Extreme Values in Financial Statistics

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par

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Mihi cura futuri
— Ovide, Métamorphoses, 13, 363

To my friends and loved ones. . .

Acknowledgements

TO BE FILLED

Lausanne, 14 Août 2015 K. M–H.

Preface

A preface is not mandatory. It would typically be written by some other person (eg your thesis director).

TO BE FILLED

Lausanne, 14 Août 2015 K. M–H.

Abstract

Key words:

Résumé

Mots clefs :

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1 Introduction

1.1 A few words to set the scene

In real life, it is not uncommon to have at one's disposal data about a phenomenon occurring through time. It may be as simple as daily rainfall data in a cit for the past two years, or it could be the weekly opening prices of a stock for the past decade.

Most of the time, people would like to use the data at their disposal to make predictions to answer questions, from the prosaic ones such as 'Will it rain tomorrow ?' to more consequential ones such as 'Will I make a profit if I cling to my shares today and sell them only tomorrow ?'. Of course, those are only vaguely worded questions : it is impossible to answer them satisfactorily without knowing the context, the objectives etc. behind them.

Yet, what these questions have in common is that they focus on the normal 'behaviour' that is to be expected in the future. Depending on the specific issue that is considered, the 'average behaviour' may not be the most interesting thing. For instance, suppose that a government wants to build a network of dams[1]. The dams are meant to protect the country from future floods for the next one hundred years, therefore the question that needs to be answered is one of 'worst case event' : "Over the next century, how severe may be the worst flood ?".

Extreme events are the kind of events we will be interested in this master thesis project. Although Extreme Value Theory has applications in many fields[2], we will here apply it more specifically to financial data.

1.2 Formalising the settings

Let $(X_n)_{n\geq 0}$ be a sequence of independent identically distributed random variables with common cumulative distribution function F_X. The sequence of maxima is defined by M_0=X_0 and $\forall n \geq 1$, M_n=$\max_{0\leq i\leq n}(X_i)$. We would like to determine the limiting distribution of the

[1]As was done in The Netherlands beginning in the fifties
[2]including climate science, seismology, insurance etc.

sequence $(M_n)_{n\geq 0}$.[3] This is a matter that will keep us busy quite a long time but the first thing to do is to re-formulate it.

Indeed, let us do a quick and simple computation :

$$\begin{aligned} F_n(t) &= \Pr(\{M_n \leq t\}) \\ &= \Pr(\{\max_{0\leq i\leq n}(X_i) \leq t\}) \\ &= \Pr(\{X_1 \leq t\} \cap \cdots \cap \{X_n \leq t\}) \\ &= (F_X(t))^n \end{aligned} \tag{1.1}$$

Here we see that not much information will be drawn from this result by taking the limit n→ $+\infty$. The limiting distribution will be degenerate. Indeed, let us consider the upper end-point of F_X[4], z^+. Then,

$$\begin{aligned} &\forall z < z^+ \lim_{z\to\infty} F_n(z) = 0 \\ &\forall z \geq z^+ \lim_{z\to\infty} F_n(z) = 1 \end{aligned} \tag{1.2}$$

It turns out we cannot use the limiting distribution directly. A common approach[5] is to consider a sequence of the maxima, centred and normalised.

We will thus consider in all what follows the sequence defined by $(M_n^*)_{n\geq 0} = (\frac{M_n - b_n}{a_n})_{n\geq 0}$ where $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are a sequence of real numbers and positive real numbers respectively. Finding a result on whether such a sequence admits a limiting distributions, and the conditions under which the result holds, will be one of our goals.

[3] If we can determine the limiting distribution of the maxima from the data, then we will have a means to make predictions on the occurrence of future extreme events.

[4] that is the smallest z such that $F_X(z)$ be equal to one. For the Normal distribution, z will be $+\infty$, by contrast for a continuous Uniform Distribution $U([a,b])$ it will be b.

[5] adopted by the mathematicians that laid the grounds for EVT

2 Investigating results on the limiting distribution

2.1 Playing with the (original) sequence of maxima

Here, we will generate finite size sequences (N = 10000) of independent identically distributed random variables following respectively :

- a standard Normal Distribution $\mathcal{N}(0,1)$
- a Cauchy Distribution *Cauchy(0,1)*
- an Exponential Distribution *Exp(1)*

We will compute the sequence of maxima, neither centred nor normalised, and draw the scatter plot as well as the plot of the maxima M_n as a function of the time steps n. We will also draw the $\frac{1}{n}$-quantiles of the distributions (distributions of the sample, not of the maxima) as a function of the time steps n. This will lead us to make an interesting observation.

2.1.1 Sample following a Normal distribution

Computing the quantiles The Normal distribution is a particular case because, unlike in the cases of the Cauchy and the Exponential distribution, there is no explicit form to the cumulative distribution function. We will thus use a "well-known"[1] inequality, holding $\forall\, t > 0$:

$$(\frac{1}{t} - \frac{1}{t^3}) * \frac{\exp(-\frac{t^2}{2})}{\sqrt{2*\pi}} < 1 - \Phi(t) < \frac{1}{t} * \frac{\exp(-\frac{t^2}{2})}{\sqrt{2*\pi}} \tag{2.1}$$

[1] Many textbooks mention it, though it is not necessarily what springs to the mind when thinking about the properties of Gaussian RVs.

From there, it is easy to see that the following holds :

$$1 - \Phi(t) \sim_{t \to +\infty} \frac{1}{t} * \frac{\exp(-\frac{t^2}{2})}{\sqrt{2 * \pi}} \tag{2.2}$$

When n grows large, the $\frac{1}{n}$-quantile grows very large so it is valid to replace $1 - \Phi(t)$ by its equivalent in the equation satisfied by the quantiles :

$$\begin{aligned} & F_X(q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\ \iff \quad & \frac{1}{q_{\frac{1}{n}}} * \frac{\exp(-\frac{q^2_{\frac{1}{n}}}{2})}{\sqrt{2 * \pi}} = \frac{1}{n} \\ \iff \quad & \log(q_{\frac{1}{n}}) + \log(\exp(-\frac{q^2_{\frac{1}{n}}}{2})) + \log(\sqrt{2 * \pi}) = \log(n) \end{aligned} \tag{2.3}$$

This equation cannot be solved analytically, we will resolve it iteratively. Th starting point is $\log(n) = \frac{t_0^2}{2}$, which gives us $t_0 = \sqrt{(2 * \log(n))}$. If we then run the Newton-Raphson algorithm, we see that the corrections to t_0 from the next iterations are small enough that we can keep t_0 as solution.[2].

Figure 2.1 – Below, a realisation of the sequence of maxima for i.i.d. standard unit Gaussian RVs

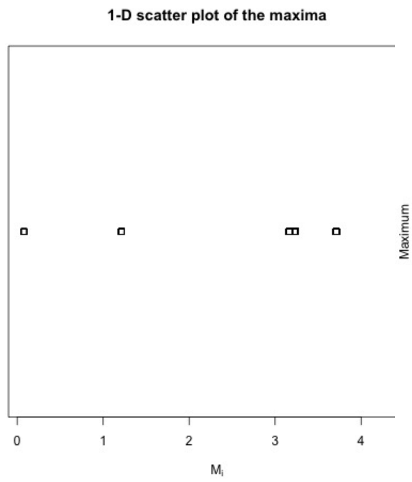


Figure 2.2 – Scatter Plot of the Maxima, n = 10000

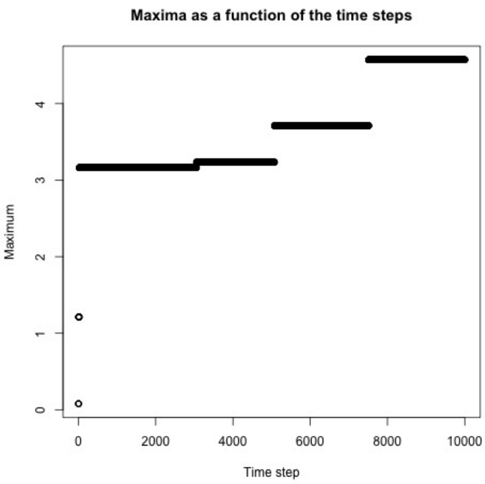


Figure 2.3 – Maxima against the time steps

[2]See the fourth of the figures below

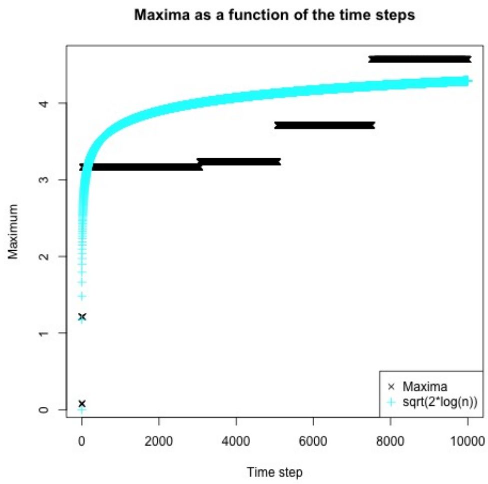


Figure 2.4 – Maxima against the time steps and function n→ $\sqrt{2 * \log(n)}$

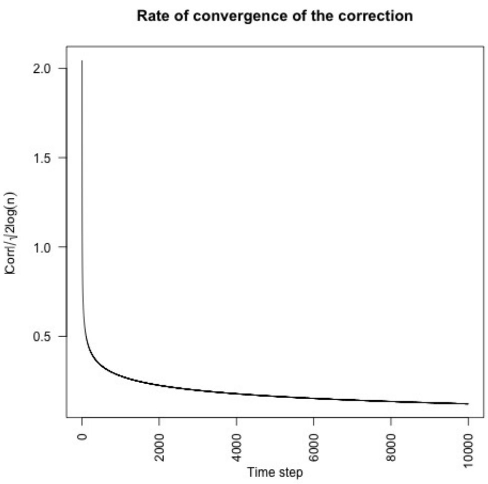


Figure 2.5 – The correction becomes negligible compared to the starting term as n grows large

2.1.2 Sample following a Cauchy distribution

Computing the quantiles Let $X_1, \cdots, X_n$ be i.i.d. RVs ∼ *Cauchy(0,1)*. The distribution function is $F_X(t) = \frac{1}{\pi} * \arctan(x) - \frac{1}{2}$. The $\frac{1}{n}$-quantiles satisfy the equation :

$$
\begin{aligned}
& F_X(q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\
\iff \quad & \frac{\arctan(q_{\frac{1}{n}})}{\pi} + \frac{1}{2} = 1 - \frac{1}{n} \\
\iff \quad & \frac{\arctan(q_{\frac{1}{n}})}{\pi} = \frac{2-n}{n} \\
\iff \quad & q_{\frac{1}{n}} = \tan(\frac{\pi}{2} * \frac{2-n}{n})
\end{aligned}
\tag{2.4}
$$

Figure 2.6 – Below, a realisation of the sequence of maxima for i.i.d. *Cauchy(0,1)* RVs

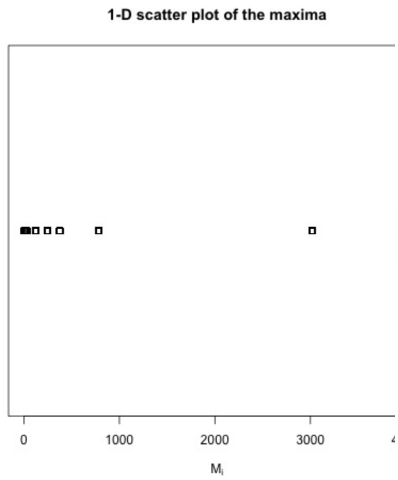


Figure 2.7 – Scatter Plot of the Maxima, n = 10000

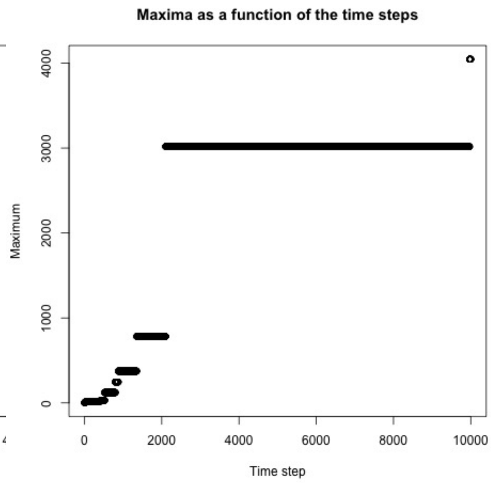


Figure 2.8 – Maxima against the time steps

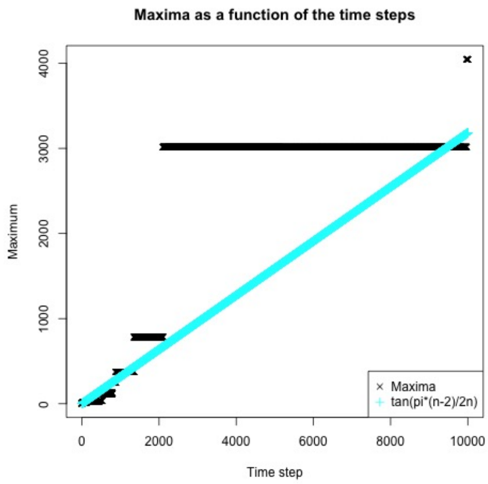


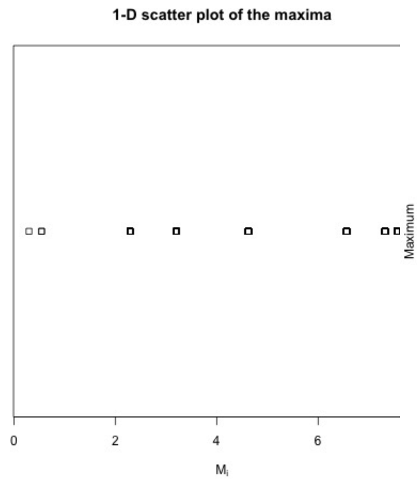
Figure 2.9 – Maxima against the time steps and function n→ $\tan(pi * \frac{2-n}{2*n})$

2.1.3 Sample following an Exponential Distribution

Computing the quantiles Let $X_1, \cdots, X_n$ be i.i.d. RVs ~ *Exponential(λ)*. The distribution function is $F_X(t) = 1 - \exp(-\lambda * t)$. The $\frac{1}{n}$-quantiles satisfy the equation :

$$\begin{aligned} & F_X(q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\ \Longleftrightarrow \quad & 1 - \exp(-\lambda * q_{\frac{1}{n}}) = 1 - \frac{1}{n} \\ \Longleftrightarrow \quad & q_{\frac{1}{n}} = \frac{1}{\lambda} * \log(n) \end{aligned} \tag{2.5}$$

Figure 2.10 – Below, a realisation of the sequence of maxima for i.i.d. *Exp(1)* RVs



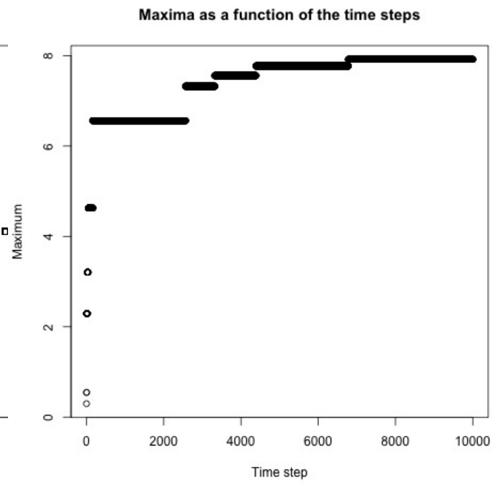


Figure 2.11 – Scatter Plot of the Maxima, n = 10000

Figure 2.12 – Maxima against the time steps

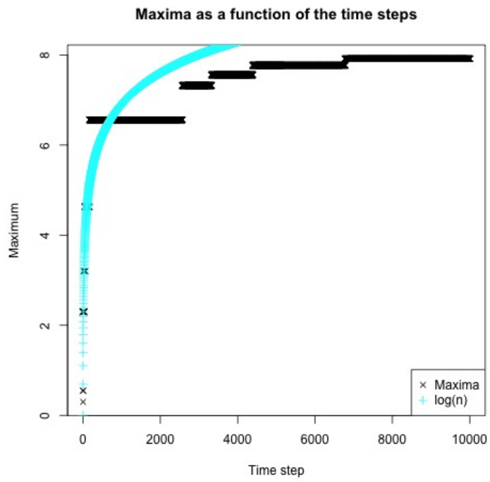


Figure 2.13 – Maxima against the time steps and function n→ $\log(n)$

2.1.4 Why does this work ?

3 Looking into real-world data

3.1 Five real-world stocks and their evolution over 15 years

We have chosen to study five stocks listed on the Paris Stock Exchange : BNP Paribas, Carrefour, LVMH, Sanofi and Total stocks.[1] The evolution of the stock prices has been studied over the past 15 years, on a weekly basis. We first draw the data itself, then the net returns and the gross log returns on the stocks[2]

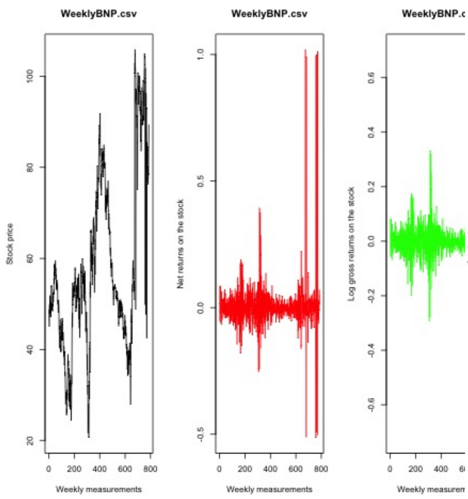


Figure 3.1 – 15 years of weekly BNP Stock Price Data

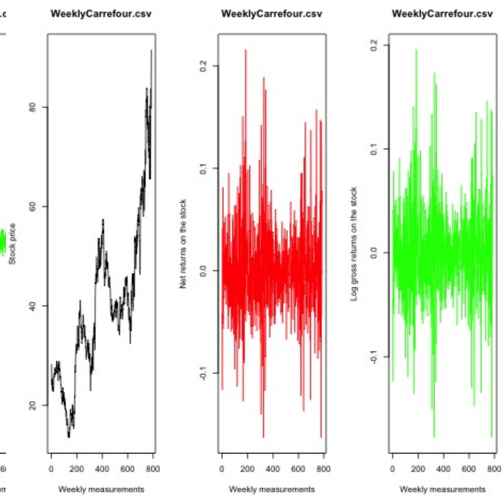


Figure 3.2 – 15 years of weekly Carrefour Stock Price Data

[1] We have chosen companies positioned on different domains, otherwise, information from different stock might more easily be redundant.

[2] Both quantities are widely used in Finance.

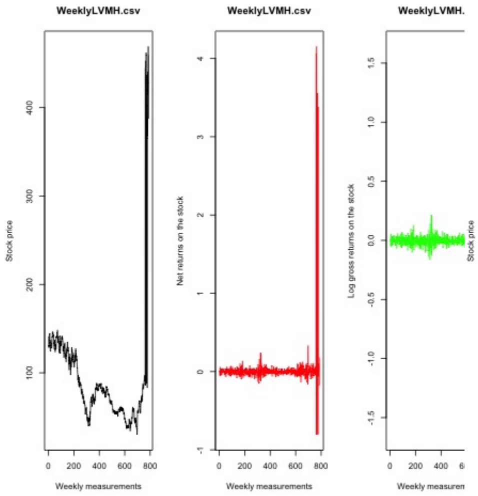


Figure 3.3 – 15 years of weekly LVMH Stock Price Data

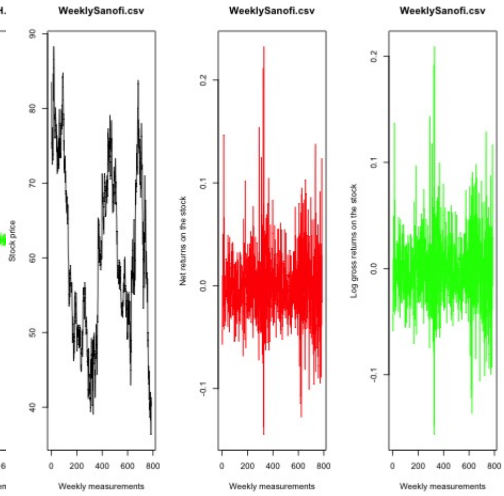


Figure 3.4 – 15 years of weekly Sanofi Stock Price Data

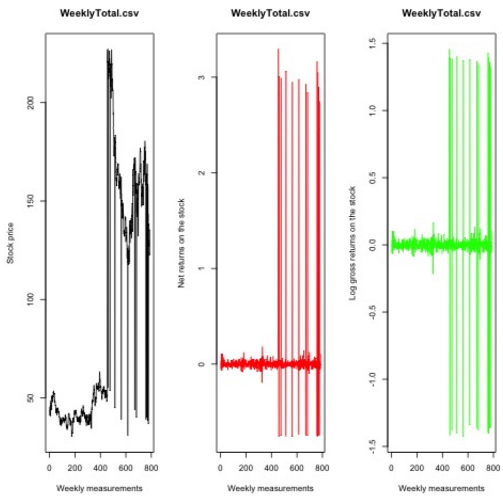


Figure 3.5 – 15 years of weekly Total Stock Price Data

Let X_t be the price of a stock at time t, the gross return at time t + 1 is defined as the ratio $\frac{X_{t+1}}{X_t}$, the net return at time t + 1 is defined as the ratio $r_t = \frac{X_{t+1}-X_t}{X_t}$ and the log gross return at time t +1 is defined as the log of the gross return at time t + 1 i.e. $R_t = \log(\frac{X_{t+1}}{X_t})$. The latter two quantities are of particular interest in Finance.
Let us observe that the relationship between R_t, X_t and X_{t+1} can be rewritten as $X_{t+1} = \exp(R_{t+1}) * X_t$. An approximation would be to take $X_{t+1} = (1 + R_{t+1}) * X_t$ by taking the expansion of the exponential, cut at order 1. Below are the plots of the quantities $\exp(R_t)$ and $1 + R_t$ for the five stocks previously considered. As we can see from the value of the residuals, this is in practice a very good approximation !

3.2 A détour around Brownian Motion

3.3 Back to the data

A An appendix

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