# Extreme Values in Financial Statistics

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par

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Mihi cura futuri
— Ovide, Métamorphoses, 13, 363

To my friends and loved ones...

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First and foremost, I would like to address my wholehearted thanks to Professor Stephan Morgenthaler who accepted to be the supervisor of my master thesis project. His guidance was steady throughout the project. At the same time, he left me free to go into and explore many different directions. As a consequence, I have learnt many things, and the need to work efficiently and set priorities prominent amongst them.

I am most grateful to Doctor Olivier Lévêque (EPFL) for his excellent stochastic calculus course as well as the exercises that go with it. In my opinion, this material is nothing short of the best for anyone who wants to get a grasp of the topic. He was always available whenever I had a question, and I want to thank him for that too.

My most sincere thanks go to Professor Joseph Gergaud, the head of the Applied Mathematics & Computer Science at ENSEEIHT, for allowing me to extend my exchange at EPFL and to pursue my master thesis project there.

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Finally, I seize this opportunity to tell my friends and my family how much I am beholden to them !

*Lausanne, 14 Août 2015* Killian Martin–Horgassan

# Preface

I am currently a third-year engineering student at ENSEEIHT, a French 'Grande Ecole d'Ingénieurs' located in Toulouse. Like most students in my class, my first two years of higher education were spent in a 'Classe Prépa Maths Sup-Maths Spé'. There, I began to acquaint myself with advanced mathematics. It was mainly theoretical mathematics, and a good foundation upon which build further knowledge and skills.
Then, I passed a competitive entrance exam and I became an Applied Mathematics & Computer Science engineering student at ENSEEIHT. My training there took a more practical dimension, as the problems we were asked to solve were closer to those encountered in real life. My life took a turn - from the academic perspective at least - when I came across two books : Options, Futures and Other Derivatives by John C. Hull and The Signal and the Noise by Nate Silver. I had always had an interest in Finance, and then I realised that I was woefully ignorant of probability, statistics and stochastic calculus. Not only would I need them in my professional life, they were fascinating branches of Mathematics as well !

I set out to right that wrong and remedy this issue ; during my first semester as an exchange student at EPFL I took courses in those fields. It was not always easy but whatever difficulties I encountered only fed my interest in the subjects. I was given the opportunity to extend my exchange at EPFL to do a master thesis project under the supervision of Prof. Stephan Morgenthaler. I could not quite make up my mind between statistics and finance, and that is why I decided to do my master thesis project on extreme value statistics applied to finance. Extreme value statistics is very different from the kind of statistics I had so far encountered, and from the early days of the project I realised that the topic I had picked was vast. It may be that I shall have only scratched the surface of it at the end of the project. In any case, this project has been a good introduction to extreme value statistics and a incentive to explore further the field of financial mathematics !

*Lausanne, 14 Août 2015* Killian Martin–Horgassan

# Abstract

The topic of this project is twofold. It is both an introduction to extreme value theory and an application to finance.

First, we give an introduction to extreme value theory by making appear the extreme values distributions from the maxima of sequences of independent identically distributed random variables. This leads us to the two extremal problems and to the Fisher-Tippett-Gnedenko and Von Mises' theorems which provide answers to those problems.

Then, we study the theory of the Black-Scholes model for stock prices. We use a Black-Scholes model with time independent coefficients to simulate the prices over time of five stocks listed on the Paris stock exchange (BNP Paribas, Carrefour, LVMH, Sanofi and Total). Finally, we show that if we consider the stock prices above a high threshold, we can fit extreme values distributions to the data.

# Résumé

Notre sujet a une thématique duale, il est conçu comme une introduction à la théorie des valeurs extrêmes en statistique doublée d'une application au domaine de la finance.

Notre présentons d'abord l'approche en théorie des valeurs extrêmes, suivie d'une découverte des distributions des valeurs extrêmes à travers l'étude des maxima d'échantillons de variables aléatoires indépendantes identiquement distribuées. Cela nous amène aux théorèmes de Von Mises et de Fisher-Tippett-Gnedenko, ils constituent les réponses aux problèmes extrêmaux c'est-à-dire le coeur de la théorie des valeurs extrêmes.

Nous nous intéressons alors aux évolutions des prix de 5 actions listées à la bourse de Paris (BNP Paribas, Carrefour, LVMH, Sanofi et Total). Nous étudions d'abord la théorie du modèle de Black-Scholes, nous appliquons ensuite un modèle de Black-Scholes avec coefficients indépendants du temps à nos données. Enfin, nous réalisons la jonction entre les valeurs extrêmes et nos données financières en étudiant, pour chacune des 5 actions, les valeurs des prix dépassant un certain seuil. Nous procédons à un ajustement de courbes à l'aide de distributions généralisées des valeurs extrêmes sur ces données.

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# 1 Introduction

## 1.1 A few words to set the scene

In real life, it is not uncommon to have at one's disposal data about a phenomenon occurring through time. It may be as simple as daily rainfall data in a city for the past two years, or it could be the weekly opening prices of a stock for the past decade.

Most of the time, people would like to use the data at their disposal to make predictions to answer questions, from the prosaic ones such as 'Will it rain tomorrow ?' to more consequential ones such as 'Will I make a profit if I cling to my shares today and sell them only tomorrow ?'. Of course, those are only vaguely worded questions : it is impossible to answer them satisfactorily without knowing the context, the objectives etc. behind them.

Yet, what these questions have in common is that they focus on the normal 'behaviour' that is to be expected in the future. Depending on the specific issue that is considered, the 'average behaviour' may not be the most interesting thing. For instance, suppose that a government wants to build a network of dams[1]. The dams are meant to protect the country from future floods for the next one hundred years, therefore the question that needs to be answered is one of 'worst case event' : "Over the next century, how severe may be the worst flood ?".

Extreme events are the kind of events we will be interested in this master thesis project. Although Extreme Value Theory has applications in many fields[2], we will here apply it more specifically to financial data.

## 1.2 Formalising the settings

Let $(X_n)_{n\geq 0}$ be a sequence of independent identically distributed random variables with common cumulative distribution function $F_X$. The sequence of maxima is defined by $M_0$=$X_0$ and $\forall n \geq 1$, $M_n = \max_{0\leq i\leq n}(X_i)$. We would like to determine the limiting distribution of the

[1]As was done in The Netherlands beginning in the fifties
[2]including climate science, seismology, insurance etc.

sequence $(M_n)_{n\geq 0}$.[3] This is a matter that will keep us busy quite a long time but the first thing to do is to re-formulate it.

Indeed, let us do a quick and simple computation :

$$\begin{aligned} F_n(t) &= \Pr(\{M_n \leq t\}) \\ &= \Pr(\{\max_{0\leq i\leq n}(X_i) \leq t\}) \\ &= \Pr(\{X_1 \leq t\} \cap \cdots \cap \{X_n \leq t\}) \\ &= (F_X(t))^n \end{aligned} \tag{1.1}$$

Here we see that little information will be drawn from this result by taking the limit $n \longrightarrow +\infty$. The limiting distribution will be degenerate. Indeed, let us consider the upper end-point of $F_X$[4], $z^+$. Then,

$$\begin{aligned} &\forall z < z^+ \lim_{z\to\infty} F_n(z) = 0 \\ &\forall z \geq z^+ \lim_{z\to\infty} F_n(z) = 1 \end{aligned} \tag{1.2}$$

It turns out we cannot use the limiting distribution directly. A common approach[5] is to consider a sequence of the maxima, standardized this time (i.e. centred and rescaled).

We will thus consider in all what follows the sequence defined by $(M_n^*)_{n\geq 0} = (\frac{M_n - b_n}{a_n})_{n\geq 0}$ where $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are a sequence of real numbers and positive real numbers respectively. Finding a result on whether such a sequence admits a limiting distributions, and the conditions under which the result holds, will be one of our goals.

**The two fundamental problems of extreme value theory** More specifically, assuming that there exists a non-degenerate distribution $G$, what may G be ? That is the ***extremal limit problem***. Additionally, what conditions do we have to impose on the common distribution of the random variables making up the sample, $F_X$, for the sequence $(M_n^*)_{n\geq 0}$ to converge to a non-degenerate distribution function G ? That is the ***domain of attraction problem***.

[3]If we can determine the limiting distribution of the maxima from the data, then we will have a means to make predictions on the occurrence of future extreme events.

[4]that is the smallest z such that $F_X(z)$ be equal to one. For the Normal distribution, z will be $+\infty$, by contrast for a continuous Uniform Distribution $U([a,b])$ it will be b. The definition, properly speaking, of the upper end-point of $F_X$ is the following : $z^+ = \inf\{z : F_X(z) \geq 1\}$.

[5]adopted by the mathematicians that laid the grounds of Extreme Value Theory.

# 2 Investigating results on the limiting distribution

## 2.1 Playing with the (original) sequence of maxima

Here, we will generate finite size sequences (N = 10000) of independent identically distributed random variables following respectively :

- a standard Normal Distribution $\mathcal{N}(0,1)$
- a Cauchy Distribution *Cauchy(0,1)*
- an Exponential Distribution *Exp(1)*

We will compute the sequence of maxima, neither centred nor normalised, and draw the scatter plot as well as the plot of the maxima $M_n$ as a function of the time steps $n$. We will also draw the $1-\frac{1}{n}$-quantiles of the distributions (distributions of the sample, not of the maxima), which we will denote by $q_n$, as a function of the time steps $n$. This will lead us to make an interesting observation.

## 2.2 Sample following a Normal distribution

**Computing the quantiles** The Normal distribution is a particular case because, unlike in the cases of the Cauchy and the Exponential distribution, there is no explicit form to the cumulative distribution function. We will thus use a "well-known"[1] inequality, holding $\forall\, t > 0$ :

$$\left(\frac{1}{t} - \frac{1}{t^3}\right)\frac{\exp(-\frac{t^2}{2})}{\sqrt{2\pi}} < 1 - \Phi(t) < \frac{1}{t}\frac{\exp(-\frac{t^2}{2})}{\sqrt{2\pi}} \tag{2.1}$$

[1] Many textbooks mention it, though it is not necessarily what springs to the mind when thinking about the properties of Gaussian RVs.

From there, it is easy to see that the following holds :

$$1-\Phi(t) \sim_{t\to+\infty} \frac{1}{t}\frac{\exp(-\frac{t^2}{2})}{\sqrt{2\pi}} \tag{2.2}$$

When $n$ grows large, the $1-\frac{1}{n}$-quantile grows very large so it is valid to replace $1-\Phi(t)$ by its equivalent in the equation satisfied by the quantiles :

$$\begin{aligned} & F_X(q_n) = 1-\frac{1}{n} \\ \iff \quad & \frac{1}{q_n}\frac{\exp(-\frac{q_n^2}{2})}{\sqrt{2\pi}} = \frac{1}{n} \\ \iff \quad & \log(q_n)+\log(\exp(-\frac{q_n^2}{2}))+\log(\sqrt{2\pi}) = \log(n) \end{aligned} \tag{2.3}$$

This equation cannot be solved analytically, we will resolve it iteratively. The starting point is $\log(n)=\frac{t_0^2}{2}$, which gives us $t_0=\sqrt{(2\log(n))}$. If we then run the Newton-Raphson algorithm, we see that the corrections to $t_0$ from the next iterations are small enough that we can keep $t_0$ as solution.[2].

Figure 2.1 – Below, a realisation of the sequence of maxima for i.i.d. standard unit Gaussian RVs

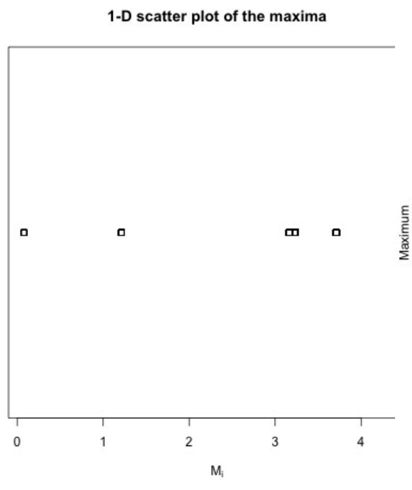



Figure 2.2 – Scatter Plot of the Maxima, n = 10000

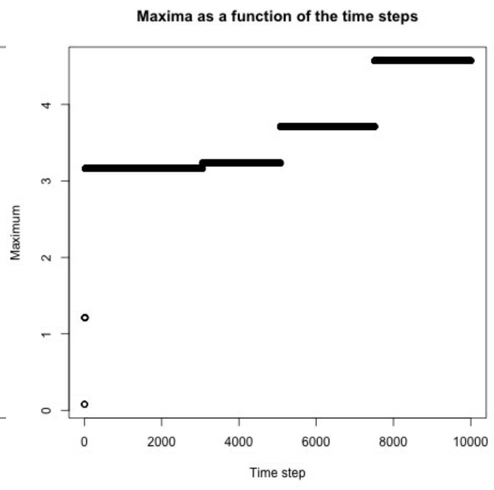



Figure 2.3 – Maxima against the time steps

[2]See the fourth of the figures below

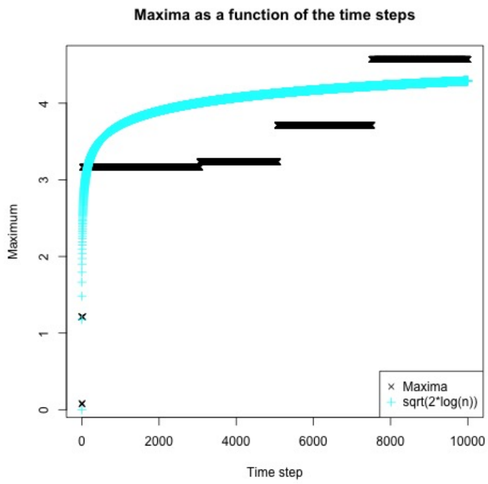



Figure 2.4 – Maxima against the time steps and function $n \to \sqrt{2\log(n)}$

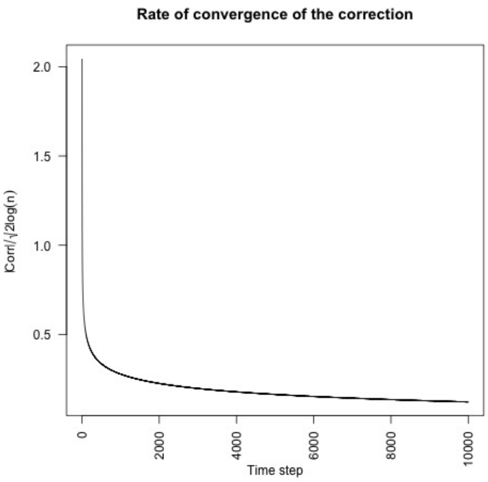



Figure 2.5 – The correction becomes negligible compared to the starting term as n grows large

## 2.3 Sample following a Cauchy distribution

**Computing the quantiles** Let $X_1, \cdots, X_n$ be i.i.d. RVs $\sim$ *Cauchy(0,1)*. The distribution function is $F_X(t) = \frac{1}{\pi}\arctan(x) - \frac{1}{2}$. The $1-\frac{1}{n}$-quantiles satisfy the equation :

$$
\begin{aligned}
& & F_X(q_n) &= 1 - \frac{1}{n} \\
&\iff & \frac{\arctan(q_n)}{\pi} + \frac{1}{2} &= 1 - \frac{1}{n} \\
&\iff & \frac{\arctan(q_n)}{\pi} &= \frac{2-n}{n} \\
&\iff & q_n &= \tan\left(\frac{\pi}{2}\frac{2-n}{n}\right)
\end{aligned}
\tag{2.4}
$$

Figure 2.6 – Below, a realisation of the sequence of maxima for i.i.d. *Cauchy(0,1)* RVs

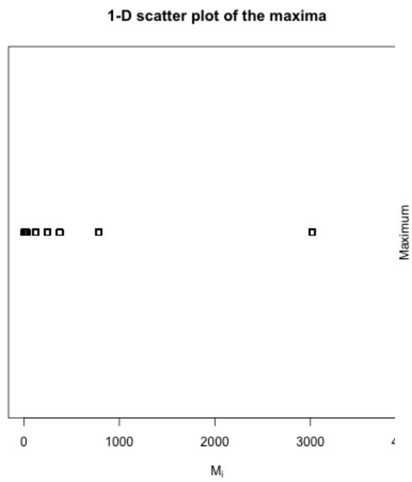



Figure 2.7 – Scatter Plot of the Maxima, n = 10000

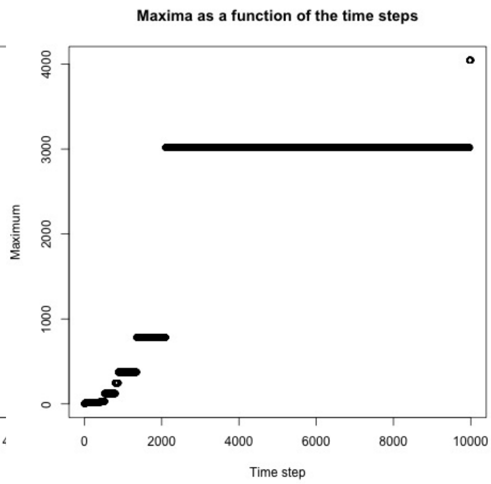



Figure 2.8 – Maxima against the time steps

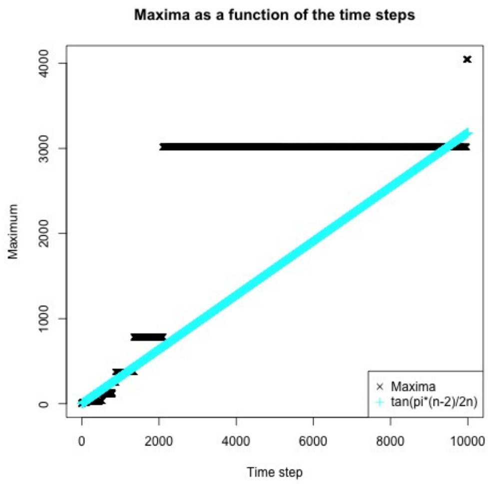



Figure 2.9 – Maxima against the time steps and function $n \to \tan(\pi\frac{2-n}{2n})$

$n \to \tan(\pi\frac{2-n}{2n})$ is roughly linear in $n$. We know that if $|x| < \frac{\pi}{2}$, $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$ so if we take the first order approximation, we get $\tan(x) \approx x$ which confirms what the plot seems to suggest.[3]

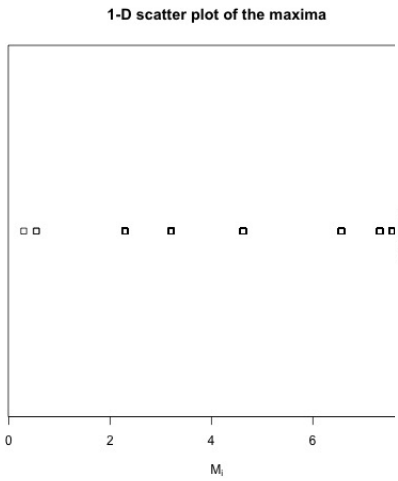
## 2.4 Sample following an Exponential Distribution

**Computing the quantiles** Let $X_1, \cdots, X_n$ be i.i.d. RVs ~ *Exponential(λ)*. The distribution function is $F_X(t) = 1 - \exp(-\lambda t)$. The $1 - \frac{1}{n}$-quantiles satisfy the equation :

$$\begin{aligned} & F_X(q_n) = 1 - \frac{1}{n} \\ \iff \quad & 1 - \exp(-\lambda q_n) = 1 - \frac{1}{n} \\ \iff \quad & q_n = \frac{1}{\lambda}\log(n) \end{aligned} \tag{2.5}$$

Figure 2.10 – Below, a realisation of the sequence of maxima for i.i.d. *Exp(1)* RVs

[3]Of course the expansion is valid as for a number of observations $n$ greater than 1, $-1 < \frac{2}{n} - 1 < 1$ and thus $|\frac{\pi}{2}\frac{2-n}{n}| = |\frac{\pi}{2}(\frac{2}{n} - 1)| < \frac{\pi}{2}$.



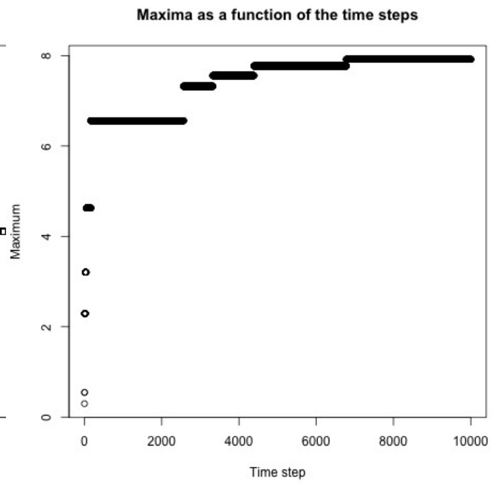




Figure 2.11 – Scatter Plot of the Maxima, n = 10000

Figure 2.12 – Maxima against the time steps

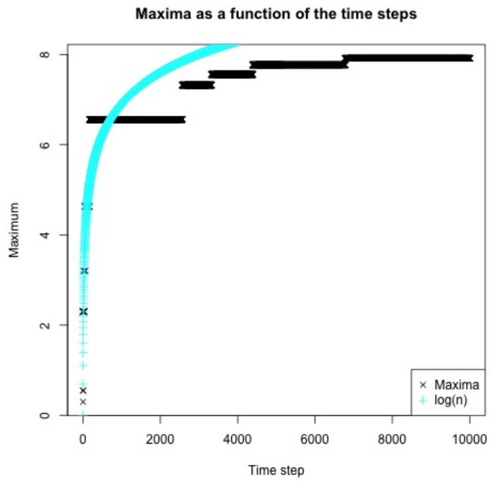



Figure 2.13 – Maxima against the time steps and function $n \rightarrow \log(n)$

## 2.5 Why does this work ?

**(The underlying idea)** Why have we made a link between the $\frac{1}{n}$-quantiles of the common distribution of the $X_i$ and the sequence of the $M_n$ ? Actually, the link is that if the sample is made up of independent realizations, then $M_n$ is an estimate of the $1-\frac{1}{n}$ quantile.
The idea behind this is as follows, that the maxima will get closer to the upper end-point of the distribution of the $X_i$, $F_{X_i}$. That is also the case for the $1-\frac{1}{n}$-quantiles of $F_{X_i}$. Of course, it is only an intuition !

## 2.6 What are the limiting distributions in theses cases ?

**Preliminary remark** In what follows we are using theoretical results exposed in the next chapter, chapter 3, concerning the three possible limiting distributions as well as Von Mises' theorem.

**Fréchet limiting distribution** Let us assume that a random variable $X$ follows a $Cauchy(0,1)$ distribution, $x^+ = +\infty$. The limiting distribution can only be either a Gumbel or a Fréchet-type distribution.

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2} \tag{2.6}$$

$$F_X(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2} \tag{2.7}$$

$$\begin{aligned}
r(x) &= \frac{f_X(x)}{1-F_X(x)} \\
&= \frac{\frac{1}{1+x^2}}{\frac{\pi}{2} - \arctan(x)} \\
&= \frac{\frac{1}{1+x^2}}{\frac{\pi}{2} - (\frac{\pi}{2} - \arctan(\frac{1}{x}))} \\
&= \frac{1}{(1+x^2)\arctan(\frac{1}{x})} \\
&= \frac{1}{(1+x^2)(\frac{1}{x} + o(\frac{1}{x^2}))} \\
&= \frac{1}{\frac{1+x^2}{x} + o(1)} \\
\implies xr(x) &= \frac{x^2}{x^2+1+o(1)} \\
\implies xr(x) &\xrightarrow[x\to+\infty]{} 1
\end{aligned} \tag{2.8}$$

Finally, by Von Mises' theorem, we can conclude that the limiting distribution for the standardized maxima of a $Cauchy(0,1)$ sample is a Fréchet distribution.

**Fréchet distribution** Let us assume that a random variable $X$ follows a $Exp(1)$ distribution, $x^+ = +\infty$. The limiting distribution here again can only be either a Gumbel or a Fréchet-type distribution.

$$f_X(x) = \lambda\exp(-\lambda x) \tag{2.9}$$

$$F_X(x) = 1 - \exp(-\lambda x) \tag{2.10}$$

$$\begin{aligned} r(x) &= \frac{f_X(x)}{1 - F_X(x)} \\ &= \frac{\lambda \exp(-\lambda x)}{1 - (1 - \exp(-\lambda x))} \\ &= \frac{\lambda \exp(-\lambda x)}{\exp(-\lambda x)} \\ &= \lambda \\ \implies \frac{\mathrm{d}r}{\mathrm{d}x}(x) &= 0 \end{aligned} \tag{2.11}$$

Finally, by Von Mises' theorem, we can conclude that the limiting distribution for the standardized maxima of an $Exp(1)$ sample is a Gumbel distribution.

**Gumbel distribution - bis** Let us assume that a random variable $X$ follows a $\mathcal{N}(0,1)$ distribution, $x^+ = +\infty$. The limiting distribution here again can only be either a Gumbel or a Fréchet-type distribution. It turns out that in that case, the limiting distribution is a Gumbel-type distribution. The computation is simple, as shown below :

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}x}(x) &= \frac{\frac{\mathrm{d}f}{\mathrm{d}x}(x)(1 - F_X(x)) - f(x)(-f(x))}{(1 - F_X(x))^2} \\ &= \frac{f(x)^2 + \frac{\mathrm{d}f}{\mathrm{d}x}(x)(1 - F_X(x))}{(1 - F_X(x))^2} \end{aligned} \tag{2.12}$$

It yields an indeterminate form. The key to solve the issue here is to use an expansion of the cumulative distribution function of a standard Gaussian random variable :
$\Phi(x) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \exp(\frac{-x^2}{2})(x + \frac{x^3}{3} + \frac{x^5}{15})$

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}x}(x) &= \frac{\frac{\exp(-x^2)}{2\pi} - \frac{x}{2\sqrt{2\pi}} \exp(\frac{-x^2}{2}) + \frac{x\exp(-x^2)}{2\pi}(x + \frac{x^3}{3} + \frac{x^5}{15})}{(\frac{1}{2} - \frac{1}{\sqrt{2\pi}} \exp(\frac{-x^2}{2})(x + \frac{x^3}{3} + \frac{x^5}{15}))^2} \\ &= \frac{\frac{\exp(-x^2)}{2\pi} - \frac{x}{2\sqrt{2\pi}} \exp(\frac{-x^2}{2}) + \frac{x\exp(-x^2)}{2\pi}(x + \frac{x^3}{3} + \frac{x^5}{15})}{\frac{1}{4} - \frac{\exp(\frac{-x^2}{2})}{\sqrt{2\pi}}(x + \frac{x^3}{3} + \frac{x^5}{15}) + \frac{\exp(-x^2)}{2\pi}(x + \frac{x^3}{3} + \frac{x^5}{15})^2} \end{aligned} \tag{2.13}$$

Now we can see that if we take the limit when $x$ goes to $+\infty$, we get a 0. By Von Mises' theorem, we can conclude that the limiting distribution for the standardized maxima of a $\mathcal{N}(0,1)$ sample is a Gumbel distribution.

**Weibull distribution** Cauchy, Normal and Exponential distributions all have an infinite upper end-point, thus we will never get a Weibull distribution as limiting distribution. Let us assume that a random variable $X$ follows a $Unif([0,1])$ distribution, $x^+ = 1 < +\infty$.

$$f_X(x) = 1 \tag{2.14}$$

$$F_X(x) = x \tag{2.15}$$

$$\begin{aligned} r(x) &= \frac{f_X(x)}{1 - F_X(x)} \\ &= \frac{1}{1-x} \\ \implies (x^+ - x)r(x) &= (x^+ - x)\frac{1}{1-x} \\ \implies (1-x)r(x) &= (1-x)\frac{1}{1-x} = 1 \\ \implies (x^+ - x)r(x) &\xrightarrow[x \to x^+]{} 1 > 0 \end{aligned} \tag{2.16}$$

Finally, by Von Mises' theorem, we can conclude that the limiting distribution for the standardized maxima of a $Unif([0,1])$ sample is a Weibull distribution.

# 3 The two extremal problems

## 3.1 The extremal limit problem

**The answer to the extremal limit problem** It turns out that all possible non-degenerate limiting distributions i.e. all extreme values distribution make up a one-parameter family $G_\gamma(x) = \exp(-(1+\gamma x)^{-\frac{1}{\gamma}})$, where the support of G is the set $\{x : 1+\gamma x > 0$ and $\gamma \in \mathbb{R}$ is the **E**xtreme **V**alue **I**nded or **EVI**. The three sub-cases are the following :

- $\underline{\gamma = 0 :}$ **Gumbel distribution**
  $G_\gamma(u) = \exp(-\exp(-u))$, $u \in \mathbb{R}$
- $\underline{\gamma > 0 :}$ **Fréchet distribution**
  $G_\gamma(u) = \exp(-(1+\gamma u)^{-\frac{1}{\gamma}})$, $u \in ]-\gamma^{-1}, +\infty[$
- $\underline{\gamma > 0 :}$ **Weibull distribution**
  $G_\gamma(u) = \exp(-(1+\gamma u)^{-\frac{1}{\gamma}})$, $u \in ]-\infty, -\gamma^{-1}[$

For the derivations and proofs relative to this section, please report to the appendix.

## 3.2 The domain of attraction problem

**Definition** The domain of attraction of an extreme value distribution family (i.e. Gumbel, Fréchet-type or Weibull-type) is the set of distribution functions $F_X$[1] such that the sequence of standardized maxima $(M_n^*)_{n\geq 0}$ will converge in distribution to that extreme value distribution family.

**Remark** There are many approaches to characterize the domains of attraction of the extreme value distribution families. We have decided to use Von Mises' theorem to characterize

[1] $F_X$ being the distribution of the $X_i$ of the sample.

them. This is the historical approach, and a rather straightforward one, still by no means are alternative approaches uninteresting[2].

**Hazard function** Let $X$ be a random variable with probability density function/mass function $f_X$ and distribution function $F_X$, then we define the hazard function $r$ as follows :
$r(x) = \frac{f_X(x)}{1-F_X(x)}$.

**A few preliminary notations** $\Phi_\alpha$, $\Psi_\alpha$, $\Delta$ are respectively the symbols used to denote a Fréchet, a Weibull and a Gumbel distributions, with :

- $\Phi_\alpha(x) = \exp(-x^\alpha)$
- $\Psi_\alpha(x) = \exp(-|x|^\alpha)$ (let us bear in mind that this is a notation, due to historical reasons).
- $\Delta(x) = \exp(-\exp(-x))$

**Von Mises' theorem**

1. If $x^+ = +\infty$ and $xr(x) \xrightarrow[x\to+\infty]{} \alpha > 0$, then $F_X \in \mathscr{D}(\Phi_\alpha)$.
2. If $x^+ < +\infty$ and $(x^+ - x)r(x) \xrightarrow[x\to x^+]{} \alpha > 0$, then $F_X \in \mathscr{D}(\Psi_\alpha)$.
3. If $r(x)$ is ultimately positive in the neighbourhood of $x^+$ (with $x^+ \leq +\infty$), is differentiable on that neighbourhood and is such that $\frac{\mathrm{d}r}{\mathrm{d}x}(x) \xrightarrow[x\to x^+]{} 0$, then $F_X \in \mathscr{D}(\Delta)$.

## 3.3 Conclusion

**Fisher-Tippett-Gnedenko theorem** The Fisher-Tippett-Gnedenko theorem, also known as the **extremal theorem**, states that if the sequence of standardized maxima converges in distribution to a non-degenerate distribution, then this distribution belongs to one of the three aforementioned extreme value distribution families. The theorem thus provides an answer to the *extremal limit problem.* Von Mises' theorem, encountered in the previous section, provides a complementary answer, that to the *domain of attraction problem.*

[2] In particular, conditions based on the sole behaviour of $F_X$ can be formulated.

# 4 Looking into financial data

## 4.1 Five stocks and their evolution over 15 years

We have chosen to study five stocks listed on the Paris Stock Exchange : BNP Paribas, Carrefour, LVMH, Sanofi and Total stocks.[1] The evolution of the stock prices has been studied over the past 15 years, on a weekly basis. We first draw the data itself, then the net returns and the gross log returns on the stocks[2]

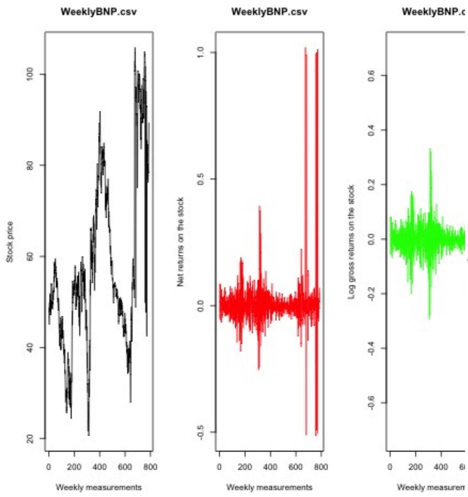



Figure 4.1 – 15 years of weekly BNP Stock Price Data

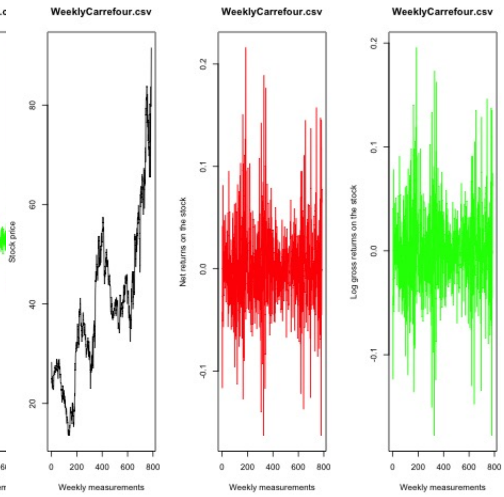



Figure 4.2 – 15 years of weekly Carrefour Stock Price Data

[1] We have chosen companies positioned on different domains, otherwise, information from different stock might more easily be redundant.

[2] Both quantities are widely used in Finance.

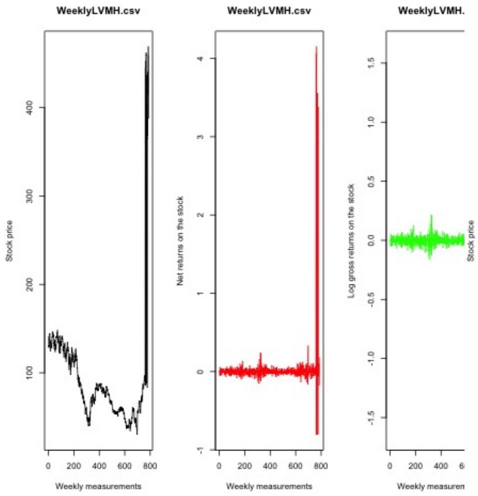



Figure 4.3 – 15 years of weekly LVMH Stock Price Data

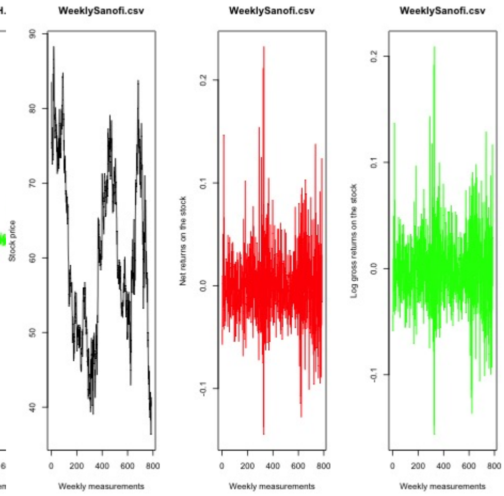



Figure 4.4 – 15 years of weekly Sanofi Stock Price Data

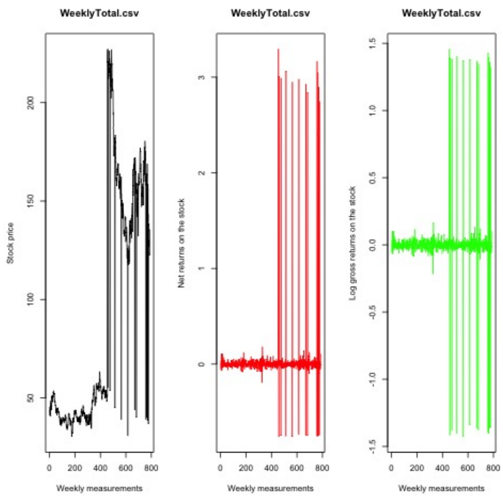



Figure 4.5 – 15 years of weekly Total Stock Price Data

Let $X_t$ be the price of a stock at time $t$, the gross return at time $t+1$ is defined as the ratio $\frac{X_{t+1}}{X_t}$, the net return at time $t+1$ is defined as the ratio $r_t = \frac{X_{t+1}-X_t}{X_t}$ and the log gross return at time $t+1$ is defined as the log of the gross return at time $t+1$ i.e. $R_t = \log(\frac{X_{t+1}}{X_t})$. The latter two quantities are of particular interest in Finance.
Let us observe that the relationship between $R_t$, $X_t$ and $X_{t+1}$ can be rewritten as $X_{t+1} = \exp(R_{t+1})X_t$. An approximation would be to take $X_{t+1} = (1+R_{t+1})X_t$ by taking the expansion of the exponential, cut at order 1. Below are the plots of the quantities $\exp(R_t)$ and $1+R_t$ for the five stocks previously considered. As we can see from the value of the residuals (of order

1E-15[3]), this is in practice a very good approximation !

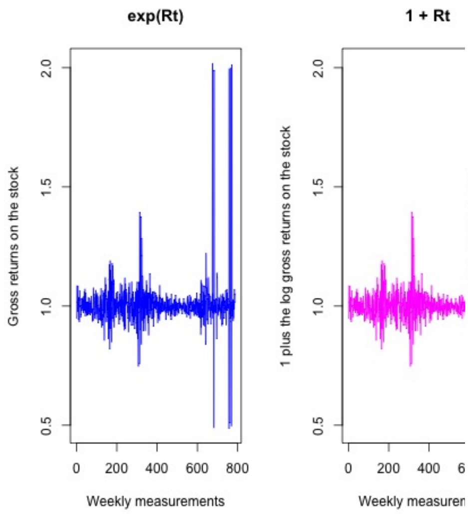



Figure 4.6 – $\exp(R_t)$ and $1+R_t$ for BNP Stock Price Data

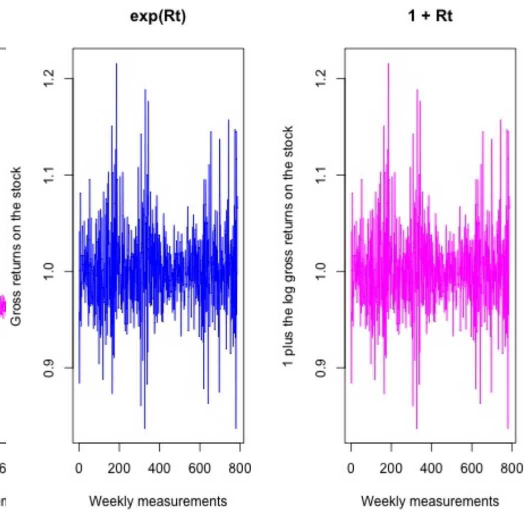



Figure 4.7 – $\exp(R_t)$ and $1+R_t$ for Carrefour Stock Price Data

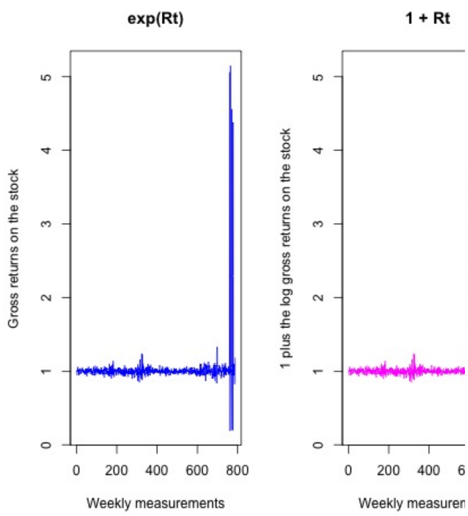



Figure 4.8 – $\exp(R_t)$ and $1+R_t$ for LVMH Stock Price Data

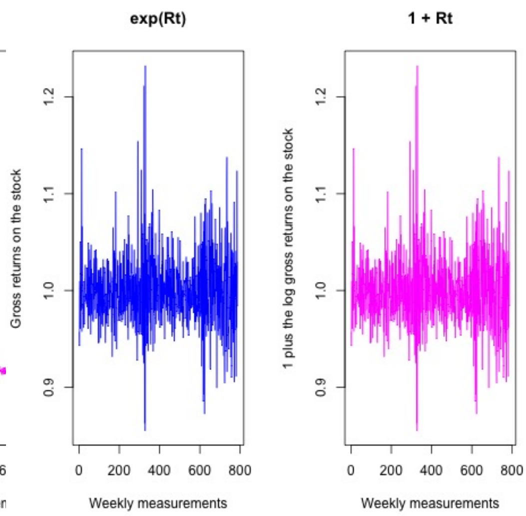



Figure 4.9 – $\exp(R_t)$ and $1+R_t$ for Sanofi Stock Price Data

[3] report to the appendix 'Additional figures' for the details.

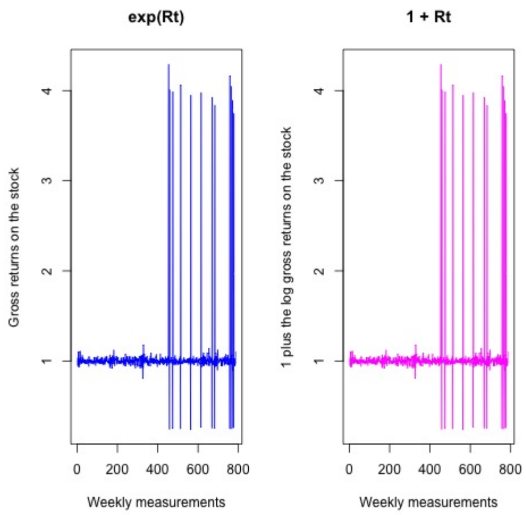



Figure 4.10 – $\exp(R_t)$ and $1 + R_t$ for Total Stock Price Data

## 4.2 A détour around Stochastic Calculus

### 4.2.1 The Black-Scholes Stochastic Differential Equation

**Presentation** It is customary to model the evolution of stock prices by a stochastic process $(S_t)_{t\geq 0}$ satisfying the Black-Scholes stochastic differential equation[4], which reads as follows :

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \tag{4.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$. Let us observe that the stochastic differential equation above is the Black-Scholes SDE with time independent coefficients : both the drift $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$ are constant.[5] An initial condition must be specified : $S_0 = s_0 > 0$.

**Resolution - existence of a solution to the Black-Scholes SDE** Let us consider the following generic stochastic differential equation :

$$\begin{cases} \mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}B_t \\ X_0 = x_0 \end{cases}$$

[4]Actually, a first model could be as simple as using arithmetical random walks. Two problems would however arise. The first is that if we used an ARW we could get negative stock prices, the second problem is that stocks selling at low prices usually have low price increments while stocks selling at high prices usually have larger price increments. In the Black-Scholes model, the price increment at time $t+1$ is made up of a component proportional to the price at time $t$ and a stochastic component. The stochastic component is also proportional to the price at time $t$. Both problems are thus solved. The process $(S_t)_{t\geq 0}$ is continuous and when $S_t$ reaches 0, the infinitesimal increment which is $S_t(\mu \mathrm{d}t + \sigma \mathrm{d}B_t)$ is also 0. Therefore, $S_t$ cannot go below 0.

[5]In the next subsection, we will deal with the Black-Scholes SDE with time dependent coefficients.

where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$, $x_0 \in \mathbb{R}$, $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz functions. Then, by a theorem from Stochastic Calculus[6], we know that there exists a unique process that is continuous and adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Here, f and g are respectively the functions $x \longrightarrow \mu x$ and $x \longrightarrow \sigma t$. These functions are Lipschitz functions, therefore we know the SDE admits a solution. And fortunately, in the case of the Black-Scholes equation, the solution can be made explicit[7]).

**Resolution - step 1** Let us set $S_t = \phi_t Z_t$ where $\phi_t$ is the (deterministic) solution to the ordinary differential equation :

$$\begin{cases} \mathrm{d}\phi_t = \mu\phi_t \mathrm{d}t \\ \phi_0 = 1 \end{cases}$$

Solving this ODE is elementary and yields the solution $\phi_t = \exp(\mu t)$. Now, let us differentiate $X_t$ under its form as a product $S_t = \phi_t Z_t$.

$$\begin{aligned} \mathrm{d}(S_t) &= \mathrm{d}(\phi_t Z_t) \\ &= \phi_t \mathrm{d}Z_t + Z_t \mathrm{d}\phi_t + \mathrm{d} < \phi, Z >_t \\ &= \phi_t \mathrm{d}Z_t + \mu\phi_t Z_t \mathrm{d}t \\ &= \phi_t \mathrm{d}Z_t + \mu S_t \mathrm{d}t \end{aligned} \tag{4.2}$$

where the infinitesimal quadratic covariation between $\phi$ and $Z$ is zero as $\phi$ has bounded variations. If we compare the last right-hand term of the series of equations just above to the original Black-Scholes SDE, we see that $\sigma S_t \mathrm{d}B_t = \phi_t \mathrm{d}Z_t$. Hence, $\mathrm{d}Z_t = \sigma Z_t \mathrm{d}B_t$. Here, we must be very careful as this differential equation does not integrate as it would in the settings of 'usual' calculus, in particular, integrating it to $\log(Z_t) - \log(Z_0) = \sigma(B_t - B_0)$ is totally wrong !

**Resolution - step 2** Now, let us set $Y_t = \log(Z_t)$. Using Ito-Doeblin's formula, we get :
$d(\log(Z_t)) = \frac{1}{Z_t}\mathrm{d}Z_t + \frac{1}{2}(\frac{-1}{Z_t^2})\mathrm{d} < Z >_t$ $(\star)$

[6] readers wanting to get ahold of an excellent course on Stochastic Calculus are advised to refer to Dr. Lévêque's course. It can be found at http://ipg.epfl.ch/~leveque/

[7] This is not always the case : solving SDEs in general is not that simple, and finding an explicit solution is not guaranteed in the general case, even though we know one exists.

→ How to compute $\mathrm{d} < Z >_t$ ?

$$
\begin{aligned}
& S_t = \phi_t Z_t \\
\implies & Z_t = \frac{S_t}{\phi_t} \\
\implies & \mathrm{d}Z_t = \frac{1}{\phi_t}\mathrm{d}S_t + S_t d(\frac{1}{\phi_t}) + \mathrm{d} < \frac{1}{\phi}, S >_t \\
& = \frac{1}{\phi_t}\mathrm{d}S_t - \frac{S_t}{\phi_t^2} d(\phi_t) \\
& = \frac{1}{\phi_t}\mathrm{d}S_t - \frac{\mu S_t}{\phi_t}\mathrm{d}t \\
& = \frac{1}{\phi_t}(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t) - \frac{\mu S_t}{\phi_t}\mathrm{d}t \\
& = \frac{\sigma S_t}{\phi_t}\mathrm{d}B_t
\end{aligned}
\tag{4.3}
$$

where the infinitesimal quadratic covariation between $\frac{1}{\phi}$ and $Z$ is zero as $\frac{1}{\phi}$ has bounded variations, in the third equality from the top.
Hence, using the Isometry formula, we have that :

$$
\begin{aligned}
& < Z >_t = \int_0^t \frac{\sigma^2 S_s^2}{\phi_s^2} \mathrm{d}s \\
& = \int_0^t \sigma^2 Z_s^2 \mathrm{d}s \\
\implies & \mathrm{d} < Z >_t = \sigma^2 Z_t^2 \mathrm{d}t
\end{aligned}
$$

Back to (⋆), we now have :

$$
\begin{aligned}
& \mathrm{d}(\log(Z_t)) = \frac{1}{Z_t}\mathrm{d}Z_t + \frac{1}{2}(\frac{-1}{Z_t^2})\sigma^2 Z_t^2 \mathrm{d}t \\
\iff & \frac{1}{Z_t}\mathrm{d}Z_t = d(\log(Z_t)) + \frac{\sigma^2}{2}\mathrm{d}t
\end{aligned}
\tag{4.4}
$$

**Resolution - step 3** Combining the previous equation with $\mathrm{d}Z_t = \sigma Z_t \mathrm{d}B_t$, we get :

$$\begin{aligned}
\sigma \mathrm{d}B_t &= \mathrm{d}(log(Z_t)) + \frac{\sigma^2}{2}\mathrm{d}t \\
\implies \log(Z_t) - \log(Z_0) &= -\frac{\sigma^2}{2}(t-0) + \sigma(B_t - B_0) \\
\implies \log(Z_t) - \log(\frac{S_0}{\phi_0}) &= -\frac{\sigma^2}{2}t + \sigma B_t \\
\implies \log(Z_t) &= \log(s_0) - \frac{\sigma^2}{2}t + \sigma B_t \\
\implies Z_t &= s_0 \exp(-\frac{\sigma^2}{2}t + \sigma B_t)
\end{aligned} \tag{4.5}$$

Finally, by remembering that $S_t = \phi_t Z_t = \exp(\mu t) Z_t$, we get :
$\forall t \geq 0, S_t = s_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$
The stochastic process $(S_t)_{t\geq 0}$, made explicit above, that is solution to the Black-Scholes equation is generally called Geometric Brownian Motion in the literature.

### 4.2.2 Black-Scholes SDE with time-dependent coefficients

**Presentation** In the simple Black-Scholes SDE, the drift $\mu$ and the volatility $\sigma$ were time-independent constants. Let us now consider a more general version of the Black-Scholes SDE :
$\begin{cases} \mathrm{d}S_t = \mu(t) S_t \mathrm{d}t + \sigma(t) S_t \mathrm{d}B_t \\ S_0 = s_0 > 0 \end{cases}$
where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$, $\mu, \sigma$ two continuous functions such that there exists $K_1 > 0$, $K_2 > 0$, such that $\forall t \geq 0$, $|\mu(t)| \leq K_1$, $K_2 \leq |\sigma(t)| \leq K_1$.

**Resolution - existence of a solution to the generalized Black-Scholes SDE** Let us consider the following generic stochastic differential equation :
$\begin{cases} \mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}B_t \\ X_0 = x_0 \end{cases}$
where $(B_t)_{t\geq 0}$ is a standard Brownian Motion with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$, $x_0 \in \mathbb{R}$, $f, g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are jointly continuous in $(t, x)$ and Lipschitz in x. Then, by a theorem from Stochastic Calculus, we know that there exists a unique solution $(X_t)_{t\geq 0}$ to the SDE. in the case of the generalized Black-Scholes SDE, the conditions are met and we can thus conclude that it admits a unique solution. The solution can be made explicit here too, fortunately !

**Resolution** Let us set $Y_t = \log(S_t)$, we then have $\mathrm{d}Y_t = \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}\mathrm{d} < S >_t$. If we remember that $\mathrm{d}S_t = \mu(t) S_t \mathrm{d}t + \sigma(t) S_t \mathrm{d}B_t$ and apply the Isometry formula, we get that $\mathrm{d} < S >_t = \sigma(t)^2 X_t^2 \mathrm{d}t$. We thus get :

$$\begin{aligned}
\sigma \mathrm{d}Y_t &= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2}\frac{1}{S_t^2}\mathrm{d} < S >_t \\
&= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2}\sigma(t)^2\mathrm{d}t \\
&= \frac{1}{S_t}(\mu(t)S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}B_t) - \frac{1}{2}\sigma(t)^2\mathrm{d}t \\
&= (\mu(t) - \frac{1}{2}\sigma(t)^2)\mathrm{d}t + \sigma(t)dBt \\
\implies Y_t &= y_0 + \int_0^t (\mu(s) - \frac{1}{2}\sigma(s)^2)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s \\
\implies Y_t &= \log(s_0) + \int_0^t (\mu(s) - \frac{1}{2}\sigma(s)^2)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s \\
\implies S_t &= s_0 \exp(\int_0^t (\mu(s) - \frac{1}{2}\sigma(s)^2)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s)
\end{aligned} \tag{4.6}$$

Let us observe that the solution found in the case of the generalized Black-Scholes SDE is coherent with the solution found for the simple Black-Scholes SDE[8].

## 4.3 Back to the data

**Fitting the data to the model** We are now going to assume the stock prices $S_t^{(i)}$[9] we have been studying follow the (simple) Black-Scholes model that is that their prices can be modeled by a geometric brownian motion. We will determine the corresponding $\mu_i$'s and $\sigma_i$'s. To do that, we will make explicit the gross return at time $t+1$ :

$$\begin{aligned}
\frac{S_{t+1}}{S_t} &= \frac{s_0 \exp((\mu - \frac{\sigma^2}{2})(t+1) + \sigma B_{t+1}}{s_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)} \\
\implies \frac{S_{t+1}}{S_t} &= \exp(\mu - \frac{\sigma^2}{2})\exp(\sigma(B_{t+1} - B_t))
\end{aligned} \tag{4.7}$$

[8]Just set functions $\mu$ and $\sigma$ equal to constants $\mu$ and $\sigma$ and we are back with the Geometric Brownian Motion previously found.

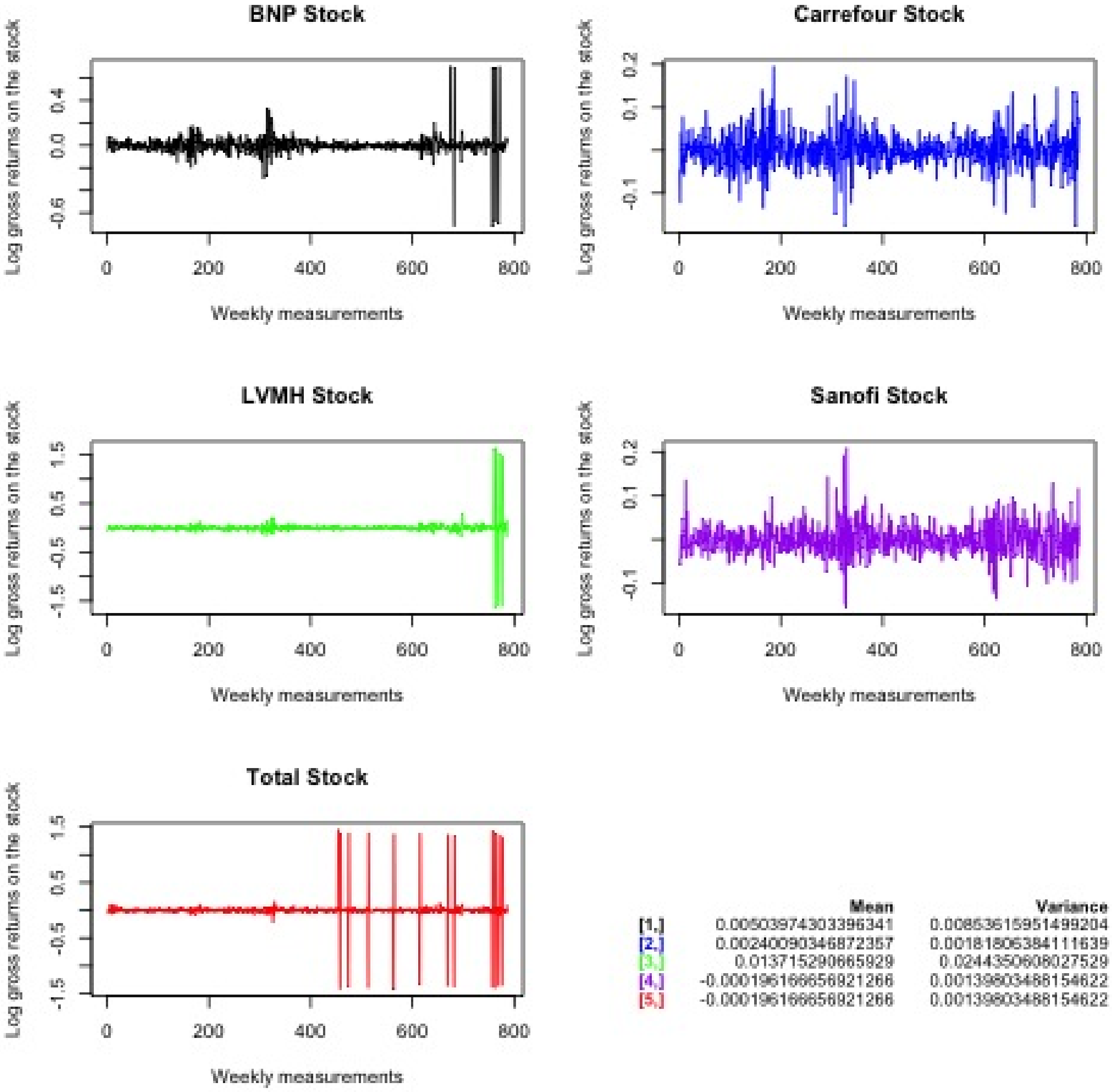
[9]The notations are the following :

- $S_t^{(1)}$ for the BNP stock.
- $S_t^{(2)}$ for the Carrefour stock.
- $S_t^{(3)}$ for the LVMH stock.
- $S_t^{(4)}$ for the Sanofi stock.
- $S_t^{(5)}$ for the Total stock.

A first observation that we could make is that as $x \longrightarrow \exp(\sigma x)$ is a Borel-measurable function (because it is continuous) and the increments of the brownian motion are independent, $\frac{S_t}{S_{t-1}} = \exp(\mu - \frac{\sigma^2}{2})\exp(\sigma(B_t - B_{t-1}))$ and $S_t = s_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t) = s_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma(B_t - B_0))$ are independent $\forall\, t \geq 0$.[10] Let us now consider the log-gross returns $R_t = (\mu - \frac{\sigma^2}{2}) + (\sigma(B_t - B_{t-1}))$. It is thus easy to see that $R_t$ has a gaussian distribution $\mathcal{N}((\mu - \frac{\sigma^2}{2}), \sigma^2)$. Therefore, to find the $\mu_i$'s and $\sigma_i$'s, we just have to compute the mean and the variance of the empirical log-gross returns which we respectively denote by $\mu_{EMP}$ and $\sigma^2_{EMP}$.

$$\begin{cases} \sigma^2 & = \sigma^2_{EMP} \\ \mu & = \mu_{EMP} + \frac{\sigma^2_{EMP}}{2} \end{cases} \tag{4.8}$$

[10]This is the formal justification behind the well-known fact that under the Black-Scholes model, the gross return at a time step is independent of the price of the stock at that date.




| | Mean | Variance |
|---|---|---|
| [1,] | 0.00503974303396341 | 0.00853615951499204 |
| [2,] | 0.00240090346872357 | 0.00181806384111639 |
| [3,] | 0.013715290665929 | 0.0244350608027529 |
| [4,] | -0.000196166656921266 | 0.00139803488154622 |
| [5,] | -0.000196166656921266 | 0.00139803488154622 |

Figure 4.11 – Log-gross returns on the stock, the $\mu$'s and the $\sigma^2$'s.

Now that we have the $\mu$'s and the $\sigma^2$'s for the stocks, we will simulate geometric Brownian motions with these parameters, and compare the results to the original data. We will run several, say 5, simulations and make a few observations. Below the corresponding plots can be found with the **yellow line representing each time the simulated geometric Brownian motion** :

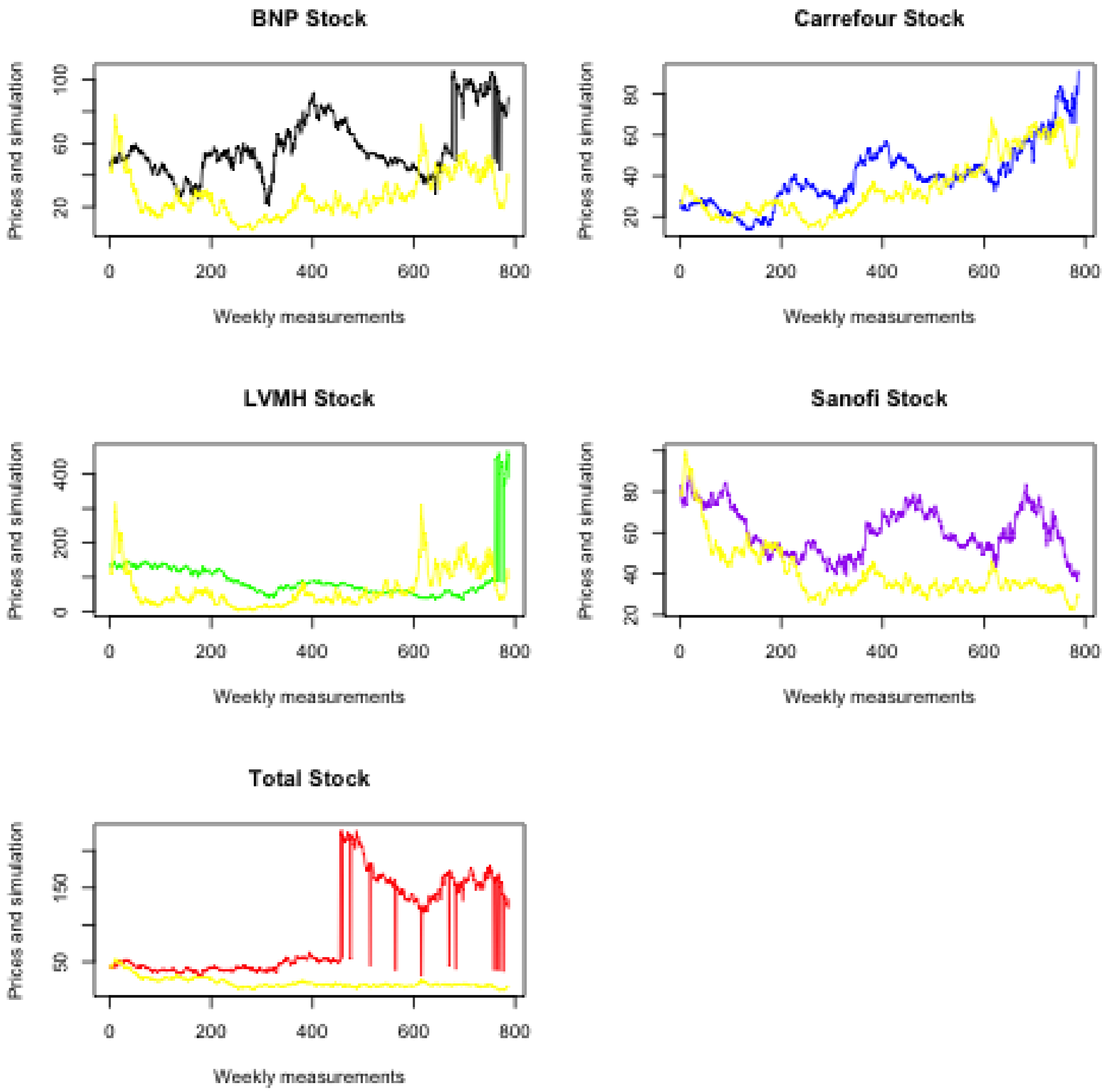



Figure 4.12 – The actual stock prices and a $1^{st}$ simulation of the corresponding geometric Brownian motions.

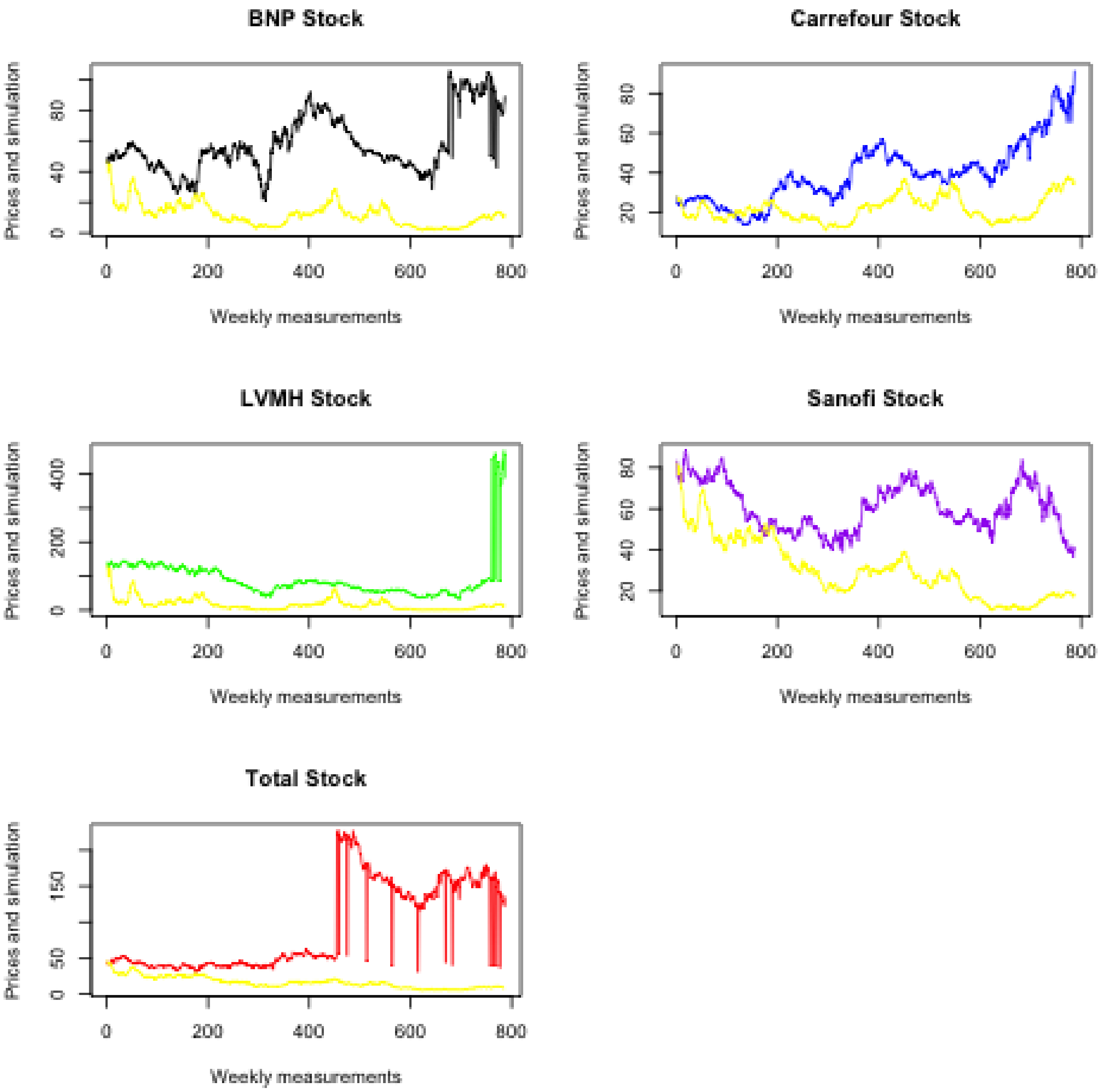



Figure 4.13 – The actual stock prices and a $2^{nd}$ simulation of the corresponding geometric Brownian motions.

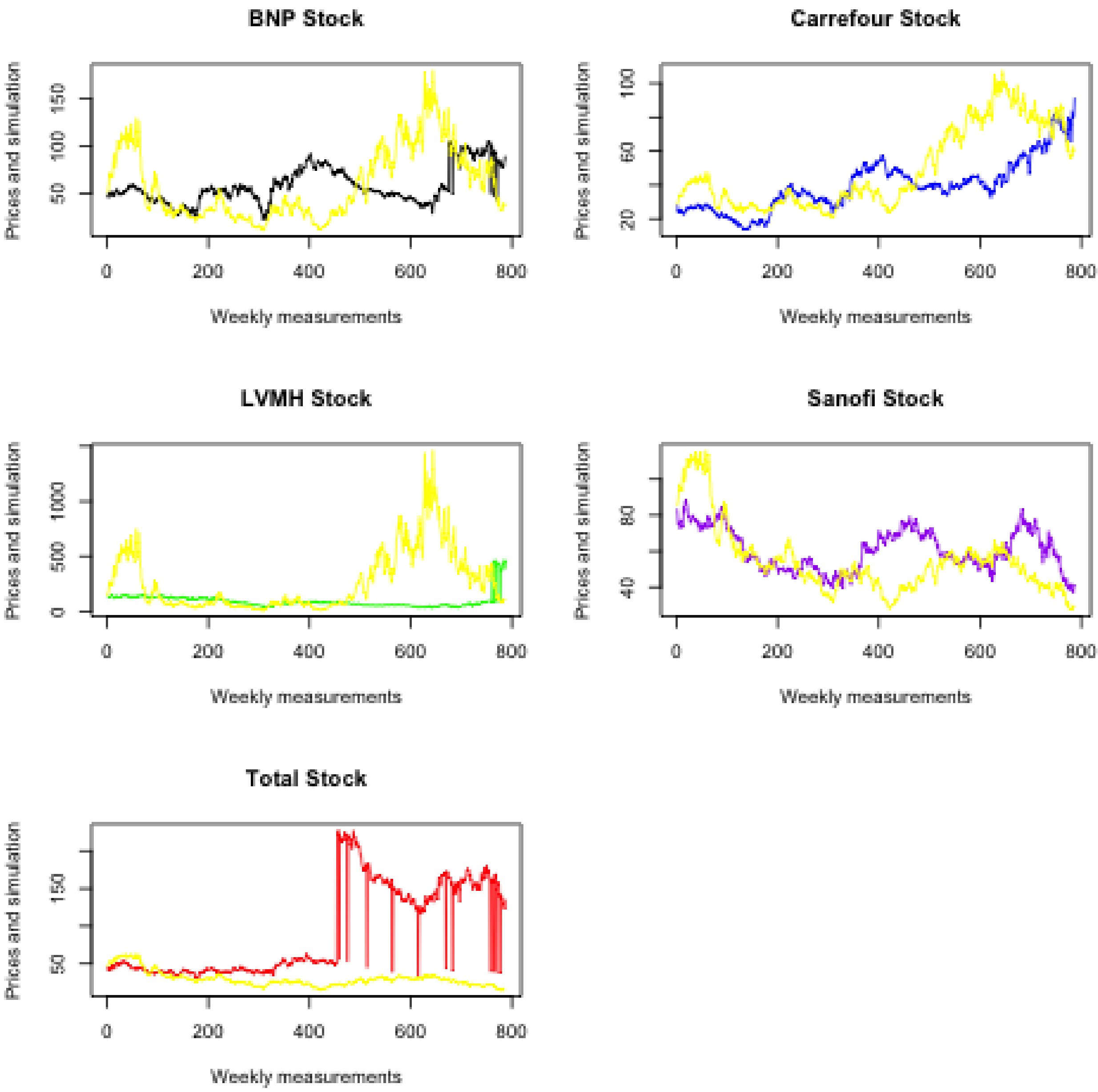



Figure 4.14 – The actual stock prices and a $3^{rd}$ simulation of the corresponding geometric Brownian motions.

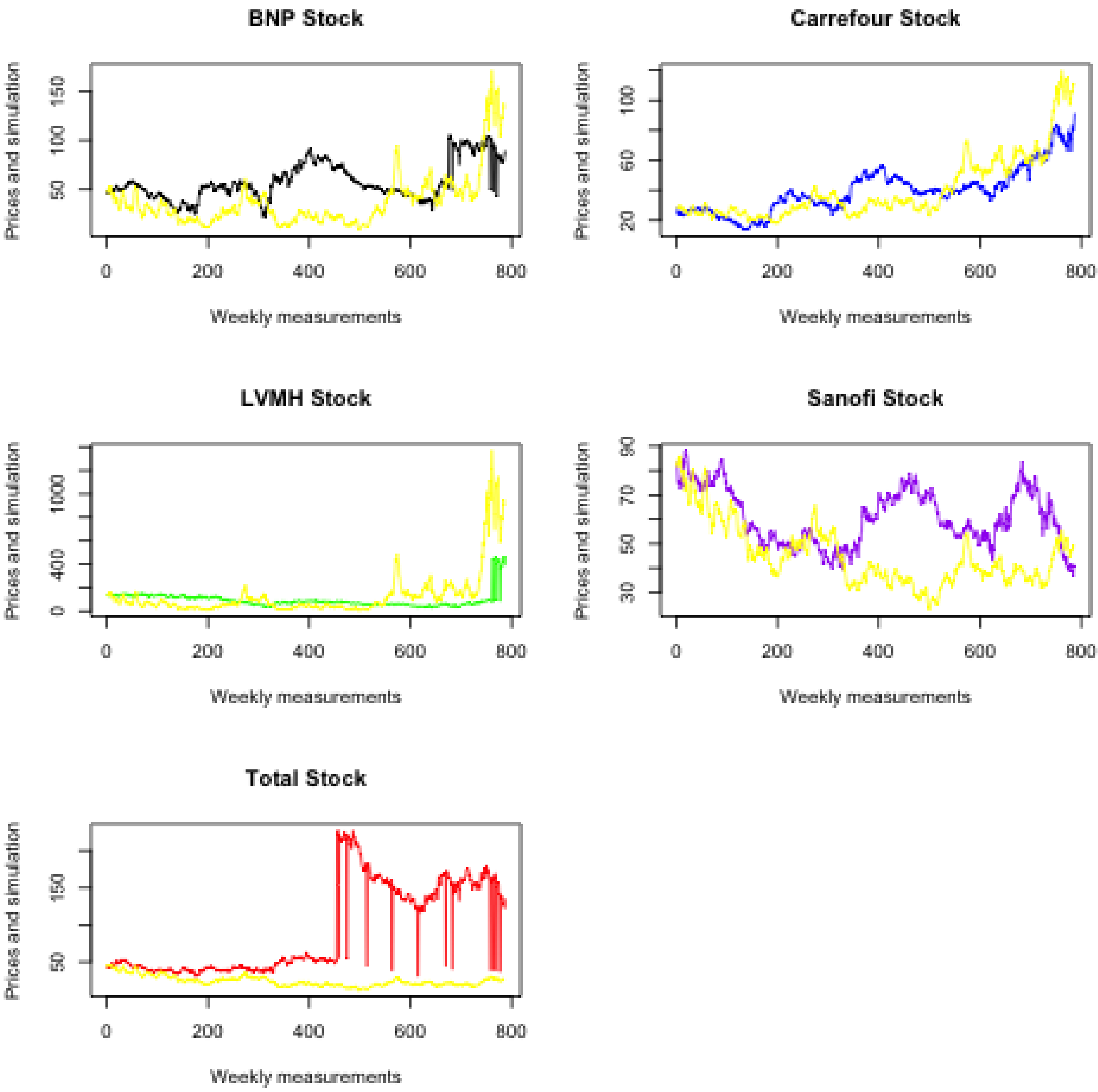



Figure 4.15 – The actual stock prices and a $4^{th}$ simulation of the corresponding geometric Brownian motions.

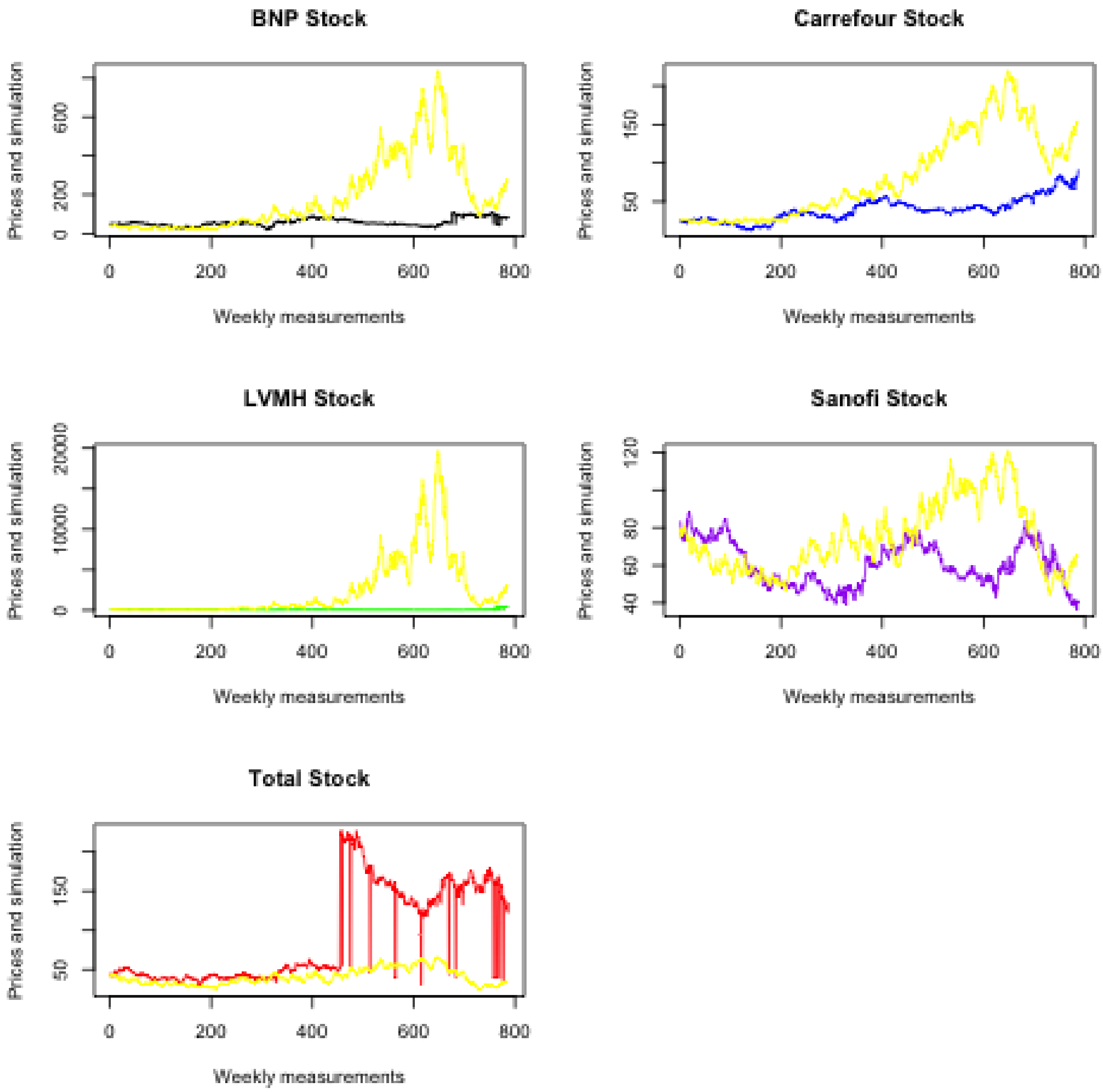



Figure 4.16 – The actual stock prices and a $5^{th}$ simulation of the corresponding geometric Brownian motions.

### 4.3.1 Observations

As we can see, using a geometric Brownian motion to model stock prices yields rather good results though the last simulation displays a case of (gross) overestimation of the prices. However, let us note that this true for the BNP, Carrefour and Sanofi stocks. If we have a look at the Total stock, we see that from week 500 or so the stock price varies quickly widely. The geometric Brownian motion is unable to cope with this behaviour, all five simulations are outright failures. There is a similar issue with the LVMH stock at the end of the series after a rather flat behaviour for most of the duration of the observations, which may explain why the simulations are poor in the case of that stock too.
As a consequence, we have decided to cut the LVMH and Total data so as to keep only the part of the time series where the stock prices are not too volatile, and then apply our simulation to the cut time series. As we can see below, the result is better[11]

[11] The LVMH stock, though, is still more volatile than the Total stock. That explains why the simulation is poorer for the former than for the latter.

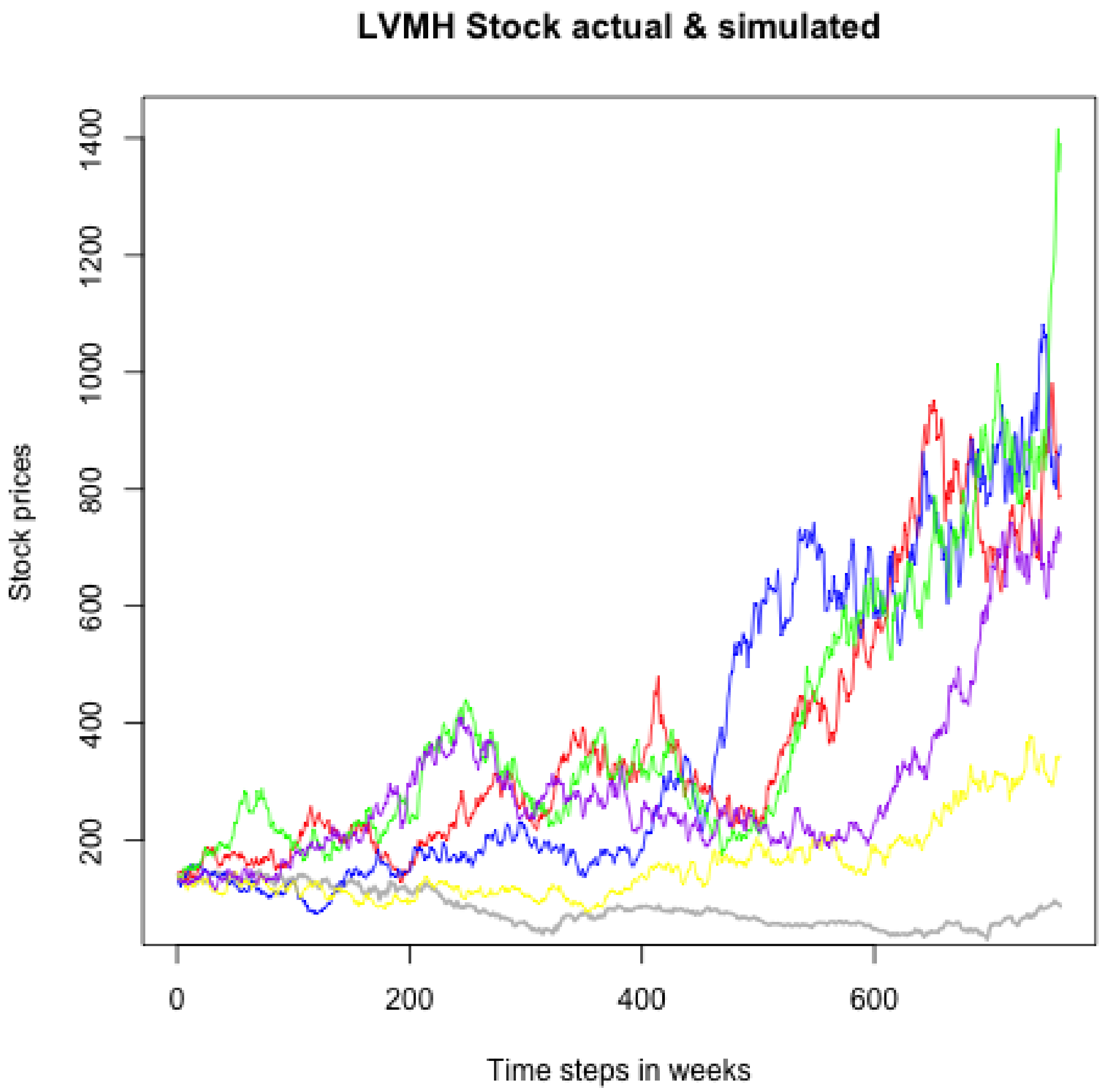



Figure 4.17 – Cut LVMH stock price time series. In **grey** the actual prices, in colour five different runs of the simulation

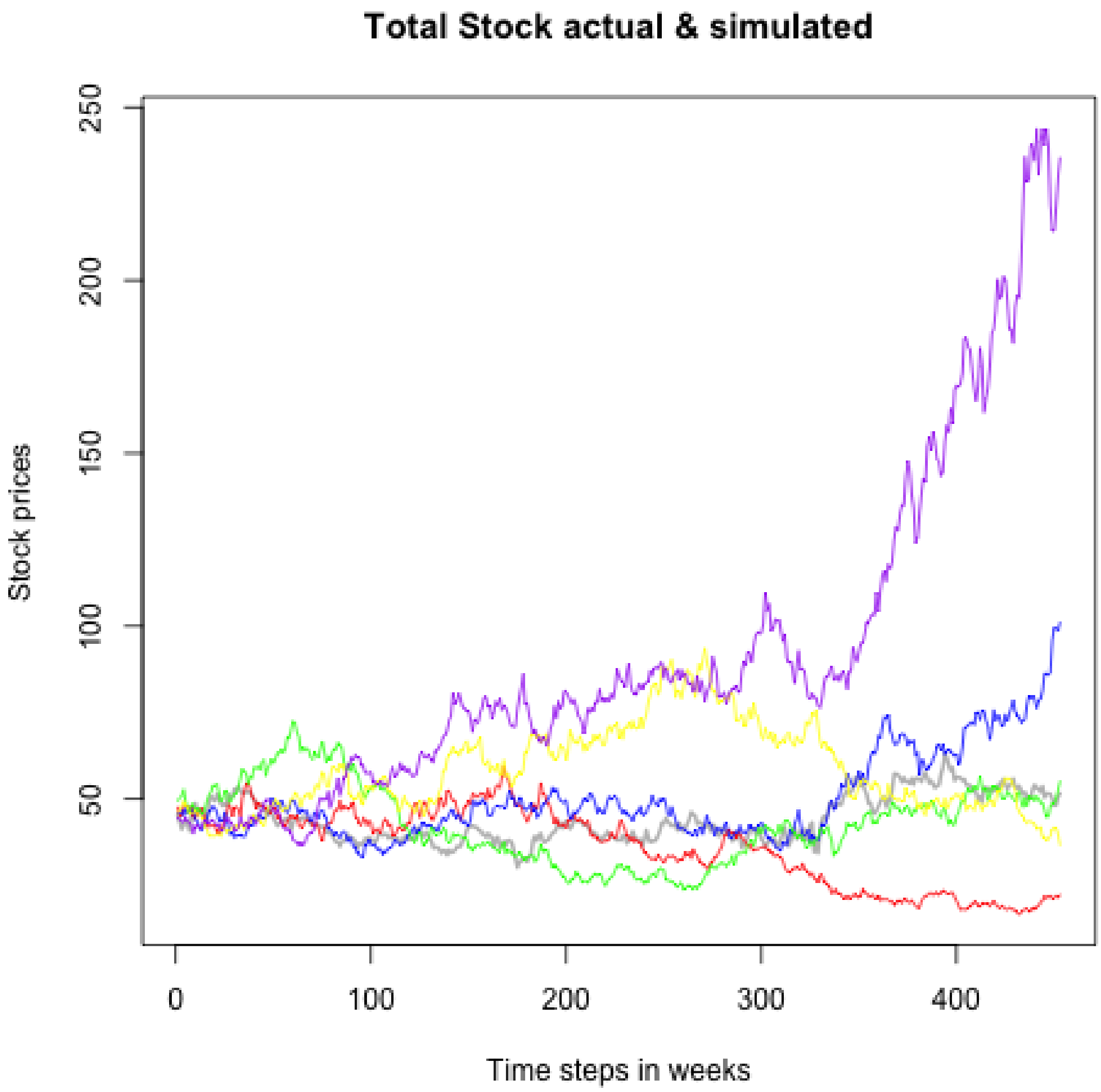



Figure 4.18 – Cut Total stock price time series. In **grey** the actual prices, in colour five different runs of the simulation

We can even get a better simulation if we choose a smaller time increment for it : the actual data is weekly, but in our simulation we can choose to take a daily time increment if we so desire. As shown below, it improves a bit the quality of the simulation.

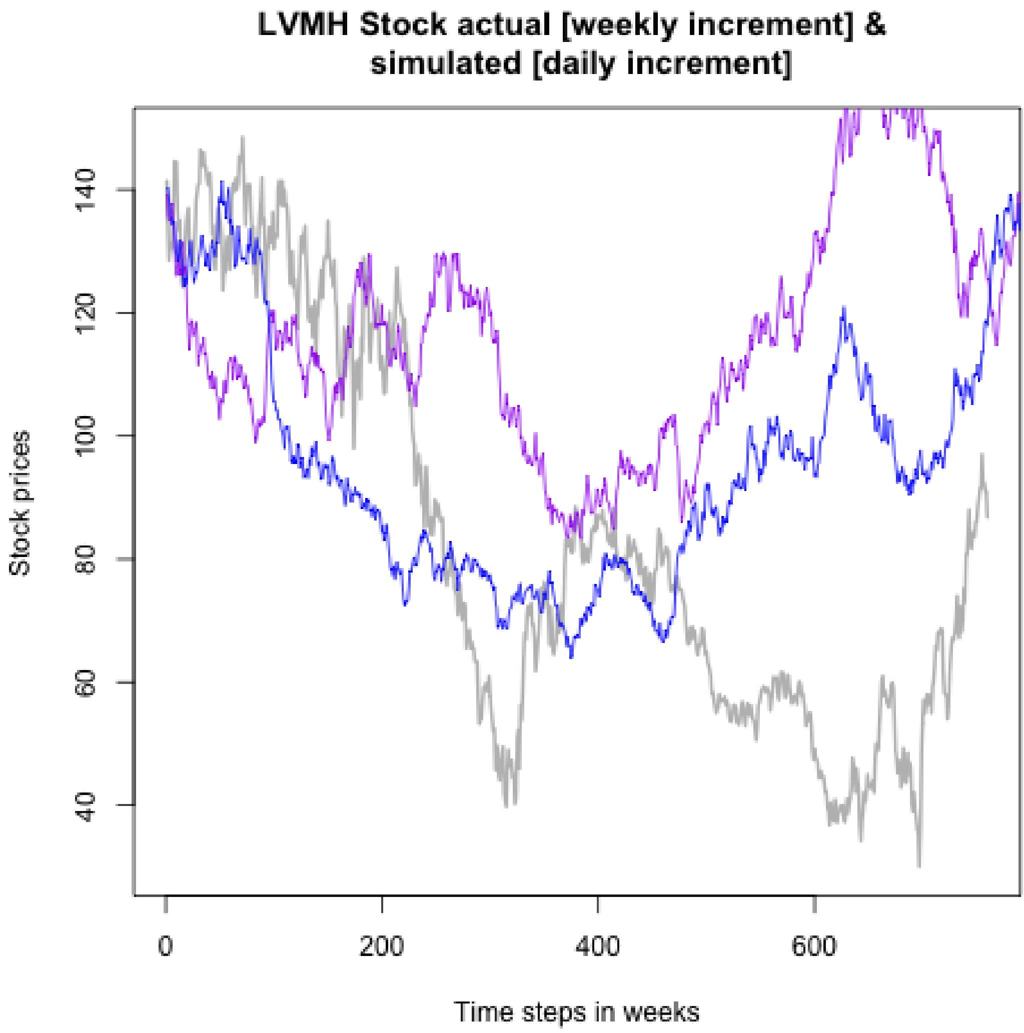



Figure 4.19 – Cut LVMH stock price time series. In **grey** the actual weekly prices, in colour two runs of the daily simulation

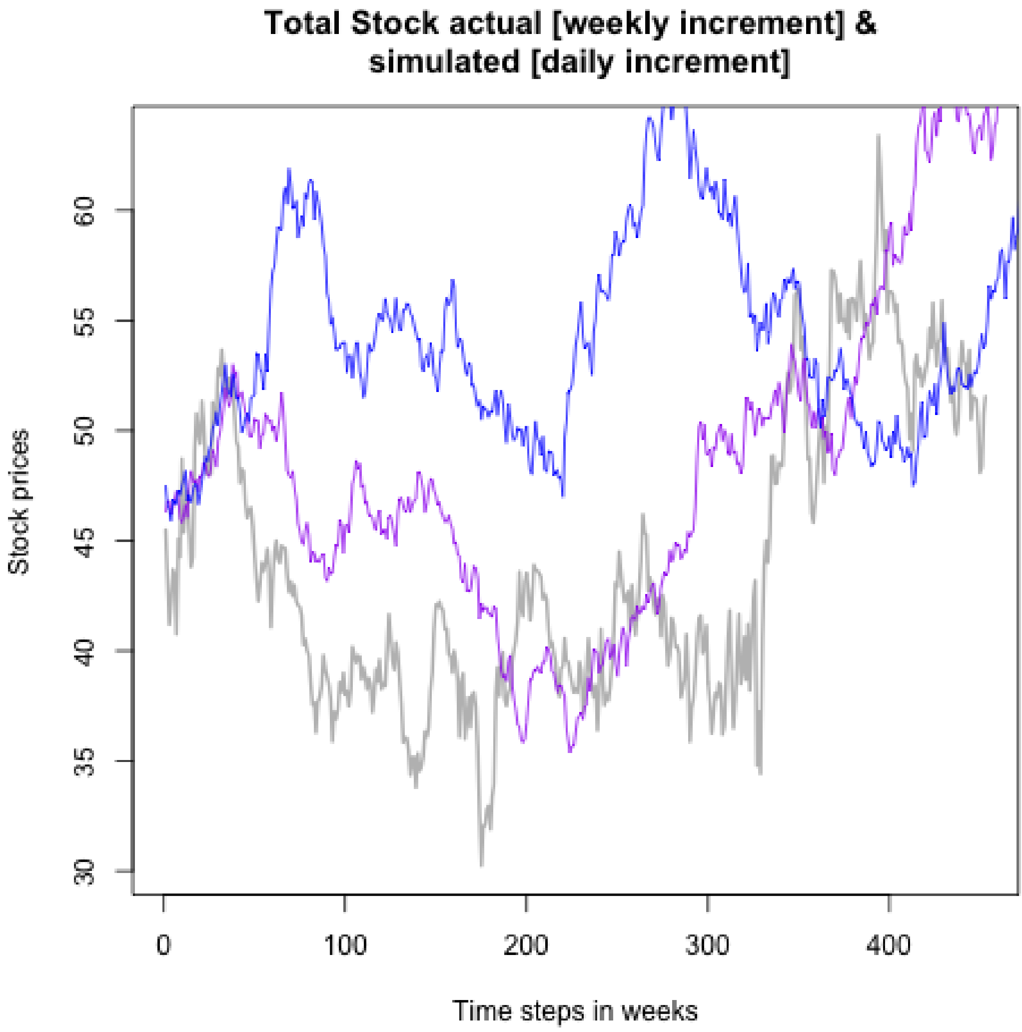



Figure 4.20 – Cut Total stock price time series. In **grey** the actual weekly prices, in colour two runs of the daily simulation

## 4.4 Fitting data above a threshold

Finally, we are going to try and fit a generalised extreme value distribution to our data. For each stock, we keep only the data that is above the empirical 95 %-quantile. We then call the function *gev.fit* and *gev.diag* from the R package *ismev*[12]. We are particularly interested in the density plot thus obtained, as they will allow us to conclude as to the distribution fitting the data.

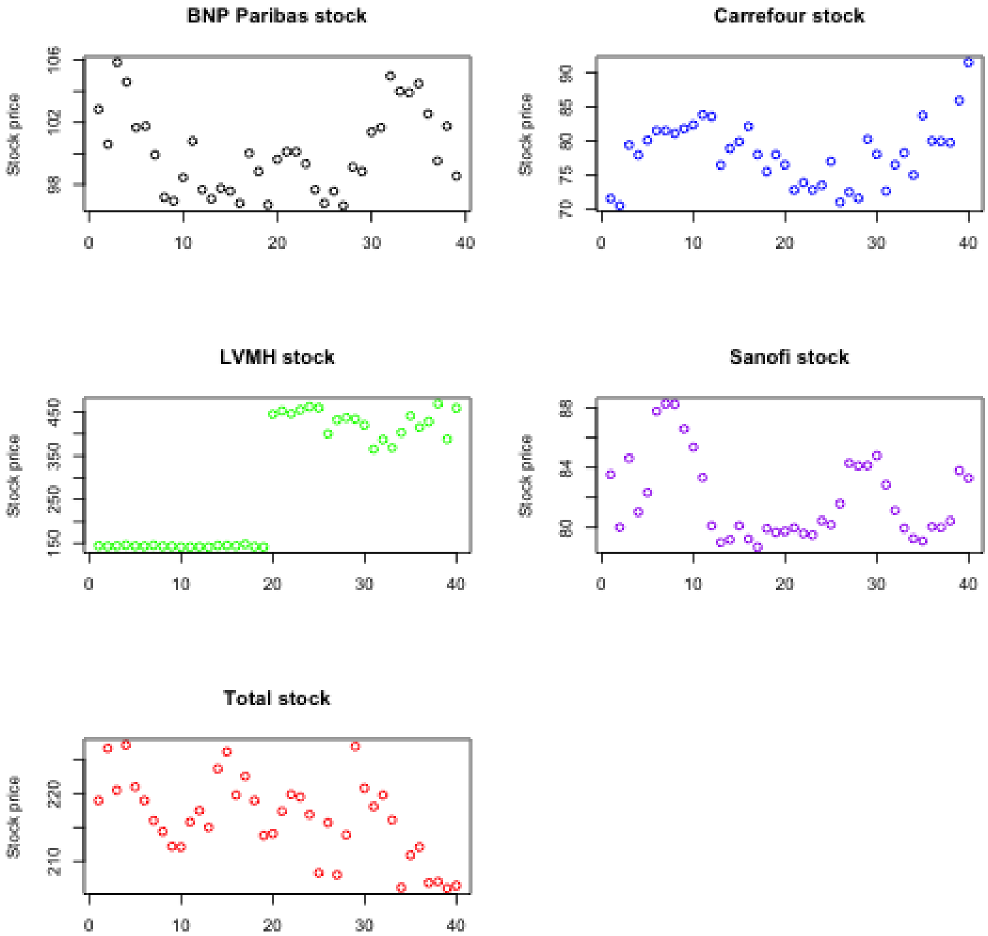



Figure 4.21 – Data above the 95 %-quantile for the BNP Paribas, Carrefour, LVMH, Sanofi and Total stocks.

The pattern followed by the LVMH data is most dissimilar to that followed by the other stocks. It suggests that the fitting may yield poor results in that case.

[12] It is based on maximum-likelihood fitting for the extreme value distribution. We have to rely on numerical resolution as there is no explicit solution to the system of maximum likelihood equations.

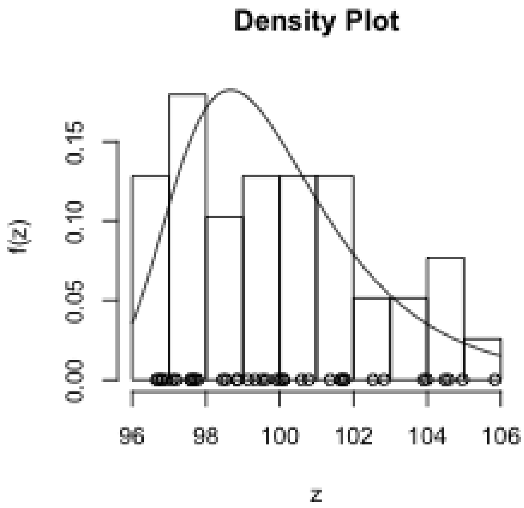



Figure 4.22 – Density fitting for the BNP Paribas data - $\hat{\gamma}_{MLE} = 0.052$

**BNP Paribas stock**

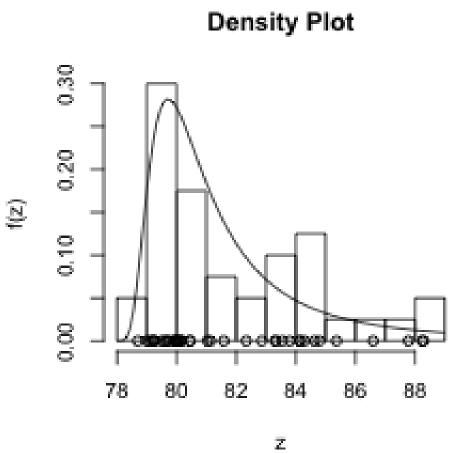



Figure 4.23 – Density fitting for the Sanofi data - $\hat{\gamma}_{MLE} = 0.48$

**Sanofi stock**

The curves are skewed to the right (it is particularly obvious in the case of the Sanofi stock). It corresponds to the case $\gamma > 0$ in the generalised extreme value distribution representation $G_\gamma$ or, equivalently, to a Fréchet-type distribution. The values of the estimate of $\gamma$ are in both cases positive, which confirms that which a naked-eye observation seemed to suggest.

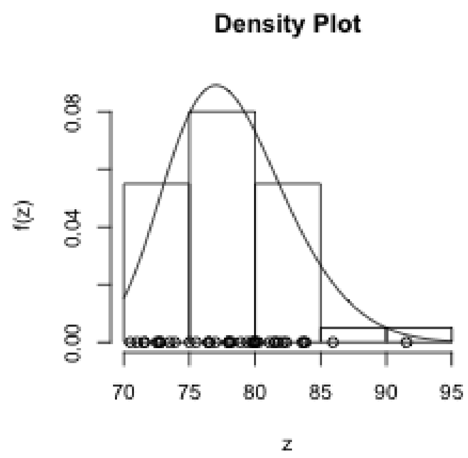



Figure 4.24 – Density fitting for the Carrefour data - $\hat{\gamma}_{MLE} = -0.16$

**Carrefour stock**

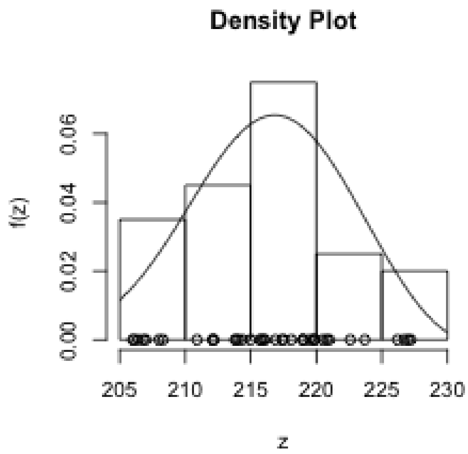



Figure 4.25 – Density fitting for the Total data - $\hat{\gamma}_{MLE} = -0.34$

**Total stock**

The values of the estimate of $\gamma$ are in both cases negative here. Both the Carrefour and the Total above-the-threshold data thus fit a Weibull distribution (corresponds to the case $\gamma < 0$ in the generalised extreme value distribution representation $G_\gamma$). Let us note that the shapes of the curves alone might have been misleading !

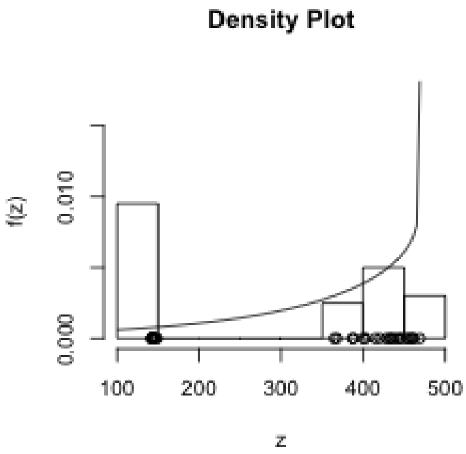



Figure 4.26 – Density fitting for the LVMH data - 95 % quantile

**LVMH stock - 95 % quantile**

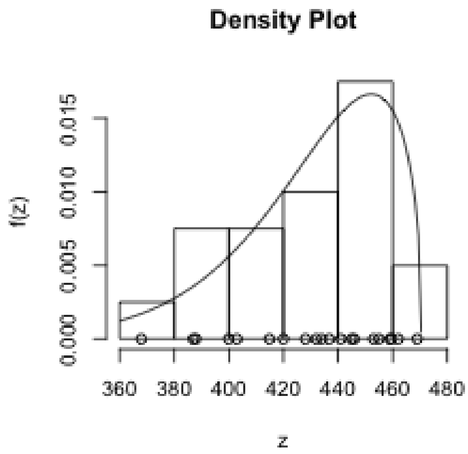



Figure 4.27 – Density fitting for the LVMH data - 97.5 % quantile - $\hat{\gamma}_{MLE} = -0.71$

**LVMH stock - 97.5 % quantile**

As we suspected the structure of the above the 95 %-quantile data for the LVMH stock makes the fitting arduous. To say the least, it is hard to recognize an extreme value distribution. If we take the data above the 97.5 %-quantile, however, the shape of the curve is that of a generalised extreme value distribution with $\gamma < 0$ that is a Weibull-type distribution. The estimate $\hat{\gamma}_{MLE} = -0.71$ supports that observation.

# 5 Conclusion

Statistics usually revolves around the behaviour of the mean. The central limit theorem is an example of this. Extreme value theory takes a very different approach, as it is the study of the behaviour of the maxima. The central limit theorem finds its extreme value counterpart in the Fisher-Tippett-Gnedenko and Von Mises' theorems. Together they make up the answer to the extremal problems : knowing what the possible limiting distributions are and knowing under which conditions there is convergence to those distributions.

Using the Black-Scholes with time independent coefficients is actually a rather good model for the simulation of stock prices, on the condition that the actual stock is not too volatile. As we have been able to see, should the actual stock display huge jumps for instance, the simulation would eventually be a poor approximation of the actual prices. Beyond the scope of the project, let us mention that stochastic jump processes are usually considered a better, more sophisticated model for stock prices. This is something we now set out to study more in-depth.
A result of note bridging the gap between on the one hand extreme value theory and finance on the other hand is that it if we consider the stock prices above a high quantile (*e.g.* 95 % or 97.5 %), the empirical distributions fit extreme values distributions.

From a more personal perspective, I must say that this project has been a wonderful experience. It was the opportunity to get back to R, to do more stochastic calculus and to get started in extreme value theory. Last but not least, following a course in class, with lecture notes that were originally conceived as such is not always easy. Working on one's own on reference books, the scope of which is often much broader than what one would need is another experience entirely however. This project has taught me precious lessons, and I will not forget them.

# Derivations and proofs

## The two extremal problems

### The extremal limit problem

#### Introduction

Hereafter are set forth the derivations for the part on the limiting distributions.

**Our approach** We are here interested in proving a result of convergence in distribution $Y_n \xrightarrow[n\to+\infty]{d} Y$. Anyone who has dabbled in probability knows that one of the classic ways to prove such a result is to use the definition and show that $\forall x$ in which $F_Y$ is continuous, $F_{Y_n}(x) \xrightarrow[n\to+\infty]{} F_Y(x)$. Anyone who has dabbled in statistics knows that in the context of discrete random variables $(Y_n)_{n\geq 0}$, Y it is easier to show the convergence of the probability mass functions i.e. that $\forall x, f_{Y_n}(x) \xrightarrow[n\to+\infty]{} f_Y(x)$ and conclude by Scheffé's lemma that $(Y_n)_{n\geq 0}$ converges in distribution to Y. In the context of Extreme Value Theory, though, we will use a result based on the convergence of expectations. It is the Helly-Bray theorem :
$Y_n \xrightarrow[n\to+\infty]{d} Y \iff \forall g$ continuous, bounded and real-valued functions, $\mathrm{E}(g(Y_n)) \xrightarrow[n\to+\infty]{} \mathrm{E}(g(Y))$.

**Remark** Our approach here is based upon the notes taken while reading "Chapter 2 : The Probabilistic side of extreme Value Theory" of Statistics of Extremes, Theory and Applications by Beirlant *et alii*. **We thus want to underline that the approach here is not our own, and that we are not engaging in plagiarism but giving account of the work done on our bibliographic sources.**

### Solving the problem - step 1

**The answer to the extremal limit problem** It turns out that all possible non-degenerate limiting distributions i.e. all extreme values distribution make up a one-parameter family $G_\gamma(x) = \exp(-(1+\gamma x)^{-\frac{1}{\gamma}})$, where the support of G is the set $\{x : 1+\gamma x > 0$ and $\gamma \in \mathbb{R}$ is the

**E**xtreme **V**alue **I**nded or **EVI**. This is what we set out to show here.

**In the context of our problem** $Y_n = M_n^* = \frac{M_n - b_n}{a_n}$ and $Y = Y_\gamma$. $Y_n \xrightarrow[n\to+\infty]{d} Y \iff \forall g$ continuous, bounded and real-valued functions, $\mathrm{E}(g(Y_n)) \xrightarrow[n\to+\infty]{} \int_{-\infty}^{+\infty} g(v)\,\mathrm{d}G_\gamma(v)$.

$$\begin{aligned} &\Pr(\{M_n^* \le x\}) = F_X(x)^n \\ \implies &\mathrm{E}(g(\frac{M_n - b_n}{a_n})) = n\int_{-\infty}^{+\infty} g(\frac{x-b_n}{a_n}) F_X(x)^{n-1}\,\mathrm{d}F_X(x) \end{aligned} \tag{1}$$

$F_X : \mathbb{R} \to [0,1]$, then if $F_X$ is assumed to be continuous[1], by the Intermediate Value Theorem there exists in particular a solution to the equation $F(x) = 1 - \frac{v}{n}$. This is equivalent to the equation $x = U(\frac{n}{v}) = Q(1-\frac{v}{n})$ where $U$ is the tail quantile function of $F_X$ and $Q$ is the quantile function of $F_X$.

$$\begin{aligned} F_X(x) &= Q(1-\frac{v}{n}) \\ \implies \mathrm{d}F_X(x) &= -\frac{1}{n}\mathrm{d}v \\ \implies n\int_{-\infty}^{+\infty} g(\frac{x-b_n}{a_n}) F_X(x)^{n-1}\,\mathrm{d}F_X(x) &= n\int_n^0 g(\frac{U(\frac{n}{v})-b_n}{a_n})(1-\frac{v}{n})^{n-1}(-\frac{1}{n})\,\mathrm{d}v \\ \implies \mathrm{E}(g(\frac{M_n-b_n}{a_n})) &= \int_0^n g(\frac{U(\frac{n}{v})-b_n}{a_n})(1-\frac{v}{n})^{n-1}\,\mathrm{d}v \end{aligned} \tag{2}$$

where $x = U(\frac{n}{v}) = Q(1-\frac{v}{n})$ goes from $-\infty$ to $+\infty$. Indeed,

- for $v = n$, $x = U(1) = Q(0) = -\infty$
- for $v = 0^+$, $x = U(+\infty) = Q(1) = +\infty$

which explains the change of bounds in the integral in the third equation. We are now working on $\int_0^n g(\frac{U(\frac{n}{v})-b_n}{a_n})(1-\frac{v}{n})^{n-1}\,\mathrm{d}v$. If we take the limit $n \to +\infty$, in particular :

- the integral will be taken between 0 and $+\infty$
- $(1-\frac{v}{n})^{n-1} \xrightarrow[n\to+\infty]{} (\exp(v))^{-1} = exp(-v)$

A limit can be obtained for $\mathrm{E}(g(\frac{M_n-b_n}{a_n}))$ when for some sequence of positive numbers $(a_n)_{n\ge0}$, $\frac{U(\frac{n}{v})-b_n}{a_n}$ is convergent $\forall v \ge 0$[2]

[1] An assumption that will be satisfied in all what follows

[2] Beirlant underlines that if we take $v = 1$, we get the idea that taking $(b_n)_{n\ge0} = (U(n))$ will guarantee it works.

**A preparatory phase** A condition that must be imposed is the condition $(\mathscr{C})$ : "For some positive function $a$ and $\forall u > 0$, $\frac{U(xu)-U(x)}{a(x)} \xrightarrow[x\to+\infty]{} h(u)$ where h is not identically equal to 0".
Proposition : The possible limits in $(\mathscr{C})$ are given by $ch_\gamma(u) = c\int_1^u v^{\gamma-1}\,\mathrm{d}v = c\frac{u^\gamma-1}{\gamma}$, where $c > 0$, $\gamma \geq 0$ and $h_0(u)$ is interpreted as $\log(u)$.
**The case c > 0 can be reduced to $c = 1$ by incorporating $c$ into $a$.**
$\forall u, v > 0$,

$$\frac{U(xuv)-U(x)}{a(x)} = \frac{U(xuv)-U(xu)}{a(xu)}\frac{a(xu)}{a(x)} + \frac{U(xu)-U(x)}{a(x)} \tag{3}$$

If $(\mathscr{C})$ holds then $\frac{a(xu)}{a(x)}$ converges to a $g(u)$.
$\forall u, v > 0$,

$$\begin{aligned} \frac{a(xuv)}{a(x)} &= \frac{a(xuv)}{a(xv)}\frac{a(xv)}{a(x)} \\ \implies \frac{a(xuv)}{a(x)} &= \frac{a((xv)u)}{a(xv)}\frac{a(xv)}{a(x)} \\ \xrightarrow[x\to+\infty]{} g(uv) &= g(u)g(v) \end{aligned} \tag{4}$$

We recognize the Cauchy Functional Equation, if a function $g$ satisfies this equation, then $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is of the form $u^\gamma$, where $\gamma$ is a real number. Now, if we write $a(x) = x^\gamma l(x)$,

$$\begin{aligned} \frac{a(xuv}{a(x)} &= \frac{(xu)^\gamma}{x^\gamma}\frac{l(xu)}{l(x)} \\ &= u^\gamma \frac{l(xu)}{l(x)} \end{aligned} \tag{5}$$

For this quantity to converge to $u^\gamma$, $\frac{l(xu)}{l(x)}$ must converge to 1. $u$ being a positive number, this is equivalent to the fact that $l$ must be a *slowly varying* function. Hence, $a(x) = x^\gamma l(x)$ is a *regularly varying* function with index of regular variation $\gamma$.

$$\begin{aligned} \frac{U(xuv)-U(x)}{a(x)} &= \frac{U(xuv)-U(xu)}{a(xu)}\frac{a(xu)}{a(x)} + \frac{U(xu)-U(x)}{a(x)} \\ \xrightarrow[x\to+\infty]{} h_\gamma(uv) &= h_\gamma(v)u^\gamma + h_\gamma(u) \end{aligned} \tag{6}$$

- If $\gamma = 0$

$$\begin{aligned} h_0(uv) &= h_0(u) + h_0(v) \\ \implies \exists c \in \mathbb{R} : h_0(u) &= c\log(u) \end{aligned} \tag{7}$$

- <u>If $\gamma \neq 0$</u>

$$h_\gamma(uv) = h_\gamma(v)u^\gamma + h_\gamma(u) \tag{8}$$

And by symmetry,

$$h_\gamma(uv) = h_\gamma(u)v^\gamma + h_\gamma(v) \tag{9}$$

Hence,

$$\begin{aligned} & h_\gamma(u)v^\gamma + h_\gamma(v) = h_\gamma(v)u^\gamma + h_\gamma(u) \\ \iff & h_\gamma(u)(v^\gamma - 1) = h_\gamma(v)(u^\gamma - 1) \\ \implies & \exists d : h_\gamma(u) = d(u^\gamma - 1) \end{aligned} \tag{10}$$

where $d$ is a constant, if we take $d = \frac{1}{c\gamma}$, we get $ch_\gamma(u) = \frac{u^\gamma - 1}{\gamma}$

To conclude,

$$\frac{U(xu) - U(x)}{a(x)} \xrightarrow[x \to +\infty]{} h(u) \implies h(u) = ch_\gamma(u) \tag{11}$$

for some constant $c$, with the auxiliary function $a$ regularly varying with index $\gamma$.

**Back to the search of an explicit form for the limiting distributions** Let us assume that the condition ($\mathscr{C}$) holds, with $b_n = U(n)$ and $a_n = a(n)$.

$$\begin{aligned} \mathrm{E}(g(\frac{M_n - b_n}{a_n})) &= \int_0^n g(\frac{U(\frac{n}{v}) - b_n}{a_n})(1 - \frac{v}{n})^{n-1}\,\mathrm{d}v \\ &= \int_0^n g(\frac{U(n\frac{1}{v}) - U(n)}{a(n)})(1 - \frac{v}{n})^{n-1}\,\mathrm{d}v \\ &\xrightarrow[x \to +\infty]{} \int_0^{+\infty} g(h_\gamma(\frac{1}{v}))\exp(-v)\,\mathrm{d}v \end{aligned} \tag{12}$$

The last integral must be equal to $\int_{-\infty}^{+\infty} g(u)\,\mathrm{d}G_\gamma(u)$

**Solving the problem - step 2 : expliciting the $G_\gamma$ in earnest**

**Case $\gamma = 0$** We make the following change of variables : $u = h_0(\frac{1}{v}) = \log(\frac{1}{v}) = -\log(v)$.
$u = \log(v) \iff v = \exp(-u)$, with $u \in \mathbb{R}$, $v \in ]0, +\infty[$.

$$\begin{aligned}
\mathrm{d}u &= -\frac{1}{v}\mathrm{d}v \\
\implies \int_0^{+\infty} g(u)\exp(-v)\,\mathrm{d}v &= \int_0^{+\infty} g(u)\exp(-\exp(-u))(-v)\,\mathrm{d}u \\
\implies \int_0^{+\infty} g(u)\exp(-v)\,\mathrm{d}v &= \int_0^{+\infty} g(u)\exp(-\exp(-u))(-\exp(-u))\,\mathrm{d}u \\
\implies \int_0^{+\infty} g(u)\exp(-v)\,\mathrm{d}v &= \int_{-\infty}^{+\infty} g(u)(-\exp(-u)\exp(-\exp(-u)))\,\mathrm{d}u \\
\implies \int_0^{+\infty} g(u)\exp(-v)\,\mathrm{d}v &= \int_{-\infty}^{+\infty} g(u)\,\mathrm{d}(\exp(-\exp(-u)))
\end{aligned} \tag{13}$$

Finally, $\underline{\gamma > 0 \implies G_\gamma(u) = \exp(-\exp(-u))}$

**Case $\gamma \neq 0$** We make the following change of variables :

$$\begin{aligned}
u &= h_\gamma(\frac{1}{v}) = \frac{(\frac{1}{v})^\gamma - 1}{\gamma} \\
\iff (\frac{1}{v})^\gamma &= \gamma u + 1 \\
\iff \frac{1}{v} &= (\gamma u + 1)^{\frac{1}{\gamma}} \\
\iff v &= (\gamma u + 1)^{-\frac{1}{\gamma}}
\end{aligned} \tag{14}$$

$$\begin{aligned}
\mathrm{d}v &= -\frac{1}{\gamma}\gamma(\gamma u + 1)^{-\frac{1}{\gamma}-1}\mathrm{d}u \\
\implies \mathrm{d}v &= -(\gamma u + 1)^{-\frac{\gamma+1}{\gamma}}\,\mathrm{d}u
\end{aligned}$$

$v$ varies from $0^+$ to $+\infty$,

- $\underline{\text{If } \gamma > 0}$, $u = \frac{(\frac{1}{v})^\gamma - 1}{\gamma}$, varying from $-\infty$ to $-\gamma^{-1}$.
- $\underline{\text{If } \gamma < 0}$, $u = -\frac{v^{-\gamma}+1}{-\gamma}$, varying from $-\gamma^{-1}$ to $-\infty$.

$$\begin{aligned}
\int_0^{+\infty} g(u)\exp(-v)\,\mathrm{d}v &= \int_{+\infty/-\gamma^{-1}}^{-\gamma^{-1}/-\infty} g(u)\exp(-(1+\gamma u)^{-\frac{1}{\gamma}})(-(1+\gamma u)^{-\frac{\gamma+1}{\gamma}})\,\mathrm{d}u \\
&= \int_{-\gamma^{-1}/-\infty}^{+\infty/-\gamma^{-1}} g(u)(1+\gamma u)^{-\frac{\gamma+1}{\gamma}}\exp(-(1+\gamma u)^{-\frac{1}{\gamma}})\,\mathrm{d}u
\end{aligned} \tag{15}$$

$$d(\exp(-(1+\gamma u)^{-\frac{1}{\gamma}})) = (1+\gamma u)^{-\frac{\gamma+1}{\gamma}} \exp(-(1+\gamma u)^{-\frac{1}{\gamma}}) du \tag{16}$$

Finally, the last integral in (2.15) is equal to,

- <u>If $\gamma > 0$</u>, $\int_{-\gamma^{-1}}^{+\infty} g(u)\, d(\exp(-(1+\gamma u)^{-\frac{1}{\gamma}}))$
- <u>If $\gamma < 0$</u>, $\int_{-\infty}^{-\gamma^{-1}} g(u)\, d(\exp(-(1+\gamma u)^{-\frac{1}{\gamma}}))$

Finally, to sum it up :

- <u>$\gamma = 0$</u> : **Gumbel distribution**
  $G_\gamma(u) = \exp(-\exp(-u)),\ u \in \mathbb{R}$
- <u>$\gamma > 0$</u> : **Fréchet distribution**
  $G_\gamma(u) = \exp(-(1+\gamma u)^{-\frac{1}{\gamma}}),\ u \in ]-\gamma^{-1}, +\infty[$
- <u>$\gamma > 0$</u> : **Weibull distribution**
  $G_\gamma(u) = \exp(-(1+\gamma u)^{-\frac{1}{\gamma}}),\ u \in ]-\infty, -\gamma^{-1}[$

# Additional figures

**Chapter 4 : Looking into financial data**

| Stock | Cumulative residuals between $\exp(R_t)$ and $1+R_t$ |
|---|---|
| BNP | 3.22E-15 |
| Carrefour | 1.44E-15 |
| LVMH | 4.75E-15 |
| Sanofi | 1.55E-15 |
| Total | 3.86E-15 |

Table 1 – Cumulative residuals between $\exp(R_t)$ and $1+R_t$ for the BNP, Carrefour, LVMH, Sanofi, Total stocks

# Bibliography


- Modelling Extremal Events, Embrechts, Kluppelberg and Milkosch
- Statistics of Extremes, Theory and Application, Beirlant, Goegebeur, Segers and Teugels
- An Introduction to Statistical Modeling of Extreme Values, Coles
- Introduction to Scientific Programming and Simulation Using R, Jones, Maillardet and Robinson

# Recommendations on the use of the bibliography

First and foremost, it must be said that **all the books included in the Bibliography section are good books** (I will focus on the books dealing with Extreme Value Theory ; the book on R programming is a very fine one though, with a nice hands-on approach). However, **they may not be intended for the same uses and the same users**. I am giving my opinion on those books in case someone chances upon the repository and they want to learn more about Extreme Value Theory. **It is only my opinion, nothing more.**

- An Introduction to Statistical Modelling of Extreme Values, by Stuart Coles (Springer) is not surprisingly an introductory book; most of the time, at best a sketch of the proof is given. It may be best used in conjunction with a traditional course given by a teacher that will go more in-depth themselves. I would not recommend it as a primary source for a master thesis project, however.
- Statistics of Extremes, Theory and Applications, by Beirlant, Goegebeur, Segers and Teugels (Wiley) is a balanced book, going enough into details. It is thus fit as a source for a master thesis project. The only thing going against it is the following : many figures to illustrate what is being explained is a good thing, but it would have been nice to detail a bit more the derivations of the proofs, at times.
- Modelling extremal events for Insurance and Finance, by Embrechts, Kluppelberg and Mikosch (Springer) is nothing short of a summa. It is extremely detailed and thorough. I would not recommend it, however, as a source for a master thesis project as it would be a bit too ambitious to use it for that purpose. If you have more time and are ready for an involved reading, all the more so if you are interested in the financial applications of Extreme Value Theory, search for a companion no further!