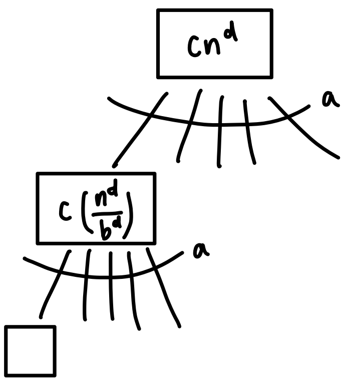
Master Theorem

$T(n) = aT\left(\frac{n}{b}\right) + cn^d$

work/level



cn^d

$ac\left(\frac{n}{b}\right)^d = cn^d\left(\frac{a}{b^d}\right)$

$a^2c\left(\frac{n}{b^2}\right)^d = cn^d\left(\frac{a}{b^d}\right)^2$

Total work: $cn^d\left(1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \ldots \left(\frac{a}{b^d}\right)^k\right)$ $\quad k = \log_b n$ (# of levels)

Cases:

1. $a = b^d$ $\quad$ * each level same work

Total $= O(cn^d(k+1)) = \boxed{O(n^d \log n)}$

2. $a < b^d$

Total $= \boxed{O(n^d)}$ $\quad$ * work exponentially decaying each level, root dominates

3. $a > b^d$

* last term dominates

$$\text{Total} = O\left(cn^d\left(\frac{a}{b^d}\right)^{\log_b n}\right) = O\left(cn^d\left(\frac{a^{\log_b n}}{\underbrace{b^{(\log_b n)d}}_{n^d}}\right)\right) = O\left(a^{\log_b n}\right)$$

$$= O\left(a^{\log_a n \log_b a}\right) = \boxed{O\left(n^{\log_b a}\right)}$$

$$\frac{\log_b n}{\log_b a} = \log_a n$$

1 Asymptotics and Limits

If we would like to prove asymptotic relations instead of just using them, we can use limits.

Asymptotic Limit Rules: If $f(n), g(n) \geq 0$:

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) = \mathcal{O}(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, for some $c > 0$, then $f(n) = \Theta(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $f(n) = \Omega(g(n))$.

Note that these are all sufficient conditions involving limits, and are not true definitions of $\mathcal{O}$, Θ, and Ω. (you should check on your own that these statements are correct!)

(a) Prove that $n^3 = \mathcal{O}(n^4)$.

$$\lim_{n\to\infty} \frac{n^3}{n^4} = \lim_{n\to\infty} \frac{1}{n} = 0 < \infty$$

(b) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$.

$f(n) = 3n$

$g(n) = 5n$

$$\lim_{n\to\infty} \frac{3n}{5n} = \frac{3}{5} < \infty$$

$f(n) = \Theta(g(n))$

but by definition it is also $f(n) = O(g(n))$

(c) Prove that for any $c > 0$, we have $\log n = \mathcal{O}(n^c)$.

Hint: Use L'Hôpital's rule: If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ (if the RHS exists)

$$\lim_{n\to\infty} \frac{\log n}{n^c} = \lim_{n\to\infty} \frac{1/n}{cn^{c-1}} = \lim_{n\to\infty} \frac{1}{cn^c} = 0 < \infty$$

(d) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist. In this case, you would be unable to use limits to prove $f(n) = \mathcal{O}(g(n))$.

$f(x) = x(\sin x + 1)$

$g(x) = x$

$$\lim_{x\to\infty} \frac{x(\sin x + 1)}{x} = \lim_{x\to\infty} \sin x + 1$$, oscillates forever

Known bound: $\sin x + 1 \leq 2$

$f(x) \leq 2g(x)$ so $f(x) = O(g(n))$

2 Asymptotic Complexity Comparisons

(a) Order the following functions so that for all i, j, if f_i comes before f_j in the order then $f_i = O(f_j)$. Do not justify your answers.

- $f_1(n) = 3^n$
- $f_2(n) = n^{\frac{1}{3}}$
- $f_3(n) = 12$
- $f_4(n) = 2^{\log_2 n}$ $= n$
- $f_5(n) = \sqrt{n}$ $= n^{1/2}$
- $f_6(n) = 2^n$
- $f_7(n) = \log_2 n$
- $f_8(n) = 2^{\sqrt{n}}$
- $f_9(n) = n^3$

As an answer you may just write the functions as a list, e.g. $f_8, f_9, f_1, \ldots$

$12,\ \log_2 n,\ n^{1/3},\ \sqrt{n},\ 2^{\log_2 n},\ n^3,\ 2^{\sqrt{n}},\ 2^n,\ 3^n$

(b) In each of the following, indicate whether $f = O(g)$, $f = \Omega(g)$, or both (in which case $f = \Theta(g)$). ***Briefly*** justify each of your answers. Recall that in terms of asymptotic growth rate, logarithmic $<$ polynomial $<$ exponential.

	$f(n)$	$g(n)$
(i)	$\log_3 n$	$\log_4(n)$
(ii)	$n \log(n^4)$	$n^2 \log(n^3)$
(iii)	$\sqrt{n}$	$(\log n)^3$
(iv)	$n + \log n$	$n + (\log n)^2$

i) $\boxed{f = \Theta(g)}$

change of base: $\log_a b = \dfrac{\log b}{\log a}$

$\log_3 n = \dfrac{\log n}{\log 3}$ $\qquad$ $\log_4 n = \dfrac{\log n}{\log 4}$

ii) $n \log(n^4) = 4n \log n$ $\qquad$ $\boxed{f = O(g)}$

$n^2 \log(n^3) = 3n^2 \log n$

iii) $n^{1/2}$ vs $(\log n)^3$

polynomial vs logarithmic

$\boxed{f = \Omega(g)}$

iv) linear term dominates

$\boxed{f = \Theta(g)}$

3 Hadamard matrices

The Hadamard matrices $H_0, H_1, H_2, \ldots$ are defined as follows:

- H_0 is the 1×1 matrix $[1]$
- For $k > 0$, H_k is the $2^k \times 2^k$ matrix

$$H_k = \left[\begin{array}{c|c} H_{k-1} & H_{k-1} \\ \hline H_{k-1} & -H_{k-1} \end{array}\right]$$

(a) Write down the Hadamard matrices H_0, H_1, and H_2.

$$H_0 = [1]$$

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(b) Compute the matrix-vector product $H_2 \cdot v$ where H_2 is the Hadamard matrix you found above, and

$$v = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Note that since H_2 is a 4×4 matrix, and the vector has length 4, the result will be a vector of length 4.

$$H_2 v = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

(c) Now, we will compute another quantity. Take v_1 and v_2 to be the top and bottom halves of v respectively. Therefore, we have that

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Compute $u_1 = H_1(v_1 + v_2)$ and $u_2 = H_1(v_1 - v_2)$ to get two vectors of length 2. Stack u_1 above u_2 to get a vector u of length 4. What do you notice about u?

$$u_1 = H_1(v_1 + v_2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_2 = H_2(v_1 - v_2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(d) Suppose that

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

is a column vector of length $n = 2^k$. v_1 and v_2 are the top and bottom half of the vector, respectively. Therefore, they are each vectors of length $\frac{n}{2} = 2^{k-1}$. Write the matrix-vector product $H_k v$ in terms of H_{k-1}, v_1, and v_2 (note that H_{k-1} is a matrix of dimension $\frac{n}{2} \times \frac{n}{2}$, or $2^{k-1} \times 2^{k-1}$). Since H_k is a $n \times n$ matrix, and v is a vector of length n, the result will be a vector of length n.

$$H_k v = \begin{bmatrix} H_{k-1}(v_1 + v_2) \\ H_{k-1}(v_1 - v_2) \end{bmatrix}$$

(e) Use your results from (c) to come up with a divide-and-conquer algorithm to calculate the matrix-vector product $H_k v$, and show that it can be calculated using $O(n \log n)$ operations. Assume that all the numbers involved are small enough that basic arithmetic operations like addition and multiplication take unit time. You do not need to prove correctness.

$H_k \in \mathbb{R}^{n \times n}$ $\quad v \in \mathbb{R}^n$

Work at each level:

Find vectors $v_1 + v_2$ and $v_1 - v_2$ $\leftarrow O(n)$

Recursive step:

Find products $H_{k-1}(v_1 + v_2)$ and $H_{k-1}(v_1 - v_2)$

$H_{k-1} \in \mathbb{R}^{n/2 \times n/2}$ $\quad v_1, v_2 \in \mathbb{R}^{n/2}$

$T(n) = 2T(n/2) + O(n) = O(n \log n)$

4 Monotone matrices

A m-by-n matrix A is *monotone* if $n \geq m$, each row of A has no duplicate entries, and it has the following property: if the minimum of row i is located at column j_i, then $j_1 < j_2 < j_3 \dots j_m$. For example, the following matrix is monotone (the minimum of each row is bolded):

$$\begin{bmatrix} \mathbf{1} & 3 & 4 & 6 & 5 & 2 \\ 7 & 3 & \mathbf{2} & 5 & 6 & 4 \\ 7 & 9 & 6 & 3 & 10 & \mathbf{0} \end{bmatrix}$$

Give an efficient (i.e., better than $O(mn)$-time) algorithm that finds the minimum in each row of an m-by-n monotone matrix A.

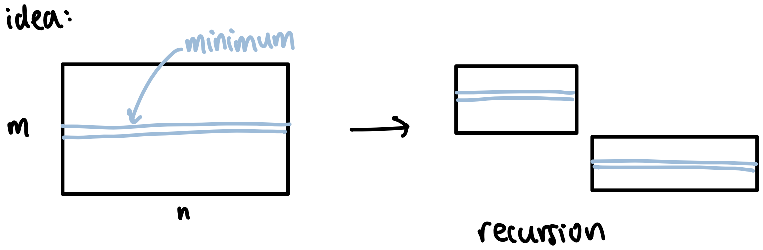
Give a 3-part solution. You do not need to write a formal recurrence relation in your runtime analysis; an informal summary of the runtime analysis such as "proof by picture" is fine.

starting thoughts:

- how to divide into smaller subproblems?
- key observation of matrix: if minimum of row i is found at column j, then minimum of each row $<i$ is in column $<j$ and minimum of each row $>i$ is in column $>j$

↳ dramatically reduce search space

main idea:



proof of correctness:

proof by induction on number of rows of A [see official solutions]

runtime analysis:

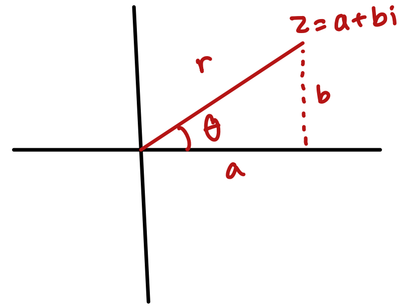
* master's theorem doesn't work since the split might not be even

key observations:

- work at each level is $O(n)$, since we scan at most the entire row
- must eventually scan every row, there are m rows

$O(mn)$

Complex Numbers:



$z = \underset{\substack{\uparrow \\ \text{real}}}{a} + \underset{\substack{\uparrow \\ \text{imaginary}}}{bi}$ (rectangular)

$z = r(\cos\theta + i\sin\theta)$ (polar)

$a = r\cos\theta$

$b = r\sin\theta$

Using Euler's Formula:

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

<u>n^{th} roots of unity</u>: n complex numbers satisfying $w^n = 1$

$w_k = e^{2\pi i k/n}$ $\quad k = 0, 1, 2, \ldots, n-1$

ex: what are the second roots of unity?

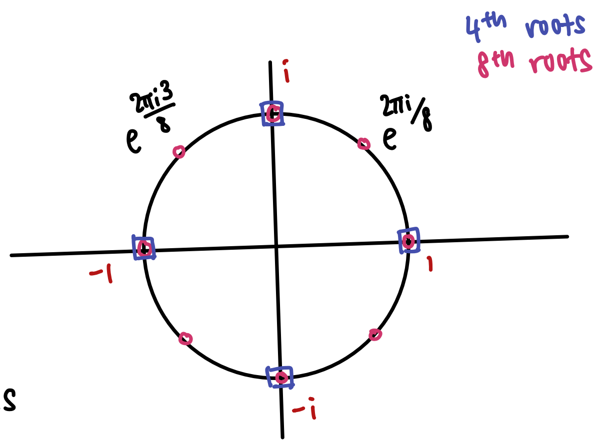
$w^2 = 1 \qquad w = +1, -1$

fourth roots of unity?

$$w^4 = 1$$

$$w^2 = 1 \qquad w^2 = -1$$

$$w = 1 \quad w = -1 \quad w = i \quad w = -i$$



Note: Squaring n^{th} roots of unity equals $\left(\frac{n}{2}\right)^{th}$ roots of unity

$$\left(e^{\frac{2\pi i 3}{8}}\right)^2 = e^{\frac{2\pi i 6}{8}} = e^{\frac{2\pi i 3}{4}}$$

* For n^{th} roots of unity, place n points evenly on unit circle

5 Complex numbers review

A *complex number* is a number that can be written in the rectangular form $a + bi$ (i is the imaginary unit, with $i^2 = -1$). The following famous equation (*Euler's formula*) relates the polar form of complex numbers to the rectangular form:

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

In polar form, $r \geq 0$ represents the distance of the complex number from 0, and θ represents its angle. Note that since $\sin(\theta) = \sin(\theta + 2\pi), \cos(\theta) = \cos(\theta + 2\pi)$, we have $re^{i\theta} = re^{i(\theta+2\pi)}$ for any r, θ.

The n-th *roots of unity* are the n complex numbers satisfying $\omega^n = 1$. They are given by

$$\omega_k = e^{2\pi ik/n}, \qquad k = 0, 1, 2, \ldots, n-1$$

(a) Let $x = e^{2\pi i3/10}, y = e^{2\pi i5/10}$ which are two 10-th roots of unity. Compute the product $x \cdot y$. Is this an n-th root of unity for some n? Is it a 10-th root of unity?

What happens if $x = e^{2\pi i6/10}, y = e^{2\pi i7/10}$?

$$xy = \exp\left(\frac{2\pi i3 + 2\pi i5}{10}\right) = \exp\left(\frac{2\pi i8}{10}\right)$$

$$xy = \exp\left(\frac{2\pi i6 + 2\pi i7}{10}\right) = \exp\left(\frac{2\pi i13}{10}\right) = \exp\left(\frac{2\pi i3}{10}\right)$$

(b) Show that for any n-th root of unity $\omega \neq 1$, $\sum_{k=0}^{n-1} \omega^k = 0$, when $n > 1$.

Hint: Use the formula for the sum of a geometric series $\sum_{k=0}^{n} \alpha^k = \frac{\alpha^{n+1}-1}{\alpha-1}$. It works for complex numbers too!

$$\sum_{k=0}^{n-1} w^k = \frac{w^n - 1}{w - 1} = 0 \qquad w^n = 1 \text{ by definition}$$

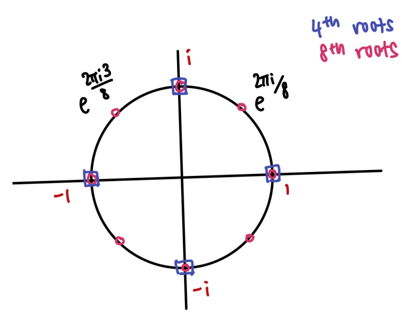
(c) (i) Find all ω such that $\omega^2 = -1$.

$w = i, -i$

(ii) Find all ω such that $\omega^4 = -1$.

$w^8 = 1$, need the 8^{th} roots of unity that aren't 4^{th} roots of unity

$w = e^{2\pi i/8}, e^{2\pi i3/8}, e^{2\pi i5/8}, e^{2\pi i7/8}$



6 Extra Divide and Conquer Practice: Quantiles

Let A be an array of length n. The boundaries for the k quantiles of A are $\{a^{(n/k)}, a^{(2n/k)}, \ldots, a^{((k-1)n/k)}\}$ where $a^{(\ell)}$ is the ℓ-th smallest element in A.

Devise an algorithm to compute the boundaries of the k quantiles in time $\mathcal{O}(n \log k)$. For convenience, you may assume that k is a power of 2.

Hint: Recall that QUICKSELECT(A, ℓ) gives $a^{(\ell)}$ in $\mathcal{O}(n)$ time.