Math 351: Abstract Algebra

# Group Theory: Art of Bell Ringing

Erlyn Garcia, Cindy Zheng

Dickinson

## Objectives

- Relate bell combinations with permutations.
- Find group theory in bell ringing
- Find the relationship between $D_4$ and $S_4$.

## Introduction

Surprisingly, before the concept of a group was introduced by mathematicians in the 19th century, it was applied into real life by English church bell ringers during the 17th century. They wanted to make the best use of every bell to extend the ringing time. So they continuously change the bells to form different sounds and to form a complete song.

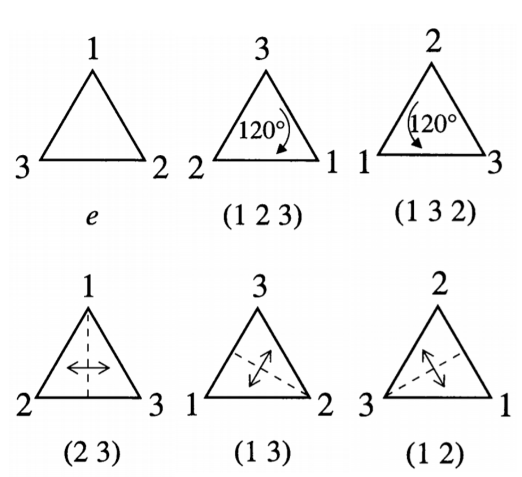
## Methods

- $1, 2, \cdots, n$: Distinct bells in church towers arranged from treble to tenor.
- Row: a sequence of bells which can be represented by a corresponding permutation.
- Change: A swap of two bells.

To make a song with $n$ bells, we need:
(a) the first row and last rows are both 1 2 3...n.
(b) no other row appears more than once.
(c) each change moves every bell by at most one position.

Consider a song with 3 bells 1, 2 and 3. By the rules above, the first row has to be 123 which is represented by the permutation $e$. Notice that when you swap any two bells, the row will be different and will be represented with a different permutation. For example, if we want to swap 1 with 2 on the row 123, the next row will be 213. Now consider the permutation of $e$, swap of 1 and 2 means (12) because we are changing the bell in first position with bell in second position. Thus the next row's permutation will be $(12)e = (12)$ which corresponds to row 213.

| *row* | *change* | *permutation* |
|---|---|---|
| **1 2 3** | | *e* |
| | (12) | |
| **2 1 3** | | (12) |
| | (23) | |
| **2 3 1** | | (123) |
| | (12) | |
| **3 2 1** | | (13) |
| | (23) | |
| **3 1 2** | | (132) |
| | (12) | |
| **1 3 2** | | (23) |
| -------- | (23) | |
| **1 2 3** | | |




## Methods

Also note that the change is always a transposition or a product of transpositions because of the (c) condition.

Now consider a song with 4 bells. Recall that a permutation can be written in cycle notation. Row 2143 represents permutation (12)(34) because 2 is in the position one, 1 is in the position two, 4 is in position three and 3 is in position four. Similarly, the permutation (1342) could be described in as the number 3 is in position one, the number 4 in position three, the number 2 in position four, and the number 1 in position two. Notice that this is an unconventional notation.

| *row* | *change* | *permutation* | | |
|---|---|---|---|---|
| **1 2 3 4** | | *e* | | |
| | (12)(34) | | | |
| **2 1 4 3** | | (12)(34) | | |
| | (23) | | | |
| **2 4 1 3** | | (1243) | | |
| | (12)(34) | | | |
| **4 2 3 1** | | (14) | | |
| | (23) | | $S(\square) = D_4$ | |
| **4 3 2 1** | | (14)(23) | | |
| | (12)(34) | | | |
| **3 4 1 2** | | (13)(24) | | |
| | (23) | | | |
| **3 1 4 2** | | (1342) | | |
| | (12)(34) | | | |
| **1 3 2 4** | | (23) | | |
| -------- | (34) | -------- | | |
| **1 3 4 2** | | (234) | | |
| | (12)(34) | | | |
| **3 1 2 4** | | (132) | | |
| | (23) | | | |
| **3 2 1 4** | | (13) | | |
| | (12)(34) | | | |
| **2 3 4 1** | | (1234) | | |
| | (23) | | $(234)S(\square)$ | $S_4$ |
| **2 4 3 1** | | (124) | | |
| | (12)(34) | | | |
| **4 2 1 3** | | (143) | | |
| | (23) | | | |
| **4 1 2 3** | | (1432) | | |
| | (12)(34) | | | |
| **1 4 3 2** | | (24) | | |
| -------- | (34) | -------- | | |
| **1 4 2 3** | | (243) | | |
| | (12)(34) | | | |
| **4 1 3 2** | | (142) | | |
| | (23) | | | |
| **4 3 1 2** | | (1423) | | |
| | (12)(34) | | | |
| **3 4 2 1** | | (1324) | | |
| | (23) | | $(243)S(\square)$ | |
| **3 2 4 1** | | (134) | | |
| | (12)(34) | | | |
| **2 3 1 4** | | (123) | | |
| | (23) | | | |
| **2 1 3 4** | | (12) | | |
| | (12)(34) | | | |
| **1 2 4 3** | | (34) | | |
| -------- | (34) | | | |
| **1 2 3 4** | | | | |

Figure 1: Song with 4 bells: $S_4$

Take note the first group, the Plain Bob minimus, $S(\square)$, gives you the symmetry group of a square $D_4$. We can actually stop after the first grouping and end it with (1234) with a change (23) and also get a song. However, to make the song as long as possible and at the same time satisfy the given rules, we need to make a new change (34).

The change (34) must occur to give you the next grouping of permutations and bells and following that grouping, another change of (34) occurs to give you the next grouping, and finally one more change of (34) gives you $e$ again. We found that the second grouping can be written as $(234)S(\square)$ and the third grouping can be written as $(243)S(\square)$. And as presented in the graphic, $S(\square)$, $(234)S(\square)$ and $(243)S(\square)$ partition $S_4$.

## Self-Reflection

We can prove that $D_4$ is a subgroup of $S_4$. And a reasonable question to ask is: Is $D_4$ a normal subgroup of $S_4$? The answer is no, because $(1243)(234) = (12) \notin (234)D_4$. Since $\{\iota, (234), (243)\}$ and $D_4$ are both subgroups of $S_4$, and we do get all elements in $S_4$ from their product, and their intersection is only $\{\iota\}$, is $S_4$ an internal direct product of $\{\iota, (234), (243)\}$ and $D_4$? The answer is also no, because $(1243)(234) = (12) \neq (13) = (234)(1243)$.

## Conclusions

- In bell ringing, rows and changes can be represented as permutations.
- By doing transposition-changes on 3 bells, we can get a song of 3 bells that consists of all elements of $S_3$ which is the symmetries of triangle.
- By doing transposition-changes on 4 bells, we can get a song of 4 bells that consists of all elements of $S_4$.
- The left cosets of $D_4$ are $D_4$, $(234)D_4$ and $(243)D_4$.
- $D_4$ is not a normal subgroup of $S_4$. And $S_4$ is not an internal direct product of$\{\iota, (234), (243)\}$ and $D_4$.

## Future questions

- Will the song with 5 bells be related to the symmetries of pentagon?
- Will the song with n bells be related to the symmetries of n-sided polygon?

## References


[1] White, Arthur, and Robin Wilson. 'The Hunting Group.' The Mathematical Gazette, vol. 79, no. 484, 1995, p. 5., doi:10.2307/3619985

[2] Earis, Philip. '24 Bell Change Ringing at Ringing World 100th Anniversary Reception.' YouTube, YouTube, 28 Mar. 2011, `http://www.youtube.com/watch?v=4-fCRBNTNp0`