Derivation of the Variance of $\hat{\tau}^{\dagger}_{\mathsf{adj},2}$ in "Covariate Adjustment in Randomized Experiments Motivated by Higher-Order Influence Functions"

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A Derivation of the Variance of the Debiased Estimator $\hat{\tau}^{\dagger}_{\mathsf{adj},2}$

The variance of $\hat{\tau}^{\dagger}_{\mathsf{adj},2}$ under the randomization-based framework is as follows,

$$\mathsf{var}^{\mathsf{rd}}(\hat{\tau}^{\dagger}_{\mathsf{adj},2}) = \mathsf{var}^{\mathsf{rd}}(\hat{\tau}_{\mathsf{unadj}}) + \mathsf{var}^{\mathsf{rd}}(\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}) - 2\mathsf{cov}^{\mathsf{rd}}(\hat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}).$$

where $\mathsf{var}^{\mathsf{rd}}(\hat{\tau}_{\mathsf{unadj}})$ has been derived before. We need to derive $\mathsf{var}^{\mathsf{rd}}(\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2})$ and $\mathsf{cov}^{\mathsf{rd}}(\hat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2})$.

A.1 Variance of $\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}$

$$\begin{aligned}
&\mathsf{var}^{\mathsf{rd}}(\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}) \equiv \mathsf{var}^{\mathsf{rd}}\left(\frac{1}{n}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\left(y_j(1)-\hat{\tau}_{\mathsf{unadj}}\right)\right)\\
&= \frac{1}{\pi_1^2}\frac{1}{n^2}\left\{\mathbb{E}\left[\left\{\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\hat{\tau}_{\mathsf{unadj}}\right)\right\}^2\right] - \left\{\mathbb{E}\left[\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\hat{\tau}_{\mathsf{unadj}}\right)\right]\right\}^2\right\}
\end{aligned}$$

We next compute the second moment and the squared mean separately.

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A.1.1 The Second Moment

To compute the second moment, we decompose it into seven terms and derive them one by one,

$$
\begin{aligned}
&\mathbb{E}\left[\left\{\sum_{1\le i\ne j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right\}^2\right]\\
&=\mathbb{E}\left[\sum_{1\le i\ne j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\sum_{1\le k\ne l\le n}\left(\frac{t_k}{\pi_1}-1\right)H_{k,l}t_l\left(y_l(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\
&=\sum_{1\le i\ne j\le n}H_{i,j}^2\left\{\begin{array}{l}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\ \quad+\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(\frac{t_j}{\pi_1}-1\right)t_i\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\end{array}\right\}\\
&\quad+\sum_{1\le i\ne j\ne k\le n}H_{i,j}H_{i,k}\left\{\begin{array}{l}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(\frac{t_k}{\pi_1}-1\right)t_i\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\ \quad+\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(\frac{t_i}{\pi_1}-1\right)t_k\left(y_k(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\end{array}\right\}\\
&\quad+\sum_{1\le i\ne j\ne k\le n}H_{i,j}H_{j,k}\left\{\begin{array}{l}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(\frac{t_k}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\ \quad+\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(\frac{t_j}{\pi_1}-1\right)t_k\left(y_k(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\end{array}\right\}\\
&\quad+\sum_{1\le i\ne j\ne k\ne l\le n}H_{i,j}H_{k,l}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(\frac{t_k}{\pi_1}-1\right)t_l\left(y_l(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\
&\coloneqq\sum_{1\le i\ne j\le n}H_{i,j}^2\left(V_{1,ij}+V_{2,ij}\right)+\sum_{1\le i\ne j\ne k\le n}H_{i,j}H_{i,k}\left(V_{3,ijk}+V_{4,ijk}\right)\\
&\quad+\sum_{1\le i\ne j\ne k\le n}H_{i,j}H_{j,k}\left(V_{5,ijk}+V_{6,ijk}\right)+\sum_{1\le i\ne j\ne k\ne l\le n}H_{i,j}H_{k,l}V_{7,ijkl}.
\end{aligned}
$$

The first term,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)^2t_j^2\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)^2\right]=\mathbb{E}\left[\left(\frac{t_it_j}{\pi_1^2}-2\frac{t_it_j}{\pi_1}+t_j\right)\left(y_j(1)^2-2y_j(1)\widehat{\tau}_{\mathsf{unadj}}+\widehat{\tau}_{\mathsf{unadj}}^2\right)\right]\\
&=\left\{\mathbb{E}\left[t_j\right]+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\mathbb{E}\left[t_it_j\right]\right\}y_j(1)^2-2\left\{\mathbb{E}\left[t_j\widehat{\tau}_{\mathsf{unadj}}\right]+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\mathbb{E}\left[t_it_j\widehat{\tau}_{\mathsf{unadj}}\right]\right\}y_j(1)\\
&\quad+\mathbb{E}\left[t_j\widehat{\tau}_{\mathsf{unadj}}^2\right]+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\mathbb{E}\left[t_it_j\widehat{\tau}_{\mathsf{unadj}}^2\right]\\
&=\left\{\pi_1+\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\right\}y_j(1)^2\\
&\quad-2\left\{\begin{array}{l}\left(\pi_1-\frac{\pi_0}{n-1}\right)\bar{\tau}+\frac{\pi_0}{n-1}y_j(1)+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\\ +\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{\pi_0}{n-2}\left(y_i(1)+y_j(1)\right)\end{array}\right\}y_j(1)
\end{aligned}
$$

$$
+\left\{\begin{array}{l}\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}^2+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\bar{\tau}\frac{2}{n-2}y_j(1)\\
\quad+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\bar{\tau}^{(2)}-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{2}{n(n-2)}y_j(1)^2+\frac{\pi_0}{\pi_1}\frac{1}{n(n-1)}y_j(1)^2\end{array}\right\}
$$

$$
+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left\{\begin{array}{l}\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}^2\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\frac{2}{n-3}\left(y_i(1)+y_j(1)\right)\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)}\\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2(y_i(1)^2+y_i(1)y_j(1)+y_j(1)^2)}{n(n-3)}\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(y_i(1)+y_j(1))^2}{n(n-2)}\end{array}\right\}
$$

$$
\begin{aligned}
=\ &\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{(n+3)\pi_1-3}{n-3}\bar{\tau}^2\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(n+4)\pi_1\pi_0+\pi_1^2-2}{(n-2)(n-3)}\bar{\tau}^{(2)}\\
&+\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2}{n-3}\bar{\tau}y_i(1)\\
&+2\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(3-n^2)\pi_1^2+(3n-2)\pi_1-2}{(n-2)(n-3)}\bar{\tau}y_j(1)\\
&+\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{n(1-2\pi_1)+1}{n(n-2)(n-3)}y_i(1)^2\\
&+\left\{\pi_0-\frac{\pi_0}{\pi_1}\frac{1}{n}+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\frac{2n(2-n)\pi_1+n+1}{n(n-2)(n-3)}-\frac{2}{n(n-2)}\right)\right\}y_j(1)^2\\
&+2\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(n-2)(1-n\pi_1)+1}{n(n-2)(n-3)}y_i(1)y_j(1)
\end{aligned}
$$

The second term,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_j}{\pi_1}-1\right)t_it_j\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\
=\ &\mathbb{E}\left[\left(\frac{t_it_j}{\pi_1^2}-2\frac{t_it_j}{\pi_1}+t_it_j\right)\left(y_i(1)y_j(1)-\left(y_i(1)+y_j(1)\right)\widehat{\tau}_{\mathsf{unadj}}+\widehat{\tau}^2_{\mathsf{unadj}}\right)\right]\\
=\ &\left(\frac{\pi_0}{\pi_1}\right)^2\mathbb{E}\left[t_it_j\right]y_i(1)y_j(1)-\left(\frac{\pi_0}{\pi_1}\right)^2\mathbb{E}\left[t_it_j\widehat{\tau}_{\mathsf{unadj}}\right]\left(y_i(1)+y_j(1)\right)+\left(\frac{\pi_0}{\pi_1}\right)^2\mathbb{E}\left[t_it_j\widehat{\tau}^2_{\mathsf{unadj}}\right]\\
=\ &\left(\frac{\pi_0}{\pi_1}\right)^2\pi_1\left(\pi_1-\frac{\pi_0}{n-1}\right)y_i(1)y_j(1)\\
&-\left(\frac{\pi_0}{\pi_1}\right)^2\left\{(\pi_1-\frac{\pi_0}{n-1})(\pi_1-\frac{2\pi_0}{n-2})\bar{\tau}+(\pi_1-\frac{\pi_0}{n-1})\frac{\pi_0}{n-2}(y_i(1)+y_j(1))\right\}\left(y_i(1)+y_j(1)\right)
\end{aligned}
$$

$$
+\left(\frac{\pi_0}{\pi_1}\right)^2\left\{\begin{array}{l}
\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}^2 \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\frac{2}{n-3}\left(y_i(1)+y_j(1)\right) \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)} \\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2(y_i(1)^2+y_i(1)y_j(1)+y_j(1)^2)}{n(n-3)} \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(y_i(1)+y_j(1))^2}{n(n-2)}
\end{array}\right\}
$$

$$
\begin{aligned}
&=\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}^2 \\
&\quad+\left(\frac{\pi_0}{\pi_1}\right)^3\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)} \\
&\quad+\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2\pi_0-(n-3)\pi_1}{n-3}\bar{\tau}\left(y_i(1)+y_j(1)\right) \\
&\quad+\left(\frac{\pi_0}{\pi_1}\right)^3\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{n(1-n)\pi_1+n+1}{n(n-2)(n-3)}\left(y_i(1)^2+y_j(1)^2\right) \\
&\quad+\left(\frac{\pi_0}{\pi_1}\right)^3\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\frac{\pi_1^2}{\pi_0}+\frac{2(1-n\pi_1)(n-2)+2}{n(n-2)(n-3)}\right\}y_i(1)y_j(1)
\end{aligned}
$$

Combine the first and the second term and the coefficients $\sum_{1\leq i\neq j\leq n}H_{i,j}^2$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)^2 t_j^2\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)^2\right]+\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_j}{\pi_1}-1\right)t_i t_j\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right] \\
&=\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{n\pi_1(-\pi_1+2)}{n-3}+\frac{6\pi_1-6}{n-3}\right)\bar{\tau}^2 \\
&\quad+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{n\pi_1(\pi_1-1)(\pi_1-2)}{(n-2)(n-3)}+\frac{-5\pi_1^2+8\pi_1-4}{(n-2)(n-3)}\right)\bar{\tau}^{(2)} \\
&\quad+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{n\pi_1(\pi_1-1)}{n-3}+\frac{-\pi_1^2-5\pi_1+4}{n-3}\right)\bar{\tau}y_i(1) \\
&\quad+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{n^2\pi_1^2(\pi_1-3)}{(n-2)(n-3)}+\frac{n\pi_1(-\pi_1^2-3\pi_1+10)}{(n-2)(n-3)}+\frac{8\pi_1^2-2\pi_1-8}{(n-2)(n-3)}\right)\bar{\tau}y_j(1) \\
&\quad+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{-n^2\pi_1\pi_0^2}{n(n-2)(n-3)}+\frac{n(\pi_1^3+3\pi_1^2-5\pi_1+2)}{n(n-2)(n-3)}+\frac{\pi_1^2-4\pi_1+2}{n(n-2)(n-3)}\right)y_i(1)^2 \\
&\quad+\left\{\pi_0-\frac{\pi_0}{\pi_1}\frac{1}{n}+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{-n^2\pi_1(\pi_1^2-6\pi_1+3)}{n(n-2)(n-3)}+\frac{n(\pi_1^3-11\pi_1^2+\pi_1+2)}{n(n-2)(n-3)} \\ +\frac{7\pi_1^2-4\pi_1+2}{n(n-2)(n-3)}\end{array}\right)\right\}y_j(1)^2
\end{aligned}
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^3\pi_1^2\pi_0}{n(n-2)(n-3)}+\frac{n^2\pi_1(3\pi_1^2+3\pi_1-4)}{n(n-2)(n-3)}\\ +\frac{n(-2\pi_1^3-8\pi_1^2+4)}{n(n-2)(n-3)}+\frac{-2\pi_1^2+8\pi_1-4}{n(n-2)(n-3)}\end{array}\right)y_i(1)y_j(1).
$$

$$
\begin{aligned}
&\sum_{1\le i\ne j\le n} H_{i,j}^2\left(V_{1,ij}+V_{2,ij}\right)\\
&=\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\left(\frac{n^2\pi_1^2(-\pi_1+2)}{(n-2)(n-3)}+\frac{2n\pi_1(4\pi_1-5)}{(n-2)(n-3)}+\frac{12\pi_0}{(n-2)(n-3)}\right)\bar{\tau}^2\\ +\left(\frac{n\pi_1(\pi_1-1)(\pi_1-2)}{(n-2)(n-3)}+\frac{-5\pi_1^2+8\pi_1-4}{(n-2)(n-3)}\right)\bar{\tau}^{(2)}\end{array}\right\}\sum_{i=1}^n H_{i,i}\left(1-H_{i,i}\right)\\
&\quad+2\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{n^2\pi_1^2(\pi_1-2)}{(n-2)(n-3)}+\frac{n\pi_1(-\pi_1^2-5\pi_1+8)}{(n-2)(n-3)}+\frac{5\pi_1^2+4\pi_1-8}{(n-2)(n-3)}\right)\bar{\tau}\sum_{i=1}^n H_{i,i}\left(1-H_{i,i}\right)y_i(1)\\
&\quad+\left\{\pi_0-\frac{\pi_0}{\pi_1}\frac{1}{n}+2\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{-n^2\pi_1(\pi_1^2-4\pi_1+2)}{n(n-2)(n-3)}+\frac{4\pi_1^2-4\pi_1+2}{n(n-2)(n-3)}\\ +\frac{n(\pi_1^3-4\pi_1^2-2\pi_1+2)}{n(n-2)(n-3)}\end{array}\right)\right\}\sum_{i=1}^n H_{i,i}\left(1-H_{i,i}\right)y_i(1)^2\\
&\quad+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^3\pi_1^2\pi_0}{n(n-2)(n-3)}+\frac{n^2\pi_1(3\pi_1^2+3\pi_1-4)}{n(n-2)(n-3)}\\ +\frac{n(-2\pi_1^3-8\pi_1^2+4)}{n(n-2)(n-3)}+\frac{-2\pi_1^2+8\pi_1-4}{n(n-2)(n-3)}\end{array}\right)\sum_{1\le i\ne j\le n} H_{i,j}^2 y_i(1)y_j(1).
\end{aligned}
$$

The third term,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_k}{\pi_1}-1\right)t_it_j\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\
&=\frac{\pi_0}{\pi_1}\mathbb{E}\left[\left(\frac{1}{\pi_1}t_it_jt_k-t_it_j\right)\left(y_i(1)y_j(1)-\left(y_i(1)+y_j(1)\right)\widehat{\tau}_{\mathsf{unadj}}+\widehat{\tau}_{\mathsf{unadj}}^2\right)\right]\\
&=\frac{\pi_0}{\pi_1}\left\{\frac{1}{\pi_1}\mathbb{E}\left[t_it_jt_k\right]-\mathbb{E}\left[t_it_j\right]\right\}y_i(1)y_j(1)+\frac{\pi_0}{\pi_1}\left\{\mathbb{E}\left[t_it_j\widehat{\tau}_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_jt_k\widehat{\tau}_{\mathsf{unadj}}\right]\right\}\left(y_i(1)+y_j(1)\right)\\
&\quad-\frac{\pi_0}{\pi_1}\mathbb{E}\left[t_it_j\left(\widehat{\tau}_{\mathsf{unadj}}\right)^2\right]+\frac{\pi_0}{\pi_1^2}\mathbb{E}\left[t_it_jt_k\left(\widehat{\tau}_{\mathsf{unadj}}\right)^2\right]\\
&=\frac{\pi_0}{\pi_1}\left\{\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)-\pi_1\left(\pi_1-\frac{\pi_0}{n-1}\right)\right\}y_i(1)y_j(1)\\
&\quad+\frac{\pi_0}{\pi_1}\left\{\begin{array}{l}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\\ \quad+\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{\pi_0}{n-2}\left(y_i(1)+y_j(1)\right)\\ -\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}\\ -\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{\pi_0}{n-3}\left(y_i(1)+y_j(1)+y_k(1)\right)\end{array}\right\}\left(y_i(1)+y_j(1)\right)
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{\pi_0}{\pi_1} \left\{ \begin{aligned}
& \frac{1}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\pi_1 - \frac{3\pi_0}{n-3}\right)\bar{\tau}^2 \\
& + \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\bar{\tau}\frac{2}{n-3}\left(y_i(1)+y_j(1)\right) \\
& + \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)} \\
& - \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\frac{2(y_i(1)^2 + y_i(1)y_j(1) + y_j(1)^2)}{n(n-3)} \\
& + \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\frac{(y_i(1)+y_j(1))^2}{n(n-2)}
\end{aligned} \right\} \\
& + \frac{\pi_0}{\pi_1^2} \left\{ \begin{aligned}
& \frac{1}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\pi_1 - \frac{3\pi_0}{n-3}\right)\left(\pi_1 - \frac{4\pi_0}{n-4}\right)\bar{\tau}^2 \\
& + \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\pi_1 - \frac{3\pi_0}{n-3}\right)\bar{\tau}\cdot\frac{2}{n-4}\left(y_i(1)+y_j(1)+y_k(1)\right) \\
& + \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\pi_1 - \frac{3\pi_0}{n-3}\right)\frac{1}{n-4}\bar{\tau}^{(2)} \\
& - \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\pi_1 - \frac{3\pi_0}{n-3}\right)\frac{2\sum_{r,s\in\{i,j,k\}} y_r(1)y_s(1)}{n(n-4)} \\
& + \frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\frac{(y_i(1)+y_j(1)+y_k(1))^2}{n(n-3)}
\end{aligned} \right\} \\
= & -4\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\pi_1 - \frac{3\pi_0}{n-3}\right)\frac{1}{n-4}\bar{\tau}^2 \\
& + \frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\frac{4\pi_1 - 3}{(n-3)(n-4)}\bar{\tau}^{(2)} \\
& + \frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\frac{2(n\pi_1 - 3) + \pi_1(n-4)}{(n-3)(n-4)}\bar{\tau}(y_i(1)+y_j(1)) \\
& + \frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\pi_1 - \frac{3\pi_0}{n-3}\right)\frac{2}{n-4}\bar{\tau}y_k(1) \\
& + \frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\frac{(n\pi_1 - 1)\pi_1}{n(n-2)} + \frac{(n\pi_1 - 2)\{-(n^2-4n+8)\pi_1 + n + 2\}}{n(n-2)(n-3)(n-4)}\right)\left(y_i(1)^2 + y_j(1)^2\right) \\
& + \frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\frac{n(1-2\pi_1)+2}{n(n-3)(n-4)}y_k(1)^2 \\
& + 2\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\frac{(n\pi_1 - 2)((n-1) - (n-2)^2\pi_1)}{n(n-2)(n-3)(n-4)} - \frac{\pi_1}{n(n-2)}\right)y_i(1)y_j(1) \\
& + \frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\frac{-n^2\pi_1 + 2n\pi_1 + 2n - 2}{n(n-3)(n-4)}y_k(1)(y_i(1)+y_j(1)).
\end{aligned}
$$

The fourth term,

$$
\mathbb{E}\left[\left(\frac{t_i}{\pi_1} - 1\right)^2 t_j t_k \left(y_j(1) - \widehat{\tau}_{\mathsf{unadj}}\right)\left(y_k(1) - \widehat{\tau}_{\mathsf{unadj}}\right)\right]
$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left(\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)t_it_jt_k+t_jt_k\right)\left\{y_j(1)y_k(1)-(y_j(1)+y_k(1))\,\widehat{\tau}_{\textsf{unadj}}+\widehat{\tau}^2_{\textsf{unadj}}\right\}\right] \\
&= \left\{\mathbb{E}\left[t_jt_k\right]+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\mathbb{E}\left[t_it_jt_k\right]\right\}y_j(1)y_k(1)-\left\{\mathbb{E}\left[t_jt_k\widehat{\tau}_{\textsf{unadj}}\right]+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\mathbb{E}\left[t_it_jt_k\widehat{\tau}_{\textsf{unadj}}\right]\right\}(y_j(1)+y_k(1)) \\
&\quad +\mathbb{E}\left[t_jt_k\widehat{\tau}^2_{\textsf{unadj}}\right]+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\mathbb{E}\left[t_it_jt_k\widehat{\tau}^2_{\textsf{unadj}}\right] \\
&= \left\{\pi_1\left(\pi_1-\frac{\pi_0}{n-1}\right)+\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\right\}y_j(1)y_k(1) \\
&\quad -\left\{\begin{array}{l}
\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}+\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{\pi_0}{n-2}(y_j(1)+y_k(1)) \\
+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau} \\
+\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{\pi_0}{n-3}(y_i(1)+y_j(1)+y_k(1))
\end{array}\right\}(y_j(1)+y_k(1)) \\
&\quad +\left\{\begin{array}{l}
\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}^2 \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\frac{2}{n-3}\left(y_j(1)+y_k(1)\right) \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)} \\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2(y_j(1)^2+y_j(1)y_k(1)+y_k(1)^2)}{n(n-3)} \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(y_j(1)+y_k(1))^2}{n(n-2)}
\end{array}\right\} \\
&\quad +\frac{1}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left\{\begin{array}{l}
\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\bar{\tau}^2 \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}\cdot\frac{2}{n-4}\left(y_i(1)+y_j(1)+y_k(1)\right) \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{1}{n-4}\bar{\tau}^{(2)} \\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{2\sum\limits_{r,s\in\{i,j,k\}}y_r(1)y_s(1)}{n(n-4)} \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{(y_i(1)+y_j(1)+y_k(1))^2}{n(n-3)}
\end{array}\right\} \\
&= \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{n\pi_1-4\pi_0}{n-4}\bar{\tau}^2 \\
&\quad +\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{-(n+4)\pi_1^2+(n+6)\pi_1-3}{(n-3)(n-4)}\bar{\tau}^{(2)} \\
&\quad +\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{2}{n-4}\bar{\tau}y_i(1) \\
&\quad -\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{n^2\pi_1^2+n\pi_1(\pi_1-5)-4\pi_1^2+6}{(n-3)(n-4)}\bar{\tau}(y_j(1)+y_k(1))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{n(1-2\pi_1)+2}{n(n-3)(n-4)}y_i(1)^2\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{-\pi_0\pi_1^2n^3}{n(n-2)(n-3)(n-4)}+\dfrac{\pi_1(\pi_1^2-3\pi_1+3)n^2}{n(n-2)(n-3)(n-4)}\\ +\dfrac{(-4\pi_1^3+\pi_1^2+2\pi_1-2)n}{n(n-2)(n-3)(n-4)}-\dfrac{4\pi_0^2}{n(n-2)(n-3)(n-4)}\end{array}\right\}\left(y_j(1)^2+y_k(1)^2\right)\\
&+\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\frac{1}{\pi_1}-2\right)\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{-\pi_1n^2+2(1+\pi_1)n-2}{n(n-3)(n-4)}y_i(1)\left(y_j(1)+y_k(1)\right)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\pi_1^3+\dfrac{\left(6\pi_1^3-4\pi_1^2\right)n^3}{n(n-2)(n-3)(n-4)}+\dfrac{\left(-28\pi_1^3+10\pi_1^2+6\pi_1\right)n^2}{n(n-2)(n-3)(n-4)}\\ +\dfrac{\left(32\pi_1^3-6\pi_1^2-6\pi_1-4\right)n}{n(n-2)(n-3)(n-4)}+\dfrac{8\pi_1^2-8\pi_1+4}{n(n-2)(n-3)(n-4)}\end{array}\right)y_j(1)y_k(1)
\end{aligned}
$$

Combine the third and the fourth term and the coefficients $\sum_{1\le i\ne j\ne k\le n}H_{i,j}H_{i,k}$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_k}{\pi_1}-1\right)t_it_j\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]+\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)^2t_jt_k\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_k(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\
=&\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\frac{(n+8)\pi_1}{n-4}-\frac{8}{n-4}\right)\bar{\tau}^2\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{n\pi_1\pi_0}{(n-3)(n-4)}+\frac{-8\pi_1^2+13\pi_1-6}{(n-3)(n-4)}\right)\bar{\tau}^{(2)}\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{n\pi_1(-7\pi_1+5)}{(n-3)(n-4)}+\frac{4\pi_1^2+14\pi_1-12}{(n-3)(n-4)}\right)\bar{\tau}y_i(1)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{-n^2\pi_1^2}{(n-3)(n-4)}+\frac{n\pi_1(-4\pi_1+8)}{(n-3)(n-4)}+\frac{8\pi_1^2+2\pi_1-12}{(n-3)(n-4)}\right)\bar{\tau}y_j(1)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{-n^2\pi_1^2}{(n-3)(n-4)}+\frac{n\pi_1(-3\pi_1+7)}{(n-3)(n-4)}+\frac{4\pi_1^2+6\pi_1-12}{(n-3)(n-4)}\right)\bar{\tau}y_k(1)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{n^2\pi_1(7\pi_1^2-9\pi_1+3)}{n(n-2)(n-3)(n-4)}+\dfrac{n(-4\pi_1^3-9\pi_1^2+13\pi_1-4)}{n(n-2)(n-3)(n-4)}\\ +\dfrac{-4\pi_1^2+16\pi_1-8}{n(n-2)(n-3)(n-4)}\end{array}\right)y_i(1)^2\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{-n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)}+\dfrac{n^2\pi_1(4\pi_1^2-8\pi_1+5)}{n(n-2)(n-3)(n-4)}\\ +\dfrac{n(-8\pi_1^3+4\pi_1^2+5\pi_1-4)}{n(n-2)(n-3)(n-4)}+\dfrac{-8\pi_0^2}{n(n-2)(n-3)(n-4)}\end{array}\right)y_j(1)^2\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{-n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)}+\dfrac{n^2\pi_1(3\pi_1^2-6\pi_1+4)}{n(n-2)(n-3)(n-4)}\\ +\dfrac{n(-4\pi_1^3-5\pi_1^2+10\pi_1-4)}{n(n-2)(n-3)(n-4)}+\dfrac{-4\pi_0(1+\pi_0)}{n(n-2)(n-3)(n-4)}\end{array}\right)y_k(1)^2
\end{aligned}
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l} \frac{n^3\pi_1^2(4\pi_1-3)}{n(n-2)(n-3)(n-4)} + \frac{n^2\pi_1(-12\pi_1^2-2\pi_1+8)}{n(n-2)(n-3)(n-4)} \\ + \frac{n(8\pi_1^3+8\pi_1^2+2\pi_1-8)}{n(n-2)(n-3)(n-4)} + \frac{8\pi_1^2-20\pi_1+8}{n(n-2)(n-3)(n-4)} \end{array}\right) y_i(1)y_j(1)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\frac{n^2\pi_1(3\pi_1-2)}{n(n-3)(n-4)} + \frac{n(-6\pi_1^2-2\pi_1+4)}{n(n-3)(n-4)} + \frac{6\pi_1-4}{n(n-3)(n-4)}\right) y_i(1)y_k(1)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l} \pi_1^3 + \frac{n^3\pi_1^2\left(7\pi_1-5\right)}{n(n-2)(n-3)(n-4)} + \frac{n^2\pi_1\left(-30\pi_1^2+8\pi_1+10\right)}{n(n-2)(n-3)(n-4)} \\ + \frac{n\left(32\pi_1^3-8\pi_1-8\right)}{n(n-2)(n-3)(n-4)} + \frac{8\pi_1^2-12\pi_1+8}{n(n-2)(n-3)(n-4)} \end{array}\right) y_j(1)y_k(1),
$$

$$
\sum_{1\le i\ne j\ne k\le n} H_{i,j}H_{i,k}\left(V_{3,ijk}+V_{4,ijk}\right)
$$

$$
= \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left\{\begin{array}{l} \left(\pi_1 - \frac{3\pi_0}{n-3}\right)\left(\frac{(n+8)\pi_1}{n-4} - \frac{8}{n-4}\right)\bar{\tau}^2 \\ + \left(\frac{n\pi_1\pi_0}{(n-3)(n-4)} + \frac{-8\pi_1^2+13\pi_1-6}{(n-3)(n-4)}\right)\bar{\tau}^{(2)} \end{array}\right\}\sum_{i=1}^n H_{i,i}(2H_{i,i}-1)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\frac{n\pi_1(-7\pi_1+5)}{(n-3)(n-4)} + \frac{4\pi_1^2+14\pi_1-12}{(n-3)(n-4)}\right)\bar{\tau}\sum_{i=1}^n H_{i,i}(2H_{i,i}-1)y_i(1)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\frac{-2n^2\pi_1^2}{(n-3)(n-4)} + \frac{n\pi_1(-7\pi_1+15)}{(n-3)(n-4)} + \frac{12\pi_1^2+8\pi_1-24}{(n-3)(n-4)}\right)\bar{\tau}
$$

$$
\times\left(\sum_{i=1}^n H_{i,i}(H_{i,i}-1)y_i(1) - \sum_{1\le i\ne j\le n} H_{i,i}H_{i,j}y_j(1)\right)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l} \frac{n^2\pi_1(7\pi_1^2-9\pi_1+3)}{n(n-2)(n-3)(n-4)} \\ + \frac{n(-4\pi_1^3-9\pi_1^2+13\pi_1-4)}{n(n-2)(n-3)(n-4)} + \frac{-4\pi_1^2+16\pi_1-8}{n(n-2)(n-3)(n-4)} \end{array}\right)\sum_{i=1}^n H_{i,i}(2H_{i,i}-1)y_i(1)^2
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l} \frac{-2n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)} + \frac{n^2\pi_1(7\pi_1^2-14\pi_1+9)}{n(n-2)(n-3)(n-4)} \\ + \frac{n(-12\pi_1^3-\pi_1^2+15\pi_1-8)}{n(n-2)(n-3)(n-4)} + \frac{-4\pi_0(1+3\pi_0)}{n(n-2)(n-3)(n-4)} \end{array}\right)
$$

$$
\times\left(\sum_{i=1}^n H_{i,i}(H_{i,i}-1)y_i(1)^2 - \sum_{1\le i\ne j\le n} H_{i,i}H_{i,j}y_j(1)^2\right)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l} \frac{n^3\pi_1^2(7\pi_1-5)}{n(n-2)(n-3)(n-4)} + \frac{n^2\pi_1(-18\pi_1^2-10\pi_1+16)}{n(n-2)(n-3)(n-4)} \\ + \frac{n(8\pi_1^3+26\pi_1^2+2\pi_1-16)}{n(n-2)(n-3)(n-4)} + \frac{8\pi_1^2-32\pi_1+16}{n(n-2)(n-3)(n-4)} \end{array}\right)
$$

$$
\times\left(-\sum_{1\le i\ne j\le n} H_{i,i}H_{i,j}y_i(1)y_j(1)-\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)+\sum_{i=1}^n H_{i,i}^2 y_i(1)^2\right)
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\pi_1^3+\dfrac{n^3\pi_1^2\left(7\pi_1-5\right)}{n(n-2)(n-3)(n-4)}+\dfrac{n^2\pi_1\left(-30\pi_1^2+8\pi_1+10\right)}{n(n-2)(n-3)(n-4)}\\ +\dfrac{n\left(32\pi_1^3-8\pi_1-8\right)}{n(n-2)(n-3)(n-4)}+\dfrac{8\pi_1^2-12\pi_1+8}{n(n-2)(n-3)(n-4)}\end{array}\right)
$$

$$
\times\sum_{1\le i\ne j\le n} H_{i,j}(1-2H_{j,j})y_i(1)y_j(1).
$$

The fifth term,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_k}{\pi_1}-1\right)t_j^2\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)^2\right]\\
=&\ \mathbb{E}\left[\left(\frac{1}{\pi_1^2}t_it_jt_k-\frac{1}{\pi_1}t_it_j-\frac{1}{\pi_1}t_jt_k+t_j\right)\left(y_j(1)^2-2y_j(1)\widehat{\tau}_{\mathsf{unadj}}+\widehat{\tau}_{\mathsf{unadj}}^2\right)\right]\\
=&\ \left\{\mathbb{E}\left[t_j\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_j\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_jt_k\right]+\frac{1}{\pi_1^2}\mathbb{E}\left[t_it_jt_k\right]\right\}y_j(1)^2\\
&-2\left\{\mathbb{E}\left[t_j\widehat{\tau}_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_j\widehat{\tau}_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_jt_k\widehat{\tau}_{\mathsf{unadj}}\right]+\frac{1}{\pi_1^2}\mathbb{E}\left[t_it_jt_k\widehat{\tau}_{\mathsf{unadj}}\right]\right\}y_j(1)\\
&+\mathbb{E}\left[t_j\widehat{\tau}_{\mathsf{unadj}}^2\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_j\widehat{\tau}_{\mathsf{unadj}}^2\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_jt_k\widehat{\tau}_{\mathsf{unadj}}^2\right]+\frac{1}{\pi_1^2}\mathbb{E}\left[t_it_jt_k\widehat{\tau}_{\mathsf{unadj}}^2\right]\\
=&\ \left\{\pi_1-2\left(\pi_1-\frac{\pi_0}{n-1}\right)+\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\right\}y_j(1)^2\\
&-2\left\{\begin{array}{l}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\bar{\tau}+\dfrac{\pi_0}{n-1}y_j(1)\\ -2\dfrac{1}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\bar{\tau}\\ -\dfrac{1}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\dfrac{\pi_0}{n-2}\left(y_i(1)+2y_j(1)+y_k(1)\right)\\ +\dfrac{1}{\pi_1^2}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\left(\pi_1-\dfrac{3\pi_0}{n-3}\right)\bar{\tau}\\ +\dfrac{1}{\pi_1^2}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\dfrac{\pi_0}{n-3}\left(y_i(1)+y_j(1)+y_k(1)\right)\end{array}\right\}y_j(1)\\
&+\left\{\begin{array}{l}\dfrac{1}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\bar{\tau}^2+\dfrac{\pi_0}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\bar{\tau}\dfrac{2}{n-2}y_j(1)\\ +\dfrac{\pi_0}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\dfrac{1}{n-2}\bar{\tau}^{(2)}\\ -\dfrac{\pi_0}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\dfrac{2}{n(n-2)}y_j(1)^2+\dfrac{\pi_0}{\pi_1}\dfrac{1}{n(n-1)}y_j(1)^2\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\pi_1}\left\{\begin{aligned}
&2\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}^2\\
&+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\frac{2}{n-3}\left(y_i(1)+2y_j(1)+y_k(1)\right)\\
&+2\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)}\\
&-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2(y_i(1)^2+2y_j(1)^2+y_i(1)y_j(1)+y_j(1)y_k(1)+y_k(1)^2)}{n(n-3)}\\
&+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(y_i(1)+y_j(1))^2+(y_j(1)+y_k(1))^2}{n(n-2)}
\end{aligned}\right\}\\
&+ \frac{1}{\pi_1^2}\left\{\begin{aligned}
&\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\bar{\tau}^2\\
&+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}\cdot\frac{2}{n-4}\left(y_i(1)+y_j(1)+y_k(1)\right)\\
&+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{1}{n-4}\bar{\tau}^{(2)}\\
&-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{2\sum_{r,s\in\{i,j,k\}}y_r(1)y_s(1)}{n(n-4)}\\
&+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{(y_i(1)+y_j(1)+y_k(1))^2}{n(n-3)}
\end{aligned}\right\}\\
=\;& \frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{12\frac{\pi_0}{\pi_1}-n}{(n-3)(n-4)}\bar{\tau}^2\\
&+\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{2\pi_1-(n+4)\pi_0+6\frac{\pi_0^2}{\pi_1}}{(n-2)(n-3)(n-4)}\bar{\tau}^{(2)}\\
&+2\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1-3\frac{\pi_0}{\pi_1}}{(n-3)(n-4)}\bar{\tau}\left(y_i(1)+y_k(1)\right)\\
&+2\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{6\left(\frac{\pi_0}{\pi_1}\right)^2-7n\frac{\pi_0}{\pi_1}+20\frac{\pi_0}{\pi_1}+n^2-4n+2}{(n-2)(n-3)(n-4)}\bar{\tau}y_j(1)\\
&+\frac{1}{\pi_1}\frac{\pi_0}{\pi_1^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(-8\pi_1^2+9\pi_1-2)n-4\pi_0}{n(n-2)(n-3)(n-4)}\left(y_i(1)^2+y_k(1)^2\right)\\
&+\frac{\pi_0}{\pi_1}\left\{\frac{1-n\pi_1}{n(n-1)}+\frac{1}{\pi_1^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{aligned}&\frac{3n^2\pi_1(1-2\pi_1)}{n(n-2)(n-3)(n-4)}+\frac{2n(11\pi_1^2-2\pi_1-1)}{n(n-2)(n-3)(n-4)}\\&-\frac{4(6\pi_1^2-2\pi_1+1)}{n(n-2)(n-3)(n-4)}\end{aligned}\right)\right\}y_j(1)^2\\
&-2\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{aligned}&\frac{n^2\pi_1(3\pi_1-2)}{n(n-2)(n-3)(n-4)}+\frac{2n(-4\pi_1^2+\pi_1+1)}{n(n-2)(n-3)(n-4)}\\&+\frac{2(2\pi_1-1)}{n(n-2)(n-3)(n-4)}\end{aligned}\right\}\left(y_i(1)+y_k(1)\right)y_j(1)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2n\pi_0-2}{n(n-3)(n-4)}y_i(1)y_k(1)
\end{aligned}
$$

The sixth term,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_j}{\pi_1}-1\right)t_jt_k\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_k(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\
=&\ \mathbb{E}\left[\left(\frac{\pi_0}{\pi_1^2}t_it_jt_k-\frac{\pi_0}{\pi_1}t_jt_k\right)\left(y_j(1)y_k(1)-(y_j(1)+y_k(1))\widehat{\tau}_{\mathsf{unadj}}+\widehat{\tau}^2_{\mathsf{unadj}}\right)\right]\\
=&\ \frac{\pi_0}{\pi_1}\left\{\frac{1}{\pi_1}\mathbb{E}\left[t_it_jt_k\right]-\mathbb{E}\left[t_jt_k\right]\right\}y_j(1)y_k(1)+\frac{\pi_0}{\pi_1}\left\{\mathbb{E}\left[t_jt_k\widehat{\tau}_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_jt_k\widehat{\tau}_{\mathsf{unadj}}\right]\right\}(y_j(1)+y_k(1))\\
&-\frac{\pi_0}{\pi_1}\mathbb{E}\left[t_jt_k\widehat{\tau}^2_{\mathsf{unadj}}\right]+\frac{\pi_0}{\pi_1^2}\mathbb{E}\left[t_it_jt_k\widehat{\tau}^2_{\mathsf{unadj}}\right]\\
=&\ \frac{\pi_0}{\pi_1}\left\{\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)-\pi_1\left(\pi_1-\frac{\pi_0}{n-1}\right)\right\}y_j(1)y_k(1)\\
&+\frac{\pi_0}{\pi_1}\left\{\begin{array}{l}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}+\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{\pi_0}{n-2}\left(y_j(1)+y_k(1)\right)\\
-\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}\\
-\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{\pi_0}{n-3}\left(y_i(1)+y_j(1)+y_k(1)\right)\end{array}\right\}(y_j(1)+y_k(1))\\
&-\frac{\pi_0}{\pi_1}\left\{\begin{array}{l}\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}^2\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\frac{2}{n-3}\left(y_j(1)+y_k(1)\right)\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)}\\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2(y_j(1)^2+y_j(1)y_k(1)+y_k(1)^2)}{n(n-3)}\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(y_j(1)+y_k(1))^2}{n(n-2)}\end{array}\right\}\\
&+\frac{\pi_0}{\pi_1^2}\left\{\begin{array}{l}\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\bar{\tau}^2\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}\cdot\frac{2}{n-4}\left(y_i(1)+y_j(1)+y_k(1)\right)\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{1}{n-4}\bar{\tau}^{(2)}\\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{2\sum\limits_{r,s\in\{i,j,k\}}y_r(1)y_s(1)}{n(n-4)}\\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{(y_i(1)+y_j(1)+y_k(1))^2}{n(n-3)}\end{array}\right\}\\
=&\ -\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{4}{n-4}\bar{\tau}^2\\
&+\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{4\pi_1-3}{(n-3)(n-4)}\bar{\tau}^{(2)}
\end{aligned}
$$

$$+\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{2}{n-4}\bar{\tau}y_i(1)$$
$$+\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{3n-10-6\frac{\pi_0}{\pi_1}}{(n-3)(n-4)}\bar{\tau}\left(y_j(1)+y_k(1)\right)$$
$$+\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{n(1-2\pi_1)+2}{n(n-3)(n-4)}y_i(1)^2$$
$$+\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(n\pi_1-2)((\frac{1}{\pi_1}-3)(n-4)-2+6\frac{\pi_0}{\pi_1})+(n-3)(n-4)}{n(n-2)(n-3)(n-4)}\left(y_j(1)^2+y_k(1)^2\right)$$
$$+\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{-\pi_1n^2+2(\pi_1+1)n-2}{n(n-3)(n-4)}y_i(1)\left(y_j(1)+y_k(1)\right)$$
$$-2\frac{1}{\pi_1}\left(\frac{\pi_0}{\pi_1}\right)^2\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{n^3\pi_1^2-2n^2\pi_1(2\pi_1+1)+2n(2\pi_1^2+\pi_1+1)+4\pi_1-2}{n(n-2)(n-3)(n-4)}\right)y_j(1)y_k(1)$$

Combining the fifth and the sixth term and the coefficients $\sum_{1\leq i\neq j\neq k\leq n}H_{i,j}H_{j,k}$, we have

$$\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_k}{\pi_1}-1\right)t_j^2\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)^2\right]+\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_j}{\pi_1}-1\right)t_jt_k\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_k(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]$$
$$=\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{n\pi_1(4\pi_1-5)}{(n-3)(n-4)}+\frac{24\pi_0}{(n-3)(n-4)}\right)\bar{\tau}^2$$
$$+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{-4n\pi_1\pi_0^2}{(n-2)(n-3)(n-4)}+\frac{20\pi_1^2-30\pi_1+12}{(n-2)(n-3)(n-4)}\right)\bar{\tau}^{(2)}$$
$$+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{2n\pi_1\pi_0}{(n-3)(n-4)}+\frac{14\pi_1-12}{(n-3)(n-4)}\right)\bar{\tau}y_i(1)$$
$$+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{n^2\pi_1^2(-3\pi_1+5)}{(n-2)(n-3)(n-4)}+\frac{n\pi_1(4\pi_1^2+14\pi_1-26)}{(n-2)(n-3)(n-4)}+\frac{-32\pi_1^2+12\pi_1+24}{(n-2)(n-3)(n-4)}\right)\bar{\tau}y_j(1)$$
$$+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{3n\pi_1\pi_0}{(n-3)(n-4)}+\frac{4\pi_1^2+10\pi_1-12}{(n-3)(n-4)}\right)\bar{\tau}y_k(1)$$
$$+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\frac{n^2\pi_1(2\pi_1^2-3\pi_1+1)}{n(n-2)(n-3)(n-4)}+\frac{n(-14\pi_1^2+17\pi_1-4)}{n(n-2)(n-3)(n-4)}+\frac{-8\pi_0}{n(n-2)(n-3)(n-4)}\right)y_i(1)^2$$
$$+\left\{\frac{\pi_0}{\pi_1}\frac{1-n\pi_1}{n(n-1)}++\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\begin{pmatrix}\frac{n^2\pi_1(3\pi_1^2-11\pi_1+5)}{n(n-2)(n-3)(n-4)}+\frac{n(-4\pi_1^3+25\pi_1^2-\pi_1-4)}{n(n-2)(n-3)(n-4)}\\+\frac{-28\pi_1^2+16\pi_1-8}{n(n-2)(n-3)(n-4)}\end{pmatrix}\right\}y_j(1)^2$$
$$+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\begin{pmatrix}\frac{n^2\pi_1(3\pi_1^2-5\pi_1+2)}{n(n-2)(n-3)(n-4)}+\frac{n(-4\pi_1^3-5\pi_1^2+12\pi_1-4)}{n(n-2)(n-3)(n-4)}\\+\frac{-4\pi_1^2+12\pi_1-8}{n(n-2)(n-3)(n-4)}\end{pmatrix}y_k(1)^2$$
$$+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\begin{pmatrix}\frac{-n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)}+\frac{n^2\pi_1(-2\pi_1^2-8\pi_1+8)}{n(n-2)(n-3)(n-4)}\\+\frac{n(22\pi_1^2-6\pi_1-8)}{n(n-2)(n-3)(n-4)}+\frac{-12\pi_1+8}{n(n-2)(n-3)(n-4)}\end{pmatrix}y_i(1)y_j(1)$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{-n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)} + \frac{n^2\pi_1(-2\pi_1^2 - 4\pi_1 + 6)}{n(n-2)(n-3)(n-4)} \\ + \frac{n(6\pi_1^2 - 8)}{n(n-2)(n-3)(n-4)} + \frac{-4\pi_1 + 8}{n(n-2)(n-3)(n-4)}\end{array}\right) y_i(1) y_k(1)
$$

$$
- 2\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)} + \frac{n^2\pi_1(4\pi_1^2 + \pi_1 - 4)}{n(n-2)(n-3)(n-4)} \\ + \frac{n(-4\pi_1^3 - 6\pi_1^2 + 2\pi_1 + 4)}{n(n-2)(n-3)(n-4)} + \frac{-4\pi_1^2 + 10\pi_1 - 4}{n(n-2)(n-3)(n-4)}\end{array}\right) y_j(1) y_k(1)
$$

$$
\sum_{1\le i\ne j\ne k\le n} H_{i,j}H_{j,k}\left(V_{5,ijk} + V_{6,ijk}\right)
$$

$$
= \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\frac{n\pi_1(4\pi_1 - 5)}{(n-3)(n-4)} + \frac{24\pi_0}{(n-3)(n-4)}\right)\bar{\tau}^2 \\ + \left(\frac{-4n\pi_1\pi_0^2}{(n-2)(n-3)(n-4)} + \frac{20\pi_1^2 - 30\pi_1 + 12}{(n-2)(n-3)(n-4)}\right)\bar{\tau}^{(2)}\end{array}\right\}\sum_{i=1}^n H_{i,i}\left(2H_{i,i} - 1\right)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)\left(\frac{5n\pi_1\pi_0}{(n-3)(n-4)} + \frac{4\pi_1^2 + 24\pi_1 - 24}{(n-3)(n-4)}\right)\bar{\tau}
$$

$$
\times\left(\sum_{i=1}^n H_{i,i}\left(H_{i,i} - 1\right) y_i(1) - \sum_{1\le i\ne j\le n} H_{i,i}H_{i,j}y_j(1)\right)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^2\pi_1^2(-3\pi_1 + 5)}{(n-2)(n-3)(n-4)} \\ + \frac{n\pi_1(4\pi_1^2 + 14\pi_1 - 26)}{(n-2)(n-3)(n-4)} + \frac{-32\pi_1^2 + 12\pi_1 + 24}{(n-2)(n-3)(n-4)}\end{array}\right)\bar{\tau}\sum_{i=1}^n H_{i,i}\left(2H_{i,i} - 1\right) y_i(1)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^2\pi_1(5\pi_1^2 - 8\pi_1 + 3)}{n(n-2)(n-3)(n-4)} + \frac{n(-4\pi_1^3 - 19\pi_1^2 + 29\pi_1 - 8)}{n(n-2)(n-3)(n-4)} \\ + \frac{-4\pi_1^2 + 20\pi_1 - 16}{n(n-2)(n-3)(n-4)}\end{array}\right)
$$

$$
\times\left(\sum_{i=1}^n H_{i,i}\left(H_{i,i} - 1\right) y_i(1)^2 - \sum_{1\le i\ne j\le n} H_{i,i}H_{i,j}y_j(1)^2\right)
$$

$$
+ \left\{\frac{\pi_0}{\pi_1}\frac{1 - n\pi_1}{n(n-1)} + \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^2\pi_1(3\pi_1^2 - 11\pi_1 + 5)}{n(n-2)(n-3)(n-4)} + \frac{n(-4\pi_1^3 + 25\pi_1^2 - \pi_1 - 4)}{n(n-2)(n-3)(n-4)} \\ + \frac{-28\pi_1^2 + 16\pi_1 - 8}{n(n-2)(n-3)(n-4)}\end{array}\right)\right\}
$$

$$
\times\sum_{i=1}^n H_{i,i}\left(2H_{i,i} - 1\right) y_i(1)^2
$$

$$
\begin{aligned}
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{-3n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)}+\dfrac{n^2\pi_1(-10\pi_1^2-10\pi_1+16)}{n(n-2)(n-3)(n-4)}\\ \quad+\dfrac{n(8\pi_1^3+34\pi_1^2-10\pi_1-16)}{n(n-2)(n-3)(n-4)}+\dfrac{8\pi_1^2-32\pi_1+16}{n(n-2)(n-3)(n-4)}\end{array}\right)\\
&\quad\times\left(-\sum_{1\le i\neq j\le n}H_{i,i}H_{i,j}y_i(1)y_j(1)-\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)+\sum_{i=1}^n H_{i,i}^2y_i(1)^2\right)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{-n^3\pi_1^2\pi_0}{n(n-2)(n-3)(n-4)}+\dfrac{n^2\pi_1(-2\pi_1^2-4\pi_1+6)}{n(n-2)(n-3)(n-4)}\\ \quad+\dfrac{n(6\pi_1^2-8)}{n(n-2)(n-3)(n-4)}+\dfrac{-4\pi_1+8}{n(n-2)(n-3)(n-4)}\end{array}\right)\\
&\quad\times\sum_{1\le i\neq j\le n}H_{i,j}\left(1-2H_{j,j}\right)y_i(1)y_j(1).
\end{aligned}
$$

The seventh term,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)\left(\frac{t_k}{\pi_1}-1\right)t_jt_l\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\left(y_l(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\\
=\;&\mathbb{E}\left[\left(\frac{1}{\pi_1^2}t_it_jt_kt_l-\frac{1}{\pi_1}t_it_jt_l-\frac{1}{\pi_1}t_jt_kt_l+t_jt_l\right)\left(y_j(1)y_l(1)-\left(y_j(1)+y_l(1)\right)\widehat{\tau}_{\mathsf{unadj}}+\widehat{\tau}^2_{\mathsf{unadj}}\right)\right]\\
=\;&\left\{\mathbb{E}\left[t_jt_l\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_jt_l\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_jt_kt_l\right]+\frac{1}{\pi_1^2}\mathbb{E}\left[t_it_jt_kt_l\right]\right\}y_j(1)y_l(1)\\
&-\left\{\mathbb{E}\left[t_jt_l\widehat{\tau}_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_jt_l\widehat{\tau}_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_jt_kt_l\widehat{\tau}_{\mathsf{unadj}}\right]+\frac{1}{\pi_1^2}\mathbb{E}\left[t_it_jt_kt_l\widehat{\tau}_{\mathsf{unadj}}\right]\right\}\left(y_j(1)+y_l(1)\right)\\
&+\mathbb{E}\left[t_jt_l\widehat{\tau}^2_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_it_jt_l\widehat{\tau}^2_{\mathsf{unadj}}\right]-\frac{1}{\pi_1}\mathbb{E}\left[t_jt_kt_l\widehat{\tau}^2_{\mathsf{unadj}}\right]+\frac{1}{\pi_1^2}\mathbb{E}\left[t_it_jt_kt_l\widehat{\tau}^2_{\mathsf{unadj}}\right]\\
=\;&\left\{\begin{array}{l}\pi_1\left(\pi_1-\dfrac{\pi_0}{n-1}\right)-2\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\\ \quad+\dfrac{1}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\left(\pi_1-\dfrac{3\pi_0}{n-3}\right)\end{array}\right\}y_j(1)y_l(1)\\
&-\left\{\begin{array}{l}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\bar{\tau}+\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\dfrac{\pi_0}{n-2}\left(y_j(1)+y_l(1)\right)\\ \quad-2\dfrac{1}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\left(\pi_1-\dfrac{3\pi_0}{n-3}\right)\bar{\tau}\\ \quad-\dfrac{1}{\pi_1}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\dfrac{\pi_0}{n-3}\left(y_i(1)+2y_j(1)+y_k(1)+2y_l(1)\right)\\ \quad+\dfrac{1}{\pi_1^2}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\left(\pi_1-\dfrac{3\pi_0}{n-3}\right)\left(\pi_1-\dfrac{4\pi_0}{n-4}\right)\bar{\tau}\\ \quad+\dfrac{1}{\pi_1^2}\left(\pi_1-\dfrac{\pi_0}{n-1}\right)\left(\pi_1-\dfrac{2\pi_0}{n-2}\right)\left(\pi_1-\dfrac{3\pi_0}{n-3}\right)\dfrac{\pi_0}{n-4}\left(y_i(1)+y_j(1)+y_k(1)+y_l(1)\right)\end{array}\right\}\left(y_j(1)+y_l(1)\right)
\end{aligned}
$$

$$
+\left\{\begin{array}{l}
\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}^2 \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\bar{\tau}\frac{2}{n-3}\left(y_j(1)+y_l(1)\right) \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{1}{n-3}\bar{\tau}^{(2)} \\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2(y_j(1)^2+y_j(1)y_l(1)+y_l(1)^2)}{n(n-3)} \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{(y_j(1)+y_l(1))^2}{n(n-2)}
\end{array}\right\}
$$

$$
-\frac{1}{\pi_1}\left\{\begin{array}{l}
2\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\bar{\tau}^2 \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\bar{\tau}\cdot\frac{2}{n-4}\left(y_i(1)+2y_j(1)+y_k(1)+2y_l(1)\right) \\
+2\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{1}{n-4}\bar{\tau}^{(2)} \\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{2\sum\limits_{r,s\in\{i,j,l\}}y_r(1)y_s(1)+2\sum\limits_{r,s\in\{j,k,l\}}y_r(1)y_s(1)}{n(n-4)} \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{(y_i(1)+y_j(1)+y_l(1))^2+(y_j(1)+y_k(1)+y_l(1))^2}{n(n-3)}
\end{array}\right\}
$$

$$
+\frac{1}{\pi_1^2}\left\{\begin{array}{l}
\frac{1}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\left(\pi_1-\frac{5\pi_0}{n-5}\right)\bar{\tau}^2 \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\bar{\tau}\frac{2}{n-5}\left(y_i(1)+y_j(1)+y_k(1)+y_l(1)\right) \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\frac{1}{n-5}\bar{\tau}^{(2)} \\
-\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\left(\pi_1-\frac{4\pi_0}{n-4}\right)\frac{2\sum\limits_{r,s\in\{i,j,k,l\}}y_r(1)y_s(1)}{n(n-5)} \\
+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{(y_i(1)+y_j(1)+y_k(1)+y_l(1))^2}{n(n-4)}
\end{array}\right\}
$$

$$
\begin{aligned}
=&\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{-n+20\frac{\pi_0}{\pi_1}}{(n-4)(n-5)}\bar{\tau}^2 \\
&+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{12\left(\frac{\pi_0}{\pi_1}\right)^2-(n+6)\frac{\pi_0}{\pi_1}+2}{(n-3)(n-4)(n-5)}\bar{\tau}^{(2)} \\
&+2\frac{1}{\pi_1}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{1-4\frac{\pi_0}{\pi_1]}}{(n-4)(n-5)}\bar{\tau}\left(y_i(1)+y_k(1)\right) \\
&+\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{n(n-5)+4+\frac{\pi_0}{\pi_1}\left(24\frac{\pi_0}{\pi_1}-14n+48\right)}{(n-3)(n-4)(n-5)}\bar{\tau}\left(y_j(1)+y_l(1)\right) \\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\frac{2\pi_1(6n+5-5n\pi_1)-3(n+3)}{n(n-3)(n-4)(n-5)}\left(y_i(1)^2+y_k(1)^2\right)
\end{aligned}
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n(n-2)}\left\{\begin{array}{l}\frac{(-\pi_1^3+\pi_1^2)n^3}{(n-3)(n-4)(n-5)}+\frac{\left(-9\pi_1^3+14\pi_1^2-7\pi_1\right)n^2}{(n-3)(n-4)(n-5)}\\ +\frac{\left(20\pi_1^3-9\pi_1^2-9\pi_1+6\right)n}{(n-3)(n-4)(n-5)}+\frac{20\pi_1^2-40\pi_1+18}{(n-3)(n-4)(n-5)}\end{array}\right\}\left(y_j(1)^2+y_l(1)^2\right)
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\begin{array}{l}\frac{(3\pi_1-4\pi_1^2)n^2}{n(n-3)(n-4)(n-5)}\\ +\frac{(\pi_1+10\pi_1^2-6)n-10\pi_1+6}{n(n-3)(n-4)(n-5)}\end{array}\right)\left(y_i(1)+y_k(1)\right)\left(y_j(1)+y_l(1)\right)
$$

$$
+\frac{2}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\pi_1-\frac{3\pi_0}{n-3}\right)\frac{n\pi_0-1}{n(n-4)(n-5)}y_i(1)y_k(1)
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}-\frac{n^4\pi_1^3}{n(n-2)(n-3)(n-4)(n-5)}+\frac{n^3(\pi_1^3+8\pi_1^2)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n^2(18\pi_1^3-28\pi_1^2-14\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}+\frac{n(-40\pi_1^3+38\pi_1^2+10\pi_1+12)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{-40\pi_1^2+40\pi_1-12}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right)y_j(1)y_l(1)
$$

Combining the seventh term with the coefficient $\sum_{1\le i\ne j\ne k\ne l\le n}H_{i,j}H_{k,l}$, we have

$$
\begin{aligned}
&\sum_{1\le i\ne j\ne k\ne l\le n}H_{i,j}H_{k,l}V_{7,ijkl}\\
=&\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left\{\begin{array}{l}\frac{-n^2\pi_1^2+n\pi_1(-20\pi_1+23)-60\pi_0}{(n-3)(n-4)(n-5)}\bar{\tau}^2\\ +\frac{-n\pi_1\pi_0+20\pi_1^2-30\pi_1+12}{(n-3)(n-4)(n-5)}\bar{\tau}^{(2)}\end{array}\right\}\left(p(2+p)-6\sum_{i=1}^nH_{i,i}^2\right)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{n^2\pi_1^2+n\pi_1(19\pi_1-22)}{(n-3)(n-4)(n-5)}+\frac{-20\pi_1^2-30\pi_1+48}{(n-3)(n-4)(n-5)}\right)\\
&\times 2\bar{\tau}\left(\sum_{i=1}^nH_{i,i}\left((2+p)-4H_{i,i}\right)y_i(1)+2\sum_{1\le i\ne j\le n}H_{i,j}H_{j,j}y_i(1)\right)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n(n-2)}\left\{\begin{array}{l}\frac{n^3\pi_1^2\pi_0}{(n-3)(n-4)(n-5)}+\frac{n^2\left(-19\pi_1^3+26\pi_1^2-10\pi_1\right)}{(n-3)(n-4)(n-5)}\\ +\frac{n\left(20\pi_1^3+21\pi_1^2-42\pi_1+12\right)}{(n-3)(n-4)(n-5)}+\frac{20\pi_1^2-60\pi_1+36}{(n-3)(n-4)(n-5)}\end{array}\right\}\\
&\times 2\left(\sum_{i=1}^nH_{i,i}\left((2+p)-4H_{i,i}\right)y_i(1)^2+2\sum_{1\le i\ne j\le n}H_{i,j}H_{j,j}y_i(1)^2\right)\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left(\frac{n^2(3\pi_1-4\pi_1^2)}{n(n-3)(n-4)(n-5)}+\frac{n(\pi_1+10\pi_1^2-6)-10\pi_1+6}{n(n-3)(n-4)(n-5)}\right)\\
&\times 2\left(-2\sum_{i=1}^nH_{i,i}^2y_i(1)^2+2\cdot\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)+\sum_{1\le i\ne j\le n}H_{i,j}\left(4H_{j,j}-p\right)y_i(1)y_j(1)\right)
\end{aligned}
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l} -\dfrac{n^4\pi_1^3}{n(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^3\pi_1^2(-9\pi_1+16)}{n(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{n^2\pi_1(38\pi_1^2-2\pi_1-48)}{n(n-2)(n-3)(n-4)(n-5)}+\dfrac{n(-40\pi_1^3-22\pi_1^2+16\pi_1+48)}{n(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{-40\pi_1^2+80\pi_1-48}{n(n-2)(n-3)(n-4)(n-5)} \end{array}\right)
$$

$$
\times\left(-2\sum_{i=1}^n H_{i,i}^2 y_i(1)^2+\mathsf{tr}^2\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)+\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)+\sum_{1\le i\ne j\le n}H_{i,j}\left(4H_{j,j}-1\right)y_i(1)y_j(1)\right).
$$

Combining these terms, the second moment is as follows,

$$
\mathbb{E}\left[\left\{\sum_{1\le i\ne j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right\}^2\right]
$$

$$
=\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left\{\begin{array}{l} \dfrac{2n^3\pi_1\pi_0}{(n-3)(n-4)(n-5)}+\dfrac{n^2(-p\pi_1^2+4\pi_1-6)}{(n-3)(n-4)(n-5)} \\ +\dfrac{n(-20p\pi_1^2+23p\pi_1+6)}{(n-3)(n-4)(n-5)}+\dfrac{-60p\pi_0}{(n-3)(n-4)(n-5)} \end{array}\right\}\bar{\tau}^2\sum_{i=1}^n H_{i,i}
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \dfrac{2n^3\pi_1\pi_0^2}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^2(p\pi_1^3+(2-p)\pi_1^2+2\pi_1-4)}{(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{n(20p\pi_1^3-32p\pi_1^2+(14p-8)\pi_1+12}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{-40p\pi_1^2+60p\pi_1-24p-8}{(n-2)(n-3)(n-4)(n-5)} \end{array}\right\}\bar{\tau}^{(2)}\sum_{i=1}^n H_{i,i}
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left\{\begin{array}{l} \dfrac{n^3\pi_1(3\pi_1^2-2)}{(n-3)(n-4)(n-5)}+\dfrac{n^2(11\pi_1^2-20\pi_1+6)}{(n-3)(n-4)(n-5)} \\ +\dfrac{n(20\pi_1^2-60\pi_1+42)}{(n-3)(n-4)(n-5)} \end{array}\right\}\bar{\tau}^2\sum_{i=1}^n H_{i,i}^2
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \dfrac{n^3(-3\pi_1^3+5\pi_1^2-2\pi_1)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^2(-11\pi_1^3+20\pi_1^2-14\pi_1+4)}{(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{n(-20\pi_1^3+49\pi_1^2-44\pi_1+12}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{-20\pi_1^2+40\pi_1-16}{(n-2)(n-3)(n-4)(n-5)} \end{array}\right\}\bar{\tau}^{(2)}\sum_{i=1}^n H_{i,i}^2
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \dfrac{n^4(4\pi_1^3-4\pi_1^2)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^3((2p-4)\pi_1^3-8\pi_1^2+16\pi_1)}{(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{n^2(38p\pi_1^3-(48p+4)\pi_1^2-16)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{80p\pi_1^2+120p\pi_1-192p+16}{(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{n(-40p\pi_1^3-136p\pi_1^2+(184p+16)\pi_1)}{(n-2)(n-3)(n-4)(n-5)} \end{array}\right\}\bar{\tau}\sum_{i=1}^n H_{i,i}y_i(1)
$$

$$
+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \dfrac{n^4(-4\pi_1^3+4\pi_1^2)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^3(-10\pi_1^3+18\pi_1^2-16\pi_1)}{(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{n^2(-18\pi_1^3+92\pi_1^2-48\pi_1+16)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n(40\pi_1^3+22\pi_1^2-160\pi_1+48)}{(n-2)(n-3)(n-4)(n-5)} \\ +\dfrac{40\pi_1^2-160\pi_1+128}{(n-2)(n-3)(n-4)(n-5)} \end{array}\right\}\bar{\tau}\sum_{i=1}^n H_{i,i}^2 y_i(1)
$$

$$
\begin{aligned}
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\frac{2n^4\pi_1^3}{(n-2)(n-3)(n-4)(n-5)}+\frac{n^3(6\pi_1^3-24\pi_1^2)}{(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n^2(-32\pi_1^2+88\pi_1)}{(n-2)(n-3)(n-4)(n-5)}+\frac{n(40\pi_1^2-8\pi_1-96)}{(n-2)(n-3)(n-4)(n-5)}\\ +\frac{-80\pi_1+96}{(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\\
&\quad\times\bar{\tau}\sum_{1\le i\ne j\le n}H_{i,i}H_{i,j}y_j(1)\\
&+\left\{\pi_0-\frac{\pi_0^2}{\pi_1}\frac{1}{n-1}+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^4(-4\pi_1^3+10\pi_1^2-4\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}+\\ \frac{n^3((-2p+4)\pi_1^3+(2p-44)\pi_1^2+12\pi_1+4)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n^2(-38p\pi_1^3+(52p+138)\pi_1^2-(20p+48)\pi_1-8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n(40p\pi_1^3+(42p-120)\pi_1^2+(-84p+24)\pi_1+24p+20)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{40p\pi_1^2-120p\pi_1+72p-16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right)\right\}\\
&\quad\times\sum_{i=1}^{n}H_{i,i}y_i(1)^2\\
&+\left\{-\pi_0+\frac{\pi_0}{\pi_1}\frac{n(-2\pi_1+1)+1}{n(n-1)}+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\frac{n^5(\pi_1^3-\pi_1^2)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n^4(4\pi_1^3-12\pi_1^2+8\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n^3(15\pi_1^3+11\pi_1^2-8\pi_1-8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n^2(-20\pi_1^3-128\pi_1^2+108\pi_1+8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n(-54\pi_1^2+116\pi_1-112)+120\pi_1^2+16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right)\right\}\\
&\quad\times\sum_{i=1}^{n}H_{i,i}^2y_i(1)^2\\
&+\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}-\frac{n^4\pi_1^3}{n(n-2)(n-3)(n-4)(n-5)}+\frac{n^3\pi_1^2(-9\pi_1+16)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{n^2\pi_1(38\pi_1^2-2\pi_1-48)}{n(n-2)(n-3)(n-4)(n-5)}+\frac{n(-40\pi_1^3-22\pi_1^2+16\pi_1+48)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\frac{-40\pi_1^2+80\pi_1-48}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right)\mathsf{tr}^2\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)
\end{aligned}
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \frac{n^5(-\pi_1^3+\pi_1^2)+n^4(\pi_1^3+2\pi_1^2-4\pi_1)}{n(n-2)(n-3)(n-4)(n-5)} \\ + \frac{n^3(4\pi_1^3-7\pi_1^2+4\pi_1+4)+n^2(4\pi_1^2-8)}{n(n-2)(n-3)(n-4)(n-5)} \\ + \frac{n(-16\pi_1+20)-16}{n(n-2)(n-3)(n-4)(n-5)} \end{array}\right\} \mathsf{tr}\left(\widehat{\mathbf{\Sigma}}_y\widehat{\mathbf{\Sigma}}^-\widehat{\mathbf{\Sigma}}_y\widehat{\mathbf{\Sigma}}^-\right)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \frac{n^5\pi_1^3+n^4(-5\pi_1^3-6\pi_1^2)+n^3((8p+8)\pi_1^3+(-6p+18)\pi_1^2+16\pi_1)}{n(n-2)(n-3)(n-4)(n-5)} \\ + \frac{n^2(-20p\pi_1^3-(18p+12)\pi_1^2+(24p-40)\pi_1-16)}{n(n-2)(n-3)(n-4)(n-5)} \\ + \frac{n(60p\pi_1^2+(-8p+8)\pi_1-24p+48)-40p\pi_1+24p-32}{n(n-2)(n-3)(n-4)(n-5)} \end{array}\right\}
$$

$$
\times \sum_{1\le i\neq j\le n} H_{i,j}y_i(1)y_j(1)
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \frac{n^4(-2\pi_1^3+2\pi_1^2)+n^3(-6\pi_1^3+16\pi_1^2-12\pi_1)}{n(n-2)(n-3)(n-4)(n-5)} \\ + \frac{n^2(14\pi_1^2-24\pi_1+16)+4n\pi_1-16}{n(n-2)(n-3)(n-4)(n-5)} \end{array}\right\} \sum_{1\le i\neq j\le n} H_{i,i}H_{i,j}y_j(1)^2
$$

$$
+ \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left\{\begin{array}{l} \frac{-2n^5\pi_1^3+n^4(-2\pi_1^3+20\pi_1^2)+n^3(12\pi_1^3-64\pi_1)}{n(n-2)(n-3)(n-4)(n-5)} \\ + \frac{n^2(-68\pi_1^2+56\pi_1+64)+n(72\pi_1-96)+32}{n(n-2)(n-3)(n-4)(n-5)} \end{array}\right\} \sum_{1\le i\neq j\le n} H_{i,i}H_{i,j}y_i(1)y_j(1)
$$

A.1.2 The Squared Mean

In this subsection we mainly compute the second term of $\mathsf{var}^{\mathsf{rd}}(\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2})$,

$$
\begin{aligned}
&\left\{\mathbb{E}\left[\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\right\}^2 \\
=& \sum_{1\le i\neq j\le n} H_{i,j}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\sum_{1\le k\neq l\le n} H_{k,l}\mathbb{E}\left[\left(\frac{t_k}{\pi_1}-1\right)t_l\left(y_l(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right] \\
=& \sum_{1\le i\neq j\le n} H_{i,j}^2\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\left\{\begin{array}{l}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right] \\ +\mathbb{E}\left[\left(\frac{t_j}{\pi_1}-1\right)t_i\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\end{array}\right\} \\
&+ \sum_{1\le i\neq j\neq k\le n} H_{i,j}H_{i,k}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\left\{\begin{array}{l}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_k\left(y_k(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right] \\ +\mathbb{E}\left[\left(\frac{t_k}{\pi_1}-1\right)t_i\left(y_i(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{1\le i\ne j\ne k\le n} H_{i,j}H_{j,k}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\left\{\begin{array}{l}\mathbb{E}\left[\left(\frac{t_j}{\pi_1}-1\right)t_k\left(y_k(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right] \\ +\mathbb{E}\left[\left(\frac{t_k}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\end{array}\right\} \\
&+ \sum_{1\le i\ne j\ne k\ne l\le n} H_{i,j}H_{k,l}\mathbb{E}\left[\left(\frac{t_i}{\pi_1}-1\right)t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\mathbb{E}\left[\left(\frac{t_k}{\pi_1}-1\right)t_l\left(y_l(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right] \\
=\ & 2\cdot\sum_{1\le i\ne j\le n} H_{i,j}^2\left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\left(2\bar{\tau}-\left(y_i(1)+y_j(1)\right)\right)\right\}^2 \\
&+2\cdot\sum_{1\le i\ne j\ne k\le n} H_{i,j}H_{i,k}\left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\right\}^2\left(2\bar{\tau}-\left(y_i(1)+y_j(1)\right)\right)\left(2\bar{\tau}-\left(y_i(1)+y_k(1)\right)\right) \\
&+2\cdot\sum_{1\le i\ne j\ne k\le n} H_{i,j}H_{j,k}\left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\right\}^2\left(2\bar{\tau}-\left(y_i(1)+y_j(1)\right)\right)\left(2\bar{\tau}-\left(y_j(1)+y_k(1)\right)\right) \\
&+\sum_{1\le i\ne j\ne k\ne l\le n} H_{i,j}H_{k,l}\left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\right\}^2\left(2\bar{\tau}-\left(y_i(1)+y_j(1)\right)\right)\left(2\bar{\tau}-\left(y_k(1)+y_l(1)\right)\right) \\
=\ & 2\cdot\left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\right\}^2\left\{\begin{array}{l}4\bar{\tau}^2\sum_{i=1}^n H_{i,i}\left(1-H_{i,i}\right)-8\bar{\tau}\sum_{i=1}^n H_{i,i}\left(1-H_{i,i}\right)y_i(1) \\ +2\sum_{i=1}^n H_{i,i}\left(1-2H_{i,i}\right)y_i(1)^2+2\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)\end{array}\right\} \\
&+2\cdot\left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\right\}^2\left\{\begin{array}{l}4\bar{\tau}^2\sum_{i=1}^n H_{i,i}\left(4H_{i,i}-2\right) \\ -8\bar{\tau}\left(\sum_{i=1}^n H_{i,i}\left(3H_{i,i}-2\right)y_i(1)-\sum_{1\le i\ne j\le n}H_{i,i}H_{i,j}y_j(1)\right) \\ +2\sum_{i=1}^n H_{i,i}\left(4H_{i,i}-1\right)y_i(1)^2-4\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right) \\ +2\sum_{1\le i\ne j\le n}H_{i,j}\left(1-4H_{j,j}\right)y_i(1)y_j(1)\end{array}\right\} \\
&+\left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\right\}^2\left\{\begin{array}{l}4\bar{\tau}^2\left(p(2+p)-6\sum_{i=1}^n H_{i,i}^2\right) \\ +8\bar{\tau}\left(\sum_{i=1}^n H_{i,i}\left(4H_{i,i}-(2+p)\right)y_i(1)-2\sum_{1\le i\ne j\le n}H_{i,i}H_{i,j}y_j(1)\right) \\ -8\sum_{i=1}^n H_{i,i}^2y_i(1)^2+4\mathsf{tr}^2\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)+4\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right) \\ +4\sum_{1\le i\ne j\le n}H_{i,j}\left(4H_{j,j}-1\right)y_i(1)y_j(1)\end{array}\right\} \\
=\ & \left\{\frac{\pi_0}{\pi_1}\left(\pi_1-\frac{\pi_0}{n-1}\right)\frac{1}{n-2}\right\}^2\left\{4\bar{\tau}^2p^2-8\bar{\tau}p\sum_{i=1}^n H_{i,i}y_i(1)+4\mathsf{tr}^2\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)\right\}
\end{aligned}
$$

A.1.3 Variance of $\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}$

Combining with the second moment and the squared mean, the variance of $\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}$ as following,

$$
\begin{aligned}
&\mathsf{var}^{\mathsf{rd}}(\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}) \equiv \mathsf{var}^{\mathsf{rd}}\left(\frac{1}{n}\sum_{1\leq i\neq j\leq n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right)\\
&= \frac{1}{\pi_1^2}\frac{1}{n^2}\left\{\mathbb{E}\left[\left\{\sum_{1\leq i\neq j\leq n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right\}^2\right]-\left\{\mathbb{E}\left[\sum_{1\leq i\neq j\leq n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_j\left(y_j(1)-\widehat{\tau}_{\mathsf{unadj}}\right)\right]\right\}^2\right\}\\
&= \frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}
\dfrac{n^6(-2\pi_1^3+2\pi_1^2)+n^5((-p+6)\pi_1^3+2\pi_1^2-10\pi_1)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n^4(-(13p+4)\pi_1^3+(21p-20)\pi_1^2+28\pi_1+12)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n^3(10p\pi_1^3+(69p+16)\pi_1^2-(102p+14)\pi_1-48)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n^2(148p\pi_1^3-390p\pi_1^2+(150p-4)\pi_1+120p+60)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n(-240p\pi_1^3+252p\pi_1^2+336p\pi_1-360p-24)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{240p\pi_1^2-480p\pi_1+240p}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}
\end{array}\right\}\bar{\tau}^2\sum_{i=1}^n H_{i,i}\\
&\quad+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}
\dfrac{2n^3\pi_1\pi_0^2}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^2(p\pi_1^3+(2-p)\pi_1^2+2\pi_1-4)}{(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{n(20p\pi_1^3-32p\pi_1^2+(14p-8)\pi_1+12}{(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{-40p\pi_1^2+60p\pi_1-24p-8}{(n-2)(n-3)(n-4)(n-5)}
\end{array}\right\}\bar{\tau}^{(2)}\sum_{i=1}^n H_{i,i}\\
&\quad+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\pi_1-\frac{2\pi_0}{n-2}\right)\left\{\begin{array}{l}
\dfrac{n^3\pi_1(3\pi_1-2)}{(n-3)(n-4)(n-5)}+\dfrac{n^2(11\pi_1^2-20\pi_1+6)}{(n-3)(n-4)(n-5)}\\
+\dfrac{n(20\pi_1^2-60\pi_1+42)}{(n-3)(n-4)(n-5)}
\end{array}\right\}\bar{\tau}^2\sum_{i=1}^n H_{i,i}^2\\
&\quad+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}
\dfrac{n^3(-3\pi_1^3+5\pi_1^2-2\pi_1)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^2(-11\pi_1^3+20\pi_1^2-14\pi_1+4)}{(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{n(-20\pi_1^3+49\pi_1^2-44\pi_1+12}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{-20\pi_1^2+40\pi_1-16}{(n-2)(n-3)(n-4)(n-5)}
\end{array}\right\}\bar{\tau}^{(2)}\sum_{i=1}^n H_{i,i}^2
\end{aligned}
$$

$$
+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{aligned}
&\frac{n^6(4\pi_1^3-4\pi_1^2)+n^5(2p\pi_1^3-16\pi_1^3+4\pi_1^2+16\pi_1)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
&+\frac{n^4((24p+20)\pi_1^3+(-40p+12)\pi_1^2-48\pi_1-16)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
&+\frac{n^3(-(54p+8)\pi_1^3-(80p+4)\pi_1^2+(176p+48)\pi_1+48)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
&+\frac{n^2(-180p\pi_1^3+(672p-8)\pi_1^2-(336p+48)\pi_1-192p-16)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
&+\frac{n(400p\pi_1^3-616p\pi_1^2+(-368p+32)\pi_1+576p-48)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
&+\frac{-320p\pi_1^2+720p\pi_1-384p+32}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}
\end{aligned}\right\}\bar{\tau}\sum_{i=1}^n H_{i,i}y_i(1)
$$

$$
+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{aligned}
&\frac{n^4(-4\pi_1^3+4\pi_1^2)}{(n-2)(n-3)(n-4)(n-5)}+\frac{n^3(-10\pi_1^3+18\pi_1^2-16\pi_1)}{(n-2)(n-3)(n-4)(n-5)}\\
&+\frac{n^2(-18\pi_1^3+92\pi_1^2-48\pi_1+16)}{(n-2)(n-3)(n-4)(n-5)}+\frac{n(40\pi_1^3+22\pi_1^2-160\pi_1+48)}{(n-2)(n-3)(n-4)(n-5)}\\
&+\frac{40\pi_1^2-160\pi_1+128}{(n-2)(n-3)(n-4)(n-5)}
\end{aligned}\right\}\bar{\tau}\sum_{i=1}^n H_{i,i}^2y_i(1)
$$

$$
+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{aligned}
&\frac{2n^4\pi_1^3}{(n-2)(n-3)(n-4)(n-5)}+\frac{n^3(6\pi_1^3-24\pi_1^2)}{(n-2)(n-3)(n-4)(n-5)}\\
&+\frac{n^2(-32\pi_1^2+88\pi_1)}{(n-2)(n-3)(n-4)(n-5)}+\frac{n(40\pi_1^2-8\pi_1-96)}{(n-2)(n-3)(n-4)(n-5)}\\
&+\frac{-80\pi_1+96}{(n-2)(n-3)(n-4)(n-5)}
\end{aligned}\right\}
$$

$$
\times\bar{\tau}\sum_{1\le i\ne j\le n} H_{i,i}H_{i,j}y_j(1)
$$

$$
+\left\{\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{n\pi_1-1}{n^2(n-1)}+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{aligned}
&\frac{n^4(-4\pi_1^3+10\pi_1^2-4\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}+\\
&\frac{n^3((-2p+4)\pi_1^3+(2p-44)\pi_1^2+12\pi_1+4)}{n(n-2)(n-3)(n-4)(n-5)}\\
&+\frac{n^2(-38p\pi_1^3+(52p+138)\pi_1^2-(20p+48)\pi_1-8)}{n(n-2)(n-3)(n-4)(n-5)}\\
&+\frac{n(40p\pi_1^3+(42p-120)\pi_1^2+(-84p+24)\pi_1+24p+20)}{n(n-2)(n-3)(n-4)(n-5)}\\
&+\frac{40p\pi_1^2-120p\pi_1+72p-16}{n(n-2)(n-3)(n-4)(n-5)}
\end{aligned}\right)\right\}
$$

$$
\times\sum_{i=1}^n H_{i,i}y_i(1)^2
$$

$$
\begin{aligned}
&+\left\{\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{-n^2\pi_1+n\pi_0+1}{n^3(n-1)}+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{n^5(\pi_1^3-\pi_1^2)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^4(4\pi_1^3-12\pi_1^2+8\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^3(15\pi_1^3+11\pi_1^2-8\pi_1-8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(-20\pi_1^3-128\pi_1^2+108\pi_1+8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n(-54\pi_1^2+116\pi_1-112)+120\pi_1^2+16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right)\right\}\\
&\times\sum_{i=1}^n H_{i,i}^2 y_i(1)^2\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{-n^6\pi_1^3+n^5(-2\pi_1^3+12\pi_1^2)+n^4(15\pi_1^3-6\pi_1^2-44\pi_1)}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\ +\dfrac{n^3(16\pi_1^3-124\pi_1^2+112\pi_1+48)}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(-44\pi_1^3+74\pi_1^2+124\pi_1-192)}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\ +\dfrac{n(-80\pi_1^3+316\pi_1^2-448\pi_1+240)-80\pi_1^2+160\pi_1-96}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\end{array}\right)\mathsf{tr}^2\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{n^5(-\pi_1^3+\pi_1^2)+n^4(\pi_1^3+2\pi_1^2-4\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^3(4\pi_1^3-7\pi_1^2+4\pi_1+4)+n^2(4\pi_1^2-8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n(-16\pi_1+20)-16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{n^5\pi_1^3+n^4(-5\pi_1^3-6\pi_1^2)+n^3((8p+8)\pi_1^3+(-6p+18)\pi_1^2+16\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(-20p\pi_1^3-(18p+12)\pi_1^2+(24p-40)\pi_1-16)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n(60p\pi_1^2+(-8p+8)\pi_1-24p+48)-40p\pi_1+24p-32}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\\
&\times\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)y_j(1)\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{n^4(-2\pi_1^3+2\pi_1^2)+n^3(-6\pi_1^3+16\pi_1^2-12\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(14\pi_1^2-24\pi_1+16)+4n\pi_1-16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\sum_{1\le i\ne j\le n} H_{i,i}H_{i,j}y_j(1)^2
\end{aligned}
$$

$$+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{-2n^5\pi_1^3+n^4(-2\pi_1^3+20\pi_1^2)+n^3(12\pi_1^3-64\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(-68\pi_1^2+56\pi_1+64)+n(72\pi_1-96)+32}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\sum_{1\le i\ne j\le n}H_{i,i}H_{i,j}y_i(1)y_j(1)$$

A.2 The Covariance $\mathsf{cov}^{\mathsf{rd}}\left(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}\right)$

For the covariance between $\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}$ and $\widehat{\tau}_{\mathsf{unadj}}$, we decompose it into four terms and compute them separetely,

$$
\begin{aligned}
&\mathsf{cov}^{\mathsf{rd}}\left(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}\right) \equiv \mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n}\sum_{l=1}^{n}\frac{t_l}{\pi_1}y_l(1), \frac{1}{n}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\left(y_j(1)-\frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\pi_1}y_k(1)\right)\right)\\
&= \mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}y_j(1), \frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\pi_1}y_k(1)\right)\\
&\quad - \mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\pi_1}y_k(1), \frac{1}{n}\sum_{l=1}^{n}\frac{t_l}{\pi_1}y_l(1)\right)\\
&= \mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}y_j(1), \frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\pi_1}y_k(1)\right)\\
&\quad - \left\{\begin{aligned}
&\mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n^2}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\frac{t_i}{\pi_1}y_i(1), \frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\pi_1}y_k(1)\right)\\
&+ \mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n^2}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\frac{t_j}{\pi_1}y_j(1), \frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\pi_1}y_k(1)\right)\\
&+ \mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n^2}\sum_{1\le i\neq j\neq k\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\frac{t_k}{\pi_1}y_k(1), \frac{1}{n}\sum_{l=1}^{n}\frac{t_l}{\pi_1}y_l(1)\right)
\end{aligned}\right\}
\end{aligned}
$$

The first term of $\mathsf{cov}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2})$,

$$
\begin{aligned}
&\mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n}\sum_{1\le i\neq j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}y_j(1), \frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\pi_1}y_k(1)\right)\\
&= \frac{1}{n^2}\frac{1}{\pi_1^2}\sum_{1\le i\neq j\neq k\le n}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_jy_j(1), t_ky_k(1)\right)\\
&\quad + \frac{1}{n^2}\frac{1}{\pi_1^2}\sum_{1\le i\neq j\le n}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_jy_j(1), t_iy_i(1)+t_jy_j(1)\right)\\
&= \frac{1}{n^2}\frac{1}{\pi_1^2}\sum_{1\le i\neq j\neq k\le n}H_{i,j}y_j(1)y_k(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_k\right)\\
&\quad + \frac{1}{n^2}\frac{1}{\pi_1^2}\sum_{1\le i\neq j\le n}H_{i,j}y_j(1)\left\{y_i(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_i\right)+y_j(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_j\right)\right\}\\
&= U_{1,1}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_k\right) + U_{1,2}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_i\right) + U_{1,3}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_j\right),
\end{aligned}
$$

where

$$U_{1,3} = \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j}y_j(1)^2 = \frac{1}{\pi_1^2}\frac{1}{n^2}\left(\sum_{i=1}^n\sum_{j=1}^n H_{i,j}y_j(1)^2 - \sum_{i=1}^n H_{i,i}y_i(1)^2\right) = -\frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}y_i(1)^2,$$

$$U_{1,2} = \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j}y_j(1)y_i(1),$$

$$\begin{aligned}
U_{1,1} &= \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_j(1)y_k(1) = \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_j(1)\left(\sum_{k=1}^n y_k(1) - y_i(1) - y_j(1)\right)\\
&= \bar{\tau}\frac{1}{\pi_1^2}\frac{1}{n}\sum_{1\le i\ne j\le n} H_{i,j}y_j(1) - \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j}y_j(1)y_i(1) - \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j}y_j(1)^2\\
&= -\bar{\tau}\frac{1}{\pi_1^2}\frac{1}{n}\sum_{i=1}^n H_{i,i}y_i(1) + \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}y_i(1)^2 - \frac{1}{\pi_1^2}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)y_j(1).
\end{aligned}$$

The second term of $\mathsf{cov}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2})$,

$$\begin{aligned}
&\mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n^2}\sum_{1\le i\ne j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\frac{t_i}{\pi_1}y_i(1), \frac{1}{n}\sum_{k=1}^n \frac{t_k}{\pi_1}y_k(1)\right)\\
&= \frac{1}{n^3}\frac{\pi_0}{\pi_1}\frac{1}{\pi_1^3}\mathsf{cov}^{\mathsf{rd}}\left(\sum_{1\le i\ne j\le n} t_iy_i(1)H_{i,j}t_j, \sum_{k=1}^n t_ky_k(1)\right)\\
&= \frac{1}{n^3}\frac{\pi_0}{\pi_1^4}\sum_{1\le i\ne j\ne k\le n}\mathsf{cov}^{\mathsf{rd}}\left(t_iy_i(1)H_{i,j}t_j, t_ky_k(1)\right) + \frac{1}{n^3}\frac{\pi_0}{\pi_1^4}\sum_{1\le i\ne j\le n}\mathsf{cov}^{\mathsf{rd}}\left(t_iy_i(1)H_{i,j}t_j, t_iy_i(1)+t_jy_j(1)\right)\\
&= \frac{1}{n^3}\frac{\pi_0}{\pi_1^4}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_i(1)y_k(1)\mathsf{cov}^{\mathsf{rd}}\left(t_it_j, t_k\right) + \frac{1}{n^3}\frac{\pi_0}{\pi_1^4}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)\{y_i(1)\mathsf{cov}^{\mathsf{rd}}\left(t_it_j, t_i\right) + y_j(1)\mathsf{cov}^{\mathsf{rd}}\left(t_it_j, t_j\right)\}\\
&= U_{2,1}\mathsf{cov}^{\mathsf{rd}}\left(t_it_j, t_k\right) + U_{2,2}\mathsf{cov}^{\mathsf{rd}}\left(t_it_j, t_i\right),
\end{aligned}$$

where

$$\begin{aligned}
U_{2,1} &= \frac{1}{n^3}\frac{\pi_0}{\pi_1^4}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_i(1)y_k(1) = \frac{\pi_0}{\pi_1^2}\frac{1}{n}U_{1,1},\\
&= -\bar{\tau}\frac{\pi_0}{\pi_1^4}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}y_i(1) + \frac{\pi_0}{\pi_1^4}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2 - \frac{\pi_0}{\pi_1^4}\frac{1}{n^3}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)y_j(1),\\
U_{2,2} &= \frac{\pi_0}{\pi_1^4}\frac{1}{n^3}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)\left\{y_i(1)+y_j(1)\right\} = \frac{\pi_0}{\pi_1^2}\frac{1}{n}\left\{U_{1,3}+U_{1,2}\right\}\\
&= -\frac{\pi_0}{\pi_1^4}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2 + \frac{\pi_0}{\pi_1^4}\frac{1}{n^3}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)y_j(1).
\end{aligned}$$

The third term of $\mathsf{cov}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2})$,

$$\mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n^2}\sum_{1\le i\ne j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\frac{t_j}{\pi_1}y_j(1), \frac{1}{n}\sum_{k=1}^n \frac{t_k}{\pi_1}y_k(1)\right)$$

$$
\begin{aligned}
&= \frac{1}{n^3}\frac{1}{\pi_1^3}\mathsf{cov}^{\mathsf{rd}}\left(\sum_{1\le i\ne j\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_jy_j(1), \sum_{k=1}^n t_ky_k(1)\right)\\
&= \frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\le n}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_jy_j(1), t_ky_k(1)\right)\\
&\quad+\frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\le n}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_jy_j(1), t_iy_i(1)+t_jy_j(1)\right)\\
&= \frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\le n}H_{i,j}y_j(1)y_k(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_k\right)\\
&\quad+\frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\le n}H_{i,j}y_j(1)\left\{y_i(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_i\right)+y_j(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_j\right)\right\}\\
&= U_{3,1}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_k\right)+U_{3,2}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_i\right)+U_{3,3}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j, t_j\right)\\
&= \left\{\bar{\tau}\frac{\pi_0}{\pi_1^2}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)-\frac{\pi_0}{\pi_1^2}\frac{1}{n^4}\sum_{i=1}^n H_{i,i}y_i(1)^2+\frac{\pi_0}{\pi_1^2}\frac{1}{n^4}\sum_{1\le i\ne j\le n}H_{i,j}y_i(1)y_j(1)\right\}\left\{1+O(\frac{1}{n})\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
U_{3,1} &= \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\ne k\le n}H_{i,j}y_j(1)y_k(1) = \frac{1}{\pi_1}\frac{1}{n}U_{1,1}\\
&= -\bar{\tau}\frac{1}{\pi_1^3}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}y_i(1)+\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2-\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\le n}H_{i,j}y_i(1)y_j(1),\\
U_{3,2} &= \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\le n}H_{i,j}y_j(1)y_i(1) = \frac{1}{\pi_1}\frac{1}{n}U_{1,2},\\
U_{3,3} &= \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\le n}H_{i,j}y_j(1)^2 = \frac{1}{\pi_1}\frac{1}{n}U_{1,3} = -\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2.
\end{aligned}
$$

The fourth term of $\mathsf{cov}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2})$,

$$
\begin{aligned}
&\mathsf{cov}^{\mathsf{rd}}\left(\frac{1}{n^2}\sum_{1\le i\ne j\ne k\le n}\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}\frac{t_j}{\pi_1}\frac{t_k}{\pi_1}y_k(1), \frac{1}{n}\sum_{l=1}^n\frac{t_l}{\pi_1}y_l(1)\right)\\
&= \frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\ne l\le n}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_jt_ky_k(1), t_ly_l(1)\right)\\
&\quad+\frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\le n}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)H_{i,j}t_jt_ky_k(1), t_iy_i(1)+t_jy_j(1)+t_ky_k(1)\right)\\
&= \frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\ne l\le n}H_{i,j}y_k(1)y_l(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_l\right)\\
&\quad+\frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\le n}H_{i,j}y_k(1)\left\{\begin{array}{l} y_i(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_i\right)+y_j(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_j\right)\\ \quad+y_k(1)\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_k\right)\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= U_{4,1}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_l\right) + U_{4,2}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_i\right) + U_{4,3}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_j\right) \\
&\quad + U_{4,4}\mathsf{cov}^{\mathsf{rd}}\left(\left(\frac{t_i}{\pi_1}-1\right)t_jt_k, t_k\right),
\end{aligned}
$$

where

$$
\begin{aligned}
U_{4,2} &= \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_k(1)y_i(1) = U_{4,3} = \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_k(1)y_j(1) = \frac{1}{\pi_1}\frac{1}{n}U_{1,1} \\
&= -\bar{\tau}\frac{1}{\pi_1^3}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}y_i(1) + \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2 - \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)y_j(1), \\
U_{4,4} &= \frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_k(1)^2 = \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\le n} H_{i,j}\left(\sum_{k=1}^n y_k(1)^2 - y_i(1)^2 - y_j(1)^2\right) \\
&= \bar{\tau}^{(2)}\frac{1}{\pi_1^3}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j} - 2\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\le n} H_{i,j}y_j(1)^2 = -\bar{\tau}^{(2)}\frac{1}{\pi_1^3}\frac{1}{n^2}\sum_{i=1}^n H_{i,i} + 2\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2 \\
&= -\bar{\tau}^{(2)}\frac{1}{\pi_1^3}\frac{p}{n^2} + 2\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2, \\
U_{4,1} &= \frac{1}{n^3}\frac{1}{\pi_1^3}\sum_{1\le i\ne j\ne k\ne l\le n} H_{i,j}y_k(1)y_l(1) = \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_k(1)\left(\sum_{l=1}^n y_l(1) - y_k(1) - y_i(1) - y_j(1)\right) \\
&= \bar{\tau}\frac{1}{\pi_1^3}\frac{1}{n^2}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_k(1) - \frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_k(1)^2 - 2\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\ne k\le n} H_{i,j}y_k(1)y_j(1) \\
&= \bar{\tau}\frac{1}{\pi_1^3}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j}\left(\sum_{k=1}^n y_k - y_i(1) - y_j(1)\right) - U_{4,4} - 2U_{4,3} \\
&= -(\bar{\tau})^2\frac{1}{\pi_1^3}\frac{p}{n} + \bar{\tau}^{(2)}\frac{1}{\pi_1^3}\frac{p}{n^2} + 4\bar{\tau}\frac{1}{\pi_1^3}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}y_i(1) - 4\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{i=1}^n H_{i,i}y_i(1)^2 + 2\frac{1}{\pi_1^3}\frac{1}{n^3}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)y_j(1).
\end{aligned}
$$

Combine with the above terms, the exact form is as follows,

$$
\begin{aligned}
&\mathsf{cov}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}) \\
&= \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{p}{n}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\frac{-n\pi_1 - 6\pi_1 + 6}{(n-2)(n-3)}\bar{\tau}^2 + \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{p}{n}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\frac{3\pi_1 - 2}{(n-2)(n-3)}\bar{\tau}^{(2)} \\
&\quad + \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\frac{2n^2\pi_1 + n(6\pi_1 - 8)}{(n-2)(n-3)}\bar{\tau}\sum_{i=1}^n H_{i,i}y_i(1) \\
&\quad + \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\frac{n^2(-4\pi_1 + 2) + 2n}{n(n-2)(n-3)}\sum_{i=1}^n H_{i,i}y_i(1)^2 \\
&\quad + \frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\frac{n^3\pi_1 + n^2(-\pi_1 - 2) + 2n}{n(n-2)(n-3)}\sum_{1\le i\ne j\le n} H_{i,j}y_i(1)y_j(1),
\end{aligned}
$$

the approximate term is,

$$\mathsf{cov}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}) = -\frac{\pi_0}{\pi_1}\frac{p}{n^2}\bar{\tau}^2 + 2\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\sum_{i=1}^{n} H_{i,i}y_i(1) + \frac{\pi_0}{\pi_1}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{1\leq i\neq j\leq n} H_{i,j}y_i(1)y_j(1).$$

A.3 The Variance of $\widehat{\tau}^{\dagger}_{\mathsf{adj},2}$

We now give the exact form of the variance of $\mathsf{var}^{\mathsf{rd}}(\widehat{\tau}^{\dagger}_{\mathsf{adj},2})$,

$$
\begin{aligned}
&\mathsf{var}^{\mathsf{rd}}(\widehat{\tau}^{\dagger}_{\mathsf{adj},2}) = \mathsf{var}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}) + \mathsf{var}^{\mathsf{rd}}(\widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}) - 2\mathsf{cov}^{\mathsf{rd}}(\widehat{\tau}_{\mathsf{unadj}}, \widehat{\mathbb{IF}}^{\dagger}_{\mathsf{unadj},2,2}) \\
&= \frac{\pi_0}{\pi_1}\frac{1}{n}V_n(y(1)) \\
&\quad + \frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{p}{n^2}\left(\pi_1 - \frac{\pi_0}{n-1}\right)
\left\{
\begin{array}{l}
\dfrac{2n^6\pi_1^2 + n^5(-(p+6)\pi_1^3 - 10\pi_1^2 - 10\pi_1)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)} \\
+ \dfrac{n^4(-(13p+50)\pi_1^3 + (21p+124)\pi_1^2 + 28\pi_1 + 12)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)} \\
+ \dfrac{n^3((10p+432)\pi_1^3 + (69p-572)\pi_1^2 - (102p+14)\pi_1 - 48)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)} \\
+ \dfrac{n^2((148p-856)\pi_1^3 + (-390p+936)\pi_1^2 + (150p-4)\pi_1 + 120p + 60)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)} \\
+ \dfrac{n((-240p+480)\pi_1^3 + (252p-480)\pi_1^2 + 336p\pi_1 - 360p - 24)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)} \\
+ \dfrac{240p\pi_1^2 - 480p\pi_1 + 240p}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}
\end{array}
\right\} \bar{\tau}^2 \\
&\quad + \frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{p}{n^2}\left(\pi_1 - \frac{\pi_0}{n-1}\right)
\left\{
\begin{array}{l}
\dfrac{n^3(-4\pi_1^3 + 2\pi_1)}{(n-2)(n-3)(n-4)(n-5)} + \dfrac{n^2((p+54)\pi_1^3 - (34+p)\pi_1^2 + 2\pi_1 - 4)}{(n-2)(n-3)(n-4)(n-5)} \\
+ \dfrac{n((20p-120)\pi_1^3 + (-32p+80)\pi_1^2 + (14p-8)\pi_1 + 12)}{(n-2)(n-3)(n-4)(n-5)} \\
+ \dfrac{-40p\pi_1^2 + 60p\pi_1 - 24p - 8}{(n-2)(n-3)(n-4)(n-5)}
\end{array}
\right\} \bar{\tau}^{(2)} \\
&\quad + \frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1 - \frac{\pi_0}{n-1}\right)\left(\pi_1 - \frac{2\pi_0}{n-2}\right)
\left\{
\begin{array}{l}
\dfrac{n^3(3\pi_1^2 - 2\pi_1)}{(n-3)(n-4)(n-5)} + \dfrac{n^2(11\pi_1^2 - 20\pi_1 + 6)}{(n-3)(n-4)(n-5)} \\
+ \dfrac{n(20\pi_1^2 - 60\pi_1 + 42)}{(n-3)(n-4)(n-5)}
\end{array}
\right\} \bar{\tau}^2 \sum_{i=1}^{n} H_{i,i}^2 \\
&\quad + \frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1 - \frac{\pi_0}{n-1}\right)
\left\{
\begin{array}{l}
\dfrac{n^3(-3\pi_1^3 + 5\pi_1^2 - 2\pi_1)}{(n-2)(n-3)(n-4)(n-5)} + \dfrac{n^2(-11\pi_1^3 + 20\pi_1^2 - 14\pi_1 + 4)}{(n-2)(n-3)(n-4)(n-5)} \\
+ \dfrac{n(-20\pi_1^3 + 49\pi_1^2 - 44\pi_1 + 12}{(n-2)(n-3)(n-4)(n-5)} + \dfrac{-20\pi_1^2 + 40\pi_1 - 16}{(n-2)(n-3)(n-4)(n-5)}
\end{array}
\right\} \bar{\tau}^{(2)} \sum_{i=1}^{n} H_{i,i}^2
\end{aligned}
$$

$$
+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}
\dfrac{-4n^6\pi_1^2+n^5((2p+20)\pi_1^3+20\pi_1^2+16\pi_1)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n^4((24p-32)\pi_1^3-(40p+180)\pi_1^2-48\pi_1-16)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n^3(-(54p+284)\pi_1^3+(-80p+780)\pi_1^2+(176p+48)\pi_1+48)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n^2((-180p+776)\pi_1^3+(672p-1256)\pi_1^2-(336p+48)\pi_1-192p-16)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{n((400p-480)\pi_1^3+(-616p+640)\pi_1^2+(-368p+32)\pi_1+576p-48)}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\
+\dfrac{-320p\pi_1^2+720p\pi_1-384p+32}{(n-1)(n-2)^2(n-3)(n-4)(n-5)}
\end{array}\right\}
$$

$$
\times\,\bar{\tau}\sum_{i=1}^{n}H_{i,i}y_i(1)
$$

$$
+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}
\dfrac{n^4(-4\pi_1^3+4\pi_1^2)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^3(-10\pi_1^3+18\pi_1^2-16\pi_1)}{(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{n^2(-18\pi_1^3+92\pi_1^2-48\pi_1+16)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n(40\pi_1^3+22\pi_1^2-160\pi_1+48)}{(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{40\pi_1^2-160\pi_1+128}{(n-2)(n-3)(n-4)(n-5)}
\end{array}\right\}\bar{\tau}\sum_{i=1}^{n}H_{i,i}^2y_i(1)
$$

$$
+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}
\dfrac{2n^4\pi_1^3}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n^3(6\pi_1^3-24\pi_1^2)}{(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{n^2(-32\pi_1^2+88\pi_1)}{(n-2)(n-3)(n-4)(n-5)}+\dfrac{n(40\pi_1^2-8\pi_1-96)}{(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{-80\pi_1+96}{(n-2)(n-3)(n-4)(n-5)}
\end{array}\right\}
$$

$$
\times\,\bar{\tau}\sum_{1\le i\ne j\le n}H_{i,i}H_{i,j}y_j(1)
$$

$$
+\left\{\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{n\pi_1-1}{n^2(n-1)}+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}
\dfrac{n^4(4\pi_1^3+6\pi_1^2-4\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}+\\
\dfrac{n^3(-(2p+68)\pi_1^3+(2p-12)\pi_1^2+12\pi_1+4)}{n(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{n^2((-38p+160)\pi_1^3+(52p+94)\pi_1^2-(20p+48)\pi_1-8)}{n(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{n(40p\pi_1^3+(42p-200)\pi_1^2+(-84p+24)\pi_1+24p+20)}{n(n-2)(n-3)(n-4)(n-5)}\\
+\dfrac{40p\pi_1^2-120p\pi_1+72p-16}{n(n-2)(n-3)(n-4)(n-5)}
\end{array}\right)\right\}
$$

$$
\times\sum_{i=1}^{n}H_{i,i}y_i(1)^2
$$

$$
\begin{aligned}
&+\left\{\frac{1}{\pi_1^2}\frac{\pi_0}{\pi_1}\frac{-n^2\pi_1+n\pi_0+1}{n^3(n-1)}+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{n^5(\pi_1^3-\pi_1^2)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^4(4\pi_1^3-12\pi_1^2+8\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^3(15\pi_1^3+11\pi_1^2-8\pi_1-8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(-20\pi_1^3-128\pi_1^2+108\pi_1+8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n(-54\pi_1^2+116\pi_1-112)+120\pi_1^2+16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right)\right\}\\
&\quad\times\sum_{i=1}^{n}H_{i,i}^2y_i(1)^2\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left(\begin{array}{l}\dfrac{-n^6\pi_1^3+n^5(-2\pi_1^3+12\pi_1^2)+n^4(15\pi_1^3-6\pi_1^2-44\pi_1)}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\ +\dfrac{n^3(16\pi_1^3-124\pi_1^2+112\pi_1+48)}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(-44\pi_1^3+74\pi_1^2+124\pi_1-192)}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\\ +\dfrac{n(-80\pi_1^3+316\pi_1^2-448\pi_1+240)-80\pi_1^2+160\pi_1-96}{n(n-1)(n-2)^2(n-3)(n-4)(n-5)}\end{array}\right)\mathsf{tr}^2\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{n^5(-\pi_1^3+\pi_1^2)+n^4(\pi_1^3+2\pi_1^2-4\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^3(4\pi_1^3-7\pi_1^2+4\pi_1+4)+n^2(4\pi_1^2-8)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n(-16\pi_1+20)-16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{-n^5\pi_1^3+n^4(15\pi_1^3-2\pi_1^2)+n^3((8p-50)\pi_1^3-(6p+22)\pi_1^2+16\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^2((-20p+40)\pi_1^3+(-18p+104)\pi_1^2+(24p-40)\pi_1-16)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n((60p-80)\pi_1^2+(-8p+8)\pi_1-24p+48)-40p\pi_1+24p-32}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\\
&\quad\times\sum_{1\le i\ne j\le n}H_{i,j}y_i(1)y_j(1)\\
&+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{n^4(-2\pi_1^3+2\pi_1^2)+n^3(-6\pi_1^3+16\pi_1^2-12\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\ +\dfrac{n^2(14\pi_1^2-24\pi_1+16)+4n\pi_1-16}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\sum_{1\le i\ne j\le n}H_{i,i}H_{i,j}y_j(1)^2
\end{aligned}
$$

$$
+\frac{1}{\pi_1^4}\frac{\pi_0}{\pi_1}\frac{1}{n^2}\left(\pi_1-\frac{\pi_0}{n-1}\right)\left\{\begin{array}{l}\dfrac{-2n^5\pi_1^3+n^4(-2\pi_1^3+20\pi_1^2)+n^3(12\pi_1^3-64\pi_1)}{n(n-2)(n-3)(n-4)(n-5)}\\+\dfrac{n^2(-68\pi_1^2+56\pi_1+64)+n(72\pi_1-96)+32}{n(n-2)(n-3)(n-4)(n-5)}\end{array}\right\}\sum_{1\le i\neq j\le n}H_{i,i}H_{i,j}y_i(1)y_j(1).
$$

The variance can be approximated as follows

$$
\begin{aligned}
\mathsf{var}^{\mathsf{rd}}(\widehat{\tau}^{\dagger}_{\mathsf{adj},2}) &= \frac{\pi_0}{\pi_1}\frac{1}{n}V_n(y(1)) \\
&+\left\{\begin{array}{l}
\left(2\frac{\pi_0}{\pi_1^2}\frac{1}{n^2}-\frac{\pi_0}{\pi_1}\frac{p}{n^3}\right)\bar{\tau}^2\sum_{i=1}^n H_{i,i}+\left(-2\frac{\pi_0}{\pi_1^2}\frac{1}{n^2}+3\frac{\pi_0}{\pi_1}\frac{1}{n^2}\right)\bar{\tau}^2\sum_{i=1}^n H_{i,i}^2\\
+\left(-4\frac{\pi_0}{\pi_1^2}\frac{1}{n^2}+2\frac{\pi_0}{\pi_1}\frac{p}{n^3}\right)\bar{\tau}\sum_{i=1}^n H_{i,i}y_i(1)+4\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\bar{\tau}\sum_{i=1}^n H_{i,i}^2y_i(1)\\
+2\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\sum_{1\le i\neq j\le n}H_{i,i}H_{i,j}y_j(1)+\frac{\pi_0}{\pi_1^2}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}y_i(1)^2\\
-\left(\frac{\pi_0}{\pi_1^2}\frac{1}{n^2}+\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\right)\sum_{i=1}^n H_{i,i}^2y_i(1)^2-\frac{\pi_0}{\pi_1}\frac{1}{n^3}\mathsf{tr}^2\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)\\
+\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\mathsf{tr}\left(\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\widehat{\boldsymbol{\Sigma}}_y\widehat{\boldsymbol{\Sigma}}^-\right)-\frac{\pi_0}{\pi_1}\frac{1}{n^2}\sum_{1\le i\neq j\le n}H_{i,j}y_i(1)y_j(1)\\
-2\frac{\pi_0}{\pi_1}\frac{1}{n^2}\sum_{1\le i\neq j\le n}H_{i,i}H_{i,j}y_i(1)y_j(1)
\end{array}\right\}\left\{1+o\left(n^{-1}\right)\right\}\\
&=\frac{\pi_0}{\pi_1}\frac{1}{n}V_n(y(1))+2\frac{\pi_0}{\pi_1^2}\frac{1}{n^2}\bar{\tau}^2\left(p-\sum_{i=1}^n H_{i,i}^2\right)+\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}^2\left(3\sum_{i=1}^n H_{i,i}^2-\frac{p^2}{n}\right)\\
&+4\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\sum_{i=1}^n H_{i,i}\left(\frac{\pi_0}{\pi_1}H_{i,i}-\frac{1}{\pi_1}\right)y_i(1)+2\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\left(\frac{p}{n}\sum_{i=1}^n H_{i,i}y_i(1)+\sum_{1\le i\neq j\le n}H_{i,i}H_{i,j}y_j(1)\right)\\
&+\frac{\pi_0}{\pi_1}\frac{1}{n^2}\sum_{i=1}^n H_{i,i}\left(\frac{1}{\pi_1}-\frac{1+\pi_0}{\pi_1}H_{i,i}\right)y_i(1)^2-\frac{\pi_0}{\pi_1}\frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^n H_{i,i}y_i(1)\right)^2\\
&+\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n H_{i,j}^2y_i(1)y_j(1)-\frac{\pi_0}{\pi_1}\frac{1}{n^2}\sum_{1\le i\neq j\le n}H_{i,j}\left(1+2H_{j,j}\right)y_i(1)y_j(1)+o\left(n^{-1}\right)\\
&=\frac{\pi_0}{\pi_1}\frac{1}{n}V_n(y(1))+2\frac{\pi_0}{\pi_1}\left(1+\frac{\pi_0}{\pi_1}\right)\frac{1}{n^2}\bar{\tau}^2\left(p-\sum_{i=1}^n H_{i,i}^2\right)+\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}^2\left(3\sum_{i=1}^n H_{i,i}^2-\frac{p^2}{n}\right)\\
&-4\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\bar{\tau}\sum_{i=1}^n H_{i,i}\left(1-H_{i,i}\right)y_i(1)-4\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\sum_{i=1}^n H_{i,i}y_i(1)\\
&+2\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\left(\frac{p}{n}\sum_{i=1}^n H_{i,i}y_i(1)+\sum_{1\le i\neq j\le n}H_{i,i}H_{i,j}y_j(1)\right)\\
&+\frac{\pi_0}{\pi_1}\left(1+\frac{\pi_0}{\pi_1}\right)\frac{1}{n^2}\sum_{i=1}^n H_{i,i}\left(1-H_{i,i}\right)y_i(1)^2-\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\sum_{i=1}^n H_{ii}^2y_i(1)^2-\frac{\pi_0}{\pi_1}\frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^n H_{i,i}y_i(1)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\pi_0}{\pi_1}\right)^2 \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} H_{i,j}^2 y_i(1)y_j(1) - \frac{\pi_0}{\pi_1}\frac{1}{n^2}\sum_{1\le i\ne j\le n} H_{i,j}\left(1+2H_{j,j}\right) y_i(1)y_j(1) + o\left(n^{-1}\right)\\
=\ & \frac{\pi_0}{\pi_1}\frac{1}{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\left(y_i(1)-\bar{\tau}\right)^2\right\} - \frac{\pi_0}{\pi_1}\frac{1}{n}\left\{\frac{1}{n}\sum_{1\le i\ne j\le n} H_{i,j}\left(1+2H_{j,j}\right)\left(y_i(1)-\bar{\tau}\right)\left(y_j(1)-\bar{\tau}\right)\right\}\\
&+2\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\sum_{i=1}^{n} H_{i,i}^2 y_i(1) - \frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}^2\left(p+2\sum_{i=1}^{n}H_{i,i}^2\right) + \frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}^2\left(3\sum_{i=1}^{n}H_{i,i}^2\right)\\
&+\frac{\pi_0}{\pi_1}\left\{1+\frac{\pi_0}{\pi_1}\right\}\frac{1}{n^2}\sum_{i=1}^{n} H_{i,i}\left(1-H_{i,i}\right)\left(y_i-\bar{\tau}\right)^2 - 2\frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}\sum_{i=1}^{n}H_{i,i}^2 y_i(1) + \frac{\pi_0}{\pi_1}\frac{1}{n^2}\bar{\tau}^2\left(p-\sum_{i=1}^{n}H_{i,i}^2\right)\\
&-\frac{\pi_0}{\pi_1}\left\{\frac{1}{n}\sum_{i=1}^{n}\left(1+H_{i,i}\right)\left(y_i(1)-\bar{\tau}\right)\right\}^2 + \left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}H_{i,j}^2\left(y_i-\bar{\tau}\right)\left(y_j-\bar{\tau}\right) - \sum_{i=1}^{n}H_{i,i}^2\left(y_i-\bar{\tau}\right)^2\right\}\\
&+o\left(n^{-1}\right)\\
=\ & \frac{\pi_0}{\pi_1}\frac{1}{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\left(y_i(1)-\bar{\tau}\right)^2 - \frac{1}{n}\sum_{1\le i\ne j\le n}H_{i,j}\left(1+2H_{j,j}\right)\left(y_i(1)-\bar{\tau}\right)\left(y_j(1)-\bar{\tau}\right)\right\}\\
&+\frac{\pi_0}{\pi_1}\left\{1+\frac{\pi_0}{\pi_1}\right\}\frac{1}{n^2}\sum_{i=1}^{n}H_{i,i}\left(1-H_{i,i}\right)\left(y_i-\bar{\tau}\right)^2 - \frac{\pi_0}{\pi_1}\left\{\frac{1}{n}\sum_{i=1}^{n}\left(1+H_{i,i}\right)\left(y_i(1)-\bar{\tau}\right)\right\}^2\\
&+\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}H_{i,j}^2\left(y_i-\bar{\tau}\right)\left(y_j-\bar{\tau}\right) - \sum_{i=1}^{n}H_{i,i}^2\left(y_i-\bar{\tau}\right)^2\right\} + o\left(n^{-1}\right)\\
=\ & \left(\frac{\pi_0}{\pi_1}\right)\frac{1}{n}V_n\left[\left(y_i(1)-\bar{\tau}\right) - \sum_{j\ne i}H_{j,i}(y_j(1)-\bar{\tau})\right]\\
&+\left(\frac{\pi_0}{\pi_1}\right)^2\frac{1}{n}\left\{\frac{1}{n}\sum_{i=1}^{n}H_{i,i}(1-H_{i,i})(y_i(1)-\bar{\tau})^2 + \frac{1}{n}\sum_{1\le i\ne j\le n}H_{i,j}^2(y_i(1)-\bar{\tau})(y_j(1)-\bar{\tau})\right\} + o\left(n^{-1}\right)\\
=\ & \nu_{\mathsf{db}}^{\mathsf{rd}} + o\left(n^{-1}\right).
\end{aligned}
$$

Lemma 1. *Under CRE, the following assertions hold:*

$$
\begin{aligned}
&\mathsf{cov}\left(t_i t_j, t_k\right) = -\frac{2\pi_1(1-\pi_1)(n\pi_1-1)}{(n-1)(n-2)},\\
&\mathsf{cov}\left(t_i t_j, t_i\right) = \frac{\pi_1(1-\pi_1)(n\pi_1-1)}{n-1},\\
&\mathsf{cov}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j t_k, t_l\right) = -\frac{(1-\pi_1)(n\pi_1-1)(n\pi_1-6(1-\pi_1))}{(n-1)(n-2)(n-3)},\\
&\mathsf{cov}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j t_k, t_i\right) = (1-2\pi_1)\frac{(n\pi_1-1)(n\pi_1-2)}{(n-1)(n-2)} + \pi_1^2\frac{n\pi_1-1}{n-1},\\
&\mathsf{cov}\left(\left(\frac{t_i}{\pi_1}-1\right)t_j t_k, t_j\right) = -\frac{2(1-\pi_1)^2(n\pi_1-1)}{(n-1)(n-2)}.
\end{aligned}
$$

Here different letters i, j, k, l denote distinct indices.

Proof. First,

$$\begin{aligned}\operatorname{cov}(t_i t_j, t_k) &= \mathbb{E}[t_i t_j t_k] - \mathbb{E}[t_i t_j]\,\mathbb{E}[t_k] = \pi_1 \frac{(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)} - \pi_1 \frac{n\pi_1 - 1}{n-1}\pi_1 \\ &= \pi_1 \frac{n\pi_1 - 1}{n-1}\left(\frac{n\pi_1 - 2}{n-2} - \pi_1\right) = -\frac{2\pi_1(1-\pi_1)(n\pi_1 - 1)}{(n-1)(n-2)}.\end{aligned}$$

Second,

$$\operatorname{cov}(t_i t_j, t_i) = \mathbb{E}\left[t_i^2 t_j\right] - \mathbb{E}[t_i t_j]\,\mathbb{E}[t_i] = \pi_1 \frac{n\pi_1 - 1}{n-1} - \pi_1^2 \frac{n\pi_1 - 1}{n-1} = \pi_1(1-\pi_1)\frac{n\pi_1 - 1}{n-1}.$$

Third,

$$\begin{aligned}&\operatorname{cov}\left(\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k, t_l\right) = \mathbb{E}\left[\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k t_l\right] - \mathbb{E}\left[\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k\right]\mathbb{E}[t_l] \\ &= \mathbb{E}\left[\frac{t_i t_j t_k t_l}{\pi_1} - t_j t_k t_l\right] - \mathbb{E}\left[\frac{t_i t_j t_k}{\pi_1} - t_j t_k\right]\pi_1 \\ &= \frac{1}{\pi_1}\pi_1 \frac{(n\pi_1 - 1)(n\pi_1 - 2)(n\pi_1 - 3)}{(n-1)(n-2)(n-3)} - \pi_1 \frac{(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)} - \left(\frac{1}{\pi_1}\pi_1 \frac{(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)} - \pi_1 \frac{n\pi_1 - 1}{n-1}\right)\pi_1 \\ &= \frac{2\pi_1(1-\pi_1)(n\pi_1 - 1)}{(n-1)(n-2)} - \frac{3(1-\pi_1)(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)(n-3)} \\ &= \frac{(1-\pi_1)(n\pi_1 - 1)}{(n-1)(n-2)}\left(2\pi_1 - \frac{3(n\pi_1 - 2)}{n-3}\right) \\ &= -\frac{(1-\pi_1)(n\pi_1 - 1)(n\pi_1 - 6(1-\pi_1))}{(n-1)(n-2)(n-3)}.\end{aligned}$$

Fourth,

$$\begin{aligned}&\operatorname{cov}\left(\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k, t_i\right) = \mathbb{E}\left[\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k t_i\right] - \mathbb{E}\left[\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k\right]\mathbb{E}[t_i] \\ &= \mathbb{E}\left[\frac{t_i^2 t_j t_k}{\pi_1} - t_i t_j t_k\right] - \mathbb{E}\left[\frac{t_i t_j t_k}{\pi_1} - t_j t_k\right]\pi_1 \\ &= \left(\frac{1}{\pi_1} - 1\right)\pi_1 \frac{(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)} - \left(\frac{1}{\pi_1}\pi_1 \frac{(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)} - \pi_1 \frac{n\pi_1 - 1}{n-1}\right)\pi_1 \\ &= (1 - 2\pi_1)\frac{(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)} + \pi_1^2 \frac{n\pi_1 - 1}{n-1}.\end{aligned}$$

Fifth,

$$\begin{aligned}&\operatorname{cov}\left(\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k, t_j\right) = \mathbb{E}\left[\left(\frac{t_i}{\pi_1} - 1\right) t_j^2 t_k\right] - \mathbb{E}\left[\left(\frac{t_i}{\pi_1} - 1\right) t_j t_k\right]\mathbb{E}[t_j] \\ &= \mathbb{E}\left[\frac{t_i t_j^2 t_k}{\pi_1} - t_j t_k\right] - \mathbb{E}\left[\frac{t_i t_j t_k}{\pi_1} - t_j t_k\right]\pi_1 \\ &= (1 - \pi_1)\left(\frac{1}{\pi_1}\pi_1 \frac{(n\pi_1 - 1)(n\pi_1 - 2)}{(n-1)(n-2)} - \pi_1 \frac{n\pi_1 - 1}{n-1}\right)\end{aligned}$$

$$= -\frac{2(1-\pi_1)^2(n\pi_1 - 1)}{(n-1)(n-2)}.$$

$\square$