

# Ordinality and Riemann Hypothesis

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## Abstract

We study a sufficient condition of the Riemann hypothesis. This condition is the existence of a special ordering on the set of finite products of distinct odd prime numbers.

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## 1 Introduction

The zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

was introduced by Euler in 1737 for real variable  $s > 1$ . In 1859, Riemann([7]) extended the function to the complex meromorphic function  $\zeta(z)$  with only simple pole at  $z = 1$  and

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$$

on  $\operatorname{Re} z > 1$ .

**Theorem 1.1** ([10]). *The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at  $z = 1$ .*

The zeta function has infinitely many zeros but there is no zero in the region  $\operatorname{Re} z \geq 1$ .

**Theorem 1.2** ([6], [10]). *The only zeros of zeta function outside the critical strip  $0 < \operatorname{Re} z < 1$  are at the negative even integers,  $-2, -4, -6, \dots$ .*

The most famous conjecture on the zeta function is the Riemann hypothesis([1], [9]).

**Riemann Hypothesis.** *The zeros of  $\zeta(z)$  in the critical strip lie on the critical line  $\operatorname{Re} z = \frac{1}{2}$ .*

Suppose that  $x$  and  $y$  are real numbers with  $0 < x < 1$ . It is known that if  $x + yi$  is a zero of the zeta function, then so are  $x - yi$ ,  $(1 - x) + yi$  and  $(1 - x) - yi$ . Riemann himself showed that if  $0 \leq y \leq 25.02$  and  $x + yi$  is a zero of the zeta function, then  $x = \frac{1}{2}$ . Therefore the Riemann Hypothesis is true up to height 25.02. In 1986, van de Lune, te Riele and Winter([3]) showed that the Riemann hypothesis is true up to height 545,439,823,215. Furthermore in 2021 Dave Platt and Tim Trudgian([5]) proved that the Riemann hypothesis is true up to height  $3 \cdot 10^{12}$ .

In this paper, we study a sufficient condition of the Riemann Hypothesis. This condition is the existence of a special ordering on the set of finite products of distinct odd prime numbers.

## 2 Preliminary lemmas and theorems

The eta function

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^z}$$

is convergent on  $\operatorname{Re} z > 0$ .

**Theorem 2.1** ([2]). *For  $0 < \operatorname{Re} z < 1$ , we have*

$$\zeta(z) = \frac{1}{1 - 2^{1-z}} \eta(z).$$

The zeros of  $1 - 2^{1-z}$  are on  $\operatorname{Re} z = 1$ . Therefore any zeros of  $\zeta(z)$  in  $0 < \operatorname{Re} z < 1$  is a zero of  $\eta(z)$ .

**Lemma 2.2.** *Let  $0 < x < 1$ . If  $x + yi$  is a zero of  $\zeta(z)$  then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x} \cos(y \ln k) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x} \sin(y \ln k) = 0.$$

*Proof.*

$$\begin{aligned} \frac{1}{k^{x+yi}} = k^{-x-yi} &= e^{(-x-yi) \ln k} \\ &= e^{-x \ln k} (\cos(y \ln k) - i \sin(y \ln k)) \\ &= \frac{1}{k^x} (\cos(y \ln k) - i \sin(y \ln k)) \end{aligned}$$

Therefore it is trivial from Theorem 2.1. □

**Lemma 2.3.** *Let  $0 < x < 1$ . If  $x + yi$  is a zero of  $\zeta(z)$  then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x} \cos(y \ln(ak)) = 0$$

*for all  $a > 0$  and*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^x} \sin(y \ln(bk)) = 0$$

*for all  $b > 0$ .*

*Proof.*

$$\begin{aligned} \cos(y \ln(ak)) &= \cos(y \ln a + y \ln k) \\ &= \cos(y \ln a) \cos(y \ln k) - \sin(y \ln a) \sin(y \ln k) \end{aligned}$$

$$\begin{aligned} \sin(y \ln(bk)) &= \sin(y \ln b + y \ln k) \\ &= \sin(y \ln b) \cos(y \ln k) + \cos(y \ln b) \sin(y \ln k) \end{aligned}$$

Therefore it is trivial from Lemma 2.2.  $\square$

**Lemma 2.4.** *Let  $0 < x < 1$ . Suppose that  $x + yi$  is a zero of  $\zeta(z)$  and  $q \geq 1$  is an odd number. Then*

$$\sum_{m=1}^{\infty} \frac{(-1)^{mq-1}}{(mq)^x} \cos(y \ln(mq)) = 0$$

*and*

$$\sum_{m=1}^{\infty} \frac{(-1)^{mq-1}}{(mq)^x} \sin(y \ln(mq)) = 0.$$

*Proof.* Since  $q$  is an odd number,  $(-1)^{mq-1} = (-1)^{m-1}$ . Therefore, from Lemma 2.3, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(-1)^{mq-1}}{(mq)^x} \cos(y \ln(mq)) &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(mq)^x} \cos(y \ln(mq)) \\ &= \frac{1}{q^x} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^x} \cos(y \ln(mq)) = 0 \end{aligned}$$

*and*

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(-1)^{mq-1}}{(mq)^x} \sin(y \ln(mq)) &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(mq)^x} \sin(y \ln(mq)) \\ &= \frac{1}{q^x} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^x} \sin(y \ln(mq)) = 0. \end{aligned}$$

$\square$

**Lemma 2.5.** *If  $0 < x < 1$ , then*

$$1 - \sum_{k=1}^{\infty} \frac{1}{2^{kx}} \cos(ky \ln 2) - i \sum_{k=1}^{\infty} \frac{1}{2^{kx}} \sin(ky \ln 2) \neq 0$$

*Proof.* Since  $0 < x < 1$ , we have

$$\begin{aligned} & 1 - \sum_{k=1}^{\infty} \frac{1}{2^{kx}} \cos(ky \ln 2) - i \sum_{k=1}^{\infty} \frac{1}{2^{kx}} \sin(ky \ln 2) \\ = & 2 - \sum_{k=0}^{\infty} \frac{\cos(ky \ln 2) + i \sin(ky \ln 2)}{2^{kx}} \\ = & 2 - \sum_{k=0}^{\infty} \frac{e^{iky \ln 2}}{2^{kx}} \\ = & 2 - \sum_{k=0}^{\infty} \left( \frac{e^{iy \ln 2}}{2^x} \right)^k \\ = & 2 - \frac{1}{1 - \frac{e^{iy \ln 2}}{2^x}} \\ = & 2 - \frac{2^x}{2^x - e^{iy \ln 2}} \\ = & \frac{2^x - 2e^{iy \ln 2}}{2^x - e^{iy \ln 2}} \\ \neq & 0. \end{aligned}$$

□

### 3 The sufficient condition of Riemann hypothesis

Let  $\mathbb{N}$  be the set of natural numbers.

**Definition 3.1.** Let  $Q$  be the set of finite products of distinct odd prime numbers.

$$Q = \{p_1 p_2 \cdots p_n \mid p_1, p_2, \dots, p_n \text{ are distinct odd prime numbers, } n \in \mathbb{N}\}$$

For each  $q = p_1 p_2 \cdots p_n \in Q$ , we define

$$\text{sgn } q = (-1)^n$$

where  $p_1, p_2, \dots, p_n$  are distinct odd prime numbers.

There are infinitely many ordering on  $Q$ .

**Definition 3.2.** Choose any ordering on  $Q$  and let

$$Q = \{q_1, q_2, \dots\}.$$

**Definition 3.3.** Let

$$\delta(k, i) = \begin{cases} 1 & \text{if } k \text{ is a multiple of } q_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(k, h) = \sum_{i=1}^h (\text{sgn } q_i) \delta(k, i), \quad f(k) = \lim_{h \rightarrow \infty} f(k, h) = \sum_{i=1}^{\infty} (\text{sgn } q_i) \delta(k, i).$$

Note that, for each  $k$ , there exist only finitely many  $i$  such that  $\delta(k, i) \neq 0$ .

**Definition 3.4.** Suppose that  $\frac{1}{2} < x < 1$  and  $x + yi$  is a zero of  $\zeta(z)$ . Let

$$a_k = \frac{(-1)^{k-1}}{k^x} \cos(y \ln k), \quad b_k = \frac{(-1)^{k-1}}{k^x} \sin(y \ln k).$$

By Lemma 2.2, we have

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k = 0. \tag{1}$$

From Lemma 2.4, we have

$$\sum_{m=1}^{\infty} a_{mq_i} = 0 \quad \text{for all } q_i \in Q$$

and therefore

$$\sum_{m=1}^{\infty} (\text{sgn } q_i) a_{mq_i} = 0 \quad \text{for all } q_i \in Q. \tag{2}$$

**Definition 3.5.** Let

$$C(n, h) = \sum_{k=1}^n \sum_{i=1}^h (\text{sgn } q_i) \delta(k, i) a_k = \sum_{k=1}^n f(k, h) a_k$$

and

$$S(n, h) = \sum_{k=1}^n \sum_{i=1}^h (\text{sgn } q_i) \delta(k, i) b_k = \sum_{k=1}^n f(k, h) b_k.$$

**Proposition 3.6.** For each  $h$ , we have

$$\lim_{n \rightarrow \infty} C(n, h) = \lim_{n \rightarrow \infty} S(n, h) = 0$$

and therefore

$$\lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} C(n, h) = \lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} S(n, h) = 0.$$

*Proof.* From eq. (2), we have

$$\begin{aligned}
0 &= \sum_{i=1}^h \sum_{m=1}^{\infty} (\text{sgn } q_i) a_{mq_i} \\
&= \sum_{i=1}^h \sum_{k=1}^{\infty} (\text{sgn } q_i) \delta(k, i) a_k \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^h (\text{sgn } q_i) \delta(k, i) a_k \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \sum_{i=1}^h (\text{sgn } q_i) \delta(k, i) a_k \\
&= \lim_{n \rightarrow \infty} C(n, h).
\end{aligned}$$

In the same way, we have

$$\lim_{n \rightarrow \infty} S(n, h) = 0.$$

□

**Lemma 3.7.** *Recall*

$$f(k) = \sum_{i=1}^{\infty} (\text{sgn } q_i) \delta(k, i).$$

*We have*

$$f(k) = \begin{cases} 0 & \text{if } k = 2^m, \ m = 0, 1, 2, \dots \\ -1 & \text{otherwise} \end{cases}$$

*Proof.* If  $k = 2^m$  for some  $m = 0, 1, 2, \dots$ , then  $k$  is not a multiple of any element in  $Q$ . Therefore  $\delta(k, i) = 0$  for all  $i$  and hence  $f(k) = 0$ .

Suppose that  $k \notin \{2^m \mid m = 0, 1, 2, \dots\}$  and

$$k = 2^m p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}, \quad m_1, m_2, \dots, m_n > 0, \quad m \geq 0, \quad n \geq 1$$

is the prime factorization of  $k$ , where  $p_1, p_2, \dots, p_n$  are distinct odd prime divisors of  $k$ . We have

$$\begin{aligned}
&\{q_i \in Q \mid \delta(k, i) = 1\} \\
&= \{p_1, \dots, p_n, p_1 p_2, \dots, p_{n-1} p_n, p_1 p_2 p_3, \dots, p_1 p_2 \cdots p_n\}.
\end{aligned}$$

Therefore

$$f(k) = -\binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = -1.$$

□

Notice that, for each  $k$ ,

$$\sum_{i=1}^{\infty} (\operatorname{sgn} q_i) \delta(k, i) a_k \quad \text{and} \quad \sum_{i=1}^{\infty} (\operatorname{sgn} q_i) \delta(k, i) b_k$$

are finite sums. For each  $n$ , from Lemma 3.7, we have

$$\begin{aligned} \lim_{h \rightarrow \infty} C(n, h) &= \lim_{h \rightarrow \infty} \sum_{k=1}^n \sum_{i=1}^h (\operatorname{sgn} q_i) \delta(k, i) a_k \\ &= \sum_{k=1}^n \sum_{i=1}^{\infty} (\operatorname{sgn} q_i) \delta(k, i) a_k \\ &= \sum_{k=1}^n f(k) a_k \\ &= \sum_{\substack{1 \leq k \leq n \\ k \neq 2^m}} (-a_k) \end{aligned}$$

In the same way we have

$$\lim_{h \rightarrow \infty} S(n, h) = \sum_{\substack{1 \leq k \leq n \\ k \neq 2^m}} (-b_k).$$

Therefore we have the following proposition.

**Proposition 3.8.**

$$\lim_{n \rightarrow \infty} \lim_{h \rightarrow \infty} C(n, h) = \sum_{k=1}^{\infty} f(k) a_k = \sum_{k \neq 2^m} (-a_k)$$

$$\lim_{n \rightarrow \infty} \lim_{h \rightarrow \infty} S(n, h) = \sum_{k=1}^{\infty} f(k) b_k = \sum_{k \neq 2^m} (-b_k)$$

Up to now, we have worked with an arbitrary ordering on  $Q$ . To prove Riemann hypothesis, we need a special ordering on  $Q$ .

**The Sufficient Condition of Riemann Hypothesis.** *There exists an ordering on  $Q$  such that*

$$\lim_{n \rightarrow \infty} \lim_{h \rightarrow \infty} C(n, h) = \lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} C(n, h) \quad (3)$$

and

$$\lim_{n \rightarrow \infty} \lim_{h \rightarrow \infty} S(n, h) = \lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} S(n, h). \quad (4)$$

**Theorem 3.9.** *If above condition is true, then the Riemann hypothesis is true.*

*Proof.* Suppose that there exists an ordering on  $Q$  satisfying eq. (3) and (4). Let  $\frac{1}{2} < x < 1$  and  $x + yi$  is a zero of  $\zeta(z)$ . We will get a contradiction.

From Proposition 3.6 and Proposition 3.8, we have

$$\sum_{k \neq 2^m} (-a_k) = \lim_{n \rightarrow \infty} \lim_{h \rightarrow \infty} C(n, h) = \lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} C(n, h) = 0$$

and

$$\sum_{k \neq 2^m} (-b_k) = \lim_{n \rightarrow \infty} \lim_{h \rightarrow \infty} S(n, h) = \lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} S(n, h) = 0.$$

Therefore, from eq.(1), we have

$$\sum_{k=0}^{\infty} a_{2^k} = \sum_{k=1}^{\infty} a_k + \sum_{k \neq 2^m} (-a_k) = 0$$

and

$$\sum_{k=0}^{\infty} b_{2^k} = \sum_{k=1}^{\infty} b_k + \sum_{k \neq 2^m} (-b_k) = 0.$$

Since  $a_1 = 1$ ,  $b_1 = 0$  and  $2^k$  is an even number for all  $k$ , we have

$$1 - \sum_{k=1}^{\infty} \frac{1}{2^{kx}} \cos(ky \ln 2) = \sum_{k=0}^{\infty} a_{2^k} = 0$$

and

$$- \sum_{k=1}^{\infty} \frac{1}{2^{kx}} \sin(ky \ln 2) = \sum_{k=0}^{\infty} b_{2^k} = 0.$$

This contradicts to Lemma 2.5. □

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