POLITECNICO
MILANO 1863

DEPARTMENT OF MECHANICAL
ENGINEERING

EXERCISE CLASS 1 (part 2/2)

Review of basic statistical concepts: assumptions check and hypotesis testing (2 samples)

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MIUR 2018-2022

Milano

Hypotesis testing with 2 samples

- One sample tests:
  - Test for mean (known variance): one-sample z-test
  - Test for mean (unknown variance): one-sample t-test
  - Test for variance: chi-squared test (variance)

- Two samples tests
  - Test for mean difference (known var): two-sample z-test
  - Test for mean difference (unknown var): two-sample t-test
  - Test for mean of paired data (unknown var): paired t-test
  - Test for equality of variances: F-test (variances)

Paired t-test

Assumptions:
- $X_{11}, X_{12}, \dots, X_{1n}$ is a random sample of size n from population 1;
- $X_{21}, X_{22}, \dots, X_{2n}$ is a random sample of size n from population 2;
- The differences between pairs, $D_j = X_{1j} - X_{2j}$, are **normal** (or central limit theorem applies);
- The variance of the differences between pairs is **unknown**

Under those assumptions, the quantity:

$$T = \frac{\overline{D} - \mu_D}{S_D/\sqrt{n}}$$

Follows a student-t distribution with $n-1$ degrees of freedom, where: $D_j = X_{1j} - X_{2j} \sim N(\mu_D, \sigma_D^2)$
(NOTICE: we are assuming differences to be random normal var.)

The Paired *t*-Test

Null hypothesis: $H_0\colon \mu_D = \Delta_0$

Test statistic: $$T_0 = \frac{\overline{D} - \Delta_0}{S_D/\sqrt{n}} \tag{5-16}$$

Alternative Hypothesis	Rejection Region
$H_1\colon \mu_D \neq \Delta_0$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1\colon \mu_D > \Delta_0$	$t_0 > t_{\alpha,n-1}$
$H_1\colon \mu_D < \Delta_0$	$t_0 < -t_{\alpha,n-1}$

The $100(1-\alpha)\%$ confidence interval on the difference between the population means is given by:

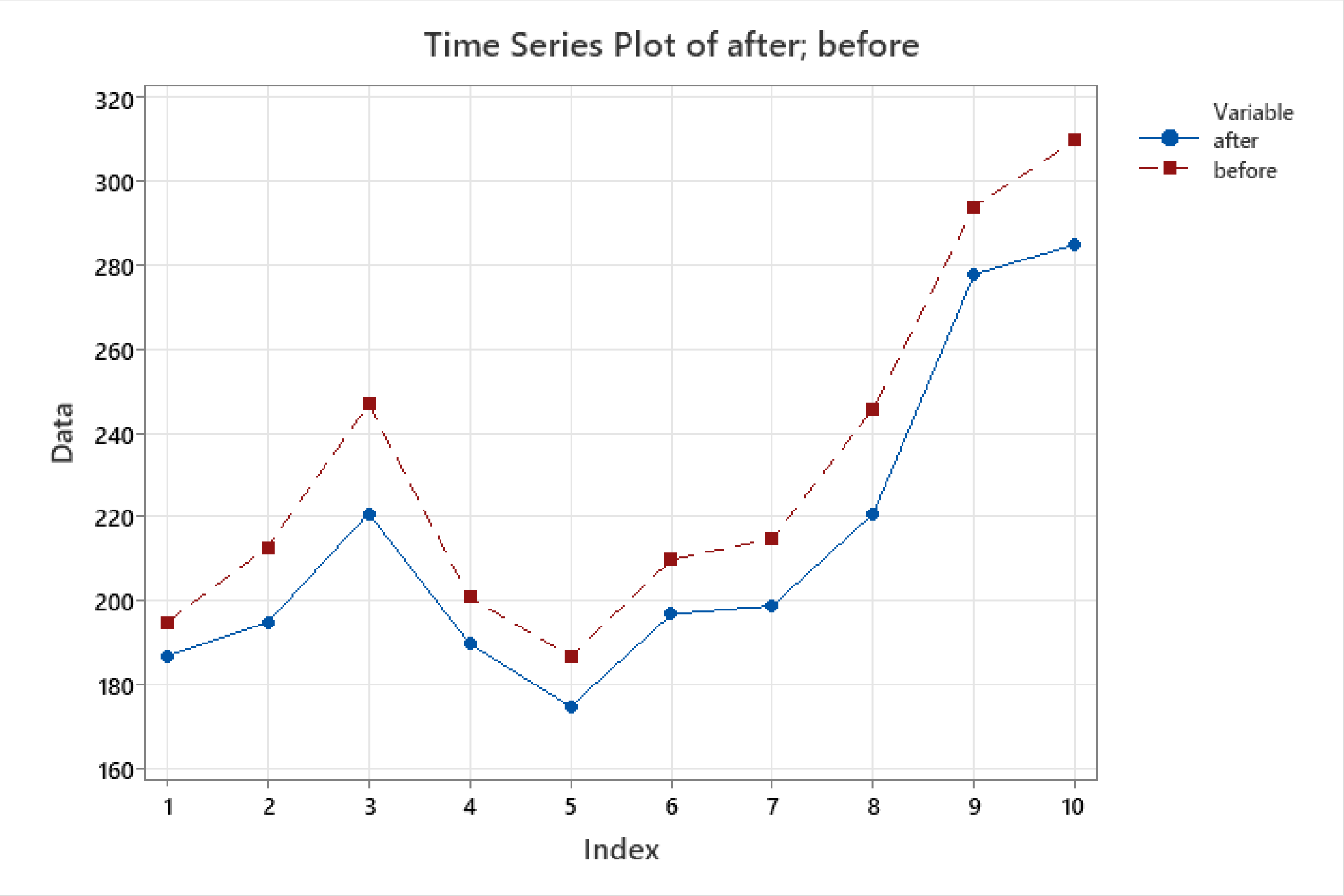
$$\overline{D} - t_{\frac{\alpha}{2},n-1} S_D/\sqrt{n} \le \mu_D \le \overline{D} + t_{\frac{\alpha}{2},n-1} S_D/\sqrt{n}$$

S_D is lower than the pooled standard deviation: paired test is more precise (smaller c.i.)

Exercise 1

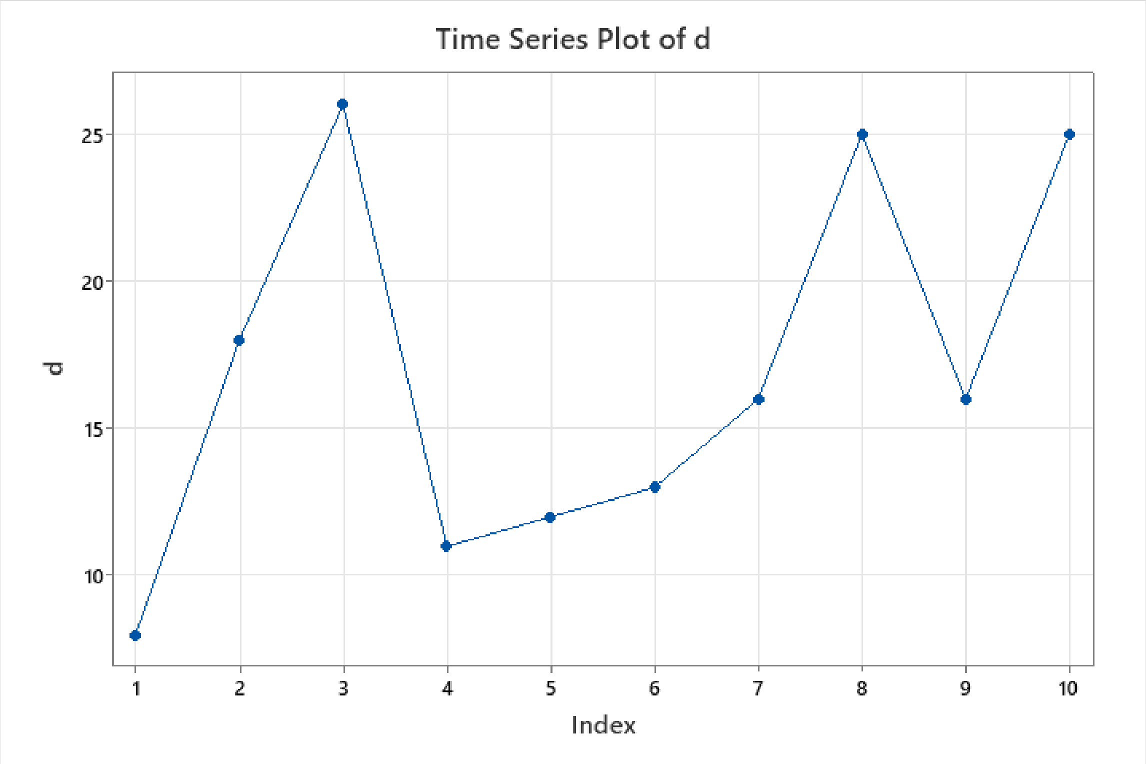
Ten people are involved in a diet program. The weights before and after the program is reported in the table (expressed in pounds, 1 lb =0.454). The data is stored in the file `weights.csv`.

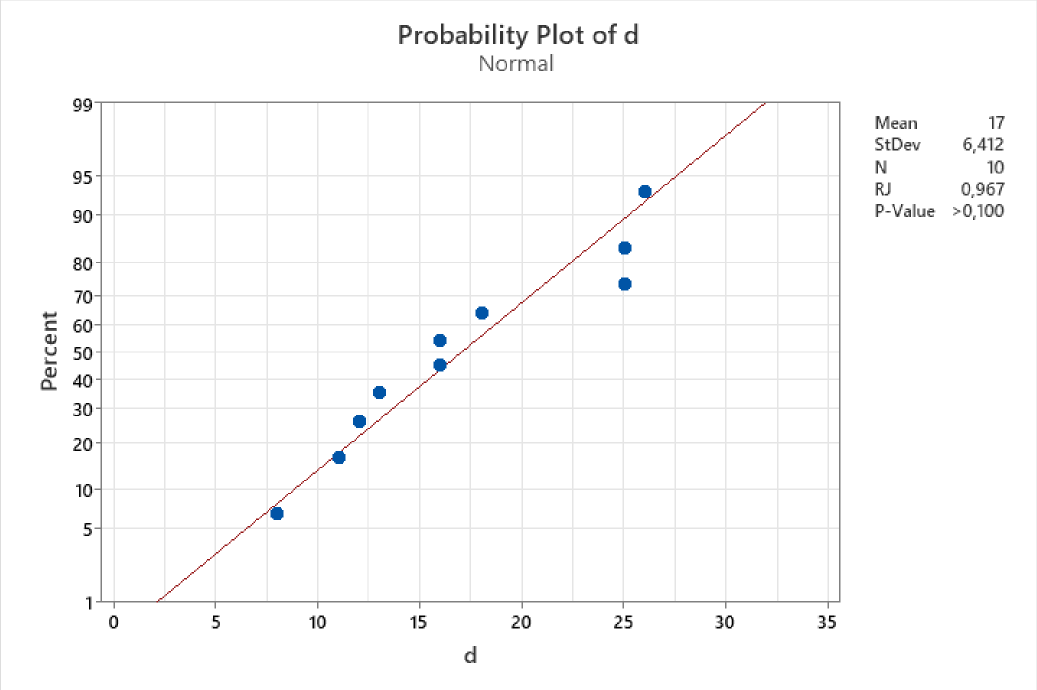
Is there statistical evidence (95%) to state that the diet program was effective? Design a two-sided confidence interval at 95% on the weight difference



Exercise 1

Preliminary activity: data visualization and **assumptions check** (on the difference “d”)





Runs test

Test

Null hypothesis H_0: The order of the data is random

Alternative hypothesis H_1: The order of the data is not random

Number of Runs

Observed	Expected	P-Value
6	5,80	0,888

The p-value may not be accurate for samples with fewer than 11 observations above K or fewer than 11 below.

Exercise 1

Is there statistical evidence (95%) to state that the diet program was effective?
Design a two-sided confidence interval at 95% on the weight difference

Knowing that the data are normally distributed, we can use the t-test to evaluate the following hypothesis: $H_0 : \mu_d = 0$ vs $H_1 : \mu_d > 0$

The t-test statistic is:

$$t_0 = \frac{\bar{d} - \Delta_0}{s_d/\sqrt{n}}$$

where $\bar{d}$ is the sample mean of the differences, s_d is the sample standard deviation of the differences and n is the number of observations.

In this case, $\Delta_0 = 0$

Test

Null hypothesis	H_0: μ = 0
Alternative hypothesis	H_1: μ > 0

T-Value	P-Value
8,38	0,000

Two-sided confidence interval at 95% on 'd':

Descriptive Statistics

N	Mean	StDev	SE Mean	95% CI for μ
10	17,00	6,41	2,03	(12,41; 21,59)

μ: population mean of d

Not paired data and known variances

Assumptions:

- $X_{11}, X_{12}, \dots, X_{1n1}$ is a random sample of size n_1 from population 1;
- $X_{21}, X_{22}, \dots, X_{2n2}$ is a random sample of size n_2 from population 2;
- The two populations are **independent**;
- Both populations are **normal** (or central limit theorem applies);
- The **variances** of the populations are **known**.

Under those assumptions, the quantity:

$$Z = \frac{\overline{X_1} - \overline{X_2} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

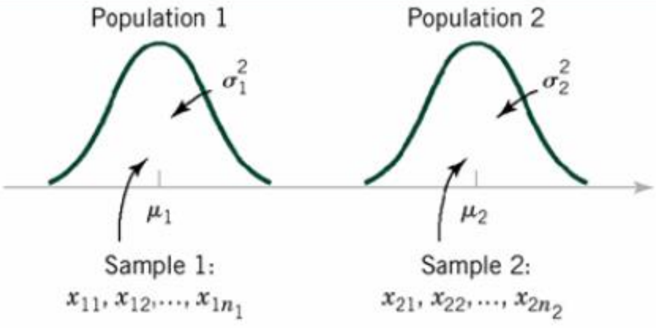
Follows a standard normal distribution, N(0,1)

Testing Hypotheses on the Difference in Means, Variances Known

Null hypothesis: $H_0: \mu_1 - \mu_2 = \Delta_0$

Test statistic: $$Z_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Alternative Hypotheses	Rejection Criterion
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	$z_0 > z_\alpha$
$H_1: \mu_1 - \mu_2 < \Delta_0$	$z_0 < -z_\alpha$



Two independent populations.

Not paired data and equal variances

Assumptions (case 1: $\sigma_1^2 = \sigma_1^2 = \sigma^2$):

- $X_{11}, X_{12}, \ldots, X_{1n1}$ is a random sample of size n_1 from population 1;
- $X_{21}, X_{22}, \ldots, X_{2n2}$ is a random sample of size n_2 from population 2;
- The two populations are **independent**;
- Both populations are **normal** (or central limit theorem applies);
- The **variances** of the populations are **unknown and equal**.

Under those assumptions, the quantity:

$$T = \frac{\overline{X_1} - \overline{X_2} - (\mu_1 - \mu_2)}{S_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

Follows a student-t distribution with $n_1 + n_2 - 2$ degrees of fredom.
Pooled variance:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Testing Hypotheses on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal[1]

Null hypothesis: $H_0\colon \mu_1 - \mu_2 = \Delta_0$

Test statistic: $$T_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{S_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \qquad (5\text{-}9)$$

Alternative Hypothesis	Rejection Criterion
$H_1\colon \mu_1 - \mu_2 \neq \Delta_0$	$t_0 > t_{\alpha/2, n_1+n_2-2}$ or $t_0 < -t_{\alpha/2, n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 > \Delta_0$	$t_0 > t_{\alpha, n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 < \Delta_0$	$t_0 < -t_{\alpha, n_1+n_2-2}$

Not paired data and not equal variances

Assumptions (case 2: $\sigma_1^2 \neq \sigma_1^2$):

- $X_{11}, X_{12}, \dots, X_{1n}$ is a random sample of size n_1 from population 1;
- $X_{21}, X_{22}, \dots, X_{2n}$ is a random sample of size n_2 from population 2;
- The two populations are **independent**;
- Both populations are **normal** (or central limit theorem applies);
- The **variances** of the populations are **unknown and not equal**.

Under those assumptions, the quantity:

$$T = \frac{\overline{X_1} - \overline{X_2} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$\nu = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(S_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(S_2^2/n_2\right)^2}{n_2 - 1}}$$

Follows a student-t distribution with ν degrees of fredom

Exercise 2

We want to evaluate the resistance of resistors provided by two different suppliers.
The data is stored in the file ` resistance.csv’.

1. What can we **infer about the mean** resistance of the resistors provided by the two different suppliers?

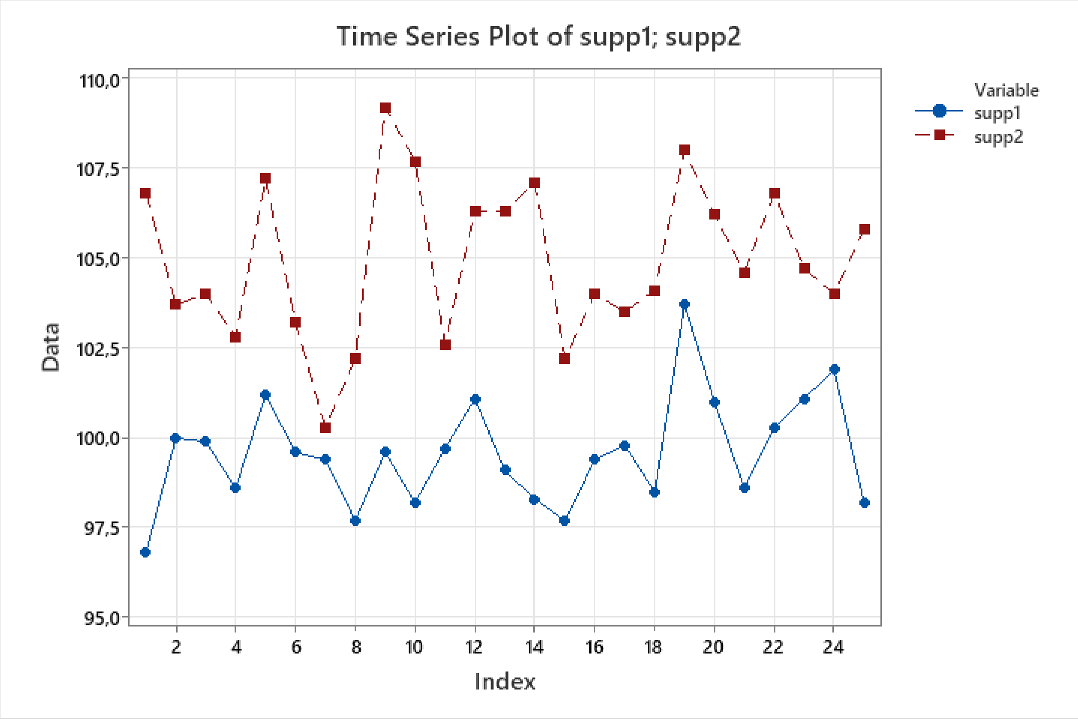
2. Compute the **Type II error** expression in the variance equality test and compute the **test power** when the true variance of the first supplier is 1.5 times larger than the one of the second supplier

Exercise 2

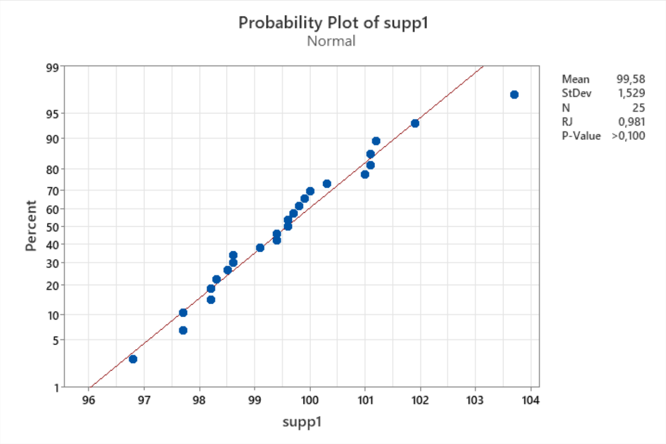
Preliminary activity: data visualization and **assumptions check**

Statistics

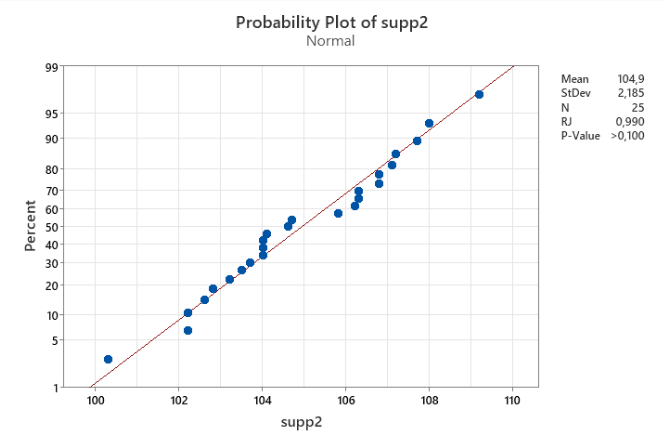
Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
supp2	35	0	105,069	0,331733	1,96256	100,3	103,7	104,7	106,8	109,2
supp1	25	10	99,576	0,305793	1,52896	96,8	98,4	99,6	100,65	103,7



Supplier 1:



Supplier 2:



Test

Runs test

Null hypothesis H_0: The order of the data is random

Alternative hypothesis H_1: The order of the data is not random

	Number of Runs		
Variable	Observed	Expected	P-Value
supp1	15	13,48	0,534
supp2	13	13,32	0,894

Point 1

What can we **infer about the mean** resistance of the resistors provided by the two different suppliers?

We want to compare the means of two populations. Variances are unknown, thus there are two possible situations:

- Equal (unknown) variances
- Different (unknown) variances

We can use the F-test to test the equality of variances.

First step: hypothesis test on the equality of variances

Null hypothesis: the two variances are equal

$$H_0 : \sigma_1^2 = \sigma_2^2$$

Alternative hypothesis: the two variances are different

$$H_1 : \sigma_1^2 \neq \sigma_2^2$$

This hypothesis test is equivalent to:

$$H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$$

$$H_1 : \frac{\sigma_1^2}{\sigma_2^2} \neq 1$$

Point 1

What can we **infer about the mean** resistance of the resistors provided by the two different suppliers?

F-TEST

Assumptions:

- $X_{11}, X_{12}, \dots, X_{1n1}$ is a random sample of size n_1 from population 1;
- $X_{21}, X_{22}, \dots, X_{2n2}$ is a random sample of size n_2 from population 2;
- The two populations are **normal** (or central limit theorem applies);
- The two populations are **independent**;
- The variances of the populations are **unknown** (obvious)

Under those assumptions, the quantity:

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

Follows an F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

Testing Hypotheses on the Equality of Variances of Two Normal Distributions

Null hypothesis: $H_0\colon \sigma_1^2 = \sigma_2^2$

Test statistic: $F_0 = \dfrac{S_1^2}{S_2^2}$ (5-21)

Alternative Hypotheses	Rejection Criterion
$H_1\colon \sigma_1^2 \neq \sigma_2^2$	$f_0 > f_{\alpha/2, n_1-1, n_2-1}$ or $f_0 < f_{1-\alpha/2, n_1-1, n_2-1}$
$H_1\colon \sigma_1^2 > \sigma_2^2$	$f_0 > f_{\alpha, n_1-1, n_2-1}$
$H_1\colon \sigma_1^2 < \sigma_2^2$	$f_0 < f_{1-\alpha, n_1-1, n_2-1}$

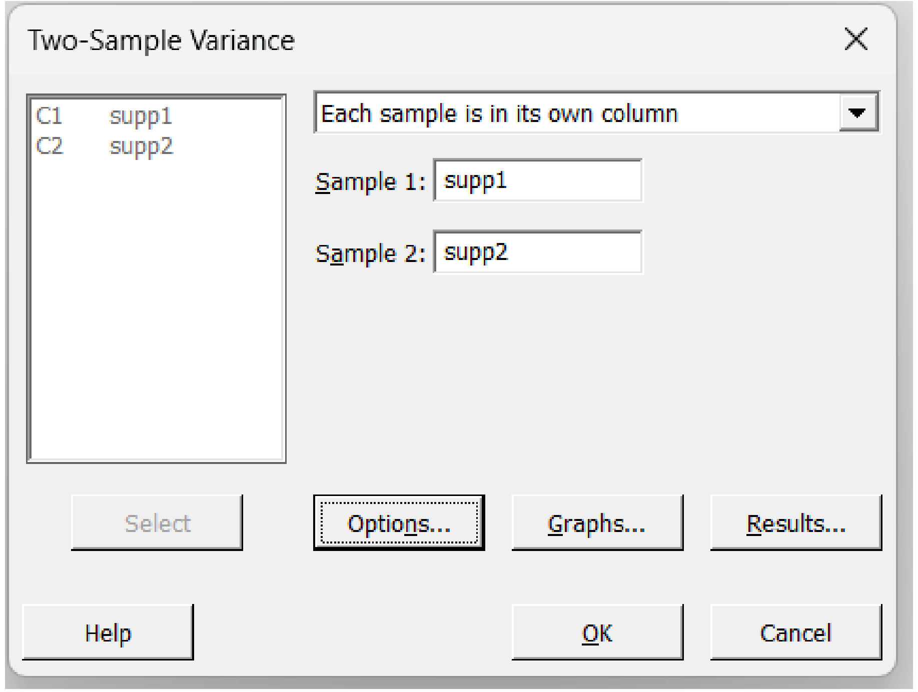
The $100(1-\alpha)\%$ confidence interval on the difference between the population means is given by:

$$\frac{S_1^2}{S_2^2} f_{1-\alpha/2, n_2-1, n_1-1} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{S_1^2}{S_2^2} f_{\alpha/2, n_2-1, n_1-1}$$

Point 1

What can we **infer about the mean** resistance of the resistors provided by the two different suppliers?

F-TEST



Stat → Basic Statistics → 2 Variances

Ratio of Standard Deviations

Estimated 95% CI for Ratio

Ratio	using F
0,699654	(0,464; 1,054)

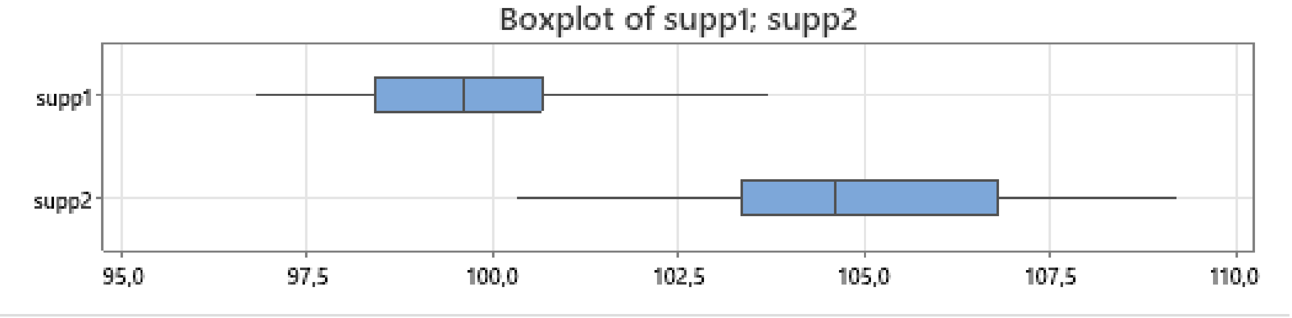
Test

Null hypothesis $H_0: \sigma_1 / \sigma_2 = 1$

Alternative hypothesis $H_1: \sigma_1 / \sigma_2 \neq 1$

Significance level $\alpha = 0{,}05$

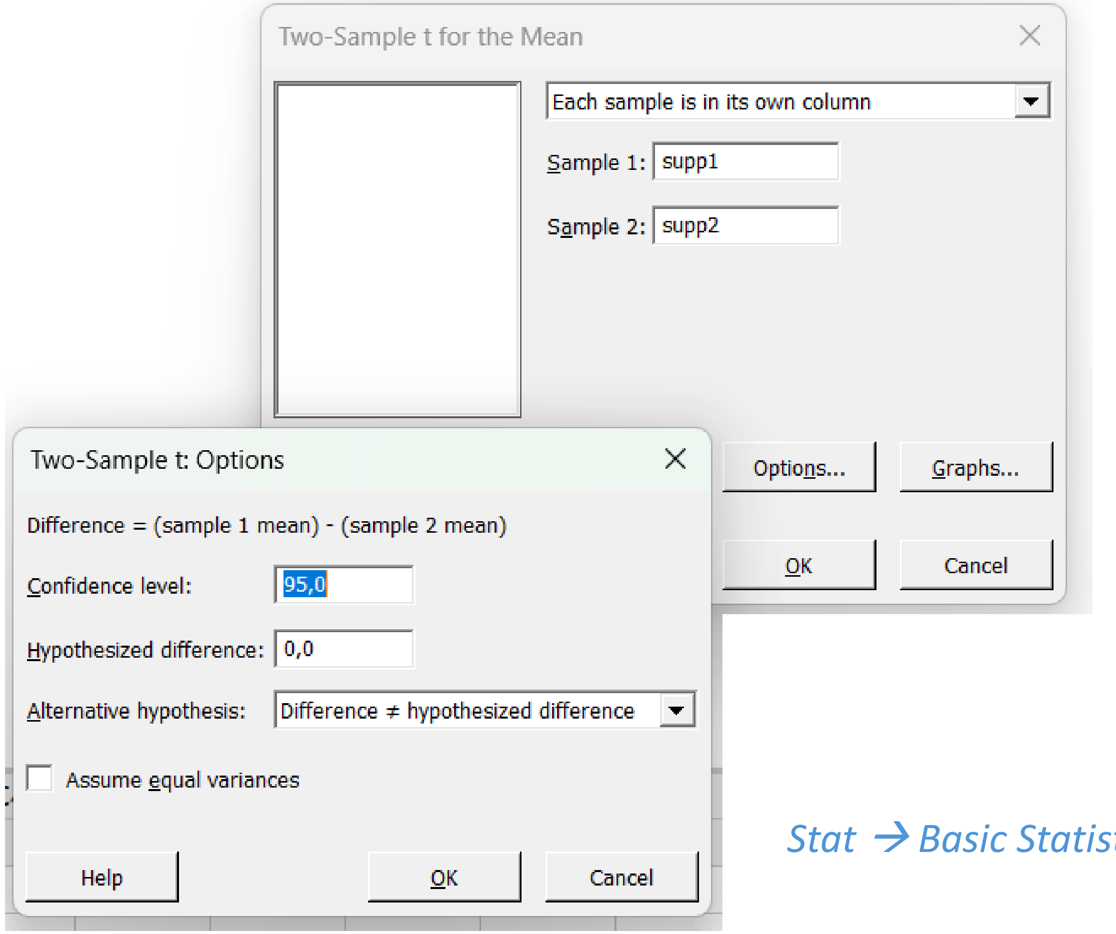
Method	Test Statistic	DF1	DF2	P-Value
F	0,49	24	24	0,087



Point 1

What can we **infer about the mean** resistance of the resistors provided by the two different suppliers?

Now that we have verified the equality of variances, we can perform the **t-test (with equal variances):**



Estimation for Difference

Difference	95% CI for Difference
-5,356	(-6,432; -4,280)

Test

Null hypothesis $H_0: \mu_1 - \mu_2 = 0$

Alternative hypothesis $H_1: \mu_1 - \mu_2 \neq 0$

T-Value	DF	P-Value
-10,04	42	0,000

Stat → Basic Statistics → 2-sample t

Point 2

Compute the **Type II error** expression in the variance equality test and compute the **test power** when the true variance of the first supplier is **1.5 times larger** than the one of the second supplier

The Type II error is the probability of accepting the null hypothesis when it is false.

$$\beta = Pr(\text{accept } H_0 \text{ when } H_1 \text{ is true})$$

Let's expand the formula for the F-test:

$$\beta = Pr\left(F_{1-\alpha/2,n_1-1,n_2-1} \leq \frac{s_1^2}{s_2^2} \leq F_{\alpha/2,n_1-1,n_2-1} \mid \frac{\sigma_1^2}{\sigma_2^2} = \delta \neq 1\right)$$

If we multiply all the terms by σ_2^2/σ_1^2 we get:

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1}$$

If we substitute σ_2^2/σ_1^2 with the ratio we want to test, we get:

$$\beta = Pr\left(\frac{F_{1-\alpha/2,n_1-1,n_2-1}}{1.5} \leq \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \leq \frac{F_{\alpha/2,n_1-1,n_2-1}}{1.5}\right)$$

Rearranging the terms we get:

$$\beta = Pr\left(\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \leq \frac{F_{\alpha/2,n_1-1,n_2-1}}{1.5}\right) - Pr\left(\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \leq \frac{F_{1-\alpha/2,n_1-1,n_2-1}}{1.5}\right)$$

Point 2

Compute the **Type II error** expression in the variance equality test and compute the **test power** when the true variance of the first supplier is **1.5 times larger** than the one of the second supplier

$$\beta = P\left(F_{1-\alpha/2, n_1-1, n_2-1} < \frac{S_1^2}{S_2^2} < F_{\alpha/2, n_1-1, n_2-1} \,\middle|\, \frac{\sigma_1^2}{\sigma_2^2} = \delta \neq 1 \right) = P\left(\frac{\sigma_2^2}{\sigma_1^2} F_{1-\alpha/2, n_1-1, n_2-1} < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < \frac{\sigma_2^2}{\sigma_1^2} F_{\alpha/2, n_1-1, n_2-1} \,\middle|\, \frac{\sigma_1^2}{\sigma_2^2} = \delta \right)$$

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1, n_2-1} \qquad \delta = 1.5 \qquad n_1 = 25 \quad n_2 = 35 \qquad \begin{cases} F_{0.025,24,34} = 2.07 \\ F_{0.975,24,34} = \dfrac{1}{F_{0.025,34,24}} = \dfrac{1}{2.18} = 0.459 \end{cases}$$

$$\beta = P\left(\frac{0.459}{1.5} < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < \frac{2.07}{1.5} \right) = P\left(0.306 < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < 1.38 \right) = P\left(\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < 1.38 \right) - P\left(\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < 0.306 \right) = 0.8073$$

```
Cumulative Distribution Function

F distribution with 24 DF in
numerator and 34 DF in denominator

      x     P( X <= x )
 0.3060          0.0018
```

```
Cumulative Distribution Function

F distribution with 24 DF in
numerator and 34 DF in denominator

      x     P( X <= x )
 1.3800          0.8091
```

The test power is
$P = 1 - \beta$ =19.27%