答案僅供參考

1. (10%) Verify that $[p\to(q \to r)] \to [(p\to q)\to(p\to r)]$ is a tautology.

p	q	r	q->r	p->q	p->r
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	0	1	1
0	1	1	1	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	0	1	0
1	1	1	1	1	1

p->(q->r)	(p->q) -> (p->r)	[p->(q->r)] -> [(p->q) -> (p->r)]
1	1	1
1	1	1
1	1	1
1	1	1
1	1	1
1	1	1
0	0	1
1	1	1

由上表可知 [p->(q->r)] -> [(p->q)->(p->r)] 為同義反覆

2. (10%) Let x, y be two positive real numbers. Prove that if the product xy exceeds 25, then x >5 or y >5.

   假設 $x \le 5$ and $y \le 5$，可得 xy>25

   但 xy 最大值為 25，與假設產生矛盾

   所以xy>25 , then x >5 or y >5

3. (15%) Determine all integer solutions to the equation

   $x_1 + x_2 + x_3 + x_4 < 10$, where $x_i \ge 0$, for all $1 \le i \le 4$.

   $X_1+X_2+X_3+X_4+X_5 \le 10$，$X_5 \ge 1$

   $X_5=Y+1$，$Y \ge 0$

   $X_1+X_2+X_3+X_4+Y=9$

   $\binom{13}{9}$

4. (10%) Three small towns, designated by A, B, and C, are interconnected by a system of two-way roads, as shown the below figure.

   (a) In how many ways can Linda travel from town A to town C?

(b) How many different round trips can Linda travel from town A to town C and back to town A?

(a)14 (b)14^2

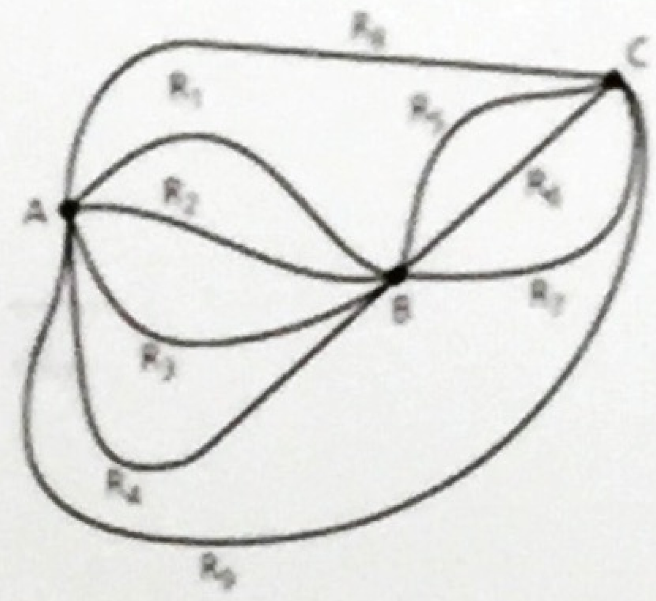

Figure 1.4

5. (15%) Consider the following program segment, where i, j , and k are integer variables.

```
for i := 2 to 20 do
  for j := 2 to i do
    for k := 2 to j do
      print (i * j + k)
```

How many times is the print statement executed in this program segment?

Sol:

第一種方法：

for(i = 2) j = 2 k = 2 → 1 種

for(i = 3) j = 2 k = 2

j = 3 k = 2,3 → 1 + 2 種 = 3 種

for(i = 4) j = 2 k = 2

j = 3 k = 2,3

j = 4 k = 2,3,4 → 1 + 2 + 3 種 = 6 種

以此類推...

總共會執行 1 + 3 + 6 +......... + 190 次 = 1330 次

第二種方法：

$C_3^{19+3-1} = C_3^{21} = 1330$

6. (10%) Let I be an index set where for each $i \in I$, $A_i \subseteq U$. Prove the *Generalized DeMorgan's Laws.* (a) $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$ (b) $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$

(a)n=1 時，$\overline{A_1} = \overline{A_1}$

n=2 時，$\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$

if n=k 時, $\overline{A_1 \cup A_2 \ldots \cup A_k} = \overline{A_1} \cap \overline{A_2} \ldots \cap \overline{A_k}$ 成立

then n=k+1 時，$\overline{A_1 \cup A_2 \ldots \cup A_k \cup A_{k+1}} = \overline{(A_1 \cup \ldots \cup A_k) \cup A_{k+1}}$

令 $S_1 = A_1 \cup \ldots \cup A_k$ 且 $S_2 = A_{k+1}$

$\overline{A_1 \cup A_2 \ldots \cup A_k \cup A_{k+1}} = \overline{(A_1 \cup \ldots \cup A_k) \cup A_{k+1}}$

$= \overline{S_1 \cup S_2}$

$= \overline{S_1} \cap \overline{S_2}$ 因為 n=2 時此公式成立

$= \overline{\overline{A_1 \cup A_2 \ldots \cup A_k} \cap \overline{A_{k+1}}}$

$= \overline{A_1} \cap \ldots \cap \overline{A_k} \cap \overline{A_{k+1}}$

(b)i = 1 時，$\overline{\cap A_1} = \cup \overline{A_1}$

i= n 時，$\overline{A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \ldots \cup \overline{A_n}$

$\overline{\bigcap_{i\in I} A_n} = \bigcup_{i\in I} \overline{A_n}$

i = n+1 時，$\overline{\overline{A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n} \cap \overline{A_{n+1}}} = \overline{\overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \ldots \cup \overline{A_n} \cap \overline{A_{n+1}}} =$

$\overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \ldots \cup \overline{A_n} \cup \overline{A_{n+1}} = \cup \overline{A_{n+1}}$

由歸納法得證$\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$ 。

7. (10%) Prove the following two equations:

(a) $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots+\binom{n}{n}=2^{n-1}$, n is an even positive integer.

(b) $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n}=2^{n-1}$, n is an odd positive integer.

Sol:

$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots+\binom{n}{n}$

$= \binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\binom{n-1}{4}+\cdots+\binom{n-1}{n-1}$

$= 2^{n-1}$

(b)小題相同方法

或使用定理$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$，$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0$

推導。

8. (10%) Let p(x), q(x) denote the following open statements.

$p(x): x \le 3$ $\quad$ $q(x): x+1$ is odd

If the universe consists of all integers, what are the truth values of the following statements?

$(a)\ q(1)$ $\quad (b)\ \neg p(3)$ $\quad (c)\ p(7) \vee q(7)$ $\quad (d)\ p(3) \wedge q(4)$ $\quad (e)\ \neg(p(-4) \vee q(-3))$

Sol:

a) False

b) False

c) False

d) True

e) False

9. Let the universe U={0,1,2,…,22} and Q be a proper subset of U of cardinality 7.

Prove that there are two subsets H and S of Q such that $\sum_{x\in H} x = \sum_{x\in S} x$. Note, $\sum_{x\in\phi} x = 0$.

Sol:

We divide this proof into two cases:

Case 1: $0\in Q$

Let n denote the number of subsets of Q. Then $n = 2^7 = 128$

Let m denote the sum of a subset of Q.

Then $0 \le m \le \sum_{x\in Q} x \le 22 + 21 + 20 + 19 + 18 + 17 = 117$

Since n ≥m+1, by the pigeonhole principle, there are two subsets H and S of Q such that $\sum_{x\in H} x = \sum_{x\in S} x$ when $0\in Q$.

Case 2: $0\notin Q$

That is, there is no proper subset S of Q s.t. $\sum_{x\in S} x = \sum_{x\in Q} x$

Also, there is no nonempty subset S of Q s.t. $\sum_{x\in S} x = \sum_{x\in\emptyset} x = 0$

Hence, we need only consider if there exist two proper and nonempty subsets S and H of Q s.t. $\sum_{x\in S} x = \sum_{x\in H} x$

Let n denote the number of proper and nonempty subsets of Q.

Then $n = 2^7 - 2 = 126$

Let m denote the sum of a proper and nonempty subset of Q.

Then $1 \le m \le 22 + 21 + 20 + 19 + 18 + 17 = 117$

Since n ≥m, by y the pigeonhole principle, there are two subsets H and S of Q such that $\sum_{x\in H} x = \sum_{x\in S} x$ when $0\notin Q$.