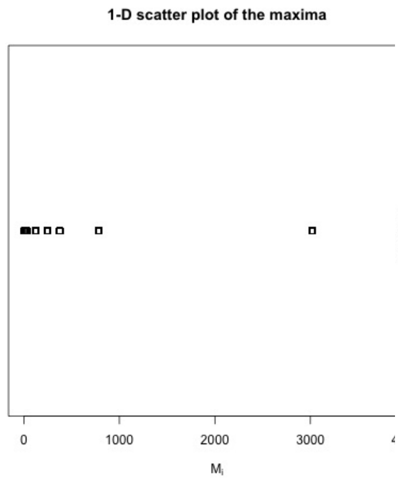

Figure 2.7 – Scatter Plot of the Maxima, n = 10000

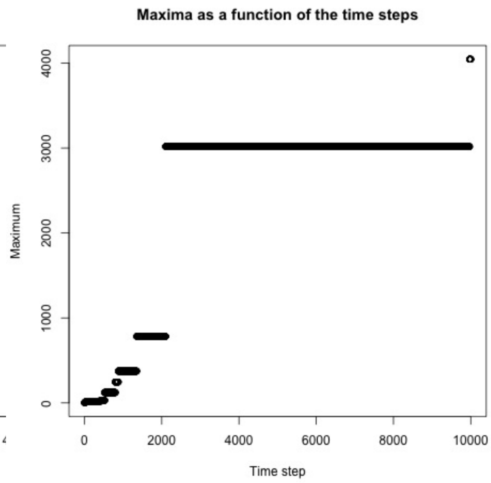

Figure 2.8 – Maxima against the time steps

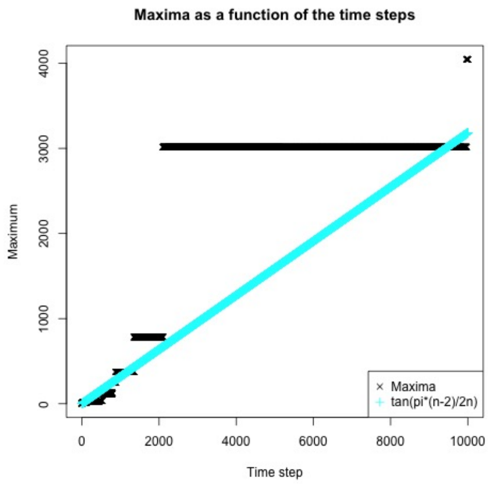

Figure 2.9 – Maxima against the time steps and function n→ $\tan(pi * \frac{2-n}{2*n})$

2.1.3 Sample following an Exponential Distribution

Computing the quantiles Let $X_1, \cdots, X_n$ be i.i.d. RVs ∼ *Exponential(λ)*. The distribution function is $F_X(t) = 1 - \exp(-\lambda * t)$. The $\frac{1}{n}$-quantiles satisfy the equation :

$$\begin{aligned} & F_X(q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\ \iff \quad & 1 - \exp(-\lambda * q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\ \iff \quad & q_{\frac{1}{n}} = \frac{1}{\lambda} * \log(n) \end{aligned} \tag{2.5}$$

Figure 2.10 – Below, a realisation of the sequence of maxima for i.i.d. *Exp(1)* RVs

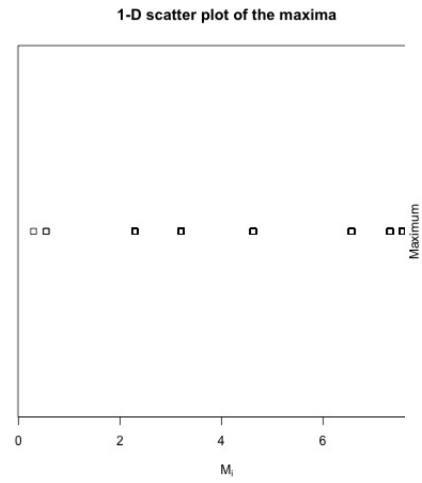

Figure 2.11 – Scatter Plot of the Maxima, n = 10000

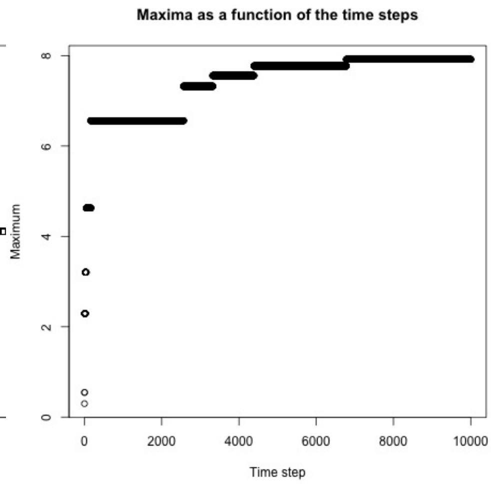

Figure 2.12 – Maxima against the time steps

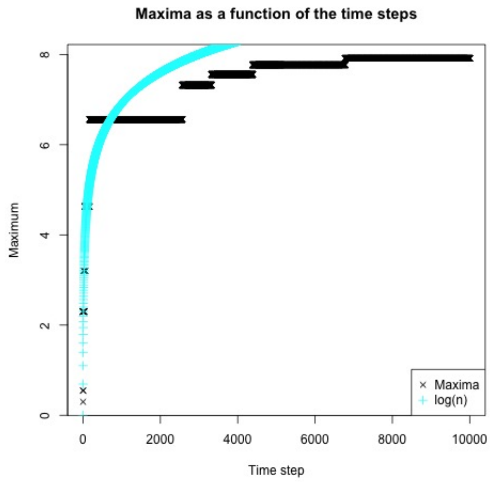

Figure 2.13 – Maxima against the time steps and function n→ $\log(n)$

2.1.4 Why does this work ?

(The underlying idea) Why have we made a link between the $\frac{1}{n}$-quantiles of the common distribution of the X_i and the sequence of the M_n ? Actually, the $\frac{1}{n}$-quantiles are a good proxy for the M_n.

The idea behind this is as follows, that the maxima will get closer to the upper end-point of the distribution of the X_i, F_{X_i}. That is also the case for the $\frac{1}{n}$-quantiles of F_{X_i}. Of course, it is only an intuition ! In the next section, we will show that it turns out to be valid (and in what precise sense it does so).

A more rigorous approach

2.2 Making appear extreme value distributions

The Extreme Value Theorem The following result is fundamental in the field of Extreme Value Theory[3], it will be found in any textbook under the name of Fisher-Tippett theorem or Extreme Value theorem.

> Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables and $(M_n)_{n\geq 1}$ be the sequence of the maxima as previously defined. If there exist a sequence of nonnegative real numbers $(a_n)_{n\geq 1}$ and a sequence of real numbers $(b_n)_{n\geq 1}$ such that $(\frac{M_n - b_n}{a_n})_{n\geq 1}$ converges in distribution to a non-degenerate distribution function F, then F belongs to the Extreme Value distribution family.

Extreme Value distribution The Extreme Value distribution sub-divides into three families of distribution :

- the Fréchet distribution family, **Fréchet(a,b,α)**
  G(z) = $\exp(-(\frac{z-b}{a})^{-\alpha})\mathbb{1}(\{z > b\})$, a > 0, b $\in \mathbb{R}$, $\alpha > 0$.
- the Gumbel distribution family, **Gumbel(a,b)**
  G(z) = $\exp(-\exp(-(\frac{z-b}{a})))$, $-\infty < z < +\infty$, a > 0, b $\in \mathbb{R}$.
- the Weibull distribution family, **Weibull(a,b,α)**
  G(z) = $\exp(-(-\frac{z-b}{a})^{-\alpha})\mathbb{1}(\{z > b\}) + \mathbb{1}(\{z \leq b\})$, a > 0, b $\in \mathbb{R}$, $\alpha > 0$.

Those three families can be combined into a single family of models, the Generalized Extreme Value Distribution.
G(z) = $\exp(-(1+\xi(\frac{z-\mu}{\sigma}))^{-\frac{1}{\xi}})$[4]

2.2.1 Making it appear in our context

Fréchet distribution Let $X_1, \cdots, X_n$ be i.i.d. RVs ~ *Exponential(λ)*. Let us remember from the previous section the form of the $\frac{1}{n}$-quantile $q_n = \frac{1}{\lambda}\log(n)$. Let us evaluate F_{M_n} in $q_{\frac{1}{n}} - t$:

[3] it is the analogue in terms of extremes of the Central Limit Theorem in classical probability theory.
[4] However, the domain of the distribution depends on the parameters μ, σ, ξ : { z / $1+\xi\frac{z-\mu}{\sigma} > 0$}, with μ, ξ in $\mathbb{R}$ and $\sigma > 0$

$$
\begin{aligned}
F_{M_n}(q_{\frac{1}{n}} - t) &= (1 - \exp(-\lambda(q_{\frac{1}{n}} - t)))^n \\
&= (1 - \exp(\lambda t)\exp(-\lambda q_{\frac{1}{n}}))^n \\
&= (1 - \frac{\exp(\lambda t)}{n})^n \\
&\longrightarrow_{n\to+\infty} \exp(-\exp(\lambda t))
\end{aligned}
\tag{2.6}
$$

Gumbel distribution

Weibull distribution

3 Looking into real-world data

3.1 Five real-world stocks and their evolution over 15 years

We have chosen to study five stocks listed on the Paris Stock Exchange : BNP Paribas, Carrefour, LVMH, Sanofi and Total stocks.[1] The evolution of the stock prices has been studied over the past 15 years, on a weekly basis. We first draw the data itself, then the net returns and the gross log returns on the stocks[2]

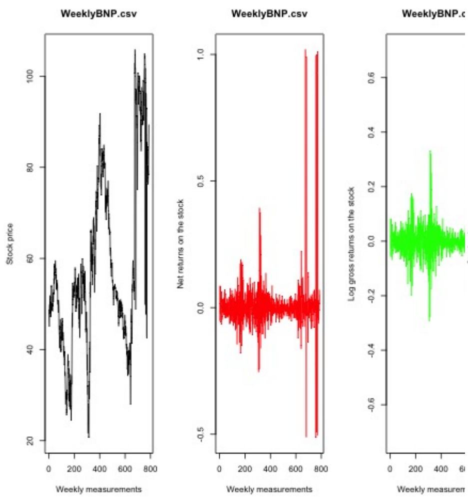

Figure 3.1 – 15 years of weekly BNP Stock Price Data

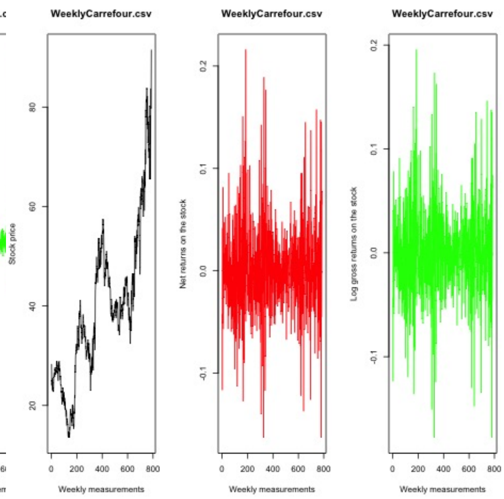

Figure 3.2 – 15 years of weekly Carrefour Stock Price Data

[1]We have chosen companies positioned on different domains, otherwise, information from different stock might more easily be redundant.
[2]Both quantities are widely used in Finance.

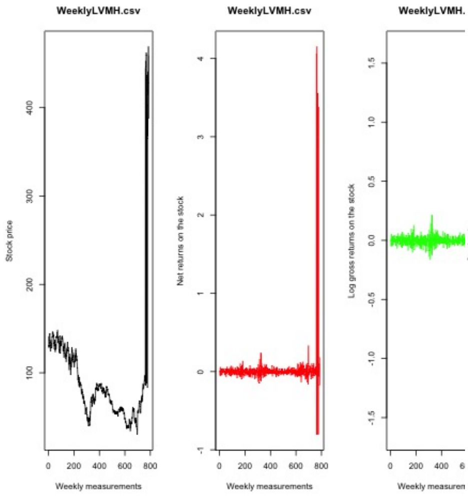

Figure 3.3 – 15 years of weekly LVMH Stock Price Data

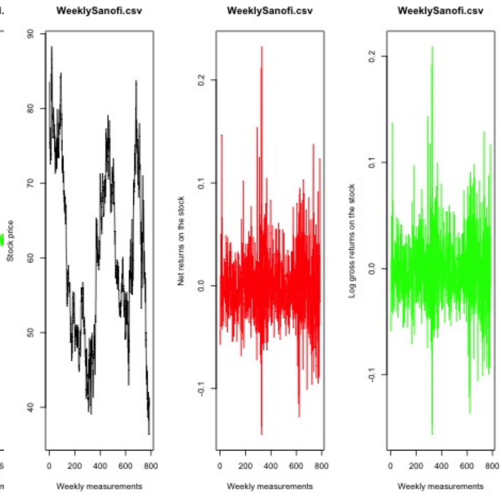

Figure 3.4 – 15 years of weekly Sanofi Stock Price Data

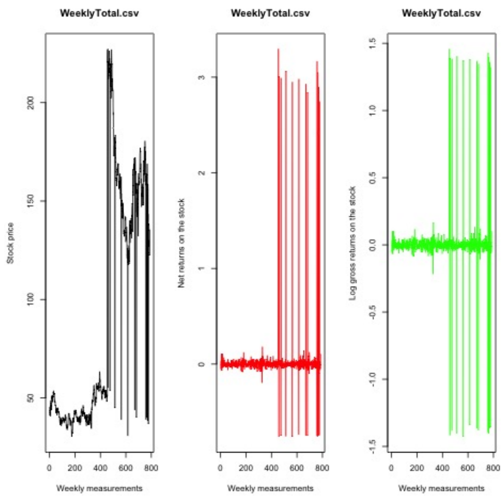

Figure 3.5 – 15 years of weekly Total Stock Price Data

Let X_t be the price of a stock at time t, the gross return at time t + 1 is defined as the ratio $\frac{X_{t+1}}{X_t}$, the net return at time t + 1 is defined as the ratio $r_t = \frac{X_{t+1}-X_t}{X_t}$ and the log gross return at time t +1 is defined as the log of the gross return at time t + 1 i.e. $R_t = \log(\frac{X_{t+1}}{X_t})$. The latter two quantities are of particular interest in Finance.
Let us observe that the relationship between R_t, X_t and X_{t+1} can be rewritten as $X_{t+1} = \exp(R_{t+1}) * X_t$. An approximation would be to take $X_{t+1} = (1 + R_{t+1}) * X_t$ by taking the expansion of the exponential, cut at order 1. Below are the plots of the quantities $\exp(R_t)$ and $1 + R_t$ for the five stocks previously considered. As we can see from the value of the residuals, this is in practice a very good approximation !

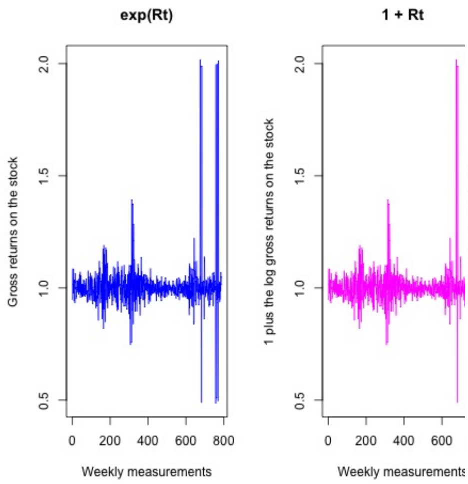

Figure 3.6 – $\exp(R_t)$ and $1 + R_t$ for BNP Stock Price Data, residual : 3.22E-15

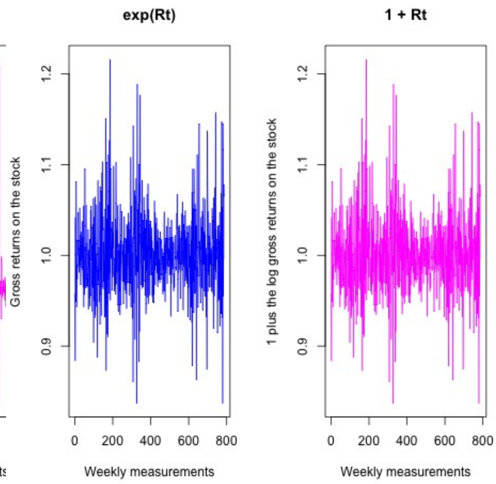

Figure 3.7 – $\exp(R_t)$ and $1 + R_t$ for Carrefour Stock Price Data, residual : 1.44E-15

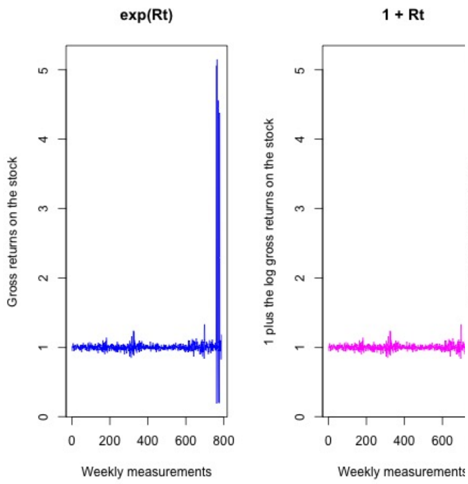

Figure 3.8 – $\exp(R_t)$ and $1 + R_t$ for LVMH Stock Price Data, residual : 4.75E-15

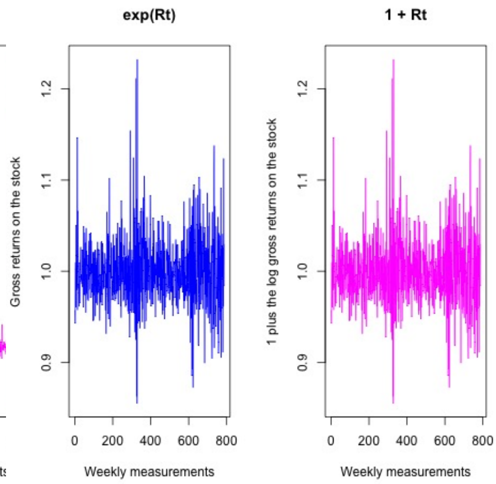

Figure 3.9 – $\exp(R_t)$ and $1 + R_t$ for Sanofi Stock Price Data, residual : 1.55E-15

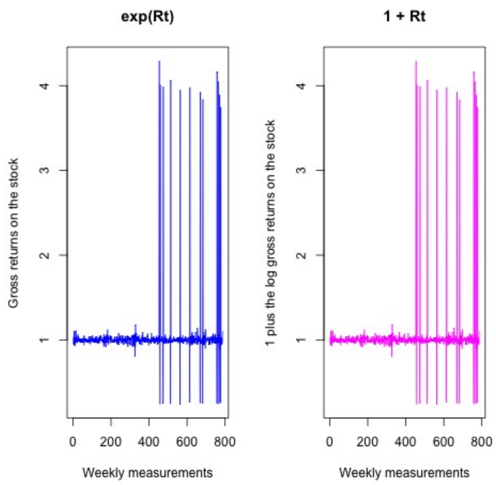

Figure 3.10 – $\exp(R_t)$ and $1 + R_t$ for Total Stock Price Data, residual : 3.86E-15

3.2 A détour around Stochastic Calculus

3.2.1 The Black-Scholes Stochastic Differential Equation

Presentation It is customary to model the evolution of stock prices by a stochastic process $(S_t)_{t\geq 0}$ satisfying the Black-Scholes stochastic differential equation, which reads as follows :

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \tag{3.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$. Let us observe that the stochastic differential equation above is the Black-Scholes SDE with time independent coefficients : both the drift $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$ are constant.[3] An initial condition must be specified : $S_0 = s_0 > 0$.

Resolution - existence of a solution to the Black-Scholes SDE Let us consider the following generic stochastic differential equation :
$$\begin{cases} \mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}B_t \\ X_0 = x_0 \end{cases}$$
where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$, $x_0 \in \mathbb{R}$, $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz functions. Then, by a theorem from Stochastic Calculus[4], we know that there exists a unique process that is continuous and adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Here, f and g are respectively the functions $x \longrightarrow \mu x$ and $x \longrightarrow \sigma t$. These functions are Lipschitz functions, therefore we know the SDE admits a solution. And fortunately, in the case of the Black-Scholes equation, the solution can be made explicit[5]).

Resolution - step 1 Let us set $S_t = \phi_t Z_t$ where ϕ_t is the (deterministic) solution to the ordinary differential equation :
$$\begin{cases} \mathrm{d}\phi_t = \mu\phi_t \mathrm{d}t \\ \phi_0 = 1 \end{cases}$$
Solving this ODE is elementary and yields the solution $\phi_t = \exp(\mu t)$. Now, let us differentiate X_t under its form as a product $S_t = \phi_t Z_t$.

$$\begin{aligned} \mathrm{d}(S_t) &= \mathrm{d}(\phi_t Z_t) \\ &= \phi_t \mathrm{d}Z_t + Z_t \mathrm{d}\phi_t + \mathrm{d} < \phi, Z >_t \\ &= \phi_t \mathrm{d}Z_t + \mu\phi_t Z_t \mathrm{d}t \\ &= \phi_t \mathrm{d}Z_t + \mu S_t \mathrm{d}t \end{aligned} \tag{3.2}$$

where the infinitesimal quadratic covariation between ϕ and Z is zero as ϕ has bounded variations. If we compare the last right-hand term of the series of equations just above to the original Black-Scholes SDE, we see that $\sigma S_t \mathrm{d}B_t = \phi_t \mathrm{d}Z_t$. Hence, $\mathrm{d}Z_t = \sigma Z_t \mathrm{d}B_t$. Here, we must be very careful as this differential equation does not integrate as it would in the settings of 'usual' calculus, in particular, integrating it to $\log(Z_t) - \log(Z_0) = \sigma(B_t - B_0)$ is <u>totally wrong !</u>

[3] In the next subsection, we will deal with the Black-Scholes SDE with time dependent coefficients.

[4] readers wanting to get ahold of an excellent course on Stochastic Calculus are advised to refer to Dr. Lévêque's course. It can be found at http://ipg.epfl.ch/~leveque/

[5] This is not always the case : solving SDEs in general is not that simple, and finding an explicit solution is not guaranteed in the general case, even though we know one exists.

Resolution - step 2 Now, let us set $Y_t = \log(Z_t)$. Using Ito-Doeblin's formula, we get :
$d(\log(Z_t)) = \frac{1}{Z_t}\mathrm{d}Z_t + \frac{1}{2}(\frac{-1}{Z_t^2})\mathrm{d} < Z >_t$ ($\star$)
→ <u>How to compute d $< Z >_t$?</u>

$$
\begin{aligned}
& S_t = \phi_t Z_t \\
\implies & Z_t = \frac{S_t}{\phi_t} \\
\implies & \mathrm{d}Z_t = \frac{1}{\phi_t}\mathrm{d}S_t + S_t d(\frac{1}{\phi_t}) + \mathrm{d} < \frac{1}{\phi}, S >_t \\
& = \frac{1}{\phi_t}\mathrm{d}S_t - \frac{S_t}{\phi_t^2} d(\phi_t) \\
& = \frac{1}{\phi_t}\mathrm{d}S_t - \frac{\mu S_t}{\phi_t}\mathrm{d}t \\
& = \frac{1}{\phi_t}(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t) - \frac{\mu S_t}{\phi_t}\mathrm{d}t \\
& = \frac{\sigma S_t}{\phi_t}\mathrm{d}B_t
\end{aligned}
\tag{3.3}
$$

where the infinitesimal quadratic covariation between $\frac{1}{\phi}$ and Z is zero as $\frac{1}{\phi}$ has bounded variations, in the third equality from the top.
Hence, using the Isometry formula, we have that :

$$
\begin{aligned}
& < Z >_t = \int_0^t \frac{\sigma^2 S_s^2}{\phi_s^2}\,\mathrm{d}s \\
& = \int_0^t \sigma^2 Z_s^2\,\mathrm{d}s \\
\implies & \mathrm{d} < Z >_t = \sigma^2 Z_t^2 \mathrm{d}t
\end{aligned}
$$

Back to ($\star$), we now have :

$$
\begin{aligned}
& \mathrm{d}(\log(Z_t)) = \frac{1}{Z_t}\mathrm{d}Z_t + \frac{1}{2}(\frac{-1}{Z_t^2})\sigma^2 Z_t^2 \mathrm{d}t \\
\iff & \frac{1}{Z_t}\mathrm{d}Z_t = d(\log(Z_t)) + \frac{\sigma^2}{2}\mathrm{d}t
\end{aligned}
\tag{3.4}
$$

Resolution - step 3 Combining the previous equation with $\mathrm{d}Z_t = \sigma Z_t \mathrm{d}B_t$, we get :

$$\begin{aligned}
\sigma \mathrm{d}B_t &= \mathrm{d}(log(Z_t)) + \frac{\sigma^2}{2}\mathrm{d}t \\
\Longrightarrow \log(Z_t) - \log(Z_0) &= -\frac{\sigma^2}{2}(t-0) + \sigma(B_t - B_0) \\
\Longrightarrow \log(Z_t) - \log(\frac{S_0}{\phi_0}) &= -\frac{\sigma^2}{2}t + \sigma B_t \\
\Longrightarrow \log(Z_t) &= \log(s_0) - \frac{\sigma^2}{2}t + \sigma B_t \\
\Longrightarrow Z_t &= s_0 \exp(-\frac{\sigma^2}{2}t + \sigma B_t)
\end{aligned} \tag{3.5}$$

Finally, by remembering that $S_t = \phi_t Z_t = \exp(\mu t) Z_t$, we get :
$\forall t \geq 0, S_t = s_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$
The stochastic process $(S_t)_{t\geq 0}$, made explicit above, that is solution to the Black-Scholes equation is generally called Geometric Brownian Motion in the literature.

3.2.2 Black-Scholes SDE with time-dependent coefficients

Presentation In the simple Black-Scholes SDE, the drift μ and the volatility σ were time-independent constants. Let us now consider a more general version of the Black-Scholes SDE :
$\begin{cases} \mathrm{d}S_t = \mu(t)S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}B_t \\ S_0 = s_0 > 0 \end{cases}$
where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$, μ,σ two continuous functions such that there exists $K_1 > 0$, $K_2 > 0$, such that $\forall t \geq 0$, $|\mu(t)| \leq K_1$, $K_2 \leq |\sigma(t)| \leq K_1$.

Resolution - existence of a solution to the generalized Black-Scholes SDE Let us consider the following generic stochastic differential equation :
$\begin{cases} \mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}B_t \\ X_0 = x_0 \end{cases}$
where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$, $x_0 \in \mathbb{R}$, $f, g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are jointly continuous in (t, x) and Lipschitz in x. Then, by a theorem from Stochastic Calculus, we know that there exists a unique solution $(X_t)_{t\geq 0}$ to the SDE. in the case of the generalized Black-Scholes SDE, the conditions are met and we can thus conclude that it admits a unique solution. The solution can be made explicit here too, fortunately !

Resolution Let us set $Y_t = \log(S_t)$, we then have $\mathrm{d}Y_t = \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}\mathrm{d} < S >_t$. If we remember that $\mathrm{d}S_t = \mu(t)S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}B_t$ and apply the Isometry formula, we get that $\mathrm{d} < S >_t = \sigma(t)^2 X_t^2 \mathrm{d}t$. We thus get :

$$
\begin{aligned}
\sigma \mathrm{d} Y_t &= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}\mathrm{d} < S >_t \\
&= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2}\sigma(t)^2\mathrm{d}t \\
&= \frac{1}{S_t}(\mu(t)S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}B_t) - \frac{1}{2}\sigma(t)^2\mathrm{d}t \\
&= (\mu(t) - \frac{1}{2}\sigma(t)^2)\mathrm{d}t + \sigma(t)dBt \qquad (3.6)\\
\implies Y_t &= y_0 + \int_0^t (\mu(s) - \frac{1}{2}\sigma(s)^2)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s \\
\implies Y_t &= \log(s_0) + \int_0^t (\mu(s) - \frac{1}{2}\sigma(s)^2)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s \\
\implies S_t &= s_0 \exp(\int_0^t (\mu(s) - \frac{1}{2}\sigma(s)^2)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s)
\end{aligned}
$$

Let us observe that the solution found in the case of the generalized Black-Scholes SDE is coherent with the solution found for the simple Black-Scholes SDE[6].

3.3 Back to the data

[6]Just set functions μ and σ equal to constants μ and σ and we are back with the Geometric Brownian Motion previously found.

Bibliography

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