Sec 3

3.1. $N_0,\ N_0+1,\ \cdots N_1$

$$f(x) = \frac{1}{N_1 - N_0 + 1}$$

$$EX = \sum N_0 \cdot \frac{1}{N_1 - N_0 + 1} + (N_0+1)\cdot f(x) + \cdots + N_1 \cdot f(x)$$

$$= \frac{1}{N_1 - N_0 + 1}\,(N_0 + N_0 + 1 + \cdots + N_1)$$

$$= \frac{1}{N_1 - N_0 + 1}\,\frac{(N_0 + N_1)\cdot(N_1 - N_0 + 1)}{2}$$

$$= \frac{N_0 + N_1}{2}$$

$$(x-1)^3 = (x-1)(x-1)^2$$
$$= (x-1)(x^2 - 2x + 1)$$
$$= x^3 - x^2 - 2x^2 + 2x + x - 1$$
$$= x^3 - 3x^2 + 3x - 1$$

$$\sum \begin{cases} \cancel{1^3} + \underbrace{(1-1)^3}_{0} = \cancel{1^3} - 3\times 1^2 + 3\times 1 - 1 \\ \cancel{2^3} + (2-1)^3 = \cancel{2^3} - 3\times 2^2 + 3\times 2 - 1 \\ \cancel{3^3} + (3-1)^3 \\ \cdots \end{cases}$$

$$\cancel{n^3} + (n-1)^3 = \cancel{n^3} - 3\times n^2 + 3\times n - 1$$

$$-n^3 = -3\times \underbrace{(1^2 + 2^2 + \cdots + n^2)}_{k} + 3\left(\frac{(1+n)n}{2}\right) - n$$

$$k = \frac{2n^3 + 3n(1+n) - 2n}{6}$$

$$= \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n+1)(n+1)}{6}$$

$$EX^2 = \sum N_0^2 \cdot \frac{1}{N_1 - N_0 + 1} + \cdots + N_1^2 f(x)$$

$$= \frac{1}{N_1 - N_0 + 1}\cdot(N_0^2 + N_0 + 1^2 + \cdots + N_1^2)$$

$$= \frac{1}{N_1 - N_0 + 1}\left(\frac{2N_1^3 + 3N_1^2 + N_1}{6} - \frac{2N_0^3 + 3N_0^2 + N_0}{6}\right)$$

$$= \frac{1}{N_1 - N_0 + 1}\left(\frac{N_1(2N_1+1)(N_1+1)}{6} - \frac{N_0(2N_0+1)(N_0+1)}{6}\right)$$

$$a^3 - b^3$$
$$= a^3 - a^2 b + a^2 b - b^3$$
$$= a^2(a-b) + b(a^2 - b^2)$$
$$= a^2(a-b) + b(a+b)(a-b)$$
$$= (a-b)[a^2 + b(a+b)]$$
$$= (a-b)(a^2 + ab + b^2)$$

$$\mathrm{Var}X = E(X - EX)^2$$
$$= E[X^2 - 2XEX + (EX)^2]$$
$$= EX^2 - (EX)^2$$
$$= \frac{1}{N_1 - N_0 + 1}\left(\frac{2N_1^3 + 3N_1^2 + N_1 - 2N_0^3 - 3N_0^2 - N_0}{6}\right) - \frac{N_0^2 + 2N_0N_1 + N_1^2}{4}$$

$$= \frac{4N_1^3 + 6N_1^2 + 2N_1 - 4N_0^3 - 6N_0^2 - 2N_0 - 3N_1^3 + 3N_1N_0^2 - 3N_1^2N_0 + 3N_0^3 - 3N_0^2 - 6N_0N_1 - 3N_1^2}{12(N_1 - N_0 + 1)}$$

$$= \frac{N_1^3 - N_0^3 + 3N_1^2 - 9N_0^2 + 3N_1N_0^2 - 3N_1^2N_0 - 6N_1N_0 + 2N_1 - 2N_0}{12(N_1 - N_0 + 1)}$$

3.2: X: the num of defective parts in the sample.

M: the num of the lot

k: sample size

(a)

If $M \geq 6$, $X = 0$

$$P(X=0 \mid M=6, N=100, k) = \frac{\binom{94}{k}\binom{6}{0}}{\binom{100}{k}} = \frac{(100-k)(99-k)\cdots(95-k)}{(100\cdot 99\cdots 95)} \leq 0.01$$

按最优秀的情况算 ($M=6$)

$k \geq 32$

if $M > 6$, 那么 k 可以更小些.

(b)

$$P(X=0 \mid M=6, N=100, k) + P(X=1 \mid M=6, N=100, k) \leq 0.01$$

$$= \frac{\binom{94}{k}\binom{6}{0}}{\binom{100}{k}} + \frac{\binom{6}{1}\binom{94}{k-1}}{\binom{100}{k}}$$

$$= \frac{(100-k)(99-k)\cdots(95-k) + (100-k)\cdots(96-k)\cdot 6}{100\cdot 99\cdots 95} \leq 0.01$$

$k = 51$

3.3

$$\left[1-(1-p)^3 - p(1-p)^3\right]\cdot(1-p)^3$$

$$- - - - \Big|_{\overline{1-p}\ \overline{1-p}\ \overline{1-p}}$$

3.4 (a) Let X be the number of keys that tried

$x-1$: fail $\quad x$ success

$$f(X) = \frac{\binom{n-1}{x-1}\binom{1}{1}}{\binom{n}{x}} = \frac{x}{n}$$

$$E(X) = \sum x \cdot f(X)$$
$$= \sum_{x=1}^{n} \frac{x^2}{n}$$
$$= \frac{1}{n}(1^2 + 2^2 + \cdots + n^2)$$
$$= \frac{(n-1)(2n+3)}{6}$$

$$(x-1)^3 = (x-1)(x^2-2x+1)$$
$$= x^3 - 2x^2 + x - x^2 + 2x - 1$$
$$= x^3 - 3x^2 + 3x - 1$$

$$\sum \begin{cases} (1-1)^3 - \cancel{1^3} = \cancel{1^3} - 3\cdot 1^2 + 3\cdot 1 - 1 \\ \cancel{(2-1)^3} - \cancel{2^3} = \cancel{2^3} - 3\times 2^2 + 3\times 2 - 1 \\ \cancel{(3-1)^3} - \cancel{3^3} = \cancel{3^3} - 3\times 3^2 + 3\times 3 - 1 \\ \vdots \\ (n-1)^3 - n^3 = \cancel{n^3} - 3\times n^2 + 3\times n - 1 \end{cases}$$

$$-n^3 = -3(1^2 + 2^2 + \cdots + n^2) + 3\frac{(1+n)n}{2} - n$$
$$\therefore 1^2 + 2^2 + \cdots + n^2 = \frac{2n^3 - 2n + 3(1+n)n}{6}$$
$$= \frac{n(2n^2 - 2n + 3 + 3n)}{6}$$
$$= \frac{n(n-1)(2n+5)}{6}$$

(b) $p = \frac{1}{n}$

$$f(X) = (1-p)^{x-1} \cdot p$$
$$= \left(1 - \frac{1}{n}\right)^{x-1} \cdot \frac{1}{n}$$

$$E(X) = \sum x \cdot f(X)$$
$$= \sum_{x=1}^{n} x\left(1 - \frac{1}{n}\right)^{x-1} \cdot \frac{1}{n}$$
$$= \frac{1}{n} \cdot \sum x\left(1 - \frac{1}{n}\right)^{x-1}$$
$$= \frac{1}{n}\left[1 - \frac{(n+1)(n-1)^n}{n^{n-2}} + \frac{(n-1)^{n+1}}{n^{n-3}}\right]$$

$$S_n = 1 \cdot a^{1-1} + 2 \cdot a^{2-1} + \cdots + n \cdot a^{n-1}$$
$$aS_n = \qquad 1 \cdot a^{2-1} + \cdots + (n-1)a^{n-1} + n \cdot a^n$$

$$S_n - aS_n = a^{1-1} + a^{2-1} + \cdots + a^{n-1} - n \cdot a^n$$
$$= a^0 + a^1 + \cdots + a^{n-1} - n \cdot a^n$$
$$= \frac{a^0(1-a^n)}{1-a} - n \cdot a^n$$
$$= \frac{1 - a^n - n \cdot a^n + n \cdot a^{n+1}}{1-a}$$
$$S_n = \frac{1 - (n+1)a^n + n \cdot a^{n+1}}{(1-a)^2} \qquad a = 1 - \frac{1}{n}$$

3.5 ※1 Assume Drug1 (D_1) and Drug2 (D_2) are equal ($P=0.8$)

Let X be the success case of D_2 among 100

$$P(X \geq 85) = \sum_{x=85}^{100} \boxed{\binom{100}{x}} 0.8^x \cdot (1-0.8)^{100-x} = 0.1285$$

※2

Not too small $\therefore$ we can not draw the conclusion that D_2 is better than D_1

※ 假设 equal 等 ≥85 的 P，不能只算 85 的 P，因为如果只算等于 85，那是一种特殊情况，排除了 86 … 100。因为这里略多的意思是，successful case 比 D_1 多的时候，是否能证明 D_2 更好。

※ 一定要有 $\binom{100}{x}$，如果没有，算出的概率就会偏小（只是一种排序）

3.6 or Binomial Dis $B(n, p)$ $p = 0.99$

(a) Norm distribution with $E(X) = 20$

(b) $P(X \le 100) = \sum_{x=0}^{100} \binom{2000}{x} \cdot p^{2000-x} \cdot (1-p)^x$

(c) $P(X \le 100) = 1$

```
R: bino <- function(n, p, N){
    s=0
    for(i in 0:n){
      s=s+choose(N,i) * p^(N-i) * (1-p)^i
    }
    return(s)
}
```

3.7. $X \sim \text{Poisson}(\lambda)$ $P(X) = \frac{e^{-\lambda} \cdot \lambda^i}{i!}$

$E(X) = \lambda$ $P(X \ge 2) \ge 0.99$

$\therefore P(X < 2) < 0.01$

$$\frac{e^{-\lambda} \cdot \lambda^0}{0!} + \frac{e^{-\lambda} \cdot \lambda^1}{1!} < 0.01$$

$$e^{-\lambda} + \lambda \cdot e^{-\lambda} < 0.01$$

$$(1+\lambda) \cdot e^{-\lambda} < 0.01$$

$$\lambda = 6.6384$$

3.8 (a) $P(T_1) = P(T_2) = \frac{1}{2}$

X: the number of customer in T_1

$$P(X) = \binom{1000}{x} \cdot \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{1000-x}$$

$$P(X > N) = \sum_{x=N+1}^{1000} \binom{1000}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{1000-x} < 0.1$$

$N = 537$

(b) $E = 1000 \times \frac{1}{2} = np = 500$ $\sigma^2 = np(1-p) = 250$

$\therefore X = n(500, 250)$

$P(X > N) = P\left(Z > \frac{N-500}{\sqrt{250}}\right) < 0.01$ $Z \sim (0,1)$

$N \approx 537$

※ 这里应是 $P(X \ge 5)$ 而不是 $P(X = 5)$

3.9 (a) $P(5 \mid 60, 1/90) = C_{60}^{5} \left(\frac{1}{90}\right)^5 \left(1 - \frac{1}{90}\right)^{45}$ X

$P(X \ge 5) = \sum_{x=5}^{60} C_{60}^{x} \left(\frac{1}{90}\right)^x \left(1 - \frac{1}{90}\right)^{60-x} = 0.0006$

(b) $P(X \ge 5 \mid \text{New York state}) = 0.0006 \times 6245$ X

$P(X \ge 1)$ $X \sim \text{binominal}(310, 0.0006)$

$P(X \ge 1) = 1 - P(X = 0) = 1 - (1 - 0.0006)^{310} = 0.1698$

※ 这是一个建模，这里依然是 binominal model

以概率 0.0006 重复 310 次。

如果直接 0.0006×310 得到的是期望：

E(the number of school has more than 5 twins)

而不是概率，要厘清.

(c). $310 \times 50 \times 10 = 155000$

$P(Z \geq 1)$ $Z \sim \text{binominal}(155000, 0.0006)$

$P(Z \geq 1) = 1 - P(Z=0) = 1 - (0.0006)^{155000} \approx 1$

亦可写作

$P(Z \geq 1)$ $Z \sim \text{binominal}(500, 0.1698)$

$P(Z \geq 1) = 1 - P(Z=0) = 1 - (0.1691)^{500} \approx 1$

3.10

(a) $P = \dfrac{\binom{N}{4}\binom{M}{2}}{\binom{N+M}{4}\binom{N+M-4}{2}}$

(b) max P R暴力循环

$P_{max} = 0.022$ $M = 165$

Prepare for 3.11

Proof: $n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}} \cdot e^{-n}$

Gamma function: $\Gamma(n+1) = \int_0^\infty x^n \cdot e^{-x}\, dx = \int_0^\infty e^{n\ln x - x}\, dx$

let $x = n+y$

$\Gamma(n+1) = \int_{-n}^\infty e^{n\ln(n+y)-n-y}\, dy$ 定义域换了

$= e^{-n}\int_{-n}^\infty e^{n\ln(n+y)-y}\, dy$

$= e^{-n}\int_{-n}^\infty e^{n\ln n + n\ln(1+\frac{y}{n})-y}\, dy$

$\because e^{n\ln n} = e^{\ln n^n} = n^n$

$\therefore \Gamma(n+1) = \left(\frac{n}{e}\right)^n \int_{-n}^\infty e^{n\ln(1+\frac{y}{n})-y}\, dy$

$= \left(\frac{n}{e}\right)^n \int_{-n}^\infty e^{\cancel{y} - \frac{y^2}{2n} + \frac{y^3}{3n^2} - \cdots - \cancel{y}}\, dy$

$= \left(\frac{n}{e}\right)^n \int_{-n}^\infty e^{-\frac{y^2}{2n} + \frac{y^3}{3n^2} - \cdots}$

let $y = \sqrt{n}\, v$ $v = \frac{y}{\sqrt{n}}$ $y \in (-n, \infty)$ $\therefore v \in (-\sqrt{n}, \infty)$

$\Gamma(n+1) = \left(\frac{n}{e}\right)^n \sqrt{n} \int_{-\sqrt{n}}^\infty e^{-\frac{v^2}{2} + \frac{v^3}{3\sqrt{n}} - \cdots}\, dv$

$\uparrow n \to \infty$

$\int_{-\sqrt{n}}^\infty \Leftrightarrow \dfrac{\int_{-\infty}^\infty e^{-\frac{v^2}{2}}\, dv}{\sqrt{2\pi}}$

$\therefore \Gamma(n+1) = \left(\frac{n}{e}\right)^n \sqrt{n} \cdot \sqrt{2\pi}$

$f(x) = \ln(1+u)$ $f'(0) = (1+u)^{-1} = \frac{1}{1+0} = 1$

$f^2(0) = -(1+u)^{-2} = -1$

$f^3(0) = 2(1+u)^{-3} = 2$

$f^4(0) = 2(-3)(1+u)^{-4} = -6$

$\vdots$

$\therefore f^n(0) = (-1)^{n+1} \cdot (n-1)!$

$\because \ln(1+u) = \sum_{n=1}^\infty \frac{(-1)^{n+1}(n-1)!}{n!} u^n$

$= \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} u^n$

$= u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + \frac{(-1)^{n+1}}{n} u^n$

$u \in (-1, 1]$

$\therefore \ln\left(1+\frac{y}{n}\right) = \frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} - \cdots$

$\boxed{y = \left(1+\frac{-x}{n}\right)^n}$

$\ln y = \boxed{n} \ln\left(1+\frac{-x}{n}\right)$

$* = \frac{\ln\left(1+\frac{-x}{n}\right)}{\boxed{\frac{1}{n}}} \to \frac{0}{0}$

$y = \left(1+\frac{-x}{n}\right)^n$

$\ln y = n \cdot \ln\left(1+\frac{-x}{n}\right)$

$= \dfrac{\ln\left(1+\frac{-x}{n}\right)}{\frac{1}{n}}$

洛必达:

$\dfrac{\frac{\cancel{-x \cdot n^2}}{1+\frac{-x}{n}}}{\cancel{-n^2}} = \dfrac{x}{1+\frac{-x}{n}} = x$ $\dfrac{\frac{x \cdot \cancel{n^2}}{1+\frac{-x}{n}}}{\cancel{-n^2}} = -\dfrac{x}{1+\frac{-x}{n}} = -x$

$\boxed{\therefore y = e^x}$

3.11.

$$P(\underline{X}=x \mid N, M, K) = \frac{M!}{x!(M-x)!} \cdot \frac{(N-M)!}{(K-x)!(N-M-K+x)!} \cdot$$

$$= \frac{K!}{x!(K-x)!} \cdot \underbrace{\frac{M!(N-M)!(N-K)!}{(M-x)!(N-M-K+x)!N!}}_{S}$$

$$M! = \boxed{\frac{M!}{\sqrt{2\pi}\, M^{M+1/2} \cdot e^{-M}}} \cdot \sqrt{2\pi}\, M^{M+1/2} \cdot e^{-M} \approx \sqrt{2\pi} \cdot M^{M+1/2} \cdot e^{-M} \qquad \frac{M!}{\sqrt{2\pi} \cdot M^{M+1/2} \cdot e^{-M}} \approx 1$$

$$\therefore S = \frac{M^{M+\frac{1}{2}} \cdot \cancel{e^{-M}} \cdot (N-M)^{N-M+\frac{1}{2}} \cdot \cancel{e^{-N+M}} \cdot (N-K)^{N-K+\frac{1}{2}} \cdot \cancel{e^{-N+K}}}{(M-x)^{M-x+\frac{1}{2}} \cancel{e^{-M}} \cdot (N-M-K+x)^{N-M-K+x+\frac{1}{2}} \cdot \cancel{e^{-x+K+M-N}} \cdot N^{N+\frac{1}{2}} \cdot \cancel{e^{-N}}}$$

$$= \frac{M^{M+\frac{1}{2}} (N-M)^{N-M+\frac{1}{2}} (N-K)^{N-K+\frac{1}{2}}}{(M-x)^{M-x+\frac{1}{2}} (N-M-K+x)^{N-M-K+x+\frac{1}{2}} N^{N+\frac{1}{2}}}$$

$\because M \to \infty \quad N \to \infty$

$$\therefore \frac{M^{M+\frac{1}{2}}}{(M-x)^{M-x+\frac{1}{2}}} = \left[\frac{M^{\frac{1}{2}}}{(M-x)^{\frac{1}{2}-x}}\right] \cdot \left(\frac{M}{M-x}\right)^{M} \qquad \left(\frac{M}{M-x}\right)^{M} = \frac{1}{\left(\frac{M-x}{M}\right)^{M}} = \frac{1}{\left(1+\frac{-x}{M}\right)^{M}} = \frac{1}{e^{-x}} = e^{x}$$

$$\frac{(N-M)^{N-M+\frac{1}{2}}}{N-M-(K-x)^{\cdots}} = \frac{(N-M)^{\frac{1}{2}}}{(\cdots\cdots)^{x+\frac{1}{2}-K}} \cdot \underbrace{\left(\frac{N-M}{N-M-(K-x)}\right)^{N-M}}_{\to e^{K-x}} \qquad \frac{(N-K)^{N-K+\frac{1}{2}}}{N^{N+\frac{1}{2}}} = \frac{(N-K)^{\frac{1}{2}-K}}{N^{\frac{1}{2}}} \cdot \underbrace{\left(\frac{N-K}{N}\right)^{N}}_{e^{-K}}$$

$$\because \lim_{M\to\infty} \left(\frac{M}{M-x}\right)^{\frac{1}{2}} = 1 \quad \therefore \lim_{M\to\infty} \frac{M^{M+\frac{1}{2}}}{(M-x)^{M-x+\frac{1}{2}}} = (M-x)^{x} \cdot e^{x}$$

$$\because \lim_{M,N\to\infty} \left(\frac{N-M}{N-M-(K-x)}\right)^{\frac{1}{2}} = 1 \quad \therefore \lim_{\cdots} \frac{(N-M)^{N-M+\frac{1}{2}}}{[N-M-(K-x)]^{N-M-K+x+\frac{1}{2}}} = [N-M-(K-x)]^{K-x} e^{K-x}$$

$$\lim_{N\to\infty} \frac{(N-K)^{N-K+\frac{1}{2}}}{N^{N+\frac{1}{2}}} = (N-K)^{-K} e^{-K}$$

$$\therefore S = \frac{(M-x)^{x} \cdot \cancel{e^{x}} \cdot [N-M-(K-x)]^{K-x} \cdot \cancel{e^{K-x}}}{(N-K)^{K} \cdot \cancel{e^{K}}}$$

$$= \left(\frac{M-x}{N-K}\right)^{x} \cdot \left[\frac{N-M-(K-x)}{(N-K)}\right]^{K-x}$$

$\because M\to\infty \; N\to\infty \quad \frac{M}{N} \to p$

$\therefore S = p^{x} \cdot (1-p)^{K-x}$

$\therefore P(\underline{X}=x \mid N, M, K) = C_{K}^{x}\, p^{K} (1-p)^{K-x}$

(b) $\because P(X=x \mid M,N,K) = C_K^x \cdot p^x (1-p)^{K-x}$

$$= \frac{K!}{x!(K-x)!} \cdot \left(\frac{M}{N}\right)^x \left(1-\frac{M}{N}\right)^{K-x} \quad \text{(from (a))}$$

$\because M/N \to 0 \quad KM/N \to pK \to \lambda \quad K \to \infty$

$$P = \frac{K!}{x!(K-x)!} \cdot \left(\frac{\lambda}{K}\right)^x \underbrace{\left(1-\frac{\lambda}{K}\right)^{K-x}}$$

$$= \frac{\lambda^x}{x!} \underbrace{\frac{K!}{(K-x)!} \cdot K^{-x}}_{=1} e^{-\lambda}$$

$$\frac{K!}{(K-x)! \cdot K^x} = \frac{K(K-1)(K-2)\cdots(K-x+1)}{\underbrace{K \cdot K \cdot K \cdots K}_{x}} = 1$$

Let $S = \left(1-\frac{\lambda}{K}\right)^{K-x}$

$\ln S = (K-x) \ln\left(1-\frac{\lambda}{K}\right) \quad \therefore \ln S = -\lambda$

$\boxed{S = e^{-\lambda}}$

$$\lim_{K\to\infty} = \frac{\ln 1-\frac{\lambda}{K}}{\frac{1}{K-x}} \quad \frac{0}{0}$$

$$= \frac{\frac{\lambda K^{-2}}{1-\frac{\lambda}{K}}}{-(K-x)^{-2}} = -\lambda$$

(c) $\because P = \frac{K!}{x!(K-x)!} \cdot \frac{(M-x)^x \cdot [N-M-(K-x)]^{K-x}}{(N-K)^K}$

$$= \frac{1}{x!} \cdot \frac{K!}{(K-x)!} \cdot \left(\frac{M-x}{N-K}\right)^x \cdot \left(\frac{N-M-(K-x)}{N-K}\right)^{K-x}$$

$$= \frac{1}{x!} \frac{K!}{(K-x)!} \frac{1}{K^x} \left(\frac{KM}{N-K}\right)^x \left(\frac{N-M-(K-x)}{N-K}\right)^{K-x}$$

$$= \frac{1}{x!} \cdot \left(\frac{\frac{KM}{N}}{1-\frac{K}{N}}\right)^x \left(\frac{N-M-(K-x)}{N-K}\right)^{K-x}$$

$$= \frac{1}{x!} \cdot \lambda^x \cdot \underbrace{\left(1-\frac{K}{N}\right)^x}_{1} \cdot \left[\frac{N-M-(K-x)}{N-K}\right]^{K-x}$$

$$\approx \frac{1}{x!} \lambda^x \cdot \left(\frac{N-M}{N}\right)^K$$

$$= \frac{1}{x!} \lambda^x \cdot \left(1-\frac{\frac{MK}{N}}{K}\right)^K$$

$$= \frac{1}{x!} \lambda^x \left(1-\frac{\lambda}{K}\right)^K$$

3.12.

$f(X=x) = C_n^x \cdot p^x \cdot (1-p)^{n-x}$ n次中成功x次

$f(Y=y) = C_{r+y-1}^{y} p^r \cdot (1-p)^y$ 第r次成功前y次失败

$F_X(r-1) = P(X \le r-1)$

$= P$ (rth success on (n+1)st or later)

$= P$ (at least $n+1-r$ failure in former n trails)

$= P(Y \ge n+1-r)$

$= 1 - P(Y \le n-r) = 1 - F_Y(n-r)$

3.13.

(a) $E(P(Z_T=x)) = \sum_{x=1}^{\infty} x \cdot P(Z_T=x) = \dfrac{E(Z)}{P(Z>0)}$

$E(P(Z_T=x^2)) = \sum_{x=1}^{\infty} x^2 \cdot P(Z_T=x) = \dfrac{E(Z^2)}{P(Z>0)}$

$Var(P(Z_T=x)) = E(x - E(P(Z_T=x)))^2$

$= E(P(Z_T=x^2)) - E^2(P(Z_T=x))$

$= \dfrac{E(Z^2)\cdot P(Z>0) - E^2(Z)}{P^2(Z>0)}$

$Z \sim Poisson(\lambda)$ $\quad f(Z=x) = \dfrac{e^{-\lambda}\lambda^k}{k!}$ $\quad x = 0, 1, 2, \cdots$

$P(Z_T=x) = \dfrac{P(Z=x)}{P(Z>0)}$ $\quad x = 1, 2, \cdots$

$P(Z>0) = \sum_{x=1}^{\infty} f(Z=x) = \sum_{x=1}^{\infty} \dfrac{e^{-\lambda}\cdot\lambda^x}{x!}$

$= e^{-\lambda}\cdot \sum_{x=1}^{\infty} \dfrac{\lambda^x}{x!} = e^{-\lambda}(e^{\lambda}-1)$

$E(Z) = \sum k \dfrac{e^{-\lambda}\cdot\lambda^k}{k!} = e^{-\lambda}\sum \dfrac{\lambda^k}{(k-1)!}$

$= \lambda\cdot e^{-\lambda}\cdot \sum \dfrac{\lambda^{k-1}}{(k-1)!} = \lambda\cdot e^{-\lambda}\cdot e^{\lambda} = \lambda$

$\therefore E(P(Z_T=x)) = \dfrac{\lambda}{e^{\lambda}(e^{\lambda}-1)} = \dfrac{\lambda}{1-e^{-\lambda}}$

$E(Z^2) = \sum x^2 \cdot \dfrac{e^{-\lambda}\cdot\lambda^x}{x!} = \lambda e^{-\lambda}\sum \dfrac{x\cdot\lambda^{x-1}}{(x-1)!}$

$= \lambda e^{-\lambda}\cdot\sum\left[\dfrac{(x-1)\cdot\lambda^{x-1}}{(x-1)!} + \dfrac{\lambda^{x-1}}{(x-1)!}\right]$

$= \lambda^2 e^{-\lambda}\sum_{x=2}^{\infty} \dfrac{(x-2)\cdot\lambda^{x-2}}{(x-2)!} + \lambda\cdot e^{-\lambda}\cdot\sum \dfrac{\lambda^{x-1}}{(x-1)!}$

$= \lambda^2 e^{-\lambda}\cdot e^{\lambda} + \lambda\cdot e^{-\lambda}\cdot e^{\lambda} = \lambda^2+\lambda$

$\therefore Var(P(Z_T=x)) = \dfrac{(\lambda^2+\lambda)(1-e^{-\lambda}) - \lambda^2}{(1-e^{-\lambda})^2}$

Proof 1: $\because \lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ (左右同取对数 + 洛必达)

$(1+\frac{x}{n})^n = 1 + n\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + \cdots$

$= 1 + x + (1-\frac{1}{n})\cdot\frac{x^2}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\cdot\frac{x^3}{3!} + \cdots$

$= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

Proof 2: $f(x) = e^x$ $\quad f'(x) = e^x = f(x)$ $\quad f(0) = e^0 = 1 = f(y=0)$

$f(y) = \sum_{n=0}^{\infty}\frac{y^n}{n!}$ $\quad f'(y) = \sum_{n=1}^{\infty} n\frac{y^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{y^{n-1}}{(n-1)!} = f(y)$ $\quad \begin{matrix} f'(y) \\ = f(y) \end{matrix}$

(b)

$Z \sim$ Negative binomial (r, p)

$P(Z=x) = C_{r+x-1}^{x}\cdot p^r\cdot(1-p)^x = C_{r+x-1}^{x}\cdot p^r\cdot(1-p)^x$

$P(Z>0) = 1 - P(Z=0)$

$= 1 - C_{r-1}^{0}\cdot p^r\cdot(1-p)^0 = 1 - p^r$

$E(Z) = \sum_{x=0}^{\infty} x\cdot P(Z=x)$

$= \sum x\cdot C_{r+x-1}^{x}\, p^r(1-p)^x$

$= \sum_{x=0}^{\infty} x\cdot\dfrac{(r+x-1)!}{x!(r-1)!}\, p^r\cdot(1-p)^x$

$= \sum_{x=1}^{\infty}\dfrac{[r+(x-1)-1]!}{(x-1)!(r-1)!}\, p^r(1-p)^x$

$= \dfrac{r(1-p)}{p}\sum_{x=1}^{\infty}\dfrac{[(r+1)+(x-1)-1]!}{(x-1)!\;r!}\, p^{r+1}(1-p)^{x-1}$

$= r(1-p)/p$

$\sum_{x=1}^{\infty} C_{r+x-1}^{x-1}\, p^{r+1}(1-p)^{x-1} = 1 \equiv \sum_{x=0}^{\infty}\cdots$

$\therefore E(P(Z_T=x)) = \dfrac{E(Z)}{P(Z>0)} = \dfrac{r(1-p)}{P(1-p)^r}$

$E(Z^2) = \sum_{x=0}^{\infty} x^2\, P(Z=x)$

$= \sum x^2\cdot\dfrac{(r+x-1)!}{x!(r-1)!}\cdot p^r(1-p)^x$

$= \sum x\dfrac{(r+x-1)!}{(x-1)!(r-1)!}\cdot p^r(1-p)^x$

$= \sum(x-1)\dfrac{(r+x-1)!}{(x-1)!(r-1)!}\cdot p^r(1-p)^x + \sum\dfrac{(r+x-1)!}{(x-1)!(r-1)!}\cdot p^r(1-p)^x$

$= \dfrac{(r+1)r(1-p)^2}{p^2}\sum_{x=2}^{\infty}\dfrac{[(r+2)+(x-2)-1]!}{(x-2)!(r+1)!}\cdot p^{r+2}(1-p)^{x-2} + \dfrac{r(1-p)}{p}$

$= \dfrac{r(r+1)(1-p)^2}{p^2} + \dfrac{r(1-p)}{p} = \dfrac{r^2(1-p)^2}{p^2} + \dfrac{r(1-p)}{p^2}$

$\therefore Var(P(Z_T=x)) = \dfrac{r^2(1-p)^2 + r(1-p)}{p^2(1-p^r)} - \dfrac{r^2(1-p)^2}{p^2(1-p^r)^2}$

3.14

(a). Aim: proof $\sum_{x=1}^{\infty} P(X=x)=1$

Proof: $\sum_{x=1}^{\infty} \frac{-(1-p)^x}{x\cdot\log p} = \frac{1}{\log p}\sum_{x=1}^{\infty}\frac{-(1-p)^x}{x} = \frac{1}{\ln p}\cdot \ln p = 1$

等比数列求和:

$S = a^1 + a^2 + \cdots + a^n$

$aS = a^2 + \cdots + a^n + a^{n+1}$

$\therefore (a-1)S = a^{n+1} - a^1$

$S = \frac{a^{n+1}-a^1}{a-1}$

Let $f(p) = \sum \frac{-(1-p)^x}{x}$

$\therefore f'(p) = \sum \frac{-1}{x}\cdot x(1-p)^{x-1}(-1) = \sum_{x=1}^{\infty}(1-p)^{x-1} = \sum_{x=0}^{\infty}(1-p)^x = \frac{1}{p}$

$(1-p)^0 + (1-p)^1 + \cdots + (1-p)^n = \frac{(1-p)^{n+1}-1}{(1-p)-1} = \frac{-1}{-p} = \frac{1}{p}$ when $|p|<1$

$\therefore \sum \frac{-(1-p)^x}{x} = \int_0^p \frac{1}{t}\,dt = \ln t\,|_0^p = \ln p$

(b). $E(X) = \sum x\cdot P(x) = \sum x\cdot\frac{-(1-p)^x}{x\cdot\log p} = \sum_{x=1}^{\infty}\frac{-(1-p)^x}{\log p} = \frac{1}{\log p}\sum -(1-p)^x$

Let $S = -(1-p)^1 - (1-p)^2 - (1-p)^3 - \cdots (1-p)^n$

$= -[(1-p)^1 + (1-p)^2 + \cdots + (1-p)^n]$

$(1-p)S = -[(1-p)^2 + \cdots + (1-p)^n + (1-p)^{n+1}]$

$(1-p-1)S = (1-p) - (1-p)^{n+1}$

$S = \frac{1-p-(1-p)^{n+1}}{-p}$ $\quad \underset{\lim n\to\infty}{S} = \frac{1-p}{-p}$

$\therefore E(X) = \frac{1-p}{-p\log p}$

$E(X^2) = \sum x^2\cdot P(x) = -\frac{1}{\log p}\sum x(1-p)^x$

Let $S = 1\cdot(1-p)^1 + 2(1-p)^2 + 3(1-p)^3 + \cdots + n(1-p)^n$

$(1-p)S = 1(1-p)^2 + 2(1-p)^3 + \cdots + (n-1)(1-p)^n + n(1-p)^{n+1}$

$(1-1+p)S = \underbrace{(1-p)^1 + (1-p)^2 + \cdots + (1-p)^n}_{} \underbrace{- n(1-p)^{n+1}}_{0}$

$= \frac{1-p}{p}$

$S = \frac{1-p}{p^2}$ $\quad \therefore E(X^2) = \frac{1-p}{-p^2\log p}$

$\therefore Var(X) = E(X-E(X))^2$

$= E(X^2) - E^2(X)$

$= \frac{1-p}{-p^2\log p} - \frac{(1-p)^2}{p^2(\log p)^2}$

$= -\frac{1-p}{p^2\log p}\left(1 + \frac{1-p}{\log p}\right)$

3.15.

Poisson : $f(x) = \dfrac{e^{-\lambda} \cdot \lambda^k}{k!}$

$$M_X(t) = E(e^{tX}) = \sum e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} = e^{-\lambda} \cdot \boxed{\sum \frac{e^{tx} \cdot \lambda^x}{x!}}$$

$$\sum \frac{e^{tx} \cdot \lambda^x}{x!} = \sum \frac{(\lambda \cdot e^t)^x}{x!} = e^{\lambda \cdot e^t}$$

$$\therefore M_X(t) = e^{-\lambda} \cdot e^{\lambda \cdot e^t} = e^{\lambda(e^t - 1)}$$

$r \to \infty \quad p \to 1 \quad r(1-p) \to \lambda$

Negative binomial $\quad f(x) = C^x_{r+x-1} \cdot p^r \cdot (1-p)^x$

$$M_X(t) = E(e^{tX}) = \sum e^{tx} \cdot f(x)$$

$$= \sum e^{tx} \cdot \frac{(r+x-1)!}{x!\,(r-1)!} \cdot p^r \cdot (1-p)^x$$

$$= p^r \sum C^x_{r+x-1} \left[e^t(1-p)\right]^x$$

$$= p^r \cdot \sum (-1)^x \cdot C^x_{-r} [\quad]^x$$

$$= p^r \cdot [\quad]+1]^{-r}$$

$$= p^r \cdot \frac{1}{[1 - e^t(1-p)]^r} = \left[\frac{p}{1 - e^t(1-p)}\right]^r$$

$$\binom{-r}{y} = \frac{(-r)(-r-1)(-r-2)\cdots(-r-y+1)}{y!}$$

$$= \frac{(-1)^y (r+y-1)(r+y)\cdots(r)}{y!} = (-1)^y \cdot \binom{r+y-1}{y}$$

$$\therefore (-1)^y \cdot \binom{-r}{y} = \binom{r+y-1}{y}$$

Newton's binomial theorem

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} \cdot y^2 + \cdots + \binom{n}{n} y^n$$

when $y = 1$

$$(x+1)^n = \binom{n}{0} x^n + \binom{n}{1} \cdot x^{n-1} + \binom{n}{2} x^{n-2} + \cdots + \binom{n}{n} \cdot x^0$$

$$\frac{p}{1 - e^t(1-p)} = 1 + \frac{1}{r} \underbrace{\frac{r(1-p)(e^t - 1)}{1 - e^t(1-p)}}_{\lambda(e^t-1)} = 1 + \frac{1}{r} \lambda (e^t - 1)$$

$$\therefore M_X(t) = \left[1 + \frac{\lambda}{r} \cdot (e^t - 1)\right]^r \overset{\lim_{r \to \infty}}{=} e^{\lambda(e^t - 1)} \qquad \square$$

3.16

(a) Gamma Function : $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$

Aim : $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

$$\Gamma(\alpha+1) = \int_0^\infty t^{\alpha+1-1} \cdot e^{-t}\, dt$$

$$= \int_0^\infty t^\alpha \cdot e^{-t}\, dt$$

let $u = t^\alpha \quad v' = e^{-t}\, dt$

$u' = \alpha \cdot t^{\alpha-1}\, dt \quad v = -e^{-t}$

$$(uv)' = (-t^\alpha e^{-t})' = t^\alpha \cdot e^{-t}\, dt - \alpha \cdot t^{\alpha-1} \cdot e^{-t}\, dt$$

$$\therefore \int_0^\infty t^\alpha \cdot e^{-t}\, dt = \int_0^\infty (t^\alpha \cdot e^{-t})'\, dt + \alpha \cdot \int_0^\infty t^{\alpha-1} \cdot e^{-t}\, dt$$

$$= t^\alpha e^{-t} \Big|_0^\infty + \alpha \cdot \Gamma(\alpha)$$

$$= \alpha \cdot \Gamma(\alpha) \qquad \square$$

(b) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$t^{-\frac{1}{2}} = \frac{1}{\sqrt{t}}$$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} \cdot e^{-t}\, dt \quad ①$$

Let $z = \sqrt{2t}$ $\quad t = \frac{z^2}{2}$ $\quad t^{-\frac{1}{2}} = \frac{\sqrt{2}}{z}$

换元消 t，只余 e^x

$$dt = \frac{1}{2} \cdot 2z\, dz = z\, dz$$

$$\therefore \Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{\sqrt{2}}{z} \cdot e^{-\frac{z^2}{2}} z\, dz$$

$$= \sqrt{2} \int_0^\infty e^{-z^2/2}\, dz = \sqrt{\pi}$$

Let $I = \int_0^\infty e^{-z^2/2}\, dz = \frac{\sqrt{\pi}}{\sqrt{2}}$

$$I^2 = \int_0^\infty \int_0^\infty e^{-z^2/2 - s^2/2}\, dz\, ds = \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}(r^2)}\, r\, d\theta\, dr$$

$$= \frac{\pi}{2} \int_0^\infty r \cdot e^{-\frac{1}{2}r^2}\, dr = \frac{\pi}{2}$$

这样换元也可以

$t = z^2 \quad t^{-\frac{1}{2}} = \frac{1}{z}$

$dt = dz^2 = 2\, dz$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{z} \cdot e^{-z^2} 2z\, dz$$

$$= 2 \underbrace{\int_0^\infty e^{-z^2}\, dz}_{I}$$

$$I^2 = \frac{\pi}{2} \int_0^\infty e^{-r^2} r\, dr = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$

3.17.

$Z \sim \text{gamma}(\alpha, \beta)$

$$f(x) = \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

Aim: $E Z^v = \frac{\beta^v \Gamma(v+\alpha)}{\Gamma(\alpha)}$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx$$

$$E Z^v = \int x^v \cdot f(x) = \int \frac{x^{\alpha-1+v} \cdot e^{-x/\beta}}{\Gamma(\alpha) \cdot \beta^\alpha}\, dx$$

$$= \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \int x^{\alpha+v-1} \cdot e^{-x/\beta}\, dx$$

$$\because \int x^{\alpha+v-1} \cdot e^{-x/\beta}\, dx$$

Let $x = \beta y$，原式 $= \int_0^\infty y^{\alpha+v-1} \cdot \beta^{\alpha+v-1} \cdot e^{-y} \beta\, dy$

$$= \beta^{\alpha+v} \cdot \Gamma(\alpha+v)$$

$$\therefore E Z^v = \frac{\beta^v \Gamma(\alpha+v)}{\Gamma(\alpha)}$$

3.18

$Y \sim P(r, p) \quad P(y \mid r, p) = C^y_{y+r-1} \cdot p^r \cdot (1-p)^y$

$$M_Y(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$$

$$M_{pY}(t) = \left(\frac{p}{1-(1-p)e^{pt}}\right)^r$$

$\lim_{p \to 0} M_{pY}(t) \quad \because \frac{0}{0}$ 且可导 $\therefore$ 洛必达法则

$$\lim_{p \to 0} M_{pY}(t) = \frac{1}{-(1-p) \cdot e^{pt} \cdot t + e^{pt}}$$

$$= \left(\frac{1}{-t+1}\right)^r$$

$Y \sim \text{gamma}(r, 1)$

$$P(y \mid r, 1) = \frac{y^{r-1} \cdot e^{-y}}{\Gamma(r)}$$

$$M_Y(t) = \int_{y=0}^\infty e^{ty} \cdot \frac{y^{r-1} \cdot e^{-y}}{\Gamma(r)}\, dy$$

$$= \frac{1}{\Gamma(r)} \int_0^\infty e^{y(t-1)} \cdot y^{r-1}\, dy$$

Let $y(1-t) = x$

$$y = \frac{x}{1-t}$$

$$\int_0^\infty e^{y(t-1)} \cdot y^{r-1}\, dy = \int_0^\infty e^{-x} \cdot \left(\frac{x}{1-t}\right)^{r-1} \cdot \frac{1}{1-t}\, dx$$

$$= (1-t)^{-r} \cdot \Gamma(r)$$

$$\therefore M_Y(t) = (1-t)^{-r}$$

换元

$\Gamma(r) = \int_0^\infty x^{r-1} \cdot e^{-x}\, dx$

凑 $\Gamma(\alpha)$

3.19 $P(Z \geq x)$

Aim: $\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} \cdot e^{-z} dz = \sum_{y=0}^{\alpha-1} \frac{x^y \cdot e^{-x}}{y!}$ $P(Y \leq \alpha-1)$

gamma$(\alpha,1)$ Poisson

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x} dx = (\alpha-1)!$

$\sum_{y=0}^{\alpha-1} \frac{x^y \cdot e^{-x}}{y!}$

$\frac{1}{\Gamma(\alpha)} \cdot \underbrace{\int_x^\infty z^{\alpha-1} \cdot e^{-z} dz}_{R}$

$= e^{-x} \cdot \underbrace{\sum_{y=0}^{\alpha-1} \frac{x^y}{y!}}_{S} = e^{-x}\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \cdots + \frac{x^{\alpha-1}}{(\alpha-1)!}\right)$

Let $u = z^{\alpha-1}$ $v' = e^{-z} dz$

$u' = (\alpha-1) z^{\alpha-2} dz$ $v = -e^{-z}$

$(uv)' = (-e^{-z} z^{\alpha-1})'$

$= z^{\alpha-1} e^{-z} dz - (\alpha-1) z^{\alpha-2} \cdot e^{-z} dz$

$\therefore \int z^{\alpha-1} e^{-z} dz = -e^{-z} \cdot z^{\alpha-1} \Big|_x^\infty + \int (\alpha-1) z^{\alpha-2} \cdot e^{-z} dz$

$= -(0 - e^{-x} x^{\alpha-1}) + (\alpha-1) \int z^{\alpha-2} e^{-z} dz$

$= e^{-x} \cdot x^{\alpha-1} + (\alpha-1) \int z^{\alpha-2} e^{-z} dz$

$\therefore \int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} \cdot e^{-z} dz$

$= \frac{e^{-x} x^{\alpha-1}}{(\alpha-1)!} + \int \frac{1}{\Gamma(\alpha-1)} z^{\alpha-2} \cdot e^{-z} dz$

$\frac{e^{-x} \cdot x^{\alpha-2}}{(\alpha-2)!} + \int \frac{1}{\Gamma(\alpha-2)} z^{\alpha-3} e^{-z} dz$

$\therefore$ 原式 $= \frac{e^{-x} x^{\alpha-1}}{(\alpha-1)!} + \frac{e^{-x} \cdot x^{\alpha-2}}{(\alpha-2)!} + \cdots + \frac{e^{-x} x^{\alpha-\alpha}}{(\alpha-\alpha)!} + \frac{1}{\Gamma(0)} \int e^{-z} dz$

$= e^{-x}\left(\frac{x^{\alpha-1}}{(\alpha-1)!} + \frac{x^{\alpha-2}}{(\alpha-2)!} + \cdots + \frac{x^0}{0!}\right)$ $\square$

3.20 $f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$ $0 < x < \infty$

(a) $E(Z) = \int_0^\infty x \cdot f(x) dx$

$= \frac{2}{\sqrt{2\pi}} \underbrace{\int_0^\infty x \cdot e^{-x^2/2} dx}_{S}$

$S = -e^{-x^2/2} \Big|_0^\infty$

$= -(0-1)$

$= 1$

$\therefore E(Z) = \frac{2}{\sqrt{2\pi}}$

$E(Z^2) = \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty x^2 \cdot e^{-x^2/2} dx$

Let $u = -x$ $v' = -x \cdot e^{-x^2/2} dx$

$u' = +dx$ $v = e^{-x^2/2}$

$\therefore (-x e^{-x^2/2})' = x^2 \cdot e^{-x^2/2} dx - e^{-x^2/2} dx$

$\therefore \int x^2 e^{-x^2/2} dx = -x \cdot e^{-x^2/2} \Big|_0^\infty + \int e^{-x^2/2} dx = \frac{\sqrt{2\pi}}{2}$

$\therefore E(Z^2) = \frac{2}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{\sqrt{2}} = 1$

$Var(Z) = E(Z^2) - E^2(Z)$

$= 1 - \frac{4}{2\pi}$

$= 1 - \frac{2}{\pi}$

(b) gamma dis: $f(x) = \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)}$

$E(Z) = \int_0^\infty x \cdot f(x) dx$

$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^\alpha \cdot e^{-x/\beta} dx$

Let $y = x/\beta$ $x = \beta y$ $dx = \beta dy$

$\therefore \int_0^\infty x^\alpha \cdot e^{-x/\beta} dx = \int_0^\infty \beta^\alpha \cdot y^\alpha \cdot e^{-y} \beta dy$

$= \beta^{\alpha+1} \Gamma(\alpha+1)$

$\therefore E(Z) = \alpha\beta$

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$

$E(Z^2) = \frac{1}{\beta^\alpha \cdot \Gamma(\alpha)} \cdot \int_0^\infty x^{\alpha+1} \cdot e^{-x/\beta} dx$

$= \beta^{\alpha+2} \cdot \Gamma(\alpha+2) \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)}$

$= \frac{\beta^2 \cdot (\alpha+1)\alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)}$

$= \beta^2 (\alpha+1)\alpha$

$Var(Z) = E(Z^2) - E^2(Z)$

$= \beta^2(\alpha+1)\alpha - \alpha^2\beta^2$

$= \beta^2\alpha$

Let $\begin{cases} \beta^2\alpha = 1 - \frac{2}{\pi} \\ \beta\alpha = \frac{2}{\sqrt{2\pi}} \end{cases}$ $\therefore \begin{cases} \alpha = \frac{2}{\pi - 2} \\ \beta = \frac{\pi - 2}{\sqrt{2\pi}} \end{cases}$

3.21. $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$ (interate the mgf of $f(x)$)

$\int_0^\infty e^{tx} \cdot f(x)\, dx$

$= \frac{1}{\pi} \cdot \int_0^\infty e^{tx} \cdot \frac{1}{1+x^2}\, dx$

$\int_0^\infty \frac{e^{tx}}{1+x^2}\, dx > \int_0^\infty \frac{x}{1+x^2}\, dx = \infty$ $\qquad \int_0^\infty \frac{x}{1+x^2}\, dx = \frac{1}{2} \ln 1+x^2 \Big|_0^\infty = \infty$

$\therefore$ the mgf (moment generating function) of $f(x)$ doesn't exist.

3.22

(a) $X \sim$ Poisson (λ) $\quad E X(X-1) = E X^2 - E X$

Aim $\operatorname{Var} X$

$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$ $\qquad \operatorname{Var}(X) = E X^2 - (E X)^2$

$E(x) = \sum_{x=0}^{\infty} x \cdot P(x) = \sum \frac{e^{-\lambda} \cdot \lambda^x}{(x-1)!}$

$= e^{-\lambda} \cdot \lambda \sum \frac{\lambda^{x-1}}{(x-1)!}$

$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$

$= \lambda$

$\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots$

$f(x) = e^x \approx P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$f(0) = P(0) = a_0 + a_1 x + \cdots + a_n x^n$

$f'(0) = P'(0) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$

$\vdots$

$f^n(0) = P^n(0) = n!\, a_n$

$\because f(0) = f'(0) = f''(0) = \cdots = f^n(0) = 1$

$\therefore a_n = \frac{1}{n!}$

$P(x) = \frac{x^0}{0!} + \frac{x^1}{1!} + \cdots + \frac{x^n}{n!} = e^x$

$E(X(X-1)) = \sum X(X-1) \cdot P(X)$

$= \sum X(X-1) \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$= e^{-\lambda} \sum \frac{\lambda^x}{(x-2)!}$

$= e^{-\lambda} \cdot \lambda^2 \cdot \sum \frac{\lambda^{x-2}}{(x-2)!}$

$= e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda}$

$= \lambda^2 = E X^2 - E X$

$\therefore E X^2 = E(X(X-1)) + E X = \lambda^2 + \lambda$

$\therefore \operatorname{Var} X = E X^2 - (E X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

(b) $X \sim$ negative binomial (r, p)

$P(x) = C_{r+x-1}^{x}\, p^r \cdot (1-p)^x$

$E(X) = \sum_{x=0}^{\infty} x \cdot P(x) = \sum C_{r+x-1}^{x}\, x \cdot p^r (1-p)^x$

$= \sum \frac{(r+x-1)!}{x!\,(r-1)!}\, x \cdot p^r (1-p)^x$

$x \to x-1$

$r \to r+1$

$= r \frac{1-p}{p} \sum \frac{[(r+1)+(x-1)-1]!}{(x-1)!\,(r)!} \cdot p^{r+1} (1-p)^{x-1}$

$= \frac{r(1-p)}{p} \cdot \underbrace{C_{(r+1)+(x-1)-1}^{x-1} \cdot p^{r+1} (1-p)^{x-1}}_{1}$

$= \frac{r(1-p)}{p}$

$E(X(X-1)) = \sum C_{r+x-1}^{x}\, x(x-1)\, p^r (1-p)^x$

$x \to x-2$

$r \to r+2$

$= \sum \frac{(r+x-1)!}{x!\,(r-1)!}\, x(x-1) \cdot p^r (1-p)^x$

$= \frac{r(r-1)(1-p)^2}{p^2} \sum \frac{[(r+2)+(x-2)-1]!}{(x-2)!\,(r+1)!}\, p^{r+2} (1-p)^{x-2}$

$= \frac{r(r-1)(1-p)^2}{p^2} = E X^2 - E X$

$\operatorname{Var} X = E X^2 - (E X)^2$

$= E(X(X-1)) + E X - (E X)^2$

$= \frac{r(r-1)(1-p)^2}{p^2} + \frac{r(1-p)}{p} - \frac{r^2(1-p)^2}{p^2}$

$= \frac{r(1-p)^2(r-2)}{p^2} + \frac{r(1-p)}{p}$

(c) $X \sim \text{gamma}(\alpha, \beta)$

$$P(x) = \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)} \qquad \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} \cdot e^{-t}\,dt$$

$$E(X) = \int_0^{\infty} x \cdot \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^{\alpha} \cdot \Gamma(\alpha)}\,dx$$

$$= \frac{1}{\beta^{\alpha} \cdot \Gamma(\alpha)} \int_0^{\infty} x^{\alpha} \cdot e^{-x/\beta}\,dx$$

let $y = x/\beta \quad x = \beta y \quad dx = \beta\,dy$

$$\therefore \int_0^{\infty} (\beta y)^{\alpha} \cdot e^{-y}\,\beta\,dy$$

$$= \beta^{\alpha+1} \int_0^{\infty} y^{\alpha} e^{-y}\,dy$$

$$= \beta^{\alpha+1} \cdot \Gamma(\alpha+1)$$

$$\therefore E(X) = \frac{\beta^{\alpha+1}\,\alpha \cdot \Gamma(\alpha)}{\beta^{\alpha} \cdot \Gamma(\alpha)} = \beta\alpha$$

$$E(X^2) = \int_0^{\infty} x^2 \cdot \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}\,dx$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} \cdot e^{-x/\beta}\,dx$$

$$\int_0^{\infty} x^{\alpha+1} \cdot e^{-x/\beta}\,dx$$

let $y = x/\beta \quad x = \beta y \quad dx = \beta\,dy$

$$\int_0^{\infty} (\beta y)^{\alpha+1} \cdot e^{-y}\,\beta\,dy$$

$$= \beta^{\alpha+2} \int_0^{\infty} y^{\alpha+1} \cdot e^{-y}\,dy$$

$$= \beta^{\alpha+2} \cdot \Gamma(\alpha+2)$$

$$= \beta^{\alpha+2} \cdot (\alpha+1)\,\Gamma(\alpha+1)$$

$$= \beta^{\alpha+2}\,(\alpha+1)\,\alpha \cdot \Gamma(\alpha)$$

$$\therefore E(X^2) = \beta^2\,\alpha(\alpha+1)$$

$$\therefore \text{Var}\,X = E(X^2) - E^2 X$$

$$= \beta^2\alpha(\alpha+1) - \beta^2\alpha^2$$

$$= \beta^2\alpha$$

(d) $X \sim \text{beta}(\alpha, \beta)$

$$P(x) = \frac{1}{B(\alpha,\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1} \qquad x \in (0,1)$$

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$\therefore EX^n = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{n+\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{B(\alpha+n,\beta)}{B(\alpha,\beta)}$$

$$\because B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\therefore EX^n = \frac{\Gamma(\alpha+n) \cdot \cancel{\Gamma(\beta)}}{\Gamma(\alpha+n+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\cancel{\Gamma(\beta)}}$$

$$= \frac{\Gamma(\alpha+n) \cdot \Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\,\Gamma(\alpha)}$$

Let $n=1$

$$EX = \frac{\Gamma(\alpha+1) \cdot \Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1) \cdot \Gamma(\alpha)}$$

$$= \frac{\alpha \cdot \Gamma(\alpha) \cdot \Gamma(\alpha+\beta)}{(\alpha+\beta)\,\Gamma(\alpha+\beta) \cdot \Gamma(\alpha)} = \frac{\alpha}{\alpha+\beta}$$

let $n=2$

$$EX^2 = \frac{\Gamma(\alpha+2) \cdot \Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\alpha+\beta+2)}$$

$$= \frac{(\alpha+1) \cdot \alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} \cdot \frac{\cancel{\Gamma(\alpha+\beta)}}{(\alpha+\beta+1)(\alpha+\beta) \cdot \cancel{\Gamma(\alpha+\beta)}}$$

$$= \frac{(\alpha+1)\,\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\therefore \text{Var}\,X = EX^2 - (EX)^2$$

$$= \frac{(\alpha+1)\,\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\frac{\alpha}{\alpha+\beta}\left(\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right)$$

$$\frac{\cancel{\alpha^2 + \beta\alpha + \alpha + \beta - \alpha^2 - \alpha\beta - \alpha}}{(\quad)(\quad)}$$

(e) $X \sim$ double exponential (μ, σ)

$$P(x) = \frac{1}{2\sigma} \cdot e^{-|x-\mu|/\sigma} \qquad x \in (-\infty, \infty) \quad \mu \in (-\infty, +\infty)$$
$$\sigma > 0$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot P(x)\, dx$$
$$= \int_{-\infty}^{\mu} \frac{x}{2\sigma} \cdot e^{-(\mu-x)/\sigma}\, dx + \int_{\mu}^{+\infty} \frac{x}{2\sigma} \cdot e^{-(x-\mu)/\sigma}\, dx$$
$$= \frac{1}{2\sigma}\Big[\underbrace{\int_{-\infty}^{0} x \cdot e^{x/\sigma}\, dx + \mu \int_{-\infty}^{0} e^{x/\sigma}\, dx}_{x \to x+\mu} + \int_{0}^{+\infty} x \cdot e^{-x/\sigma}\, dx + \mu \cdot \int_{0}^{+\infty} e^{-x/\sigma}\, dx\Big]$$
$$\because \int_{-\infty}^{0} x \cdot e^{x/\sigma} + \int_{0}^{+\infty} x \cdot e^{-x/\sigma} = 0$$
$$\therefore E(X) = \frac{1}{2\sigma} \mu \Big[\int_{-\infty}^{0} e^{x/\sigma}\, dx + \int_{0}^{\infty} e^{-x/\sigma}\, dx\Big]$$
$$= \frac{\mu}{2\sigma} \cdot \Big(\sigma e^{x/\sigma}\Big|_{-\infty}^{0} - \sigma e^{-x/\sigma}\Big|_{0}^{\infty}\Big)$$
$$= \frac{\mu}{2\sigma} \cdot \big[\sigma - (0 - \sigma)\big]$$
$$= \mu$$

$$E(X^2) = \int_{-\infty}^{\mu} \frac{x^2}{2\sigma} \cdot e^{-(\mu-x)/\sigma}\, dx + \int_{\mu}^{+\infty} \frac{x^2}{2\sigma} \cdot e^{-(x-\mu)/\sigma}\, dx$$
$$= \frac{1}{2\sigma}\Big[\int_{-\infty}^{0} (x+\mu)^2 \cdot e^{x/\sigma}\, dx + \int_{0}^{\infty} (x+\mu)^2 \cdot e^{-x/\sigma}\, dx\Big]$$
$$= \frac{1}{2\sigma}\Big[\int_{-\infty}^{0} x^2 \cdot e^{x/\sigma}\, dx + \cancel{2\mu \int_{-\infty}^{0} x\, e^{x/\sigma}\, dx} + \mu^2 \int_{-\infty}^{0} e^{x/\sigma}\, dx$$
$$+ \int_{0}^{\infty} x^2 \cdot e^{-x/\sigma}\, dx + \cancel{2\mu \int_{0}^{\infty} x \cdot e^{-x/\sigma}\, dx} + \mu^2 \int_{0}^{\infty} e^{-x/\sigma}\, dx\Big]$$
$$= \frac{1}{\cancel{2}\sigma}\Big[\cancel{2} \cdot \underbrace{\int_{0}^{\infty} x^2 \cdot e^{-x/\sigma}\, dx}_{} + \mu^2 \cdot \cancel{2} \cdot \underbrace{\int_{0}^{\infty} e^{-x/\sigma}\, dx}_{\sigma}\Big]$$
$$= \frac{1}{\sigma}\Big[2\sigma^3 + \mu^2 \cdot \sigma\Big] = 2\sigma^2 + \mu^2$$

$$\therefore Var(X) = E(X^2) - E^2(X)$$
$$= 2\sigma^2 + \mu^2 - \mu^2$$
$$= 2\sigma^2$$

$$\int_{0}^{\infty} x^2 \cdot e^{-x/\sigma}\, dx$$

Let $u = x^2$ $\qquad v' = e^{-x/\sigma}\, dx$

$u' = 2x\, dx$ $\qquad v = -\sigma \cdot e^{-x/\sigma}$

$$\therefore \left(-\sigma x^2 e^{-x/\sigma}\right)' = x^2 \cdot e^{-x/\sigma} dx - 2\sigma x\, e^{-x/\sigma} dx$$
$$\therefore \int x^2 \cdot e^{-x/\sigma} dx = -\sigma \cdot x^2 \cdot e^{-x/\sigma}\Big|_{0}^{\infty} + 2\sigma \int x \cdot e^{-x/\sigma}\, dx$$
$$= 0 + 2\sigma \cdot \sigma^2 = 2\sigma^3$$

Let $u = x$ $\qquad v' = e^{-x/\sigma}\, dx$

$u' = 1\, dx$ $\qquad v = -\sigma \cdot e^{-x/\sigma}$

$$\left(-\sigma x \cdot e^{-x/\sigma}\right)' = x \cdot e^{-x/\sigma} dx - \sigma \cdot e^{-x/\sigma} dx$$
$$\therefore \int x \cdot e^{-x/\sigma}\, dx = \underbrace{-\sigma x \cdot e^{-x/\sigma}\Big|_{0}^{\infty}}_{0} + \underbrace{\sigma \int e^{-x/\sigma} dx}_{} = \sigma^2$$

3.23 $f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}$ $x \in (\alpha, +\infty)$ $\alpha > 0$ $\beta > 0$

(a) $f(x)$ is a pdf

$$\int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}}\,dx = \beta\alpha^\beta \cdot \int_\alpha^\infty x^{-(\beta+1)}\,dx$$
$$= \beta\alpha^\beta \cdot \left(\frac{1}{-\beta}\right) x^{-\beta} \Big|_\alpha^\infty$$
$$= -\alpha^\beta \cdot (0 - \alpha^{-\beta})$$
$$= 1$$

$\therefore$ $f(x)$ is a pdf.

(b) Mean & Var

$$E(X) = \int_\alpha^\infty x \cdot \frac{\beta\alpha^\beta}{x^{\beta+1}}\,dx$$
$$= \beta \cdot \alpha^\beta \int_\alpha^\infty x^{-\beta}\,dx$$
$$= \beta \cdot \alpha^\beta \left(\frac{1}{1-\beta}\right) x^{-\beta+1} \Big|_\alpha^\infty$$
$$= \frac{\beta \cdot \alpha^\beta}{1-\beta} \cdot (0 - \alpha^{1-\beta})$$
$$= \frac{\beta\alpha}{\beta-1}$$

$$E(X^2) = \int_\alpha^\infty x^2 \cdot \frac{\beta\alpha^\beta}{x^{\beta+1}}\,dx$$
$$= \beta\alpha^\beta \cdot \int_\alpha^\infty x^{1-\beta}\,dx$$
$$= \beta \cdot \alpha^\beta \cdot \frac{1}{2-\beta} x^{2-\beta} \Big|_\alpha^\infty$$
$$= \frac{\beta \cdot \alpha^\beta}{2-\beta} \cdot (0 - \alpha^{2-\beta})$$
$$= \frac{\beta \cdot \alpha^2}{\beta-2}$$

$$Var(X) = E(X^2) - E^2(X)$$
$$= \frac{\beta \cdot \alpha^2}{\beta-2} - \frac{\beta^2\alpha^2}{(\beta-1)^2}$$
$$= \beta\alpha^2\left(\frac{1}{\beta-2} - \frac{\beta}{(\beta-1)^2}\right)$$

(c) $\beta \le 2$ $Var(X)$ × exist

If $\beta \le 2$ $E(X^2)$ doesn't exist

3.24 (a) $X \sim$ exponential(β) $Y = X^{1/r}$ $Y \sim$ Weibull (r, β)

? Pdf , mean , var

$$f(x, \beta) = \frac{1}{\beta} \cdot e^{-x/\beta}$$

$$f(Y, \beta, r) = \frac{1}{\beta} \cdot e^{-y^r/\beta} \cdot r \cdot y^{r-1}$$
$$= \frac{r}{\beta} \cdot e^{-y^r/\beta} \cdot y^{r-1}$$

let $y^r/\beta = z$ $y = (\beta z)^{1/r}$

$$\therefore f(Y, \beta, r) = \frac{r}{\beta} \cdot e^{-z} \cdot (\beta z)^{(1-\frac{1}{r})}$$
$$= r \cdot \beta^{-\frac{1}{r}} \cdot e^{-z} z^{(1-\frac{1}{r})}$$

$$dy = d(\beta z)^{1/r} = \frac{\beta}{r}(\beta z)^{1/r - 1}\,dz$$

$$E(Y) = \int y \cdot f(Y, \beta, r)\,dy$$
$$= \int (\beta z)^{1/r} \cdot r \cdot \beta^{-\frac{1}{r}} \cdot e^{-z} \cdot z^{(1-\frac{1}{r})} \cdot \frac{\beta}{r}(\beta z)^{\frac{1}{r}-1}\,dz$$
$$= \int \cancel{\beta^{\frac{1}{r}}} \cdot \cancel{z^{\frac{1}{r}}} \cdot \cancel{r} \cdot \cancel{\beta^{-\frac{1}{r}}} \cdot e^{-z} \cdot z^{1-\cancel{\frac{1}{r}}} \cdot \beta^{\frac{1}{r}} \cdot \cancel{\frac{1}{r}} \cdot z^{\frac{1}{r}-1}\,dz$$
$$= \beta^{\frac{1}{r}} \int e^{-z} \cdot z^{\frac{1}{r}}\,dz$$
$$= \beta^{\frac{1}{r}} \cdot \Gamma\left(\frac{1}{r}+1\right)$$

$y = X^{1/r}$

$X = y^r$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\,dx$$
$$\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$$
$$= \int_0^\infty x^\alpha \cdot e^{-x}\,dx$$

$$E(Y^2) = \int y^2 \cdot f(Y, \beta, r)\,dy$$
$$= \int \beta^{2/r} \cdot z^{2/r} \cdot \cdots$$
$$= \beta^{2/r} \cdot \int e^{-z} \cdot z^{\frac{2}{r}}\,dz$$
$$= \beta^{2/r}\,\Gamma\left(\frac{2}{r}+1\right)$$

$$\therefore Var(Y) = E(Y^2) - E^2(Y)$$
$$= \beta^{2/r}\,\Gamma\left(\frac{2}{r}+1\right) - \left[\beta^{1/r}\,\Gamma\left(\frac{1}{r}+1\right)\right]^2$$

(b) $X \sim \text{exponential}(\beta)$ $\quad Y = (2X/\beta)^{1/2}$ $\quad Y \sim$ Rayleigh dis.

$P(x) = \frac{1}{\beta} \cdot e^{-x/\beta}$ $\qquad y = (2x/\beta)^{\frac{1}{2}} \Rightarrow \frac{\beta}{2} y^2 = x$

$P(Y) = \frac{1}{\beta} \cdot e^{-\frac{y^2}{2}} \cdot \beta y = y \cdot e^{-\frac{y^2}{2}}$ $\qquad$ P79

$E(Y) = \int_0^\infty y\, f(Y)\, dy$

$= \int y^2 \cdot e^{-y^2/2}\, dy = \int_0^\infty y^2 e^{-y^2/2}\, dy = \frac{\sqrt{\pi}}{\sqrt{2}}$

Let $y^2/2 = z$ $\qquad y = (2z)^{\frac{1}{2}}$

$dy = \frac{1}{2}(2z)^{-\frac{1}{2}} = (2z)^{-\frac{1}{2}}\, dz$ $\qquad z \in (0, +\infty)$

$E(Y) = \int_0^\infty 2z\, e^{-z} \cdot (2z)^{-\frac{1}{2}}\, dz$

$= \sqrt{2} \int_0^\infty z^{\frac{1}{2}} \cdot e^{-z}\, dz$

$= \sqrt{2}\ \Gamma(\frac{3}{2})$

$\Gamma(d) = \int_0^\infty t^{d-1} e^{-t}\, dt$

$\therefore \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$

Let $u = y$ $\qquad v' = y \cdot e^{-y^2/2}\, dy$

$u' = dy$ $\qquad v = -e^{-y^2/2}$

$\therefore (-y \cdot e^{-y^2/2})' = y^2 \cdot e^{-y^2/2}\, dy - e^{-y^2/2}\, dy$

$\therefore \underbrace{-y \cdot e^{-y^2/2}\Big|_0^\infty}_{0} + \int_0^\infty e^{-y^2/2}\, dy = \int_0^\infty y^2 \cdot e^{-y^2/2}\, dy$

$\int_0^\infty \int_0^\infty e^{-x^2/2 - y^2/2}\, dy\, dx$

$\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2/2}\, r\, d\theta\, dr$

$= \frac{\pi}{2}\left(-e^{-r^2/2}\Big|_0^\infty\right) = \frac{\pi}{2}$

$\frac{\sqrt{\pi}}{\sqrt{2}}$

$E(Y^2) = \int_0^\infty y^3 \cdot e^{-y^2/2}\, dy$

$= \int_0^\infty (2z)^{\frac{3}{2}} \cdot e^{-z} (2z)^{-\frac{1}{2}}\, dz$

$= 2\int_0^\infty z \cdot e^{-z}\, dz$

$= 2 \cdot \Gamma(2)$

$= 2$

$\therefore Var(Y) = E(Y^2) - E^2(Y)$

$= 2 - 2\,\Gamma^2(\frac{3}{2})$

$= 2 - \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)^2$

$= 2 - \frac{\pi}{2}$

(c) $X \sim \text{gamma}(a, b)$ $\qquad Y = 1/X$ $\qquad Y \sim$ Inverted $IG(a, b)$

$P(x) = \frac{x^{a-1} \cdot e^{-x/b}}{b^a\, \Gamma(a)}$ $\qquad Y = \frac{1}{X}$ $\quad \therefore X = (Y)^{-1}$

$\therefore f(y) = \frac{y^{1-a} \cdot e^{-(yb)^{-1}}}{b^a\, \Gamma(a)} \cdot \left|(-1) \cdot y^{-2}\right| = \frac{y^{-a-1} \cdot e^{-(yb)^{-1}}}{b^a\, \Gamma(a)}$

$(a-1)\,\Gamma(a-1) = \Gamma(a)$

$a\ \Gamma(a) = \Gamma(a+1)$

$E(Y) = \int_0^\infty y \cdot f(y)\, dy$

$= \frac{1}{b^a \Gamma(a)} \int_0^\infty y^{-a} \cdot e^{-\frac{1}{yb}}\, dy = \frac{b^{a-1}\Gamma(a-1)}{b^a\, \Gamma(a)} = \frac{\Gamma(a-1)}{b(a-1)\,\Gamma(a-1)} = \frac{1}{b(a-1)}$

Let $\frac{1}{yb} = z$ $\qquad y = (zb)^{-1}$ $\qquad dy = -b\,(zb)^{-2}$

$\int_\infty^0 y^{-a} \cdot e^{-\frac{1}{yb}}\, dy = \int_\infty^0 (zb)^a \cdot e^{-z} \cdot (-b) \cdot (zb)^{-2}\, dz$

$= -b^{a-1} \int_\infty^0 z^{a-2} \cdot e^{-z}\, dz$

$= b^{a-1} \int_0^\infty z^{a-2} \cdot e^{-z}\, dz$

$= b^{a-1} \cdot \Gamma(a-1)$

$E(Y^2) = \int_0^\infty y^2 f(y)\, dy$

$= \frac{1}{b^a \Gamma(a)} \int_0^\infty y^{1-a} \cdot e^{-\frac{1}{yb}}\, dy = \frac{b^{a-2} \cdot \Gamma(a-2)}{b^a (a-1)(a-2)\,\Gamma(a-2)} = \frac{1}{b^2 (a-1)(a-2)}$

$\int_\infty^0 y^{1-a} \cdot e^{-\frac{1}{yb}}\, dy = \int_\infty^0 (zb)^{a-1} \cdot e^{-z} \cdot (-b)(zb)^{-2}\, dz = b^{a-2} \int_0^\infty z^{a-3} e^{-z}\, dz = b^{a-2} \cdot \Gamma(a-2)$

$\therefore Var(Y) = E(Y^2) - E^2(Y)$

$= \frac{1}{b^2(a-1)(a-2)} - \frac{1}{b^2(a-1)^2}$

$= \frac{1}{b^2(a-1)} \left(\frac{1}{a-2} - \frac{1}{a-1}\right)$

$= \frac{1}{b^2(a-1)} \cdot \frac{1}{(a-2)(a-1)}$

$= \frac{1}{b^2 (a-2)(a-1)^2}$

(d) $X \sim \text{gamma}\left(\frac{3}{2}, \beta\right)$, $Y = (X/\beta)^{\frac{1}{2}}$ $Y \sim$ Maxwell dis

$$f(x) = \frac{x^{\frac{1}{2}} \cdot e^{-x/\beta}}{\beta^{\frac{3}{2}}\,\Gamma(\alpha)} \qquad Y = (X/\beta)^{\frac{1}{2}} \qquad X = \beta \cdot Y^2$$

$$f(y) = \frac{\beta^{\frac{1}{2}} y \cdot e^{-Y^2}}{\beta^{\frac{3}{2}}\,\Gamma(\alpha)} \cdot 2\beta Y = \frac{2}{\Gamma(\alpha)} \cdot y^2 \cdot e^{-Y^2}$$

$$E(Y) = \int_0^\infty y \cdot f(y)\,dy$$

$$= \frac{2}{\Gamma(\alpha)} \int_0^\infty y^3 \cdot e^{-y^2}\,dy = \frac{1}{\Gamma(\alpha)}$$

$$\int_0^\infty y^3 \cdot e^{-y^2}\,dy \qquad = \frac{1}{\Gamma(\frac{3}{2})} = \frac{2}{\sqrt{\pi}}$$

Let $u = -y^2$ $\quad v' = -y \cdot e^{-y^2}\,dy$

$u' = -2y\,dy$ $\quad v = \frac{1}{2} e^{-y^2}$

$$(uv)' = \left(-\frac{1}{2} y^2 \cdot e^{-y^2}\right)'$$

$$= y^3 \cdot e^{-y^2}\,dy - y \cdot e^{-y^2}\,dy$$

$$\therefore \int y^3 \cdot e^{-y^2}\,dy = \underbrace{-\frac{1}{2} y^2 \cdot e^{-y^2}\Big|_0^\infty}_{0} + \int y \cdot e^{-y^2}\,dy$$

$$= -\frac{1}{2} e^{-y^2}\Big|_0^\infty$$

$$= -\frac{1}{2}(0 - 1)$$

$$= \frac{1}{2}$$

$$E(Y^2) = \int_0^\infty y^2 f(y)\,dy$$

$$= \frac{2}{\Gamma(\alpha)} \int_0^\infty y^4 \cdot e^{-y^2}\,dy$$

Let $y^2 = z$ $\quad y = z^{\frac{1}{2}}$ $\quad dy = \frac{1}{2} z^{-\frac{1}{2}}\,dz$

$$\therefore \int_0^\infty y^4 \cdot e^{-y^2}\,dy$$

$$= \frac{1}{2} \int_0^\infty z^2 \cdot e^{-z} \cdot z^{-\frac{1}{2}}\,dz$$

$$= \frac{1}{2} \int_0^\infty z^{\frac{3}{2}} e^{-z}\,dz$$

$$= \Gamma\left(\frac{5}{2}\right) \cdot \frac{1}{2}$$

$$\because \Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$$

$$\therefore \Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2} - 1\right) \cdot \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right)$$

$$\therefore E(Y^2) = \frac{2}{\Gamma(\frac{3}{2})} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right) \cdot \frac{1}{2} = \frac{3}{2}$$

$$\therefore Var(Y) = E(Y^2) - E^2(Y)$$

$$= \frac{3}{2} - \frac{4}{\pi}$$

(e) $X \sim \text{exponential}(1)$, $Y = \alpha - \gamma \log X$ $\quad Y \sim \text{Gumbel}(\alpha, \gamma)$

$\alpha \in (-\infty, \infty)$ $\quad \gamma > 0$

$$P(x) = \frac{1}{\beta} \cdot e^{-x/\beta} = e^{-x} \qquad Y = \alpha - \gamma \log X$$

$$\therefore f(y) = e^{-e^{\frac{\alpha - y}{\gamma}}} \cdot \frac{1}{\gamma} e^{\frac{\alpha - y}{\gamma}} \qquad X = e^{\frac{\alpha - y}{\gamma}}$$

$$= \frac{1}{\gamma} \cdot e^{\frac{\alpha - y}{\gamma}\left(-e^{\frac{\alpha - y}{\gamma}}\right)}$$

Let $I_1 = \int_0^\infty \log x \cdot e^{-x}\,dx$ $\qquad \left[E(\log(x)) = \int \log(x) \cdot f(x)\,dx\right]$ - ※

$$E(Y) = \int_0^\infty y\,f(y)\,dy = E(\alpha - \gamma \log X) = \alpha - \gamma E(\log X) = \alpha - \gamma I_1$$

$$E(Y^2) = E(\alpha - \gamma \log X)^2 = \alpha^2 - 2\alpha\gamma E(\log X) + \gamma^2 E(\log^2 X)$$

Let $I_2 = \int_0^\infty (\log x)^2 \cdot e^{-x}\,dx$ $\qquad \square$

3.25 $h_T(t) = \lim_{\delta \to 0} \dfrac{P(t \le T < t+\delta \mid T \ge t)}{\delta}$

Aim: $h_T(t) = \dfrac{f_T(t)}{1 - F_T(t)} = -\dfrac{d}{dt} \log(1 - F_T(t))$

$P(t \le T < t+\delta \mid T \ge t) = \dfrac{F_T(t+\delta) - F_T(t)}{1 - F_T(t)}$

$\because \lim_{\delta \to 0} \dfrac{F_T(t+\delta) - F_T(t)}{\delta}$

$= f(t)$

$\therefore h_T(t) = \dfrac{f_T(t)}{1 - F_T(t)}$

$P(A \mid B) = \dfrac{P(AB)}{P(B)}$

$\dfrac{d \log(1 - F_T(t))}{dt} = \dfrac{1}{1 - F_T(t)} (-1) \cdot F_T'(t)$

$= -\dfrac{f_T(t)}{1 - F_T(t)}$

$\therefore h_T(t) = -\dfrac{d \log(1 - F_T(t))}{dt}$ $\square$

3.26.

(a) $T \sim$ exponential (β) Aim: $h_T(t) = \dfrac{1}{\beta}$

$p(t) = \dfrac{1}{\beta} \cdot e^{-t/\beta}$

$F_T(t) = \int_0^t p(x)\, dx$

$= \int_0^t \dfrac{1}{\beta} \cdot e^{-x/\beta}\, dx$

$= \dfrac{1}{\beta} (-\beta) e^{-x/\beta} \Big|_0^t$

$= 1 - e^{-t/\beta}$

$1 - F_T(t) = e^{-t/\beta}$

$\log(1 - F_T(t)) = -t/\beta$

$\therefore -\dfrac{d}{dt} \log(1 - F_T(t)) = +\dfrac{1}{\beta}$ $\square$

(b) $T \sim$ Weibull (γ, β) Aim $h_T(t) = (\gamma/\beta)\, t^{\gamma - 1}$

$f(y, \beta, \gamma) = \dfrac{1}{\beta} \cdot e^{-y^\gamma/\beta} \cdot \gamma \cdot y^{\gamma-1}$

$= \dfrac{\gamma}{\beta} \cdot e^{-y^\gamma/\beta} \cdot y^{\gamma-1}$

$F_T(t) = \int_0^t f(y, \beta, \gamma)\, dy$

$= \dfrac{\gamma}{\beta} \underline{\int_0^t y^{\gamma-1} \cdot e^{-y^\gamma/\beta}\, dy} = 1 - e^{-t^\gamma/\beta}$

Let $y^\gamma/\beta = z$ $\quad y = (z\beta)^{\frac{1}{\gamma}}$ $\quad dy = \dfrac{1}{\gamma}(z\beta)^{\frac{1}{\gamma}-1} \cdot \beta\, dz$

$y = 0 \quad z = 0$

$y = t \quad z = t^\gamma/\beta$

$= \int_0^{t^\gamma/\beta} (z\beta)^{\frac{\gamma-1}{\gamma}} \cdot e^{-z} \dfrac{\beta}{\gamma} (z\beta)^{\frac{1-\gamma}{\gamma}}\, dz$

$= \dfrac{\beta}{\gamma} \int_0^{t^\gamma/\beta} e^{-z}\, dz$

$= \dfrac{\beta}{\gamma} (-1) e^{-z} \Big|_0^{t^\gamma/\beta} = \dfrac{\beta}{\gamma} (-1) \left(e^{-t^\gamma/\beta} - 1 \right)$

$\therefore 1 - F_T(t) = e^{-t^\gamma/\beta}$

$\log(1 - F_T(t)) = -t^\gamma/\beta$

$-\dfrac{d}{dt} \log(1 - F_T(t)) = \dfrac{1}{\beta} \cdot \gamma \cdot t^{\gamma-1}$ $\square$

(c). $T \sim \text{logistic}(\mu, \beta)$ $\quad F_T(t) = \dfrac{1}{1+e^{-(t-\mu)/\beta}}$

Aim $h_T(t) = \frac{1}{\beta} F_T(t)$

$$1 - F_T(t) = \frac{e^{-(t-\mu)/\beta}}{1+e^{-(t-\mu)/\beta}}$$

$$\log(1-F_T(t)) = \log\frac{a}{b} = \log a - \log b$$

$$= -(t-\mu)/\beta - \log(1+e^{-(t-\mu)/\beta})$$

$$\frac{-d}{dt}\log(1-F_T(t)) = \frac{1}{\beta} + \frac{-\frac{1}{\beta}e^{-(t-\mu)/\beta}}{1+e^{-(t-\mu)/\beta}}$$

$$= \frac{\frac{1}{\beta} + \frac{1}{\beta}\cdot e^{\cdots} - \frac{1}{\beta}e^{\cdots}}{1+e^{\cdots}}$$

$$= \frac{1}{\beta}\cdot\frac{1}{1+e^{\cdots}}$$

$$= \frac{1}{\beta}\cdot F_T(t) \quad \square$$

3.27

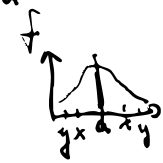
If the most frequently occur value is single $\Rightarrow$ unimodal

(a). Uniform (a,b) ✓

is unimodel

if $a \geq x \geq y$ $\quad f(a) \geq f(x) \geq f(y)$

& $a \leq x \leq y$ $\quad f(a) \geq f(x) \geq f(y)$



(b) gamma (α, β)

$$f(x) = \frac{x^{\alpha-1}\cdot e^{-x/\beta}}{\beta\cdot\Gamma(\alpha)}$$

$$f'(x) = \frac{1}{\beta\Gamma(\alpha)}\left[x^{\alpha-1}\cdot e^{-x/\beta}\cdot\left(-\frac{1}{\beta}\right) + (\alpha-1)\cdot x^{\alpha-2}\cdot e^{-x/\beta}\right]$$

$$= \frac{1}{\beta\Gamma(\alpha)}\left[x^{\alpha-2}\cdot e^{-x/\beta}\left(-\frac{x}{\beta}+\alpha-1\right)\right]$$

when $x = \beta(\alpha-1)$ $\quad f'(x) = 0$

$x <$ $\cdots$ $\quad f'(x) > 0$

$x >$ $\cdots$ $\quad f'(x) < 0$

$a = \beta(\alpha-1)$

$\therefore$ gamma (a, β) is unimodel $\square$

(c) $n(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\cdot\left(-\frac{1}{2\sigma^2}\right)2(x-\mu)$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}(x-\mu)$$

$$\therefore f'(x)\begin{cases} > 0 & x < \mu \\ = 0 & x = \mu \\ < 0 & x > \mu \end{cases}$$

$\therefore$ when $a = \mu$ $\quad f(a) > f(x)$ $x \neq \mu$

$\therefore n(\mu, \sigma^2)$ is unimodal. $\square$

(d) beta (α, β) $\quad x \in (0,1)$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$$

$$f'(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \left[x^{\alpha-1}\cdot(\beta-1)\cdot(1-x)^{\beta-2} + (\alpha-1)\cdot x^{\alpha-2}\cdot(1-x)^{\beta-1} \right]$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \left[x^{\alpha-2}\cdot(1-x)^{\beta-2} \left(x(\beta-1) + (\alpha-1)(1-x) \right) \right]$$

Let $x(\beta-1) + \alpha - 1 + (\alpha-1)(-x) = 0$

$$x(\beta-1+1-\alpha) = 1-\alpha$$

$$x = \frac{1-\alpha}{\beta-\alpha}$$

$$\therefore f'(x) \begin{cases} >0 & x < \cdots \\ =0 & x = \frac{1-\alpha}{\beta-\alpha} \\ <0 & x > \cdots \end{cases}$$

3.28. exponential family

$$f(x|\theta) = h(x)\, g(\theta) \cdot e^{\eta(\theta)\cdot T(x)}$$

(a) $n(\mu, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$T(x) = -(x-\mu)^2 \quad \eta(\theta) = \frac{1}{2\sigma^2} \quad g(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \quad h(x) = 1$

1° μ known $\quad f(x|\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$h(x) = 1 \quad c(\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \quad w_1(\sigma^2) = -\frac{1}{2\sigma^2} \quad t_1(x) = (x-\mu)^2$

2° σ^2 known $\quad f(x|\mu) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2\sigma^2}} \cdot e^{\frac{\mu x}{\sigma^2}} \cdot e^{-\frac{\mu^2}{2\sigma^2}}$

$h(x) = e^{-x^2/2\sigma^2} \quad c(\mu) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{\mu^2}{2\sigma^2}} \quad w_1(\mu) = \mu \quad t_1(x) = \frac{x}{\sigma^2}$

(b) gamma (α, β)

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta\,\Gamma(\alpha)}$$

$h(x) = x^{\alpha-1} \quad g(\theta) = \frac{1}{\beta\cdot\Gamma(\alpha)}$

$T(x) = -x \quad \eta(\theta) = \frac{1}{\beta}$

1° α known $\quad f(x|\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-x/\beta}$

$h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \quad c(\beta) = \frac{1}{\beta^\alpha} \quad w_1(\beta) = \frac{1}{\beta} \quad t_1(x) = -x$

2° β known $\quad f(x/\alpha) = e^{-x/\beta} \frac{1}{\beta^\alpha\Gamma(\alpha)} e^{(\alpha-1)\log x}$

$h(x) = e^{-x/\beta} \quad c(\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \quad w_1(\alpha) = \alpha-1 \quad t_1(x) = \log x$

3° α, β unknow

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{(\alpha-1)\log x - x/\beta}$$

$h(x) = 1 \quad c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \quad w_1(\alpha) = \alpha-1 \quad t_1(x) = \log x$

$w_2(\alpha,\beta) = -\frac{1}{\beta} \quad t_2(x) = x$

(c) beta (α, β) $\quad f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$

1° α known $\quad f(x|\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1} e^{(\beta-1)\log(1-x)}$

$h(x) = x^{\alpha-1} \quad c(\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \quad w_1(\beta) = \beta-1 \quad t_1(x) = \log(1-x)$

2° β known $\quad f(x|\alpha) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} e^{(\alpha-1)\log x}(1-x)^{\beta-1}$

$h(x) = (1-x)^{\beta-1} \quad c(\alpha) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \quad w_1(\alpha) = \alpha-1 \quad t_1(x) = \log x$

3° α, β unknow $\quad f(x|\alpha) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} e^{(\alpha-1)\log x + (\beta-1)\log(1-x)}$

$h(x) = 1 \quad c(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \quad w_1(\alpha) = \alpha-1 \quad t_1(x) = \log x$

$w_2(\beta) = \beta-1 \quad t_2(x) = \log 1-x$

(d) Poisson family

$$f(x) = \frac{e^{-\lambda}\cdot\lambda^x}{x!} \qquad \lambda^x = e^{x\log\lambda}$$

$h(x) = \frac{1}{x!} \quad c(\lambda) = e^{-\lambda}$

$w_1(\lambda) = \log\lambda \quad t_1(x) = x$

(e) negative binomial r know r

$f(x) = C^x_{r+x-1}\, p^r (1-p)^x \quad f(x|p) = C^x_{r+x-1}\, p^r\, e^{x\cdot\log(1-p)}$

$h(x) = C^x_{r+x-1} \quad c(p) = p^r \quad w_1(p) = \log(1-p) \quad t_1(x) = x$

3.29 (a) $n(\mu, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad f(x) = h(x) \cdot e^{\eta T(x) - A(\eta)}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2\sigma^2} + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2}}$$

$$\therefore \eta_1 = -\frac{1}{2\sigma^2} \qquad \eta_2 = \frac{\mu}{\sigma^2}$$

$$\eta_1 \in (-\infty, 0) \qquad \eta_2 \in (-\infty, +\infty)$$

(b) gamma (α, β) $\qquad \alpha, \beta > 0$

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta\,\Gamma(\alpha)} = \frac{1}{\beta\,\Gamma(\alpha)} \cdot e^{-x/\beta + (\alpha-1)\log x}$$

$$\therefore \eta_1 = -\frac{1}{\beta} \qquad \eta_2 = \alpha - 1$$

$$\eta_1 \in (-\infty, 0) \qquad \eta_2 \in (-1, +\infty)$$

(c) beta (α, β) $\qquad f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot e^{(\alpha-1)\log x + (\beta-1)\cdot\log(1-x)}$$

$$\therefore \eta_1 = \alpha - 1 \qquad \eta_2 = \beta - 1$$

$$\eta_1 \in (-1, +\infty) \qquad \eta_2 \in (-1, +\infty)$$

(d) Poisson family.

$$f(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} = e^{-\lambda} \cdot \lambda^x \cdot (x!)^{-1} = \frac{e^{-\lambda}}{x!} e^{x \cdot \log\lambda} \qquad \therefore \eta = \log\lambda \in (-\infty, +\infty)$$

(e) negative binomial r know r

$$f(x) = C^x_{r+x-1}\, p^r (1-p)^x = \binom{r+x-1}{x} p^r \cdot e^{x\log(1-p)} \qquad \therefore \eta = \log(1-p) \in (-\infty, 0]$$

Note: Natural parameter

Suppose $z \sim$ exponential family

$$p(x|\eta) = h(x) \cdot e^{[\eta T(x) - A(\eta)]} \qquad x \in \mathcal{X} \subset \mathbb{R}^d$$

If η is an interior point of $\mathcal{E}$ 有界, then

- the mgf of $T(z)$ exists:

① $M_T(s) = E[e^{sT(z)} \mid \eta] = e^{A(s+\eta) - A(\eta)}$

s in some neighborhood of 0

- $E[T(z) \mid \eta] = A'(\eta)$
- $Var[T(z) \mid \eta] = A''(\eta)$

Proof ①

$$M_T(s) = \int e^{sT(x)} \cdot h(x) \cdot e^{\eta T(x) - A(\eta)}\, dx$$

$$= \int h(x) \cdot e^{(s+\eta)T(x) - A(\eta)}\, dx$$

$$= \underbrace{\int h(x) \cdot e^{(s+\eta)T(x) - A(s+\eta)}\, dx}_{1} \cdot e^{A(s+\eta) - A(\eta)}$$

$$= e^{A(s+\eta) - A(\eta)}$$

330. Theorem 3.4.2 P140

If $X \sim h(x|\theta)$, $h(x|\theta) = h(x)\, c(\theta) \cdot \exp\left(\sum w_i(\theta)\, t_i(x)\right)$, then

$$E\left(\sum \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(X)\right) = -\frac{\partial}{\partial \theta_j} \log c(\theta)$$

$$Var\left(\sum \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(X)\right) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) - E\left(\sum \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(X)\right)$$

(a) $X \sim$ binomial dis, Var?

$f(x) = C_n^x \cdot p^x (1-p)^{n-x} = \binom{n}{x} \cdot e^{(n-x)\log(1-p) + x \cdot \log p}$

$\therefore w_1(p) = \log(1-p)$ $\quad t_1(x) = n-x$ $\quad h(x) = \binom{n}{x}$

$w_2(p) = \log p$ $\quad t_2(x) = x$ $\quad c(p) = 1$

$E\left[-\frac{1}{1-p}(n-x) + \frac{1}{p} \cdot x\right] = E\left(\frac{x-np}{p(1-p)}\right) = -\frac{\partial}{\partial p} \log 1 = 0$

$\therefore E(X) = np$ $\square$

$Var\left(\frac{x-np}{p(1-p)}\right) = 0 - E\left(-\frac{1}{(1-p)^2}(n-x) - \frac{1}{p^2} x\right) = -E\left(\frac{n}{(1-p)^2} + \frac{x(1-2p)}{(1-p)^2 \cdot p^2}\right)$

$= \frac{n}{(1-p)^2} \quad \frac{1-2p}{(1-p)^2 \cdot p^2} E(X)$

$= \frac{n}{p(1-p)}$

$Var\left(\frac{x-np}{p(1-p)}\right) = Var\left(\frac{x}{p(1-p)}\right)$

$= \left[\frac{1}{p(1-p)}\right]^2 \cdot Var(X)$

$\therefore Var(X) = np(1-p)$ $\square$

(b) $X \sim$ Poisson (λ) $\quad E(X)$? $Var(X)$?

$f(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} = e^{-\lambda} \frac{1}{x!} e^{x \cdot \log \lambda}$

$\therefore w(\lambda) = \log \lambda$ $\quad t(x) = x$ $\quad h(x) = \frac{1}{x!}$ $\quad c(\lambda) = e^{-\lambda}$

$E\left(\frac{1}{\lambda} \cdot x\right) = -\left(\log e^{-\lambda}\right)' = -(-\lambda)' = -(-1) = 1$

$\therefore E(X) = \lambda$

$Var\left(\frac{1}{\lambda} x\right) = 0 - E\left(-\frac{1}{\lambda^2} \cdot x\right) = \frac{1}{\lambda^2} \cdot \lambda = \frac{1}{\lambda}$

$\frac{1}{\lambda^2} Var(X) = \frac{1}{\lambda}$

$\therefore Var(X) = \lambda$ $\square$

$w'(\lambda) = \lambda^{-1}$

$w''(\lambda) = -1 \quad (\lambda^{-2})$

$\log c(\lambda) = -\lambda$

$\log' c(\lambda) = -1$

$\log'' c(\lambda) = 0$

3.31. (a) (3.44) P140 $E\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(Z)\right) = -\frac{\partial}{\partial \theta_j} \log c(\theta)$;

(3.45) $$Var\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(Z)\right) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) - E\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(Z)\right)$$

$$\int h(x)\, c(\theta) \cdot \exp\left(\sum_{i=1}^{k} w_i(\theta) \cdot t_i(x)\right) dx = 1$$

$$\frac{d}{dx} \log g(x) = \frac{g'(x)}{g(x)} \qquad c'(\theta) = \boxed{c(\theta) \frac{\partial}{\partial(\theta)} \log c(\theta)}$$

$$\frac{\partial}{\partial \theta} \int h(x)\, c(\theta)\, e^{\sum w_i(\theta) \cdot t_i(x)} dx = 0$$

① $$\int h(x) \cdot c'(\theta)\, e^{\sum w_i(\theta) \cdot t_i(x)} dx +$$

$$\int h(x) \cdot c(\theta)\, e^{\sum w_i(\theta) \cdot t_i(x)} \cdot \frac{\partial w_i(\theta)}{\partial \theta} t_i(x)\, dx = 0 \qquad E(\cdots)$$

① $$\int h(x) \cdot c(\theta) \cdot \frac{\partial}{\partial(\theta_j)} \log c(\theta) \cdot e^{\sum w_i(\theta) \cdot t_i(x)} dx = -E\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(Z)\right)$$

$$\therefore \frac{\partial}{\partial \theta_j} \log c(\theta) = -E\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(Z)\right)$$

(b). $$\frac{d^2}{dx^2} \log g(x) = g'(x) \cdot (-1) \cdot g^{-2}(x) + g''(x) \cdot g^{-1}(x)$$

$$= \frac{g''(x)}{g(x)} - \frac{g'(x)}{g^2(x)} \qquad c'(\theta) = c(\theta) \frac{\partial}{\partial \theta} \log c(\theta)$$

$$\frac{\partial^2}{\partial^2 \theta_j} \log c(\theta) = \frac{c''(\theta)}{c(\theta)} - \frac{c'(\theta)}{c^2(\theta)} = \frac{c''(\theta)}{c(\theta)} - \frac{\partial}{\partial \theta} \log c(\theta) \Big/ c(\theta)$$

$$\int h(x) \cdot c'(\theta) \cdot e^{\sum \cdots} \cdot \sum \frac{\partial w_i(\theta)}{\partial \theta} t_i(Z)\, dx +$$

$$\int h(x) \cdot c''(\theta) \cdot e^{\sum \cdots} dx +$$

$$\int h(x) \cdot c(\theta)\, e^{\sum \cdots} \frac{\partial^2 w_i(\theta)}{\partial^2 \theta} t_i(x) dx + \int h(x) \cdot c(\theta) \cdot e^{\sum \cdots} \sum \frac{\partial w_i(\theta)}{\partial \theta} t_i(x) \cdot \frac{\partial w_i(\theta)}{\partial \theta} t_i(Z)\, dx$$

$$+ \int h(x) \cdot c'(\theta) \cdot e^{\sum \cdots} \frac{\partial w_i(\theta)}{\partial \theta} t_i(x)\, dx$$

$$= \int h(x)\, c(\theta) \cdot \frac{\partial}{\partial \theta} \log c(\theta) \cdot e^{\sum \cdots} \sum \frac{\partial w_i(\theta)}{\partial \theta} t_i(Z) dx +$$

$$+ \int h(x) \cdot \left[c(\theta) \frac{\partial^2}{\partial^2 \theta} \log c(\theta) + \frac{\partial}{\partial \theta} \log c(\theta)\right] \cdot e^{\sum \cdots} dx +$$

$$Var(Z) = E[(Z - E(Z))^2] = EZ^2 - (EZ)^2$$

$$E\left(\sum \frac{\partial^2 w_i(\theta)}{\partial^2 \theta} t_i(Z)\right) + E\left[\left(\sum \frac{\partial w_i(\theta)}{\partial \theta} t_i(Z)\right)^2\right] +$$

$$E^2\left(\sum \frac{\partial w_i(\theta)}{\partial \theta} t_i(Z)\right)$$

$$\int h(x) \cdot c(\theta) \cdot \frac{\partial}{\partial \theta} \log c(\theta) \cdot e^{\sum \cdots} \sum \frac{\partial w_i(\theta)}{\partial \theta} t_i(x) dx$$

$$= \frac{\partial^2}{\partial^2 \theta} \log c(\theta) + \left[\frac{c'(\theta)}{c(\theta)}\right]^2 +$$

$$E\left(\sum \frac{\partial^2 w_i(\theta)}{\partial^2 \theta} t_i(Z)\right) + E\left[\left(\sum \frac{\partial w_i(\theta)}{\partial \theta} \cdot t_i(Z)\right)^2\right]$$

$$+ \sum \frac{\partial}{\partial \theta} \log c(\theta) \cdot E\left(\sum \frac{\partial w_i(\theta)}{\partial \theta} \cdot t_i(Z)\right)$$

$$= \frac{\partial^2}{\partial^2 \theta} \log c(\theta) + \left(\frac{\partial}{\partial \theta} \log c(\theta)\right)^2$$

$$-2E^2\left(\sum \frac{\partial w_i(\theta)}{\partial \theta} t_j(\theta)\right)$$

$$+ E\left(\sum \frac{\partial^2 w_i(\theta)}{\partial^2 \theta} t_i(Z)\right)$$

$$+ E\left[\left(\sum \frac{\partial w_i(\theta)}{\partial \theta} \cdot t_i(Z)\right)^2\right]$$

(a)

$$= \frac{\partial^2}{\partial^2 \theta} \log c(\theta)$$

$$+ Var\left(\sum \frac{\partial w_i(\theta)}{\partial \theta} t_i(Z)\right)$$

$$+ E\left(\sum \frac{\partial^2 w_i(\theta)}{\partial^2 \theta} t_i(Z)\right)$$

3.32 (a) $\underline{X} \sim$ exponential family

3.4.7 $f(x|\eta) = h(x) \cdot c(\eta) \cdot e^{\sum_i \eta_i t_i(x)}$ P142

Aim: $E(t_j(\underline{X})) = -\frac{\partial}{\partial \eta_j} \cdot \log c(\eta)$ ①

$Var(t_j(\underline{X})) = -\frac{\partial^2}{\partial \eta_j^2} \log c(\eta)$ ②

Theorem 3.4.2 P140

If $\underline{X} \sim h(x|\theta)$, $h(x|\theta) = h(x)\, c(\theta) \cdot \exp(\sum w_i(\theta) t_i(x))$, then

$E(\sum \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(\underline{X})) = -\frac{\partial}{\partial \theta_j} \log c(\theta)$ ③

$Var(\sum \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(\underline{X})) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) - E(\sum \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(\underline{X}))$ ④

$w_i(\theta) = \eta_i$

$\frac{\partial w_i(\theta)}{\partial \theta_j} = \frac{d}{d\eta_i} \eta_i = 1$

$\therefore E(\sum \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(\underline{X})) = E(t_i(\underline{X}))$ $\therefore$ ① = ③

$(\log x)'' = (\frac{1}{x})' = (x^{-1})'$

$= -1 \cdot \frac{1}{x^2}$

$=$

$\because E(\sum \frac{\partial^2 w_i(\theta)}{\partial \theta^2} t_i(\underline{X})) = E(\sum \frac{d^2}{d^2 \eta_j} \eta_i \cdot t_i(\underline{X})) = E(0) = 0$

$\therefore -\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) - E(\sum \frac{\partial^2 w_i(\theta)}{\partial \theta_j^2} t_i(\underline{X})) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\theta) = -\frac{\partial^2}{\partial \eta_j^2} \log c(\eta)$

$\therefore$ ② = ④

(b) use (a) solve the mean & Var of gamma(a, b)

$\because \underline{X} \sim$ gamma(a, b)

$\therefore f(\underline{X}) = \frac{x^{a-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} = \frac{1}{\beta^\alpha \cdot \Gamma(\alpha)} \cdot x^{a-1} \cdot e^{-x/\beta}$

$h(x) = x^{a-1}$ $\eta = \frac{1}{\beta}$ $c(\frac{1}{\beta}) = (\frac{1}{\beta})^\alpha$ $t(x) = -x$

$\log c(\eta) = \log c(\frac{1}{\beta}) = \alpha \cdot \log \frac{1}{\beta}$

$\therefore E(-\underline{X}) = -\frac{d}{d\frac{1}{\beta}} \alpha \cdot \log \frac{1}{\beta} = -\alpha \cdot \beta$

$\therefore E(\underline{X}) = \alpha\beta$

$\therefore Var(\underline{X}) = \alpha\beta^2$

$Var(-\underline{X}) = -\frac{d^2}{d^2\frac{1}{\beta}} \alpha \cdot \log \frac{1}{\beta} = +\alpha\beta^2 = Var(\underline{X})$

$Var(\underline{X}) = E(\underline{X}^2) - E^2(\underline{X})$

$Var(k\underline{X}) = E(k^2\underline{X}^2) - E^2(k\underline{X})$

$= k^2 \cdot E(\underline{X}^2) - [kE(\underline{X})]^2 = k^2 \cdot Var(\underline{X})$

3.33. (a) $n(\theta, \theta)$

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$= \frac{1}{\sqrt{\theta}\sqrt{2\pi}} \cdot e^{-\frac{(x-\theta)^2}{2\theta}}$

$f(x|\theta) = h(x) \cdot c(\theta) \cdot e^{\sum_i \theta_i t_i(x)}$

$f(x) = \frac{1}{\sqrt{\theta}\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2\theta} + x - \frac{\theta}{2}} = \frac{1}{\sqrt{\theta}\sqrt{2\pi}\, e^{\frac{\theta}{2}}} e^{-\frac{x^2}{2\theta} + x}$

$w(\theta) = \frac{1}{2\theta}$ $t(x) = -x^2$ $\underline{c(\theta) = \frac{1}{\sqrt{2\pi\theta}\, e^{\frac{\theta}{2}}}}$ $h(x) = e^x$ $\theta > 0$

nonnegative real line

(b) $n(\theta, a\theta^2)$ a known

$$f(x) = \frac{1}{\sqrt{a}\,\theta\sqrt{2\pi}} \cdot e^{-\frac{(x-\theta)^2}{2a\theta^2}} \qquad a\theta^2 = \sigma^2$$

$$\sigma = \sqrt{a\theta^2} = \sqrt{a}\,\theta$$

$$= \frac{1}{\sqrt{2a\pi}\,\theta} \cdot e^{-\frac{x^2}{2a\theta^2} + \frac{x}{a\theta} - \frac{1}{2a}} \qquad 2\sigma^2$$

$$= \frac{e^{\frac{1}{2a}}}{\sqrt{2a\pi}\,\theta} \cdot e^{-\frac{x^2}{2a\theta^2} + \frac{x}{a\theta}}$$

$\therefore$ $w_1 = -\frac{1}{2a\theta^2}$ $\quad t_1(x) = x^2$

$w_2 = \frac{1}{a\theta}$ $\quad t_2(x) = x$

A parabola

抛物线

$h(x) = 1$

$c(\theta) = e^{\frac{1}{2a}} \cdot \left(\sqrt{2a\pi}\,\theta\right)^{-1}$ $\qquad \theta \in (-\infty, \infty) \quad a > 0$

(c) gamma $(\alpha, 1/\alpha)$

$$f(x) = \frac{x^{\alpha-1} \cdot e^{-\alpha x}}{\alpha^{1/\alpha}\,\Gamma(\alpha)} = \frac{1}{\alpha^{1/\alpha}\,\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\alpha x}$$

$h(x) = x^{\alpha-1}$ $\quad c(\alpha) = \alpha^{-1/\alpha} \cdot \Gamma^{-1}(\alpha)$ $\quad w_1 = -\alpha$ $\quad t_1(x) = x$

$\alpha \in (0, +\infty)$

a line

(d) $f(x|\theta) = C \cdot e^{-(x-\theta)^4}$

$$f(x) = C \cdot e^{x^4 - 4\theta x^3 + 6\theta^2 x^2 - 4\theta^3 x + \theta^4}$$

$$= C \cdot e^{x^4} \cdot e^{\theta^4} \cdot e^{-4\theta x^3 + 6\theta^2 x^2 - 4\theta^3 x}$$

$\therefore H(x) = C \cdot e^{x^4}$ $\quad c(\theta) = e^{\theta^4}$

$w_1 = -4\theta$ $\quad t_1(x) = x^3$ $\qquad \theta \in (-\infty, +\infty)$

$w_2 = 6\theta^2$ $\quad t_2(x) = x^2$

$w_3 = -4\theta^3$ $\quad t_3(x) = x$ $\qquad$ The curve is a spiral in 3-space

3.34. (a) $\bar{X} \sim n(\lambda, \lambda/n)$ poisson approximation

$$f(x) = \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{\lambda}} e^{-\frac{(x-\lambda)^2 n^2}{2\lambda^2}} \qquad \sigma^2 = \lambda/n$$

$$\sigma = \frac{\sqrt{\lambda}}{\sqrt{n}}$$

$$= \sqrt{\frac{\lambda}{2\pi n}} \cdot e^{-\frac{n^2}{2\lambda^2}x^2 + \frac{n^2}{\lambda}x - n^2}$$

$$= \sqrt{\frac{\lambda}{2\pi n}} \cdot e^{-n^2} \cdot e^{-\frac{n^2}{2\lambda^2}x^2 + \frac{n^2}{\lambda}x}$$

$\therefore h(x) = 1$

$c(\lambda) = e^{-n^2} \cdot \sqrt{\frac{\lambda}{2\pi n}}$

$w_1 = \frac{n^2}{2\lambda^2}$ $\quad t_1(x) = -x^2$

$w_2 = \frac{n^2}{\lambda}$ $\quad t_2(x) = x$

non-negative real line

(b) $\bar{X} \sim n(p, p(1-p)/n)$ binomial approximation

$$f(x) = \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{p(1-p)}} \cdot e^{-\frac{(x-p)^2 n}{2p(1-p)}} \qquad \sigma^2 = \frac{p(1-p)}{n}$$

$$\sigma = \frac{\sqrt{p(1-p)}}{\sqrt{n}}$$

$$= \sqrt{\frac{n}{2\pi p(1-p)}} \cdot e^{-\frac{n}{2p(1-p)}x^2 + \frac{n}{1-p}x - \frac{pn}{2(1-p)}}$$

$$= \sqrt{\frac{n}{2\pi p(1-p)}} \cdot e^{-\frac{pn}{2(1-p)}} \cdot e^{-\frac{n}{2p(1-p)}x^2 + \frac{n}{1-p}x}$$

$\therefore h(x) = 1$

$c(p) = e^{-\frac{np}{2(1-p)}} \sqrt{\frac{n}{2\pi p(1-p)}}$

$w_1 = \frac{n}{2p(1-p)}$ $\quad t_1(x) = -x^2$

$w_2 = \frac{n}{1-p}$ $\quad t_2(x) = x$

(c) $\bar{X} \sim n(r(1-p)/p, r(1-p)/n\,p^2)$ negative ~

$r(1-p)/n$

3.35.

(a) $n(e^{\theta}, e^{\theta})$ $\quad \theta \in (-\infty, +\infty)$

$$f(x) = \frac{1}{\sqrt{2\pi e^{\theta}}} \cdot e^{-\frac{(x-e^{\theta})^2}{2e^{\theta}}} \qquad \sigma^2 = e^{\theta}$$

$$\sigma = e^{\frac{\theta}{2}}$$

$$= \frac{1}{\sqrt{2\pi e^{\theta}}} \cdot e^{-\frac{x^2}{2e^{\theta}} + x - \frac{e^{\theta}}{2}}$$

$$= \frac{1}{\sqrt{2\pi e^{\theta} e^{e^{\theta}}}} \cdot e^{x} e^{-\frac{x^2}{2e^{\theta}}}$$

$h(x) = e^{x}$ $\qquad \underbrace{w_1 = \frac{1}{2e^{\theta}}}_{\text{a positive line}}$ $\qquad t_1(x) = -x^2$

$c(\theta) = \frac{1}{\sqrt{2\pi e^{\theta} e^{e^{\theta}}}}$

(b) $X \sim \text{gamma}(\alpha, \beta)$ $\quad EX = \mu$

$E(X) = \alpha \cdot \beta = \mu$ $\qquad \alpha = \frac{\mu}{\beta}$

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)}$$

$$= \frac{x^{\mu/\beta} \cdot e^{-x/\beta}}{(\beta)^{\mu/\beta} \cdot \Gamma(\frac{\mu}{\beta})}$$

$h(x) = x^{\mu/\beta}$ $\qquad w_1(\beta) = \frac{1}{\beta}$ $\qquad T_1(x) = -x$

$c(\beta) = \frac{1}{\beta^{\mu/\beta} \Gamma(\frac{\mu}{\beta})}$

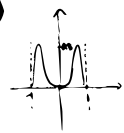
(c) $(\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \cdots, \beta_n) = (\frac{\mu}{\beta_1}, \frac{\mu}{\beta_2}, \cdots, \frac{\mu}{\beta_n}, \frac{1}{\beta_1}, \frac{1}{\beta_2}, \cdots \frac{1}{\beta_n})$

3.36. $f(x) = \frac{63}{4}(x^6 - x^8)$ $\quad x \in (-1, 1)$ 偶

$g(x) = 1/\sigma \, f((x-\mu)/\sigma)$ $\qquad (\frac{1}{2})^6 - (\frac{1}{2})^8$

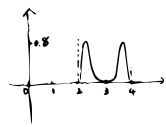
(a) $\mu = 0$ $\quad \sigma = 1$

$g(x) = f(x)$



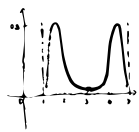
(b) $\mu = 3$ $\quad \sigma = 1$

$g(x) = f(x-3)$



(c) $\mu = 3$ $\quad \sigma = 2$

$g(x) = \frac{1}{2} f(\frac{x-3}{2})$



3.37. Aim: $g(\mu+\varepsilon) = g(\mu-\varepsilon)$

$g(\mu+\varepsilon) = \frac{1}{\sigma} f(\frac{(\mu+\varepsilon)-\mu}{\sigma}) = \frac{1}{\sigma} f(\frac{\varepsilon}{\sigma})$

Assume $g(x)$ is symmetric about μ

$g(\mu-\varepsilon) = \frac{1}{\sigma} f(\frac{(\mu-\varepsilon)-\mu}{\sigma}) = \frac{1}{\sigma} f(-\frac{\varepsilon}{\sigma})$

$\because f(x)$ is symmetric about 0 $\quad \therefore f(-x) = f(x)$

$\therefore g(\mu+\varepsilon) = g(\mu-\varepsilon)$ $\quad \therefore g(x)$ is symmetric about μ. Q

3.38 $\alpha = P(Z > z_\alpha) = \int_{z_\alpha}^{\infty} f(z)\,dz$

Aim: If $X \sim g(x) = \frac{1}{\sigma} f((x-\mu)/\sigma)$

$x_\alpha = \sigma z_\alpha + \mu$, then $P(X > x_\alpha) = \alpha$

$P(X > x_\alpha) = \int_{\sigma z_\alpha + \mu}^{\infty} f(x)\,dx$

Let $\frac{x-\mu}{\sigma} = k$ $\quad dk = \frac{1}{\sigma} dx$

$P\left(\frac{x-\mu}{\sigma} > \frac{x_\alpha - \mu}{\sigma}\right) = P\left(\frac{x-\mu}{\sigma} > z_\alpha\right) = P(K > z_\alpha)$

$= \int_{z_\alpha}^{\infty} f(k)\,dk$

$= \int_{z_\alpha}^{\infty} f\left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sigma}\,dx$

$= \alpha$

3.39 $f(x \mid \mu, \sigma) = \dfrac{1}{\sigma\pi\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)}$ $\quad x \in (-\infty, +\infty)$

(a). $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$

$P(X \geq \mu) = \int_{-\infty}^{\mu} f(x)\,dx$

$= \frac{1}{\sigma\pi} \underbrace{\int_{-\infty}^{\mu} \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}\,dx}$

Let $y = \frac{x-\mu}{\sigma}$ $\quad x = \sigma y + \mu$

$dx = \sigma\,dy$

$x = -\infty \quad y = -\infty$

$x = \mu \quad y = 0$

$\therefore \int_{-\infty}^{\mu} \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+y^2}\,\sigma\,dy = \sigma \int_{-\infty}^{0} \frac{1}{1+y^2}\,dy$

$= \sigma \cdot \arctan y \Big|_{-\infty}^{0}$

$= \sigma \cdot \left[0 - \left(-\frac{\pi}{2}\right)\right]$

$= \sigma \cdot \frac{\pi}{2}$

$\therefore P(X \geq \mu) = \frac{1}{\sigma\pi} \cdot \sigma \cdot \frac{\pi}{2} = \frac{1}{2}$

Similarly

$P(X \leq \mu) = \frac{1}{2}$ $\square$

$\int \frac{1}{1+x^2}\,dx$ $\quad$ Let $x = \tan(u)$

$dx = d\,\frac{\sin u}{\cos u} = \frac{\cos^2 u - \sin u(-\sin u)}{\cos^2 u} = \frac{1}{\cos^2 u}\,du$

$\therefore \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1 + \frac{\sin^2 u}{\cos^2 u}} \cdot \frac{1}{\cos^2 u}\,du$

$= \int \frac{1}{\frac{\cos^2 u + \sin^2 u}{\cos^2 u}} \cdot \frac{1}{\cos^2 u}\,du$

$= \int 1\,du$

$= u$

$= \arctan x$

(b) $P(X \geq \mu + \sigma) = P(X \leq \mu - \sigma) = \frac{1}{4}$

Aim $\mu = 0$, $\sigma = 1$ ✓ + 3.38

When $\mu = 0$, $\sigma = 1$

$f(x) = \frac{1}{\pi(1 + x^2)}$

$P(X \geq 1) = \int_1^{+\infty} f(x)\,dx = \frac{1}{\pi} \int_1^{+\infty} \frac{1}{1+x^2}\,dx = \frac{1}{\pi} \cdot \arctan x \Big|_1^{+\infty}$

$= \frac{1}{\pi} \cdot \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$= \frac{1}{4}$

$P(X \leq -1) = \int_{-\infty}^{-1} f(x)\,dx = \frac{1}{\pi} \int_{-\infty}^{-1} \frac{1}{1+x^2}\,dx$

$= \frac{1}{\pi}\left(-\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$

$= \frac{1}{4}$

$= \int_1^{\infty} \frac{1}{\pi(1+z^2)}\,dz$

$\frac{1}{4} = P(Z > 1) = \int_1^{\infty} f(z)\,dz$

$X \sim g(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\pi(1 + \cdots)}$

$x_\alpha = \sigma z_\alpha + \mu$

$\int_{\sigma + \mu}^{\infty} g(x)\,dx = \frac{1}{4}$ $\square$

3.40 $f^{*}\left(\frac{x-\mu}{\sigma^2}\right)$ ×

$Z = \frac{X-\mu}{\sigma}$ $\quad E(Z) = \frac{1}{\sigma}E(X-\mu) = 0$

$Var(Z) = \frac{1}{\sigma^2}\cdot Var(X-\mu) = \frac{\sigma^2}{\sigma^2} = 1$

3.41

(a) Aim if $\mu_1 > \mu_2$ $\quad X_1 \sim n(\mu_1, \sigma^2)$ $\quad F(X_1) > F(X_2)$

$X_2 \sim n(\mu_2, \sigma^2)$

$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$F(x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}\,\sigma\cdot e^{\frac{1}{2\sigma^2}}}\int_{-\infty}^{x_0} e^{(x-\mu)^2}\,dx$

$\int_{-\infty}^{x_0} e^{(x-\mu)^2}\,dx$ $\quad$ let $z = x-\mu$ $\quad dx = dz$

$x = x_0 \quad z = x_0 - \mu$

$= \int_{-\infty}^{x_0-\mu} e^{z^2}\,dz$

$\because \int_{-\infty}^{x_0-\mu_1} e^{z^2}\,dz < \int_{-\infty}^{x_0-\mu_2} e^{z^2}\,dz$

$\therefore F(x_0|\theta_1) < F(x_0|\theta_2)$ $\quad \square$

(b) $\beta_1 > \beta_2 > 0$ $\quad X_1 \sim gamma(\alpha, \beta_1)$ $\quad F(X_1) > F(X_2)$

$X_2 \sim gamma(\alpha, \beta_2)$

$f(x) = \frac{x^{\alpha-1}\cdot e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$ $\quad \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\cdot e^{-x}\,dx$

$F(x_0) = \int_{-\infty}^{x_0} f(x)\,dx = \frac{e^{\frac{1}{\beta}}}{\beta^{\alpha}\cdot\Gamma(\alpha)}\cdot\underline{\int_{-\infty}^{x_0} x^{\alpha-1}\cdot e^{-x}\,dx} = \frac{e^{\frac{1}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}\cdot\Gamma(\alpha) = \frac{e^{\frac{1}{\beta}}}{\beta^{\alpha}}$

$\because \beta_1 > \beta_2 > 0$ $\quad \therefore \frac{1}{\beta_1} < \frac{1}{\beta_2}$ $\quad \beta_1^{\alpha} > \beta_2^{\alpha}$

$\therefore e^{\frac{1}{\beta_1}} < e^{\frac{1}{\beta_2}}$ $\quad \frac{1}{\beta_1^{\alpha}} < \frac{1}{\beta_2^{\alpha}}$

$\therefore \frac{e^{\frac{1}{\beta_1}}}{\beta_1^{\alpha}} < \frac{e^{\frac{1}{\beta_2}}}{\beta_2^{\alpha}}$

that is $F(x_0|\beta_1) < F(x_0|\beta_2)$ $\quad \square$

3.42 Let $Z \sim n(\mu, \sigma^2)$ $\quad \underbrace{\mu < \mu_2}_{\xi_1} \underbrace{< \mu_1}_{\xi_2}$

(a) $F(x) = P(Z_1 \le x) = P(Z+\xi_1+\xi_2 \le x)$

$= P(Z \le x-\xi_1-\xi_2)$

$F(x_2) = P(Z_2 \le x) = P(Z+\xi_1 \le x)$

$= P(Z \le x-\xi_1)$

$\because P(Z \le x-\xi_1-\xi_2) \le P(Z \le x-\xi_1)$

$\therefore F(x_1) < F(x_2)$ $\quad \square$

(b) Aim: if $\sigma_1 > \sigma_2$, $Z_1 \sim f\left(\frac{x}{\sigma_1}\right)$, $F(x|\sigma_1) < F(x|\sigma_2)$ $\quad x \in [0, +\infty)$

$X_2 \sim f\left(\frac{x}{\sigma_2}\right)$ $\quad Z \sim f(z)$ $\quad x_1 = \sigma_1 z$

$x_2 = \sigma_2 z$

$F(x|\sigma_1) = P(Z_1 \le x) = P(\sigma_1 z \le x)$ $\quad \because x > 0$

$= P\left(z \le \frac{x}{\sigma_1}\right)$ $\quad \sigma_1 > \sigma_2 > 0$

$F(x|\sigma_2) = P(Z_2 \le x) = P(\sigma_2 z \le x)$ $\quad \therefore \frac{x}{\sigma_1} < \frac{x}{\sigma_2}$

$= P\left(z \le \frac{x}{\sigma_2}\right)$ $\quad \therefore F(x|\sigma_1) < F(x|\sigma_2)$ $\quad \square$

3.43. $X \sim f(x|\theta)$ $\quad \theta_1 > \theta_2 \quad F(x_1) < F(x_2)$

(a) Aim $Y = \frac{1}{X}$ $\quad (x \in (0, \infty))$ $\qquad F(y|\theta_1) > F(y|\theta_2)$

$F(y|\theta_1) = P(y_1 \le y) = P(\frac{1}{x_1} \le y) = P(x_1 \ge \frac{1}{y})$

$F(y|\theta_2) = P(y_2 \le y) = P(\frac{1}{x_2} \le y) = P(x_2 \ge \frac{1}{y})$

$\theta_1 \overset{\varepsilon_1}{>} \theta_2 \overset{\varepsilon_2}{>} \theta$

$Z \sim f(z|\theta)$

$P(x_1 \ge \frac{1}{y}) = P(Z + \varepsilon_1 + \varepsilon_2 \ge \frac{1}{y}) = P(Z \ge \frac{1}{y} - \varepsilon_1 - \varepsilon_2)$ ①

$P(x_2 \ge \frac{1}{y}) = P(Z + \varepsilon_2 \ge \frac{1}{y}) = P(Z \ge \frac{1}{y} - \varepsilon_2)$ ②

$\because P(Z \ge \frac{1}{y} - \varepsilon_1 - \varepsilon_2) > P(Z \ge \frac{1}{y} - \varepsilon_2)$

$\therefore F(y|\theta_1) > F(y|\theta_2)$ $\square$

(b) $F(x_1|\frac{1}{\theta_1}) > F(x_2|\frac{1}{\theta_2})$ $\qquad \theta_1 > \theta_2 > \theta$, $\frac{1}{\theta_1} \overset{\varepsilon_1}{<} \frac{1}{\theta_2} \overset{\varepsilon_2}{<} \frac{1}{\theta}$

$Z \sim f(z|\frac{1}{\theta})$

$F(x|\frac{1}{\theta_1}) = P(X_1 \le x) = P(Z - \varepsilon_1 - \varepsilon_2 \le x)$

$= P(Z \le x + \varepsilon_1 + \varepsilon_2)$

$F(x|\frac{1}{\theta_2}) = P(X_2 \le x) = P(Z - \varepsilon_2 \le x)$

$= P(Z \le x + \varepsilon_2)$

$\because P(Z \le x + \varepsilon_1 + \varepsilon_2) > P(Z \le x + \varepsilon_2)$

$\therefore F(x|\frac{1}{\theta_1}) > F(x|\frac{1}{\theta_2})$ $\square$

3.44. Aim: $P(|X| \ge b) \le EX^2/b^2$ or $E|X|/b$

$P(X \ge a) \le \frac{E(X)}{a}$ ... Markov's Inequality

Proof: $E(X) = \int_0^\infty x \cdot f(x)\,dx$ $\qquad X \ge 0$

$= \int_0^a x \cdot f(x)\,dx + \int_a^{+\infty} x f(x)\,dx$ $\qquad X \ge a$

$\ge 0 + \int_a^{+\infty} x f(x)\,dx$

$\ge a \cdot \int_a^{+\infty} f(x)\,dx$

$= a \cdot P(X \ge a)$ $\qquad \therefore P(x \ge a) \le \frac{E(X)}{a}$

$\because |X| \ge 0$ $\quad$ Let $|X| = Z \ge 0$ $\qquad \because |X| \ge 0$

$\therefore P(Z \ge b) \le EZ/b$ $\qquad \therefore |X| \ge b = |X|^2 \ge b^2$

$\therefore P(|X| \ge b) \le E(|X|)/b$ $\qquad \therefore P(|X| \ge b) = P(|X|^2 \ge b^2)$

$\le \frac{E(|X|^2)}{b^2}$

when $b=3$

$\frac{EX^2}{b^2} = \frac{2}{9}$ better bound. $\qquad \frac{E|X|}{b} = \frac{1}{3}$

when $b=\sqrt{2}$

$\frac{EX^2}{b^2} = \frac{2}{2} = 1$ $\qquad \frac{E|X|}{b} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ better bound

$f(x) = e^{-x}$ $\qquad b=3$ $\qquad b=\sqrt{2}$

Aim: compare $\frac{EX^2}{b^2}$? $\frac{E|X|}{b}$

$\because X \sim$ exponential distribution

$\because X \ge 0$

$f(x) = \lambda \cdot e^{-\lambda x}$

$E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty \lambda x \cdot e^{-\lambda x}\,dx$

Let $u = \lambda x$ $\qquad v' = e^{-\lambda x}\,dx$

$u' = \lambda\,dx$ $\qquad v = -\frac{1}{\lambda} e^{-\lambda x}$

$\therefore (-x \cdot e^{-\lambda x})' = \lambda x \cdot e^{-\lambda x}\,dx - e^{-\lambda x}\,dx$

$\therefore \int \lambda x\, e^{-\lambda x}\,dx = \int e^{-\lambda x}\,dx - x \cdot e^{-\lambda x}\Big|_0^{+\infty} = -\frac{1}{\lambda} \cdot e^{-\lambda x}\Big|_0^{+\infty} - 0 = \frac{1}{\lambda}$

$\therefore E|X| = EX = \frac{1}{\lambda} = \frac{1}{1} = 1$ $\qquad EX^2 = \frac{2}{\lambda^2} = 2$

$EX^2 = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty \lambda x^2 \cdot e^{-\lambda x}\,dx$

Let $u = \lambda x^2$ $\qquad v' = e^{-\lambda x}\,dx$

$u' = 2\lambda x\,dx$ $\qquad v = -\frac{1}{\lambda} e^{-\lambda x}$

$(uv)' = (-x^2 e^{-\lambda x})' = \lambda x^2 \cdot e^{-\lambda x}\,dx - 2x \cdot e^{-\lambda x}\,dx$

$\therefore \int \lambda x^2 e^{-\lambda x}\,dx = \int 2x \cdot e^{-\lambda x}\,dx + \underbrace{x^2 \cdot e^{-\lambda x}\Big|_0^{+\infty}}_{0}$

$= 2\int x \cdot e^{-\lambda x}\,dx$

$= \frac{2}{\lambda^2}$

3.45 (a) $X \sim$ mgf $M_X(t)$ $\quad t \in (-h, h)$

Aim: $P(X \geq a) \leq e^{-at} M_X(t) \quad t \in (0, h)$

$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$

$P(X \geq a) \cdot e^{at} = \int_a^{+\infty} f(x) \cdot e^{at}\, dx = \int_a^{+\infty} e^{ta} f(x)\, dx \quad \because t > 0$

$\leq \int_a^{+\infty} e^{tx} f(x)\, dx$

$\leq \int_{-\infty}^{+\infty} e^{tx} f(x)\, dx$

$= M_X(t)$

$\therefore P(X \geq a) \leq \dfrac{M_X(t)}{e^{at}}$

(b) Aim: $P(X \leq a) \leq e^{-at} M_X(t) \quad t \in (-h, 0)$

$P(X \leq a) \cdot e^{+at} = \int_{-\infty}^{a} e^{ta} f(x)\, dx \quad \because t < 0$

$\leq \int_{-\infty}^{a} e^{tx} f(x)\, dx$

$\leq \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$= M_X(t)$

$\therefore P(X \leq a) \leq \dfrac{M_X(t)}{e^{at}}$

(c) $P(X \geq 0) \leq E\, h(t, X) \quad t \in [0, +\infty)$

Aim: the general conditions on $h(t, X)$

$P(X \geq 0) = \int_0^{+\infty} f(x)\, dx$

$E h(t, X) = \int_{-\infty}^{+\infty} h(t, x) f(x)\, dx \geq \int_0^{\infty} f(x)\, dx$

$h(t, X)$ must be at least > 0

3.46. $P(|X - \mu| \geq k\sigma) \quad X \sim$ uniform $(0, 1)$

$\mu = 0.5$

$Var(X) = E(X^2) - E^2(X)$

$= \int_a^b \frac{x^2}{b-a}\, dx - \left(\frac{b-a}{2}\right)^2$

$= \frac{1}{b-a} \frac{1}{3} x^3 \Big|_a^b - \left(\frac{b-a}{2}\right)^2$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = \sigma^2$

$\therefore P(|X - \frac{1}{2}| \geq k \frac{1}{2\sqrt{3}})$

$= 1 - P(-\frac{k}{2\sqrt{3}} + \frac{1}{2} < X < \frac{k}{2\sqrt{3}} + \frac{1}{2}) = \begin{cases} 1 - \frac{k}{\sqrt{3}} & k < \sqrt{3} \\ 0 & k > \sqrt{3} \end{cases}$

$X \sim$ exponential (λ)

$f(x) = \lambda \cdot e^{-\lambda x} \quad x \in (0, +\infty)$

$E(X) = \frac{1}{\lambda} \quad E(X^2) = \frac{2}{\lambda^2} \quad Var(X) = E(X^2) - E^2(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$P(|X - \frac{1}{\lambda}| \geq \frac{k}{\lambda})$

$= 1 - P(-\frac{k-1}{\lambda} < x < \frac{k+1}{\lambda}) = \begin{cases} 1 - \int_{-\frac{k-1}{\lambda}}^{\frac{k+1}{\lambda}} f(x)\, dx & k < 1 \\ 1 - \int_0^{\frac{k+1}{\lambda}} f(x)\, dx & k > 1 \end{cases} = \begin{cases} 1 + e^{-(k+1)} - e^{k-1} & k < 1 \\ e^{-(k+1)} & k > 1 \end{cases}$

Chebychev's inequality: $P(|X - \mu| > k\sigma) \leq 1/k^2$

Markov's Inequality: $P(X \geq a) \leq \dfrac{E(X)}{a}$

3.47 $Z \sim n(\mu, \sigma)$ $\mu = 0$ $\sigma = 1$

Aim: $P(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} \cdot e^{-t^2/2}$

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$f(|z| \geq t) = 2 \cdot P(z \geq t)$$

$$= 2 \cdot \int_t^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_t^\infty e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \int_t^\infty \frac{1+x^2}{1+x^2} e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \left[\int_t^\infty \frac{x^2}{1+x^2} e^{-\frac{x^2}{2}} dx + \int_t^\infty \frac{1}{1+x^2} e^{-\frac{x^2}{2}} dx \right]$$

$$\int_t^\infty \frac{x^2}{1+x^2} e^{-\frac{x^2}{2}} dx$$

let $u = \frac{x}{1+x^2}$ $\quad v' = x \cdot e^{-\frac{x^2}{2}} dx$ $\qquad (1+x^2)^{-1}$

$-1 \, (1+x^2)^{-2} \cdot 2x$

$u' = \frac{1-x^2}{(1+x^2)^2}$ $\quad v = -e^{-\frac{x^2}{2}}$

$$(uv)' = \left(-\frac{xe^{-\frac{x^2}{2}}}{1+x^2}\right)' = \frac{x^2 e^{-\frac{x^2}{2}}}{1+x^2} dx - \frac{(1-x^2)e^{-\frac{x^2}{2}}}{(1+x^2)^2} dx$$

$$\therefore \int \frac{x^2 e^{-\frac{x^2}{2}}}{1+x^2} dx = -\frac{xe^{-\frac{x^2}{2}}}{1+x^2}\Big|_t^\infty + \int_t^\infty \frac{(1-x^2)e^{-\frac{x^2}{2}}}{(1+x^2)^2} dx$$

$$= \frac{te^{-t^2/2}}{1+t^2} + \int_t^\infty \frac{1-x^2}{(1+x^2)^2} \cdot e^{-x^2/2} dx$$

$$\therefore f(|z| \geq t) = \sqrt{\frac{2}{\pi}} \left(\frac{t \cdot e^{-t^2/2}}{1+t^2} + \frac{2}{(1+x^2)^2} e^{-x^2/2} \right) dx$$

$$\geq \sqrt{\frac{2}{\pi}} \cdot \frac{t \cdot e^{-t^2/2}}{1+t^2} \quad \square$$

3.48 $Z \sim$ binomial

$$P(x) = C_n^x \, p^x \cdot q^{n-x}$$

$$= \frac{n!}{x!(n-x)!} \cdot p^x \cdot q^{n-x}$$

$$P(x+1) = C_n^{x+1} \cdot p^{x+1} \cdot q^{n-x-1}$$

$$= \frac{n!}{(x+1)!(n-x-1)!} \cdot p^{x+1} \cdot q^{n-x-1}$$

$$= P(x) \cdot \frac{n-x}{x+1} \cdot \frac{p}{q}$$

$Z \sim$ negative binomial

$$P(x) = C_{x+r-1}^{x} \cdot p^r (1-p)^x$$

$$= \frac{(x+r-1)!}{x!(r-1)!} \cdot p^r (1-p)^x$$

$$P(x+1) = C_{x+1+r-1}^{x+1} \, p^r (1-p)^{x+1}$$

$$= \frac{(x+1+r-1)!}{(x+1)!(r-1)!} p^r (1-p)^{x+1}$$

$$= P(x) \cdot (1-p) \cdot \frac{x+r}{x+1}$$

$Z \sim$ hypergeometric

$$P(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \frac{\frac{k!}{x!(k-x)!} \cdot \frac{(N-k)!}{(n-x)!(N-k-n+x)!}}{\frac{N!}{n!(N-n)!}}$$

$$P(x+1) = \frac{\binom{k}{x+1}\binom{N-k}{n-x-1}}{\binom{N}{n}}$$

$$= \frac{\frac{k!}{(x+1)!(k-x-1)!} \cdot \frac{(N-k)!}{(n-x+1)!(N-k-n+x+1)!}}{\frac{N!}{n!(N-n)!}}$$

$$= P(x) \cdot \frac{k-x}{x+1} \cdot \frac{n-x}{N-k+n+x+1}$$

3.49 (a) $X \sim \text{gamma}(\alpha, \beta)$ $\quad E(g(X)(X-\alpha\beta)) = \beta E(X g'(X))$

$$f(x) = \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}$$

$$E(g(X)(X-\alpha\beta)) = \int_0^{\infty} g(x)(x-\alpha\beta) \cdot \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^{\alpha} \cdot \Gamma(\alpha)}\, dx = \frac{1}{\beta^{\alpha} \cdot \Gamma(\alpha)} \int_0^{\infty} g(x) \cdot (x-\alpha\beta) \cdot x^{\alpha-1} \cdot e^{-x/\beta}\, dx$$

Let $u = g(x)$ $\qquad v' = (x-\alpha\beta)(x^{\alpha-1} \cdot e^{-x/\beta})\, dx = x^{\alpha} \cdot e^{-x/\beta} - \alpha\beta\, x^{\alpha-1} \cdot e^{-x/\beta}\, dx$

$u' = g'(x)\, dx$ $\qquad v = -x^{\alpha} \cdot \beta \cdot e^{-x/\beta}$

$$\therefore \left(-g(x) \cdot x^{\alpha} \cdot \beta \cdot e^{-x/\beta}\right)' = g(x)(x-\alpha\beta)(x^{\alpha-1} \cdot e^{-x/\beta})\, dx - g'(x) \cdot x^{\alpha} \cdot \beta \cdot e^{-x/\beta}\, dx$$

$$\therefore \int_0^{\infty} g(x)(x-\alpha\beta)(x^{\alpha-1} e^{-x/\beta})\, dx = -g(x) \cdot x^{\alpha}\beta \cdot e^{-x/\beta}\Big|_0^{+\infty} + \beta \int g'(x) \cdot x^{\alpha} \cdot e^{-x/\beta}\, dx$$

If $\underset{x \to +\infty}{g(x)}$ is limited

$$\int_0^{\infty} g(x)(x-\alpha\beta)(x^{\alpha-1} \cdot e^{-x/\beta})\, dx = \beta \int g'(x) \cdot x^{\alpha} \cdot e^{-x/\beta}\, dx$$

$$\therefore E(g(X)(X-\alpha\beta)) = \beta \int g'(x) \cdot \frac{x^{\alpha} \cdot e^{-x/\beta}}{\beta^{\alpha} \cdot \Gamma(\alpha)}\, dx = \beta \cdot \int g'(x) \cdot x \cdot f(x)\, dx = \beta E(X \cdot g'(X)) \quad \square$$

(b). $X \sim \text{beta}(\alpha, \beta)$ $\quad E\left[g(X)\left(\beta - (\alpha-1)\frac{1-X}{X}\right)\right] = E((1-X)g'(X))$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$$

$$E\left[g(x)\left(\beta-(\alpha-1)\cdot\frac{1-X}{X}\right)\right] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 g(x) \cdot \left[\beta-(\alpha-1)\cdot\frac{1-x}{x}\right] \cdot x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

Let $u = g(x)$ $\qquad v' = \left[\beta-(\alpha-1)\frac{1-x}{x}\right] \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} dx = \beta \cdot x^{\alpha-1}(1-x)^{\beta-1} - (\alpha-1) \cdot x^{\alpha-2} \cdot (1-x)^{\beta}\, dx$

$u' = g'(x)dx$ $\qquad v = -x^{\alpha-1}(1-x)^{\beta}$

$$\therefore \left[-g(x)\, x^{\alpha-1}(1-x)^{\beta}\right]' = g(x) \cdot \left[\beta-(\alpha-1)\frac{1-x}{x}\right] \cdot x^{\alpha-1}(1-x)^{\beta-1} dx - g'(x) \cdot x^{\alpha-1}(1-x)^{\beta} dx$$

$$\therefore \int g(x) \cdot \left[\beta-(\alpha-1)\frac{1-x}{x}\right] \cdot x^{\alpha-1}(1-x)^{\beta-1} dx = \underbrace{-g(x) \cdot x^{\alpha-1}(1-x)^{\beta}\Big|_0^1}_{0} + \int g'(x) \cdot x^{\alpha-1} \cdot (1-x)^{\beta} dx$$

$$\therefore E\left[g(X)\left(\beta-(\alpha-1)\frac{1-X}{X}\right)\right] = \int_0^1 g'(x)(1-x)\, \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1}}\, dx$$

$$= \int_0^1 g'(x)(1-x) \cdot f(x)\, dx$$

$$= E[g'(X)(1-X)] \quad \square$$

3.50. $X \sim \text{negative binomial}(r, p)$

Aim: $E((1-p)g(X)) = E\left(\frac{X}{r+X-1}\, g(X-1)\right)$

$$f(x) = C_{r+x-1}^{x}\, p^{r}(1-p)^{x}$$

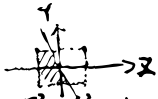
$$E((1-p) \cdot g(X)) = \int_0^{x} (1-p) \cdot g(x) \cdot \frac{(r+x-1)!}{x!\,(r-1)!} \cdot p^{r}(1-p)^{x}\, dx$$

$$E\left(\frac{Y}{r+Y-1}\, g(Y-1)\right) = \int_0^{y} \frac{y}{r+y-1} \cdot g(y-1) \cdot \underbrace{\frac{(r+y-1)!}{y!\,(r-1)!}}_{f(y)} \cdot p^{r}(1-p)^{y}\, dy$$

$$= \int_0^{y} g(y-1) \cdot f(y-1) \cdot (1-p)\, dy$$

$$= E[(1-p)\, g(Y-1)] \qquad \text{Let } Z = Y-1 \qquad \therefore (1-p)E(g(Z)) = E\left(\frac{Z}{r+Z-1}\, g(Z-1)\right) \quad \square$$

Sec4

4.1. 

$f(x,y)=\frac{1}{4}$

(a) $X^2+Y^2<1$ $\quad S(X^2+Y^2\le 1)=\pi \quad \therefore P(X^2+Y^2<1)=\frac{\pi}{4}=\frac{\pi}{4}$

(b) $2X-Y>0$ $\quad 2X-Y>0 \quad Y>-2X \quad P=\frac{1}{2}$

(c) $|X+Y|<2$ $\quad P=1$

4.2. For X and Y, $g_1(x,y)$, $g_2(x,y)$, proof

(a) $E(ag_1(X,Y)+bg_2(X,Y)+c)=aE(g_1(X,Y))+b\cdots$

$E(ag_1(X,Y)+bg_2(X,Y)+c)$

$=\iint (ag_1(x,y)+bg_2(x,y)+c)\cdot f_{X,Y}(x,y)\,dxdy$

$=a\iint g_1(x,y)\cdot f(x,y)\,dxdy+b\iint g_2(x,y)\cdot f(x,y)\,dxdy+c\iint f(x,y)\,dxdy$

$=a\cdot E(g_1(X,Y))+bE(g_2(X,Y))+c$

(b) If $g_1(x,y)\ge 0$, then $E(g_1(X,Y))\ge 0$

$E(g_1(X,Y))=\iint g_1(x,y)\,f(x,y)\,dxdy.$

$\because g_1(x,y)\ge 0 \quad f(x,y)\ge 0$

$\therefore E(g_1(X,Y))\ge 0$

(c) If $g_1(x,y)\ge g_2(x,y)$, then $E(g_1(X,Y))\ge E(g_2(X,Y))$

$E(g_1(X,Y))-E(g_2(X,Y))$

$=\iint_x [g_1(x,y)-g_2(x,y)]\,f(x,y)\,dxdy.$

$\because g_1(x,y)\ge g_2(x,y) \quad \therefore g_1(x,y)-g_2(x,y)\ge 0$

$\therefore E(g_1(X,Y))-E(g_2(X,Y))\ge 0$

(d) If $g_1(x,y)\in[a,b]$, then $E(g_1(X,Y))\in[a,b]$

$E(g_1(X,Y))=\iint_x g_1(x,y)\,f(x,y)\,dxdy \quad \because g_1(x,y)\ge a$

$\ge a\cdot\underbrace{\iint_x f(x,y)\,dxdy}_{1}$

$\ge a$

Analogously, $E(g_1(X,Y))\le b \quad \therefore E(g_1(X,Y))\in[a,b]$

4.3 Example 4.1.5

Def: $X=0$ if first die ≤ 2 $\quad X=1$ if $\cdots >2$

$Y=0$ if second is odd $\quad Y=1$ $\cdots$ even

Aim: $f(0,0)=f(0,1)=\frac{1}{6}$ $\quad f(x,y)=0$

$f(1,0)=f(1,1)=\frac{1}{3}$

$f(X=1,Y=0)=P(\{(3,1),(3,3),(3,5),(4,1),(4,3),(4,5),(5,1),(5,3),(5,5),(6,1),(6,3),(6,5)\})=\frac{12}{36}=\frac{1}{3}$

$f(X=1,Y=1)=P(\{(3,2),(3,4),(3,6),(4,2),(4,4),(4,6),(5,2),(5,4),(5,6),(6,2),(6,4),(6,6)\})=\frac{12}{36}=\frac{1}{3}$

$P(X=0,Y=0)=P(\{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5)\})=\frac{6}{36}=\frac{1}{6}$

$P(X=0,Y=1)=P(\{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6)\})=\frac{6}{36}=\frac{1}{6}$

4.4. $f(x,y) = \begin{cases} C(x+2y) & y\in(0,1) \\ 0 & x\in(0,2) \end{cases}$

(a) Find the value of C

$\because \int_y \int_x f(x,y)\,dx\,dy = 1$, that is

$\int_0^1 \int_0^2 C(x+2y)\,dx\,dy = 1$

$\int_0^1 \int_0^2 Cx\,dx\,dy + \int_0^2 \int_0^1 2Cy\,dy\,dx = 1$

$\int_0^1 \frac{C}{2}x^2\Big|_0^2 dy + \int_0^2 Cy^2\Big|_0^1 dx = 1$

$\int_0^1 2C\,dy + \int_0^2 C\,dx$

$2C + 2C\cdot 1 \quad \therefore C = \frac{1}{4}$

(b) find $f(x)$

$f(x) = \int_0^1 f(x,y)\,dy$

$= \int_0^1 \frac{1}{4}(x+2y)\,dy$

$= \frac{1}{4}\cdot y\Big|_0^1$

$= \frac{1}{4}$

$f(x) = \begin{cases} \int_0^1 \frac{1}{4}(x+2y)\,dy = \frac{1}{4}x + \frac{1}{4} & x\in(0,2) \\ 0 & \text{otherwise} \end{cases}$

常数

(c) $F(x,y)$

$F(x_1,y_1) = \int_0^{y_1}\int_0^{x_1} f(x,y)\,dx\,dy$

$= \int_0^{y_1}\int_0^{x_1} \frac{1}{4}(x+2y)\,dx\,dy$

$= \int_0^{y_1} \frac{1}{8}x^2\Big|_0^{x_1} dy + \int_0^{x_1} \frac{1}{4}y^2\Big|_0^{y_1} dx$

$= \int_0^{y_1} \frac{1}{8}x_1^2\,dy + \int_0^{x_1} \frac{1}{4}y_1^2\,dx$

$= \frac{1}{8}\cdot x_1^2\cdot y_1 + \frac{1}{4}\cdot y_1^2\cdot x_1$

let $x_1 = x$, $y_1 = y$

$F(x,y) = \begin{cases} \frac{1}{4}xy\left(\frac{x}{2}+y\right) & x\in(0,2)\ y\in(0,1) \\ 1 & x\in(2,+\infty)\ y\in(1,+\infty) \\ 0 & x<0\ \ y<0 \end{cases}$

还要 $F(x)$ $F(y)$

$f(y) = \int_0^2 f(y,x)\,dx$

$= \int_0^2 \frac{1}{4}(x+2y)\,dx$

$= \frac{1}{8}x^2\Big|_0^2 + \frac{1}{4}yx\Big|_0^2$

$= \frac{1}{2} + \frac{1}{2}y$

$F(y) = \int_0^y \frac{1}{2} + \frac{1}{2}y\,dy$

$= \frac{1}{2}y + \frac{1}{4}y^2 \quad y\in(0,1) \quad x\in(2,+\infty)$

$f(x) = \begin{cases} \frac{1}{4}x + \frac{1}{4} & x\in(0,2)\ y\in(1,+\infty) \\ 0 & \text{otherwise} \end{cases}$

$F(x) = \int_0^x \frac{1}{4}x + \frac{1}{4}\,dx$

$= \frac{1}{4}x + \frac{1}{8}x^2$

$\therefore F(x,y) = \begin{cases} 0 & x<0,\ y<0 \\ \frac{1}{4}xy\left(\frac{x}{2}+y\right) & x\in(0,2),\ y\in(0,1) \\ \frac{1}{2}y + \frac{1}{4}y^2 & y\in(0,1)\ x\in(2,+\infty) \\ \frac{1}{4}x + \frac{1}{8}x^2 & x\in(0,2)\ y\in(1,+\infty) \\ 1 & x\in(2,+\infty)\ y\in(1,+\infty) \end{cases}$

(d) $f(z)$? $z = 9/(x+1)^2$

$z = 9/(x+1)^2 > 0 \quad x\in(0,2)$

$(x+1)^2 = \frac{9}{z} \quad x = 3z^{-\frac{1}{2}} - 1 \quad x' = -\frac{1}{2}z^{-\frac{3}{2}}$

$\therefore f(z) = f(x = 3z^{-\frac{1}{2}} - 1)\ \frac{1}{2}z^{-\frac{3}{2}}$

$= \left[\frac{1}{8}(3z^{-\frac{1}{2}} - 1)^2 + \frac{1}{4}(3z^{-\frac{1}{2}} - 1)\right]\frac{1}{2}\cdot z^{-\frac{3}{2}}$

$= \frac{1}{4}(3z^{-\frac{1}{2}} - 1)\left[\frac{1}{2}(3z^{-\frac{1}{2}} - 1) + 1\right]\cdot\frac{1}{2}z^{-\frac{3}{2}}$

$= \left(\frac{1}{8}z^{-\frac{1}{2}} - \frac{1}{24}z^{-\frac{3}{2}}\right)\left[\frac{3}{2}z^{-\frac{1}{2}} + \frac{1}{2}\right]$

$= \frac{3}{16}z^{-1} - \frac{1}{16}z^{-2} + \frac{1}{4}z^{-\frac{1}{2}} - \frac{1}{48}z^{-\frac{3}{2}}$

$F(z) = F_z(Z \le z)$

$= F_z\left(\frac{9}{(X+1)^2} \le z\right) = F_z\left(X \ge \frac{3}{\sqrt{z}} - 1\right)$

$= 1 - F_z\left(X \le \frac{3}{\sqrt{z}} - 1\right)$

$f(z) = -f_z\left(\frac{3}{\sqrt{z}} - 1\right)\left(\frac{3}{\sqrt{z}} - 1\right)'$

$= \left[\frac{1}{4}\left(\frac{3}{\sqrt{z}} - 1\right) + \frac{1}{4}\right]\cdot\frac{3}{2}z^{-\frac{3}{2}}$

$= \frac{1}{4}\cdot\frac{3}{\sqrt{z}}\cdot\frac{3}{2}z^{-\frac{3}{2}}$

$= \frac{9}{8}\cdot z^{-2} \quad z\in(1,9)$

错因: ① z^{-1} 求导出错

② $f(z)$ 公式代错, 代成 $F(z)$

4.5 (a) $f(x,y) = x+y \quad x\in[0,1]\ y\in[0,1]$

? $P(Z > \sqrt{Y})$

$P(Z > \sqrt{Y}) = \int_{\sqrt{y}}^1 \int_0^1 f\,dy\,dx$

$= \int_{\sqrt{y}}^1 \int_0^1 x+y\,dy\,dx$

$= \int_0^1 \int_{\sqrt{y}}^1 x+y\,dx\,dy$

$= \int_0^1 \left(\frac{1}{2}x^2 + yx\right)\Big|_{\sqrt{y}}^1 dy$

$= \int_0^1 \frac{1}{2}(1-y) + y(1-\sqrt{y})\,dy$

$= \left(\frac{1}{2}y + \frac{1}{4}y^2 - \frac{2}{3}y^{\frac{3}{2}}\right)\Big|_0^1 = \frac{1}{2} + \frac{1}{4} - \frac{2}{3} = \frac{10+5-8}{20} = \frac{7}{20}$

(b) $f(x,y) = 2x \quad x\in[0,1]\ y\in[0,1]$

? $P(X^2 < Y < Z) \Rightarrow \sqrt{y} > x > y$

$\int_0^1 \int_y^{\sqrt{y}} 2x\,dx\,dy$ ①

$= \int_0^1 x^2\Big|_y^{\sqrt{y}} dy$

$= \int_0^1 y - y^2\,dy$

$= \frac{1}{2}y\Big|_0^1 - \frac{1}{3}y^3\Big|_0^1 = \frac{1}{2} - \frac{1}{3}$

$= \frac{1}{6}$

$\int_0^1 \int_{x^2}^x 2x\,dy\,dx$ ②

$= \int_0^1 2xy\Big|_{x^2}^x dx$

$= \int_0^1 2x^2 - 2x^3\,dx$

$= \left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^1$

$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

①.② 分析

4.6. Let x be the time A arrive. $x \sim Uni(1,2)$

y " $y \sim Uni(1,2)$

Waiting time of A is t

$$t = \begin{cases} y - x & y > x \\ 0 & y \le x \end{cases} \qquad ?\ f(t)$$

$F(x) = \frac{x-1}{2-1} \quad x \in (1,2]$

$F(y) = \frac{y-1}{2-1} \quad y \in (1,2]$

$\because X, Y$ independent

$\therefore f(x,y) = 1$

$$F(t) = \int_0^{2-t} \int_y^{y+t} f(x,y)\, dx\, dy = \int_1^{2-t} \int_y^{y+t} 1\, dx\, dy$$

$$= \int_0^{2-t} \int_y^{y+t} 1\, dx\, dy$$

$$= \int_0^{2-t} t\, dy$$

$$= t(2-t)$$

$$= 2t - t^2$$

$t=0 \quad F(0) = P(Y \le X) = \int_1^2 \int_1^x 1\, dy\, dx = \int_1^2 (x-1)\, dx = \frac{1}{2}x^2 \Big|_1^2 - 1 = \frac{1}{2}$

$1 \ge t > 0 \quad F(t) = P(X < Y \le X + t) = \int_1^{2-t} \int_x^{x+t} 1\, dy\, dx = \int_1^{2-t} t\, dx = t(2-t)$

$0 < \text{等待的时间} \le t$

$= 1 - \underline{P(2 \ge Y \ge X+t)} = 1 - \int_1^{2-t} \int_{x+t}^{2} 1\, dy\, dx = 1 - \int_1^{2-t} 2 - x - t\, dx$

$1 - \text{等待的时间} > t$

$= 1 - (2-t)(2-t-1) + \frac{1}{2}x^2 \Big|_1^{2-t}$

$= 1 - (2-t)(1-t) + \frac{1}{2}(2-t)^2 - \frac{1}{2}$

$= \frac{1}{2} - \cancel{2} + t + \cancel{2t} - t^2 + \cancel{2} - \cancel{2t} + \frac{1}{2}t^2$

$= \frac{1}{2} + t - \frac{1}{2}t^2$

? why dif

4.7 Let X be the departural time $x \sim Uni(0,30)$

if $x = 10 \Rightarrow$ leave at 8:10 min

Let Y be the arrival time $y \sim Uni(40,50)$

Z be the arrived time $Z = X + Y$

$?\ P(Z \le 60)$

$f(x) = \frac{1}{30}$

$f(y) = \frac{1}{10}$

$f(x,y) = \frac{1}{300}$

$$P(Z \le 60) = 1 - P(Z > 60) = 1 - \int_0^{30} \int_{60-x}^{50} \frac{1}{300}\, dy\, dx$$

$$= 1 - \int_0^{30} \frac{50 - 60 + x}{300}\, dx$$

$$= 1 - \frac{1}{300}\left(-10x + \frac{1}{2}x^2\right)\Big|_0^{30}$$

$$= 1 - \frac{1}{300}\left(-300 + \frac{1}{2} \cdot 900\right)$$

$$= 1 + 1 - \frac{1}{2} \cdot 3$$

$$= \frac{1}{2}$$

$$\int_0^{30} \int_{40}^{60-x} \frac{1}{300}\, dy\, dx$$

完全分析

4.8 (a) $P(\underline{X}=m \mid M=m) = P(\underline{X}=2m \mid M=m) = \frac{1}{2}$ (Proof)

$? \; P(M=x \mid \underline{X}=x) \qquad ? \; P(M=x/2 \mid \underline{X}=x)$

$\because P(A|B) = \frac{P(AB)}{P(B)} \qquad P(A) = P(AB_1) + P(AB_2) + \cdots + P(AB_n)$

$\therefore P(\underline{X}=m \mid M=m) = \frac{P(\underline{X}=m, M=m)}{P(M=m)}$

$P(\underline{X}=2m \mid M=m) = \frac{P(\underline{X}=2m, M=m)}{P(M=m)}$

$\because P(M=m) = P(\underline{X}=m, M=m) + P(\underline{X}=2m, M=m)$

$\because P(\underline{X}=m, M=m) = P(\underline{X}=2m, M=m)$

$\therefore P(M=m) = 2 \cdot P(\underline{X}=m, M=m)$

$\therefore P(\underline{X}=m \mid M=m) = \frac{1}{2}$

Analogously $P(\underline{X}=2m \mid M=m) = \frac{1}{2}$

$P(M=x \mid \underline{X}=x) = \frac{\pi(x)}{\pi(x)+\pi(\frac{x}{2})}$? $\qquad M \sim \pi(x)$

$P(M=x \mid \underline{X}=x) = \frac{P(M=x, \underline{X}=x)}{P(\underline{X}=x)} = \frac{\pi(x) \cdot P(\underline{X}=x)}{P(\underline{X}=x)}$

$= \pi(x)$

$\because \pi(x) + \pi(\frac{x}{2}) = 1$

$\therefore P(M=x \mid \underline{X}=x) = \frac{\pi(x)}{\pi(x)+\pi(\frac{x}{2})}$

Analogously $P(M=\frac{x}{2} \mid \underline{X}=x) = \frac{\pi(\frac{x}{2})}{\pi(x)+\pi(\frac{x}{2})}$

(b) Verify that one should trade only if

$\pi(x/2) < 2\pi(x)$ if π: exponential $(\lambda) \Rightarrow x < 2\log 2/\lambda$

The expected income after exchanging

$P(M=x \mid \underline{X}=x) = \frac{\pi(x)}{\pi(x)+\pi(\frac{x}{2})}$ 我得到了的那一份 $x \quad 2x$

$P(M=\frac{x}{2} \mid \underline{X}=x) = \frac{\pi(\frac{x}{2})}{\pi(x)+\pi(\frac{x}{2})}$ $\cdots$ 多 $\cdots \quad x \quad \frac{x}{2}$

$\therefore E = P(M=x \mid \underline{X}=x) \cdot 2x + P(M=\frac{x}{2} \mid \underline{X}=x) \cdot \frac{x}{2}$

$= \frac{2x\pi(x) + \frac{x}{2}\pi(\frac{x}{2})}{\pi(x)+\pi(\frac{x}{2})} > x \iff$ should trade

$\Rightarrow 2x \cdot \pi(x) + \frac{x}{2}\pi(\frac{x}{2}) > x\pi(x) + x \cdot \pi(\frac{x}{2})$

$x\pi(x) - \frac{x}{2}\pi(\frac{x}{2}) > 0$

$2\pi(x) > \pi(\frac{x}{2})$

$\because \pi(x) = \lambda \cdot e^{-\lambda \cdot x} \quad (x \geq 0)$

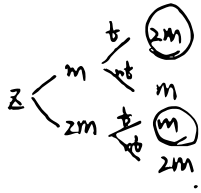
$2 \cancel{\lambda} \cdot e^{-\lambda \cdot x} > \cancel{\lambda} \cdot e^{-\lambda \cdot \frac{x}{2}}$

$\log 2 + (-\lambda x) > -\lambda \cdot \frac{x}{2}$

$\log 2 > \lambda \cdot \frac{x}{2}$

$x < \frac{2\log 2}{\lambda}$ $\square$

(c). $P(Y=2x \mid Z=m) = P(Y=x \mid Z=2m) = 1$ $\quad E(Y) = \frac{3m}{2}$



If $Z=m$ then Y must be $2x$

$\therefore P(Y=2x \mid Z=m) = 1$

Analogously

$P(Y=Z \mid Z=2m) = 1$

If change

$$E(Y_t) = P(Y=2x \mid Z=m) \cdot \underbrace{P(Z=m)}_{\frac{1}{2}} \cdot 2x + P(Y=x \mid Z=2m) \cdot \underbrace{P(Z=2m)}_{\frac{1}{2}} \cdot x$$

$$= \frac{1}{2} \cdot 2x + \frac{1}{2} \cdot x$$

$$= \frac{3}{2}x$$

$$P(Y_{not}) = P(Z=m) \cdot 2x + P(Z=2m) \cdot x$$

$$= \frac{3}{2}x \quad \square$$

4.9. If $F_{Z,Y}(x,y) = F_Z(x) \cdot F_Y(y)$

then $P(a \le Z \le b, c \le Y \le d) = P(a \le Z \le b) \cdot P(c \le Y \le d)$

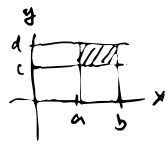
$P(a \le Z \le b, c \le Y \le d)$

$= F_{Z,Y}(b,d) + F_{Z,Y}(a,c) - F_{Z,Y}(b,c) - F_{Z,Y}(a,d)$

$= F_Z(b) \cdot F_Y(d) + F_Z(a) \cdot F_Y(c) - F_Z(b) \cdot F_Y(c) - F_Z(a) \cdot F_Y(d)$

$= [F_Z(b) - F_Z(a)][F_Y(d) - F_Y(c)]$

$= P(a \le Z \le b) \cdot P(c \le Y \le d)$



4.10 (a) show Z, Y are dependent

		Z		
		1	2	3
Y	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{6}$	0

if $f(x,y) = f(x) \cdot f(y) \Rightarrow$ independent

$f(Z=1) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$ $\qquad f(Y=2) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}$

$f(Z=2) = \frac{1}{6} + \frac{1}{3} = \frac{2+1}{6} = \frac{1}{2}$ $\qquad f(Y=3) = \frac{1}{3}$

$f(Z=3) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$ $\qquad f(Y=4) = \frac{1}{3}$

$f(\underset{X}{1}, \underset{Y}{2}) = \frac{1}{12} = f_Z(1) \cdot f_Y(2)$

$f(1,3) = \frac{1}{6} \ne f_Z(1) \cdot f_Y(3) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$

$\therefore Z, Y$ are dependent.

(b) Give a probability table for U, V, have the same marginals as Z, Y but are independent.

		U		
		1	2	3
V	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

4.11. U = the number of trials needed to get the first head.

V = ... two . . . .

Are U, V independent?

$$P(U=u;1,\tfrac{1}{2}) = \binom{u-1+1-1}{1-1}(1-\tfrac{1}{2})^{u-1}(\tfrac{1}{2})^{1}$$
$$= \frac{(u-1)!}{0!(u-1)!}\cdot(\tfrac{1}{2})^{u}$$
$$= 1\,(\tfrac{1}{2})^{u}$$
$$= (\tfrac{1}{2})^{u}$$

$$P(V=v;2,\tfrac{1}{2}) = \binom{v-2+2-1}{2-1}(1-\tfrac{1}{2})^{v-1}(\tfrac{1}{2})^{2}$$
$$= \frac{(v-1)!}{1!(v-1-1)!}\cdot(\tfrac{1}{2})^{v}$$
$$= (v-1)(\tfrac{1}{2})^{v}$$

$$P(u,v) = (\tfrac{1}{2})^{u}\underbrace{(\tfrac{1}{2})^{\cancel{v-u-1}}}_{\text{tail}}\cdot\underbrace{(\tfrac{1}{2})}_{\text{head}}\cdot(\tfrac{1}{2})^{v}$$

$\neq P(U)\cdot P(V)$

$\therefore$ not independent.

4.12. $a \quad b \quad c \qquad P(a+b > c > a-b)$? $\quad a+b+c=1$

$a, b \in (0,1]$

$c = 1-a-b$

$a+b > c \Rightarrow a+b > 1-a-b \qquad a+b > \frac{1}{2}$

$c > a-b \Rightarrow 1-a\cancel{-b} > a\cancel{-b} \qquad \frac{1}{2} > a$

$\therefore P(a+b > c > a-b)$

$= P(a+b > \frac{1}{2},\ a < \frac{1}{2})$

$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \qquad b > \frac{1}{2}-a \quad \because a < \frac{1}{2} \quad \therefore b > 0$

$\therefore P(a+b > \frac{1}{2}) = P(b > \frac{1}{2}-a) = \frac{1}{2}$

4.13 (a) $\min_{g(\cdot)} E(Y-g(Z))^2 = E(Y-E(Y|Z))^2$

拆开: $E[Y^2 - 2Yg(Z) + g^2(Z)]$

$= E(Y^2) + E[g^2(Z)] - 2EY\cdot E\,g(Z)$

$E(Y^2 - 2Y\cdot E(Y|Z) + E^2(Y|Z)]$

$= E(Y^2) + E^2(Y|Z) - 2EY\cdot E(Y|Z)$

$f(Z) = E[g^2(Z)] - 2EY\,E\,g(Z)$

$\frac{d\,f(Z)}{d\,g(Z)} = 2E(g(Z)) - 2EY = 0$

$\therefore E(g(Z)) = EY$

$\because E(Y|Z)$ is the best predictor of Y conditional on Z

$\therefore$ when $g(Z) = E(Y|Z)$

取到 min

(b) $\min_{b} E(Z-b)^2 = E(Z-EZ)^2$

Let $Z = b \qquad E(Y|Z) = E(Y) \qquad g(Z) = Z = b$

$\therefore \min_{b} E(Y-b)^2 = E(Y-EY)^2 \quad \square$

4.14. X, Y are independent, $X, Y \sim n(0,1)$

(a) Find $P(X^2+Y^2<1)$

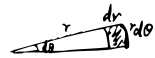
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \mu=0 \quad \sigma=1$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

Sol 1: 三角換元

$\because X, Y$ are independent

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \cdot e^{-\frac{x^2+y^2}{2}}$$



$$P(X^2+Y^2<1) = \int_0^1 \int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}} \underbrace{r\,dr\,d\theta}_{\Delta}$$

$$= \frac{1}{2\pi} \int_0^1 2\pi \cdot e^{-\frac{r^2}{2}} r\,dr$$

$$= \int_0^1 e^{-\frac{r^2}{2}} r\,dr$$

$$= -e^{-\frac{r^2}{2}} \Big|_0^1$$

$$= -e^{-\frac{1}{2}} + 1 = 1 - e^{-\frac{1}{2}} \quad \square$$

Sol 2

$\because X \sim n(0,1) \qquad Y \sim n(0,1)$

$X^2 \sim \chi_1^2 \qquad Y^2 \sim \chi_1^2$

$\chi_1^2 + \chi_1^2 \sim \chi_2^2$ two degree freedom

$X \sim \chi_2^2 \Rightarrow X \sim \Gamma(\frac{2}{2}, 2) \qquad \Gamma(1,2) \sim \exp(\frac{1}{2})$

$\chi_k^2 \sim \Gamma(\frac{k}{2}, 2)$

$$\Gamma(x;1,2) = \frac{x^0 \cdot e^{-\frac{x}{2}}}{\frac{1}{2}\Gamma(1)} \qquad \Gamma(\alpha,\beta) = \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}$$

$\Gamma(1) = 0! = 1$

$$\Gamma(x;1,\tfrac{1}{2}) = 2 \cdot e^{-\frac{x}{2}}$$

$$F(1;1,\tfrac{1}{2}) = \int_0^1 2 \cdot e^{-\frac{x}{2}}\,dx$$

$$= 2\int_0^1 e^{-\frac{x}{2}}\,dx$$

$$= -e^{-\frac{x}{2}} \Big|_0^1$$

$$= 1 - e^{-\frac{1}{2}}$$

$\exp(\frac{1}{2})$

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2} \cdot e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F(1) = \int_0^1 \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2} \int_0^1 e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2} \cdot e^{-\frac{x}{2}} (-2) \Big|_0^1$$

$$= 1 - e^{-\frac{1}{2}}$$

(b) Verifying $X^2 \sim \chi_1^2$, then find $P(X^2<1)$

$\because X^2 \sim \chi_1^2 \sim \Gamma(\frac{1}{2}, 2) \qquad \Gamma(\alpha,\beta) = \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}$

$$\Gamma(\tfrac{1}{2}, 2) = \frac{x^{-\frac{1}{2}} \cdot e^{-\frac{x}{2}}}{\sqrt{2}\,\Gamma(\frac{1}{2})}$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\Gamma(\tfrac{1}{2}, 2) = \frac{x^{-\frac{1}{2}} \cdot e^{-\frac{x}{2}}}{\sqrt{2} \cdot \sqrt{\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} \cdot e^{-\frac{x}{2}}$$

$$\therefore F(1;\tfrac{1}{2},2) = \frac{1}{\sqrt{2\pi}} \cdot \underbrace{\int_0^1 x^{-\frac{1}{2}} \cdot e^{-\frac{x}{2}}\,dx}_{I}$$

无法解下去了

換思路:

$$P(X^2<1) = P(-1<X<1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= 2\Phi(1) - 1$$

$$= 2 \times 0.84 - 1$$

$$= 1.68 - 1$$

$$= 0.68$$

Proof: $\because B(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

Let $m, n = \frac{1}{2}$

$$B(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)}$$

$\Gamma(1) = (1-1)! = 1$

$\therefore B(\frac{1}{2}, \frac{1}{2}) = \Gamma^2(\frac{1}{2})$

Beta dis: $f(x;m,n)$

$$= \frac{1}{B(m,n)} x^{m-1} (1-x)^{n-1} \qquad x \in [0,1]$$

$$= \frac{1}{\Gamma^2(\frac{1}{2})} \cdot x^{-\frac{1}{2}} \cdot (1-x)^{-\frac{1}{2}}$$

$$\therefore \frac{1}{\Gamma^2(\frac{1}{2})} \int_0^1 x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}}\,dx = 1$$

$$\therefore \Gamma^2(\tfrac{1}{2}) = \int_0^1 \underline{x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}}}\,dx$$

三角換元

Let $x = \sin^2\theta \qquad dx = 2\sin\theta \cdot \cos\theta\,d\theta$

$x=0 \qquad \theta=0$

$x=1 \qquad \theta=\frac{\pi}{2}$

$$\therefore \Gamma^2(\tfrac{1}{2}) = \int_0^{\frac{\pi}{2}} \frac{2\sin\theta \cdot \cos\theta}{\sin\theta \cdot \cos\theta}\,d\theta = \pi$$

$\therefore \Gamma(\frac{1}{2}) = \sqrt{\pi}$

4.15. $X \sim \text{Poisson}(\theta)$ $\quad Y \sim \text{Poisson}(\lambda)$ $\quad X+Y \sim \text{Poisson}(\theta+\lambda)$

a. Proof: $X \mid X+Y \sim \text{binomial}\left(\frac{\theta}{\theta+\lambda}\right)$

Let $U = X+Y$ $\quad V = X$

$\because U \sim \text{Poisson}(\theta+\lambda)$

$\therefore f(u) = \frac{(\theta+\lambda)^u \cdot e^{-(\theta+\lambda)}}{u!}$

$f(v \mid u) = \frac{f(u,v)}{f(u)}$

$f(u,v) = \frac{\lambda^{u-v} \cdot e^{-\lambda}}{(u-v)!} \cdot \frac{\theta^v \cdot e^{-\theta}}{v!} \cdot \underset{1}{|J|}$

$= \frac{u!}{(u-v)! \cdot v!} \cdot \frac{\lambda^{u-v} \cdot \theta^v}{(\theta+\lambda)^u}$

$= \binom{u}{v} \left(\frac{\theta}{\theta+\lambda}\right)^v \cdot \left(\frac{\lambda}{\theta+\lambda}\right)^{u-v}$ $\quad f_{X,Y}(u^{-1}, v^{-1}) \cdot |J|$

$\therefore X \mid X+Y \sim \text{binominal}\left(\frac{\theta}{\theta+\lambda}\right)$ □

b. $Y \mid X+Y \sim \text{binominal}\left(\frac{\lambda}{\theta+\lambda}\right)$

Proof: let $U = X+Y$ $\quad V = Y$

$f(v \mid u) = \frac{f(u,v)}{f(u)}$ $\quad f(u,v) = f_{X,Y}(u^{-1}, v^{-1}) |J|$

$= \binom{u}{v} \left(\frac{\lambda}{\theta+\lambda}\right)^v \cdot \left(\frac{\theta}{\theta+\lambda}\right)^{u-v}$ $\quad = \frac{\theta^{u-v} \cdot e^{-\lambda}}{(u-v)!} \frac{\lambda^v \cdot e^{-\lambda}}{v!}$

$\therefore Y \mid X+Y \sim \text{bionominal}\left(\frac{\lambda}{\theta+\lambda}\right)$ □

4.16. (a) X, Y are independent, with same geo distri

$U = \min(X, Y)$ $\quad V = X - Y$, U, V are independent

$X = \{0, 1, 2, 3, \cdots\}$ $\quad U = \{0, 1, 2, 3, \cdots\}$

$Y = \{0, 1, 2, 3, \cdots\}$ $\quad V = \{\cdots -3, -2, -1, 0, 1, 2, 3 \cdots\}$

$f(u,v) = P(U=u, V=v)$

$= P_{X,Y}(X = u+v, Y = u) \cdot 1$

$= (1-p)^{u+v-1} \cdot p \; (1-p)^{u-1} \cdot p$

$= (1-p)^{2u+v-2} \cdot p^2$

1° If $v > 0$, $X > Y$

let $Y = u$ $\quad X = Y + v = u + v$

$|J| = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$

2° If $v = 0$ $\quad X = Y$

$X = Y = u$

$f(u,v) = P_{X,Y}(X = Y = u)$

$= (1-p)^{u-1} \cdot p \cdot (1-p)^{u-1} \cdot p$

$= (1-p)^{2u-2} \cdot p^2$

$f(u,v) = P_{X,Y}(X = u, Y = u-v)$

$= (1-p)^{u-1} \cdot p \; (1-p)^{u-v-1} \cdot p$

$= (1-p)^{2u-v-2} \cdot p^2$

3° If $v < 0$ $\quad X < Y$

let $X = u$ $\quad Y = X - v = u - v$

$|J| = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$

$\therefore$ 综上

$f_{U,V}(u,v) = \underbrace{\left(p^2 (1-p)^{2u}\right)} \underbrace{(1-p)^{|v|-2}}$

can be factorize into two functions of u, v, respectively

U, V are independent.

(b). Find $P(Z)$ $\quad Z = X/(X+Y)$ $\quad$ if $X+Y=0$ $\quad Z=0$

Let $r = X$, $\quad S = X+Y$ $\quad r < S$ $\quad Y = S - r$ $\quad (X, Y) \sim (ir, i(s-r))$ $i = 1, 2, \cdots$

$r, s \in \mathbb{Z}^+$

$P(Z = \frac{r}{s}) = \sum_{i=1}^{\infty} P_{X,Y}(X = ir, Y = i(s-r)) = \sum p(1-p)^{ir-1} \; p(1-p)^{i(s-r)-1}$

$= p^2 (1-p)^{-2} \cdot \sum (1-p)^{is}$ 等比 $\quad p^2 (1-p)^{is-2}$

$= \frac{p^2 (1-p)^{s-2}}{1-(1-p)^s}$

(c) Find $f(X, X+Y)$

Let $U = Z$ $\quad V = Z+Y$ $\quad Y = V-U$

$X = U$

$f(u,v) = f_{Z,Y}(Z=u, Y=v-u)\cdot|J|$ $\quad |J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$= p(1-p)^{v-1}\cdot p(1-p)^{v-u-1}$

$= p^2(1-p)^{v-2}$

4.17(a) $Z \sim expo(1)$ $\qquad$ Find $P(Y)$

$Y = i+1 \quad i \le Z < i+1$

$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases} \quad \lambda = 1$

$f(x) = \begin{cases} e^{-x} & x \ge 0 \\ 0 & x < 0 \end{cases}$

$P(Y) = \int_i^{i+1} e^{-i}\, di = -e^{-i}\Big|_i^{i+1} = -e^{-i-1} + e^{-i} = e^{-i}(1-e^{-1})$

$\because Y = i+1$

$P(y) = e^{-y-1}(1-e^{-1})$ is geometric distri $p = 1-e^{-1}$

(b). Find $P(X-4 \mid Y \ge 5)$

$Y \ge 5 \Leftrightarrow Z \ge 4$

$\therefore P(Z-4 \mid Y \ge 5) = P(Z-4 \mid Z \ge 4) = P(Z-4 \le x \mid Z \ge 4) = P(x \ge 0) = e^{-x}$

4.18. $\int_0^\infty g(x)dx = 1 \quad g(x) \ge 0 \Rightarrow g(x)$ is a pdf

Proof: $f(x,y) = \dfrac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}}$ $x, y > 0$ is a pdf.

$\because g(x) \ge 0$ and $\sqrt{x^2+y^2} \ge 0$

$\therefore f(x,y) \ge 0$

$\int_y\int_x f(x,y)\, dx, y$ $\quad$ let $x = \sin\alpha \quad y = \cos\alpha \quad \sqrt{x^2+y^2} = r$

$= \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{2}{\pi}\frac{g(r)}{r} r\, dr\, d\theta$

$= \int_0^\infty g(r)\underbrace{\frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1\, d\theta}_{1}\, dr$ $\qquad \therefore \iint f(x,y)\,dx\,dy = 1$

$\therefore f(x,y)$ is a pdf.

$= \int_0^\infty g(r)\, dr$

$= 1$

4.19 (a) Z_1, Z_2 are independent $n(0,1)$ $\quad$ Find $f((Z_1-Z_2)^2/2)$

$f(Z_1 = x) = \frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad \mu = 0 \quad \sigma = 1$ $\qquad f_{Z_1Z_2}(x_1, x_2) = \frac{1}{2\pi}\cdot e^{-\frac{(x_1^2+x_2^2)}{2}}$

$f_{Z_1}(x) = f_{Z_2}(x) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}$ $\qquad (2Y_1)^{\frac{1}{2}}$

Let $Y_1 = (X_1-X_2)^2/2$ $\quad Z_1 = \sqrt{2Y_1} + Y_2$ $\quad |J| = \begin{vmatrix} \frac{1}{2}(2Y_1)^{-\frac{1}{2}}\cdot 2 & 1 \\ 0 & 1 \end{vmatrix} = (2Y_1)^{-\frac{1}{2}}$

$Y_2 = X_2 \quad Z_2 = Y_2$

$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi}\cdot e^{-\frac{(\sqrt{2y_1}+y_2)^2 + y_2^2}{2}}\cdot|(2y_1)^{-\frac{1}{2}}|$

要分 $X_1 \ge X_2$

$X_1 < X_2$

两段.

$$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi} \cdot e^{-\frac{(\sqrt{2y_1}+y_2)^2 + y_2^2}{2}} \cdot (2y_1)^{-\frac{1}{2}}$$

$$e^{+\frac{y_1}{2}} \cdot \underbrace{\int_0^\infty e^{-\left(y_2 + \frac{\sqrt{2y_1}}{2}\right)^2} dy_2}_{\frac{\sqrt{\pi}}{2}}$$

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

$$f_{Y_1}(y_1) = \int_0^\infty f_{Y_1Y_2}(y_1, y_2)\, dy_2$$

$$= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{2}} \int_0^\infty e^{-\frac{2y_1 + 2\sqrt{2}\sqrt{y_1}\, y_2 + y_2^2}{2}} \cdot (y_1)^{-\frac{1}{2}}\, dy_2$$

$$= \frac{1}{2\sqrt{2}\pi} (y_1)^{-\frac{1}{2}} \cdot e^{-y_1} \int_0^\infty e^{-\left(\sqrt{2y_1}\, y_2 + y_2^2\right)} dy_2$$

$$= \frac{1}{2\sqrt{2}\pi} (y_1)^{-\frac{1}{2}} \cdot e^{-\frac{y_1}{2}} \cdot \sqrt{\pi}$$

$$= \frac{1}{2\sqrt{2\pi}} (y_1)^{-\frac{1}{2}} \cdot e^{-\frac{y_1}{2}}$$

$$\chi_1^2 = \frac{x^{-\frac{1}{2}} \cdot e^{-\frac{x}{2}}}{2^{\frac{1}{2}} \sqrt{\pi}} \qquad k=1 \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

(b). If $X_1 \sim \text{gamma}(\alpha_1, 1)$ independent.

$X_2 \sim \text{gamma}(\alpha_2, 1)$

Find the pdf of $X_1/(X_1+X_2)$ & $X_2/(X_1+X_2)$

$$f(x_1) = \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)} = \frac{x^{\alpha_1 - 1} \cdot e^{-x}}{\Gamma(\alpha_1)}$$

$$f(x_2) = \frac{x^{\alpha_2 - 1} \cdot e^{-x}}{\Gamma(\alpha_2)}$$

$f(x_1+x_2) \sim \text{gamma}(\alpha_1 + \alpha_2, 1)$

$$\therefore f(x_1+x_2) = \frac{x^{\alpha_1+\alpha_2-1} \cdot e^{-x_1-x_2}}{\Gamma(\alpha_1+\alpha_2)}$$

Let $U = X_1 \quad V = X_1 + X_2$

$x_1 = u \qquad |J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$x_2 = v - u$

$$f(u, v) = f(U=u, V=v)$$

$$= f_{X_1X_2}(X_1 = u, X_2 = v-u) \cdot |J|$$

$\because X_1, X_2$ independent $\quad \therefore f_{X_1X_2} = f(x_1) \cdot f(x_2)$

$$\therefore f(u, v) = \frac{u^{\alpha_1-1} \cdot e^{-u}}{\Gamma(\alpha_1)} \cdot \frac{(v-u)^{\alpha_2-1} \cdot e^{-(v-u)}}{\Gamma(\alpha_2)} \cdot 1$$

[没不下去]

Let $U = \frac{X_1}{X_1+X_2} \quad V = X_1$

$x_1 = v \qquad |J| = \begin{vmatrix} 0 & 1 \\ -v \cdot u^{-2} & \cdot \end{vmatrix} = v \cdot u^{-2}$

$x_2 = \frac{v}{u} - v$

$$f(u, v) = f_{X_1X_2}\left(X_1 = v, X_2 = \frac{v}{u} - v\right) |J|$$

$$= \frac{v^{\alpha_1-1} \cdot e^{-v}}{\Gamma(\alpha_1)} \cdot \frac{\left(\frac{v}{u} - v\right)^{\alpha_2-1} \cdot e^{-\left(\frac{v}{u} - v\right)}}{\Gamma(\alpha_2)} \cdot v \cdot u^{-2}$$

$$= \frac{v^{\alpha_1} \cdot \cancel{e^{-v}}}{\Gamma(\alpha_1)} \cdot \frac{v^{\alpha_2-1} \cdot \left(\frac{1}{u} - 1\right)^{\alpha_2-1} \cdot e^{-\frac{v}{u} \cancel{+v}}}{\Gamma(\alpha_2)\, u^2}$$

$$= \frac{v^{\alpha_1+\alpha_2-1} \cdot e^{-v}}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)} \cdot \frac{\left(\frac{1}{u} - 1\right)^{\alpha_2-1} \cdot e^{\frac{1}{u}}}{u^2}$$

u, v. independent

$$\therefore f(u) = f\left(\frac{x_1}{x_1+x_2}\right) = \frac{\left(\frac{x_1}{x_1+x_2}\right)^{\alpha_1+\alpha_2-1} \cdot e^{-\frac{x_1}{x_1+x_2}}}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)}$$

$u = \frac{x_2}{x_1+x_2} \qquad V = x_2$

$x_1 = \frac{v}{u} - v \qquad |J| = \begin{vmatrix} -v \cdot u^{-2} & \cdot \\ 0 & 1 \end{vmatrix} = v \cdot u^{-2}$

$x_2 = v$

$$f(u, v) = f_{X_1X_2}\left(X_1 = \frac{v}{u} - v, X_2 = v\right) \cdot |J|$$

$$= \frac{\left(\frac{v}{u} - v\right)^{\alpha_1-1} \cdot e^{-(\ )}}{\Gamma(\alpha_1)} \cdot \frac{v^{\alpha_2-1} \cdot e^{-v}}{\Gamma(\alpha_2)}$$

α_1, α_2 交换位置，但不影响结果.

$$\therefore f\left(\frac{x_2}{x_1+x_2}\right) = f\left(\frac{x_1}{x_1+x_2}\right) = \frac{\left(\frac{x_1}{x_1+x_2}\right)^{\alpha_1+\alpha_2-1} \cdot e^{-\frac{x_1}{x_1+x_2}}}{\Gamma(\alpha_1)\, \Gamma(\alpha_2)}$$

□

4.20. $X_1, X_2 \sim n(0, \sigma^2)$

(a) Find $f(Y_1, Y_2)$

$$Y_1 = X_1^2 + X_2^2 \qquad Y_2 = \frac{X_1}{\sqrt{Y_1}}$$

$$f(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \mu = 0$$

$$\therefore f(x_1) = f(x_2) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}} \qquad f(x_1, x_2) = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}$$

$$Y_1 = X_1^2 + X_2^2 \qquad (Y_1 \geq 0)$$

$$Y_2^2 = \frac{X_1^2}{Y_1} = \frac{X_1^2}{X_1^2+X_2^2} \qquad Y_1 Y_2^2 = X_1^2 \qquad \underline{X_1 = Y_2\sqrt{Y_1}} = Y_2 \cdot Y_1^{\frac{1}{2}}$$

$$X_2 = \sqrt{Y_1 - Y_1 Y_2^2} = [Y_1(1-Y_2^2)]^{\frac{1}{2}} \qquad |J| = \begin{vmatrix} \frac{1}{2} \cdot Y_2 \cdot Y_1^{-\frac{1}{2}} & Y_1^{\frac{1}{2}} \\ \frac{1}{2} \cdot Y_1^{-\frac{1}{2}} \cdot (1-Y_2^2)^{\frac{1}{2}} & -Y_1^{\frac{1}{2}} Y_2 (1-Y_2^2)^{-\frac{1}{2}} \end{vmatrix}$$

$$f(y_1, y_2) = f_{Z_1, Z_2}(Z_1 = Y_2\sqrt{Y_1},\ Z_2 = [Y_1(1-Y_2^2)]^{\frac{1}{2}}) \cdot |J| \qquad = \left| -\frac{1}{2} Y_2^2 (1-Y_2^2)^{-\frac{1}{2}} - \frac{1}{2} \cdot (1-Y_2^2)^{-\frac{1}{2}} \right|$$

$$= \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{Y_1}{2\sigma^2}} \cdot \frac{1}{2}(1-Y_2^2)^{-\frac{1}{2}} \qquad = \frac{1}{2}\left[Y_2^2(1-Y_2^2)^{-\frac{1}{2}} + (1-Y_2^2)^{\frac{1}{2}}\right] = \underline{\frac{1}{2}(1-Y_2^2)^{-\frac{1}{2}}}$$

$$= \frac{1}{4\pi\sigma^2} \cdot e^{-\frac{Y_1}{2\sigma^2}} (1-Y_2^2)^{-\frac{1}{2}} \quad \times$$

$\because A_1 = \{-\infty < x_1 < \infty,\ x_2 > 0\} \Leftarrow B = \{0 < y_1 < \infty,\ -1 < y_2 < 1\}$

$A_2 = \{-\infty < x_1 < \infty,\ x_2 < 0\}$

$$\begin{cases} x_1 = y_2\sqrt{y_1} \\ x_2 = [y_1(1-y_2^2)]^{\frac{1}{2}} \end{cases} \quad J_1$$

$$J_1 = -J_2$$

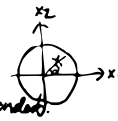
$$\begin{cases} x_1 = y_2\sqrt{y_1} \\ x_2 = -[y_1(1-y_2^2)]^{\frac{1}{2}} \end{cases} \quad J_2$$

(b) $\because f_{Y_1Y_2}(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)$

$\therefore Y_1, Y_2$ are independent.

$Y_1 = r^2 \qquad Y_2 = \cos\alpha$

these two variables are independent.



4.21. $R^2 \sim \chi_2^2$

$\theta \sim$ uniform $(0, 2\pi)$

Find $f_{X,Y}(x, y)$, $X = R\cos\theta \quad Y = R\sin\theta$

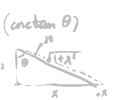
$\because R^2 \sim \chi_2^2 \qquad k=2$

$$\text{chi-square } f(\chi_k^2) = \begin{cases} \dfrac{\chi^{\frac{k}{2}-1} \cdot e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\,\Gamma(\frac{k}{2})} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\therefore f_{R^2}(r^2) = \frac{e^{-\frac{r^2}{2}}}{2 \cdot \Gamma(1)} \qquad \Gamma(1) = (0)! = 1$$

$$= \frac{1}{2} \cdot e^{-\frac{r^2}{2}} \quad r^2 > 0$$

Proof (arctan θ)



$\tan\theta = x \qquad \theta = \arctan x \qquad \frac{d}{dx}\tan\theta = \frac{d}{dx}x$

$$\cos\theta = \frac{1}{\sqrt{1+x^2}} \qquad \left[\frac{d}{d\theta}\tan\theta\right]\frac{d\theta}{dx} = 1$$

$$\cos^2\theta = \frac{1}{1+x^2} \qquad \left(\frac{\sin\theta}{\cos\theta}\right)' = \frac{\cos^2\theta + \sin\theta}{\cos^2\theta} \cdot \frac{1}{\cos^2\theta}$$

$$\frac{d\theta}{dx} = \frac{1}{1+x^2} \qquad \therefore \frac{1}{\cos^2\theta} \cdot \frac{d\theta}{dx} = 1$$

$$\therefore \frac{d\arctan x}{dx} = \frac{1}{1+x^2} \qquad \therefore \underline{\frac{d\theta}{dx} = \cos^2\theta}$$

$\theta \sim$ uniform $(0, 2\pi)$

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

$\Leftarrow f_{X,Y}(x, y) \qquad x = r \cdot \cos\theta \qquad y = r \cdot \sin\theta$

r^2, θ independent $\therefore f_{R^2,\Theta}(r^2, \theta) \Rightarrow f_{X,Y}$

$$f_{R^2\Theta} = f_{R^2}(r^2) \cdot f_\Theta(\theta) = \frac{1}{4\pi} \cdot e^{-\frac{r^2}{2}} \qquad r \in (0, \infty),\ \theta \in (0, 2\pi)$$

$x = r \cdot \cos\theta \qquad R^2 = X^2 + Y^2 \in (0, \infty)$

$y = r \cdot \sin\theta \qquad \theta = \arctan\frac{Y}{X}$

$$\therefore J = \begin{vmatrix} 2x & 2y \\ \left(\frac{1}{1+\frac{y^2}{x^2}}\right) 2 \cdot (-x^{-2}) & \left(\frac{1}{1+\frac{y^2}{x^2}}\right) \cdot \frac{1}{x} \end{vmatrix}$$

$$\therefore f_{X,Y}(x, y) = f_{R^2,\Theta}(r^2 = x^2+y^2,\ \theta = \tfrac{y}{x}) \cdot J \qquad = \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2}$$

$$= \frac{1}{4\pi} e^{-\frac{x^2+y^2}{2}} \cdot 2 \qquad = \frac{2(x^2+y^2)}{x^2+y^2}$$

$$= \frac{1}{2\pi} \cdot e^{-\frac{x^2+y^2}{2}} \qquad = 2$$

$\therefore X, Y$ are independent normal distribution.

standard.

4.22. $(X, Y) \sim f(x, y)$

$U = aX + b \qquad V = cY + d \qquad a > 0,\ c > 0$

Proof: $f_{U,V}(u,v) = \frac{1}{ac} \cdot f\left(\frac{u-b}{a}, \frac{v-d}{c}\right)$

$\because U = aX + b \quad \therefore X = \frac{U-b}{a}$

$V = cY + d \quad Y = \frac{V-d}{c}$

$J = \begin{vmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{ac}$

$f_{U,V}(u,v) = f_{X,Y}\left(X = \frac{u-b}{a}, Y = \frac{v-d}{c}\right) \cdot |J| \qquad \because a > 0,\ c > 0 \quad \therefore \frac{1}{ac} > 0$

$= f\left(\frac{u-b}{a}, \frac{v-d}{c}\right) \cdot \frac{1}{ac} \qquad \square$

4.23. $X \sim beta(\alpha, \beta) \quad Y \sim beta(\alpha+\beta, \gamma)$ find XY by making the transformation (a)/(b) and integrating out V.

(a) $U = XY \qquad V = Y$

(a) $\because X, Y$ are independent

$\therefore f_{X,Y}(X = x, Y = y) = f_X(x) \cdot f_Y(y)$

$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}$

$f(y; \alpha+\beta, \gamma) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\cdot\Gamma(\gamma)} \cdot y^{\alpha+\beta-1} (1-y)^{\gamma-1}$

$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} x^{\alpha-1} \cdot y^{\alpha+\beta-1} (1-x)^{\beta-1} (1-y)^{\gamma-1}$

Let $U = XY \qquad X = \frac{U}{V}\ [0,1] \qquad J = \begin{vmatrix} \frac{1}{v} & (-1)U \cdot V^{-2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$

$V = Y \qquad Y = V\ [0,1]$

$0 < u < v < 1$

$f_{U,V}(U = u, V = v) = f_{X,Y}\left(X = \frac{u}{v}, Y = v\right) \cdot |J|$

$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot \left(\frac{u}{v}\right)^{\alpha-1} \cdot v^{\alpha+\beta-1} \left(1 - \frac{u}{v}\right)^{\beta-1} (1-v)^{\gamma-1} \cdot \frac{1}{v}$

直接:

$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \cdot v^{\beta-1} \left(1 - \frac{u}{v}\right)^{\beta-1} (1-v)^{\gamma-1}$

$= (\alpha+\beta+\gamma-1) \cdot \Gamma(\alpha+\beta-1) \cdot \frac{\Gamma(\alpha+\beta-1+\gamma)}{\Gamma(\alpha+\beta-1)\Gamma(\gamma)} v^{\alpha+\beta-1} (1-v)^{\gamma-1} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{u}{v}\right)^{\alpha-1} \left(1 - \frac{u}{v}\right)^{\beta-1}$

$f_U(u) = \int_u^1 f_{U,V}(u,v)\, dv$

Let $k = \frac{v-u}{1-u} \qquad v = (1-u)k + u$

$v = u \quad k = 0 \qquad dv = (1-u)dk$

$v = 1 \quad k = 1$

$= \int_u^1 \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \cdot v^{\beta-1} \left(1 - \frac{u}{v}\right)^{\beta-1} (1-v)^{\gamma-1} dv$

$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \int_u^1 v^{\beta-1} (1-v)^{\gamma-1} \left(\frac{v-u}{v}\right)^{\beta-1} dv$

$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} k^{\beta-1} (1-k)^{\alpha-1} dk = 1$

$\therefore \int_0^1 k^{\beta-1} (1-k)^{\alpha-1} dk = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \int_u^1 (1-v)^{\gamma-1} (v-u)^{\beta-1} dv$

$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \int_0^1 [1-(1-u)k-u]^{\gamma-1} [(1-u)k]^{\beta-1} (1-u)\, dk$

$\int_0^1 (1-u)^{\gamma-1} (1-k)^{\gamma-1} \cdot k^{\beta-1} (1-u)^{\beta}\, dk$

$= \frac{\Gamma(\cdots)}{\Gamma\cdots\Gamma\cdots} u^{\alpha-1} \cdot (1-u)^{\gamma+\beta-1} \int_0^1 (1-k)^{\gamma-1} \cdot k^{\beta-1} dk = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} (1-u)^{\gamma+\beta-1} \cdot \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} u^{\alpha-1} \cdot (1-u)^{\gamma+\beta-1}$

(b) $U = XY \quad V = X/Y$

$\because X, Y$ are independent

$\therefore f_{X,Y}(X=x, Y=y) = f_X(x) \cdot f_Y(y)$

$X, Y \in [0,1]$
$U \in [0,1]$
$V \in [0, \infty)$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} x^{\alpha-1} \cdot y^{\alpha+\beta-1} (1-x)^{\beta-1} (1-y)^{\gamma-1}$$

Let $U = XY \qquad X = (UV)^{\frac{1}{2}}$

$V = X/Y \qquad Y = \left(\frac{U}{V}\right)^{\frac{1}{2}}$

$0 < u < v < \frac{1}{u} \qquad 0 < u < 1$

$$J = \begin{vmatrix} \frac{1}{2}(UV)^{-\frac{1}{2}} \cdot V & \frac{1}{2}(UV)^{-\frac{1}{2}} U \\ V^{-\frac{1}{2}} \cdot \frac{1}{2} \cdot U^{-\frac{1}{2}} & U^{\frac{1}{2}} \cdot (-\frac{1}{2}) \cdot V^{-\frac{3}{2}} \end{vmatrix}$$

$$= -\frac{1}{4} U^{-1} - \frac{1}{4} V^{-1}$$

$$= -\frac{1}{2} V^{-1}$$

$$f_{U,V}(U=u, V=v) = f_{X,Y}\left(X = (uv)^{\frac{1}{2}}, Y = \left(\frac{u}{v}\right)^{\frac{1}{2}}\right) |J|$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} (uv)^{\frac{\alpha-1}{2}} \left(\frac{u}{v}\right)^{\frac{\alpha+\beta-1}{2}} \left[1 - (uv)^{\frac{1}{2}}\right]^{\beta-1} \left[1 - \left(\frac{u}{v}\right)^{\frac{1}{2}}\right]^{\gamma-1} \cdot \frac{1}{2v}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1+\frac{\beta}{2}} \cdot \underbrace{v^{-\frac{\beta}{2}-1}}_{v} \cdot \frac{1}{2} \left[1 - (uv)^{\frac{1}{2}}\right]^{\beta-1} \left[1 - \left(\frac{u}{v}\right)^{\frac{1}{2}}\right]^{\gamma-1}$$

$$k = \frac{(uv)^{\frac{1}{2}} - u}{1-u}$$

$v = u \qquad k = \frac{u-u}{1-u} = 0$

$v = \frac{1}{u} \qquad k = \frac{1-u}{1-u} = 1$

$$v = \frac{[(1-u)k+u]^2}{u} \quad ①$$

$$dv = \frac{2}{u}[(1-u)k+u] \cdot (1-u)\,dk$$

$$1-(uv)^{\frac{1}{2}} = 1-[(1-u)k+u]$$

$$= (1-u)(1-k) \quad ②$$

$$\left(\frac{u}{v}\right)^{\frac{1}{2}} = u^{\frac{1}{2}} \cdot v^{-\frac{1}{2}} = u^{\frac{1}{2}} \cdot [(1-u)k+u]^{-1} \cdot u^{\frac{1}{2}}$$

$$= u \cdot [(1-u)k+u]^{-1}$$

$$= \frac{u}{(1-u)k+u}$$

$$1-\left(\frac{u}{v}\right)^{\frac{1}{2}} = 1 - \frac{u}{(1-u)k+u}$$

$$= \frac{(1-u)k}{(1-u)k}$$

$$f_U(U=u) = \int_{u}^{\frac{1}{u}} f(u,v)\,dv$$

substitute v with ①

$1-(uv)^{\frac{1}{2}}$ with ②

$1-\left(\frac{u}{v}\right)^{\frac{1}{2}}$ with ③

then:

$$f_U(U=u) = \int_0^1 \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot [(1-u)k+u]^{\alpha-1} \left[\frac{u}{(1-u)k+u}\right]^{\alpha+\beta-1} \cdot \frac{1}{2}$$

$$(1-u)^{\beta-1}(1-k)^{\beta-1} \left[1 - \frac{u}{(1-u)k+u}\right]^{\gamma-1}$$

$$\frac{2}{u}[(1-u)k+u](1-u)\,dk \qquad \cancel{\alpha-1-\alpha-\beta+1}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int u^{\alpha+\beta-2} \left[\frac{1}{(1-u)k+u}\right]^{\beta-1}$$

$$(1-u)^{\beta}(1-k)^{\beta-1} \left[1 - \frac{u}{(1-u)k+u}\right]^{\gamma-1} dk$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int u^{\alpha+\beta-2} u^{-\beta+1} (1-u)^{\beta} \underbrace{(1-k)^{\beta-1}}$$

$$\underbrace{\left[\frac{u}{(1-u)k+u}\right]^{\beta-1} \left[1 - \frac{u}{(1-u)k+u}\right]^{\gamma-1}} dk = \frac{\Gamma(\alpha+\beta+\gamma)}{\cancel{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}} u^{\alpha-1} \underbrace{(1-u)^{\beta+\gamma-1}} \cdot \frac{\cancel{\Gamma(\alpha)\Gamma(\gamma)}}{\Gamma(\alpha+\gamma)} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\gamma)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta+\gamma-1}$$

$$\frac{u^{\beta-1}}{[(1-u)k+u]^{\beta-1}} \quad \frac{(1-u)^{\gamma-1} k^{\gamma-1}}{[(1-u)k+u]^{\gamma-1}} = \frac{u^{\beta-1}(1-u)^{\gamma-1} k^{\gamma-1} (1-k)^{\beta-1}}{[(1-u)k+u]^{\beta+\gamma}}$$

$$= \left[1 - \frac{k}{(1-u)k+u}\right]^{\beta-1} \left[\frac{k}{(1-u)k+u}\right]^{\gamma-1}$$

$$\underbrace{(1-u)^{\gamma-1}}$$

4.24 $X \sim \text{gamma}(r, 1)$ Show: $Z_1 = X+Y$ are independent.

$Y \sim \text{gamma}(s, 1)$ $Z_2 = X/(X+Y)$

Find $f(Z_1)$ & $f(Z_2)$

$f_X(x) = \dfrac{x^{r-1} \cdot e^{-x}}{\Gamma(r)}$ $x>0$ gamma $f(x;\alpha,\beta) = \dfrac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)}$

$f_Y(y) = \dfrac{y^{s-1} e^{-y}}{\Gamma(s)}$ $y>0$ beta $Z_2 = \dfrac{X}{X+Y} = \dfrac{1}{1+Y} < 1$

$\because$ X, Y are independent

$\therefore f_{X,Y}(x,y) = \dfrac{x^{r-1} \cdot e^{-x} \cdot y^{s-1} \cdot e^{-y}}{\Gamma(r) \cdot \Gamma(s)}$

$\because Z_1 = X+Y > 0$ $X = Z_1 \cdot Z_2$ $J = \begin{vmatrix} z_2 & z_1 \\ 1-z_2 & -z_1 \end{vmatrix} = -z_1 z_2 - z_1 + z_1 z_2 = -z_1$

$Z_2 = X/(X+Y) \in (0,1]$ $\therefore$ $Y = z_1 - z_1 z_2$ $|J| = z_1$

$\therefore f_{Z_1, Z_2}(Z_1 = z_1, Z_2 = z_2) = f_{X,Y}(X = z_1 z_2, Y = z_1 - z_1 z_2)$

$$= \frac{(z_1 z_2)^{r-1} \cdot e^{-z_1 z_2}}{\Gamma(r)} \cdot \frac{(z_1 - z_1 z_2)^{s-1} \cdot e^{-(z_1 - z_1 z_2)}}{\Gamma(s)} \cdot z_1$$

$$= \frac{z_1^{r-1} z_2^{r-1} e^{-z_1 z_2}}{\Gamma(r)} \cdot \frac{z_1^{s-1} (1-z_2)^{s-1} \cdot e^{-z_1(1-z_2)}}{\Gamma(s)} \cdot z_1$$

$$= \frac{z_1^{r+s-1} e^{-z_1}}{\Gamma(r+s)} \cdot \frac{z_2^{r-1} (1-z_2)^{s-1} \Gamma(r+s)}{\Gamma(r)\Gamma(s)}$$

$\therefore$ Z_1, Z_2 are independent

$f_{Z_1}(Z_1 = z_1) = \dfrac{z_1^{r+s-1} e^{-z}}{\Gamma(r+s)}$ $\sim$ gamma distribution

$f_{Z_2}(Z_2 = z_2) = \dfrac{\Gamma(r+s)}{\Gamma(r) \cdot \Gamma(s)} \cdot z_2^{r-1} (1-z_2)^{s-1}$ $\sim$ beta distribution.

4.25 Find $f(x,y)$ from $f(X,Z)$

(a) 4.5.8 P201 $X \sim \text{uniform}(0,1)$ $Z \sim \text{uniform}(0, \frac{1}{10})$

$Y = X+Z$ $f(x,y)$?

$f(x) = 1$ $f(z) = \dfrac{1}{\frac{1}{10} - 1} = 10$

$Y = Z+Z$ $X = X$, $Z = Y - X$ $J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$\because$ X, Z are independent

$\therefore f_{X,Z}(X = x, Z = z) = f(x) \cdot f(z) = 10$

$f_{X,Y}(X = x, Y = y) = f_{X,Z}(X = x, Z = y - x) \cdot |J| = 10$ $\square$

(b) 4.5.9 Box $X \sim$ Uniform $(-1,1)$ $Z \sim$ uniform $(0, \frac{1}{10})$

$Y = X^2 + Z$ $f(x,y)$?

$f(x) = \frac{1}{1-(-1)} = \frac{1}{2}$ $f(z) = \frac{1}{\frac{1}{10} - 0} = 10$

$f_{X,Z}(x,z) = f(x) \cdot f(z) = 5$

$Y = X^2 + Z$ $\therefore$ $X = X$, $Z = Y - X^2$ $J = \begin{vmatrix} 1 & 0 \\ -2x & 1 \end{vmatrix} = 1$

$f_{X,Y}(x,y) = f_{X,Z}(X = x, Z = y - x^2) \cdot |J|$

$= 5$ □

4.26 $X \sim$ exponential (λ) $Y \sim$ exponential (μ)

$Z = \min\{X, Y\}$ $W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$

(a) Find $f_{Z,W}(z,w)$

$f(x) = \lambda \cdot e^{-\lambda x}$ $x \geq 0$ $f(x,y) = \lambda \cdot e^{-\lambda x} \cdot \mu \cdot e^{-\mu y}$

$f(y) = \mu \cdot e^{-\mu y}$ $y \geq 0$ $= \lambda\mu \cdot e^{-\lambda x - \mu y}$

1° $0 \leq x < y$

$z = x$ $Z = z$

$w = 1$

$f_{Z,W}(z,w) = \int_x^\infty f_{X,Y}(z,y)\, dy$

$= \int_x^\infty \lambda\mu \cdot e^{-\lambda x} e^{-\mu y}\, dy$

$= \lambda\mu e^{-\lambda x} \int_x^\infty e^{-\mu y}\, dy$ ⊗

Let $k = y - x$ $y = k + x$

$y = x$ $k = 0$ $dy = dk$

$y = \infty$ $k = \infty$

$\therefore \int_x^\infty e^{-\mu y}\, dy = \int_0^\infty e^{-\mu(k+x)}\, dk$

$= e^{-\mu x} \int_0^\infty e^{-\mu k}\, dk$

$\because \int \mu \cdot e^{-\mu k}\, dk = 1$ $\therefore \int_0^\infty e^{-\mu k}\, dk = \frac{1}{\mu}$

$\therefore \int_x^\infty e^{-\mu y}\, dy = \frac{1}{\mu} \cdot e^{-\mu x}$ ⊙

代入⊗

$f_{Z,W}(z,w) = \lambda\mu \cdot e^{-\lambda x} \cdot \frac{1}{\mu} e^{-\mu x}$

$= \lambda \cdot e^{-(\lambda+\mu)x}$

$= \lambda \cdot e^{-(\lambda+\mu)z}$

2° $0 \leq y \leq x$

$z = y$

$w = 0$

$f_{Z,W}(z,w) = \int_y^\infty f_{X,Y}(x,y)\, dx$

$= \int_y^\infty \lambda\mu \cdot e^{-\lambda x} \cdot e^{-\mu y}\, dx$

$= \lambda\mu \cdot e^{-\mu y} \cdot \underline{\int_y^\infty e^{-\lambda x}\, dx}$

$\int_y^\infty e^{-\lambda x}\, dx = \frac{1}{\lambda} \cdot e^{-\lambda y}$

$f_{Z,W}(z,w) = \mu \cdot e^{-(\lambda+\mu)y} = \mu \cdot e^{-(\lambda+\mu)z}$

$\therefore f_{Z,W}(z,w) = \begin{cases} \lambda \cdot e^{-(\lambda+\mu)z} & w = 1 \\ \mu \cdot e^{-(\lambda+\mu)z} & w = 0 \end{cases}$

(b) Prove Z, W are independent

Hint $P(Z \leq z \mid W = i) = P(Z \leq z)$

$P(Z \leq z \mid W = 1) = \int_0^z \lambda \cdot e^{-(\lambda+\mu)z}\, dz$

$= \frac{\lambda}{\lambda+\mu} \int_0^z (\lambda+\mu) e^{-(\lambda+\mu)z}\, dz$

$P(Z \leq z \mid W = 0) = \frac{\mu}{\lambda+\mu} \int_0^z (\lambda+\mu) \cdot e^{-(\lambda+\mu)z}\, dz$

$P(Z \leq z) = \int_0^z (\lambda+\mu) e^{-(\lambda+\mu)z}\, dz$

$= P(Z \leq z \mid W = 1) + P(Z \leq z \mid W = 0)$ □

4.27. $X \sim n(\mu, \sigma^2)$ $Y \sim n(\gamma, \sigma^2)$, X, Y are independent

$U = X + Y$ Proof U, V are independent n dist

$V = X - Y$ Find $f_U(u)$ $f_V(v)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\because X, Y$ are independent

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\gamma)^2}{2\sigma^2}} \qquad \therefore f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 + (x-\gamma)^2\right]}$$

$\because U = X + Y$ $\quad \therefore X = \frac{1}{2}(U+V)$

$V = X - Y$ $\quad Y = \frac{1}{2}(U-V)$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \qquad |J| = \frac{1}{2}$$

$$f_{U,V}(U=u, V=v) = f_{X,Y}\left(X = \tfrac{1}{2}(u+v), Y = \tfrac{1}{2}(u-v)\right) \cdot |J|$$

$$= \frac{1}{4\pi\sigma^2} \cdot e^{-\frac{1}{2\sigma^2}\left(\left[\frac{1}{2}(u+v) - \mu\right]^2 + \left[\frac{1}{2}(u-v) - \gamma\right]^2\right)}$$

$$\left[\tfrac{1}{2}(u+v) - \mu\right]^2 + \left[\tfrac{1}{2}(u-v) - \gamma\right]^2$$

$$= \frac{1}{4}(u+v)^2 - \mu(u+v) + \mu^2 + \frac{1}{4}(u-v)^2 - \gamma(u-v) + \gamma^2$$

$$= \frac{1}{4}u^2 + \cancel{\frac{1}{2}uv} + \frac{1}{4}v^2 + \frac{1}{4}u^2 - \cancel{\frac{1}{2}uv} + \frac{1}{4}v^2 - u(\mu+\gamma) - v(\mu-\gamma) + \mu^2 + \gamma^2$$

$$= \frac{1}{2}u^2 - u(\mu+\gamma) + \frac{1}{2}v^2 - v(\mu-\gamma)$$

$$= \left[\frac{1}{\sqrt{2}}u - \frac{\sqrt{2}}{2}(\mu+\gamma)\right]^2 - \frac{1}{2}(\mu+\gamma)^2 + \left[\frac{1}{\sqrt{2}}v - \frac{\sqrt{2}}{2}(\mu-\gamma)\right]^2 - \frac{1}{2}(\mu-\gamma)^2 + \mu^2 + \gamma^2$$

$$= \frac{1}{2}\left(u - \frac{\mu+\gamma}{2}\right)^2 + \frac{1}{2}\left(v - \frac{\mu-\gamma}{2}\right)^2$$

$$u^2 + \frac{1}{v^2} - 2\frac{u}{v} + u^2$$

$$2u^2 - 2\frac{u}{v} + \frac{1}{v^2}$$

$$\therefore f_{U,V}(U=u, V=v) = \frac{1}{4\pi\sigma^2} \cdot e^{-\frac{1}{2\sigma^2}\left[\frac{1}{2}\left(u - \frac{\mu+\gamma}{2}\right)^2 + \frac{1}{2}\left(v - \frac{\mu-\gamma}{2}\right)^2\right]}$$

$$= \frac{1}{2\sqrt{\pi}\sigma} \cdot e^{-\frac{1}{4\sigma^2}\left(u - \frac{\mu+\gamma}{2}\right)^2} \cdot \frac{1}{2\sqrt{\pi}\sigma} \cdot e^{-\frac{1}{4\sigma^2}\left(u - \frac{\mu+\gamma}{2}\right)^2}$$

$$= f_U(U=u) \cdot f_V(V=v)$$

$\therefore U, V$ are independent $\square$

$U \sim n\left(\frac{\mu+\gamma}{2}, \sqrt{2}\sigma\right)$, $V \sim n\left(\frac{\mu-\gamma}{2}, \sqrt{2}\sigma\right)$ $\square$

4.28 X, Y are independent $X \sim n(0,1)$ $Y \sim n(0,1)$

(a) Show $X/(X+Y) \sim$ Cauchy distribution.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \qquad f_{X,Y}(x, y) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x^2+y^2)} \qquad \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$$

Let $U = X$ $\quad X = U$

$V = \frac{X}{X+Y}$ $\quad Y = \left(\frac{1}{V} - 1\right)U$

$$J = \begin{vmatrix} 1 & 0 \\ \frac{1}{v} - 1 & -U \cdot V^{-2} \end{vmatrix} = -U \cdot V^{-2} = -\frac{U}{V^2}$$

$$f_{U,V}(u, v) = f_{X,Y}\left(X = u, Y = \left(\tfrac{1}{v} - 1\right)u\right) \cdot |J|$$

$$f_{U,V}(u,v) = f_{Z,Y}\left(Z=u,\ Y=\left(\tfrac{1}{v}-1\right)u\right)\cdot|J|$$

$$= f_{Z,Y}\left[u,\left(\tfrac{1}{v}-1\right)u\right]\cdot\frac{u}{v^2} \qquad u>0$$

$$+ f_{Z,Y}\left[u,\left(\tfrac{1}{v}-1\right)u\right]\cdot\frac{-u}{v^2} \qquad u<0$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du$$

$$= \int_0^{\infty}\frac{1}{2\pi}\cdot e^{-\frac{1}{2}\left(u^2+\left[\left(\frac{1}{v}-1\right)u\right]^2\right)}\frac{u}{v^2}\,du$$

$$+\int_{-\infty}^{0}\frac{1}{2\pi}\cdot e^{-\frac{1}{2}\left(u^2+\left[\left(\frac{1}{v}-1\right)u\right]^2\right)}\cdot\frac{-u}{v^2}\,du$$

$$= 2\int_0^{\infty}\frac{1}{2\pi}\cdot e^{-\frac{1}{2}\left(u^2+\left[\left(\frac{1}{v}-1\right)u\right]^2\right)}\frac{u}{v^2}\,du$$

$$= \frac{1}{\pi v^2}\int_0^{\infty} e^{-\frac{1}{2}u^2\left[\left(\frac{1}{v}-1\right)^2+1\right]}\,u\,du$$

$$= \frac{1}{\pi v^2}\cdot\frac{v^2}{(1-v)^2+v^2}$$

$$= \frac{1}{\pi}\cdot\frac{1}{(1-v)^2+v^2}$$

$$= \frac{1}{\pi}\cdot\frac{1}{2\left(v-\frac{1}{2}\right)^2+\frac{1}{2}}$$

$$= \frac{1\times 4}{\pi\cdot 2}\,\frac{\frac{1}{4}}{\left(v-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$= \frac{1}{\pi\cdot\frac{1}{2}}\cdot\frac{\left(\frac{1}{2}\right)^2}{\left(v-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$\int_0^{\infty} e^{u^2\left(-\frac{1}{2}\right)\left[\left(\frac{1}{v}-1\right)^2+1\right]}\,u\,du$$

Let $S=u^2$ $\because u\geq 0$ $\therefore u=\sqrt{S}$ $\therefore du = \frac{1}{2}\cdot S^{-\frac{1}{2}}\,ds$

$u=0$ $S=0$

$u=\infty$ $S=\infty$

$$\therefore \int_0^{\infty} e^{S\cdot\left(-\frac{1}{2}\right)[\cdots]}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{S}}\,ds$$

$$= \frac{1}{2}\int_0^{\infty} e^{S\cdot\left(-\frac{1}{2}\right)[\cdots]}\,ds$$

$$= \frac{1}{2}\cdot\frac{1}{\left(-\frac{1}{2}\right)[\cdots]}\,e^{S\left(-\frac{1}{2}\right)[\cdots]}\Big|_0^{\infty}$$

$$= -\frac{1}{[\cdots]}(0-1)$$

$$= \frac{1}{\left(\frac{1}{v}-1\right)^2+1}$$

$$= \frac{v^2}{(1-v)^2+v^2}$$

$\gamma = \frac{1}{2}$ $\quad x_0 = \frac{1}{2}$

$f_V(v) \sim f(v; x_0, \gamma) = f\left(v; \frac{1}{2}, \frac{1}{2}\right)$ □

Rule: $\dfrac{\text{norm dis}}{\text{norm dis}} \sim$ Cauchy dis

(b) Find $f_{Z/|Y|}$

Let $U = Y$ $\quad\therefore Z = UV$ $\quad J = \begin{vmatrix} v & u \\ 1 & 0 \end{vmatrix} = -u$

$V = \frac{Z}{Y}$ $\quad Y = U$

$$f_{U,V}(U=u, V=v) = f_{Z,Y}(Z=uv, Y=u)\cdot|J|$$

$$= \frac{1}{2\pi}\cdot e^{-\frac{1}{2}(u^2v^2+u^2)}\cdot|u|$$

$$f_V(v) = \int_0^{\infty} f_{U,V}(U=u, V=v)\,u\,du$$

$$+\int_{-\infty}^{0} f_{U,V}(\qquad)(-u)\,du$$

$$= 2\int_0^{\infty} f_{U,V}(\qquad)\,u\,du$$

$\because u \in (0,+\infty)$, that is $Y \in (0,+\infty)$

$\therefore V = \frac{Z}{Y} = \frac{Z}{|Y|}$

$$\therefore f_V(v) = \cancel{2}\int_0^{\infty}\frac{1}{2\pi}\cdot e^{-\frac{1}{2}u^2(v^2+1)}\cdot u\,du$$

$$= \frac{1}{\pi}\int_0^{\infty} e^{u^2\left(-\frac{1}{2}\right)(v^2+1)}\,u\,du$$

$$= \frac{1}{\pi}\,\frac{1}{\left(-\frac{1}{2}\right)(v^2+1)}\left(\frac{1}{2}\right)e^{u^2\left(-\frac{1}{2}\right)(v^2+1)}\Big|_0^{\infty}\,du$$

$$= \frac{1}{\pi}\cdot\frac{1}{v^2+1} \qquad \gamma = 1 \quad x_0 = 0$$

$f_{Z/|Y|} \sim f\left(\frac{Z}{|Y|}; 0, 1\right) = f(v; 0, 1)$ □

4.29. $Z = R\cdot\cos\theta \quad Y = R\cdot\sin\theta \quad \theta \sim U(0, 2\pi)$

(a) Show that $Z/Y \sim$ Cauchy dis

Let $U = \frac{Z}{Y} = \frac{1}{\tan\theta}$ $\qquad f(\theta) = \frac{1}{2\pi}$ $\qquad (\arctan\theta)' = \frac{1}{1+\theta^2}$ Prob

$\theta = \arctan\left(\frac{1}{U}\right) \qquad \theta' = \arctan'\left(\frac{1}{U}\right) = \frac{1}{1+\left(\frac{1}{U}\right)^2}\cdot(-1)\cdot U^{-2}$

$f_U(U=u) = 2 f_\theta\left(\theta = \arctan\left(\frac{1}{u}\right)\right)\cdot|\theta'|$

$= \frac{1}{2\pi}\cdot\frac{1}{1+\left(\frac{1}{u}\right)^2}\cdot u^{-2}$

$= \frac{1}{\pi}\cdot\frac{u^2}{u^2+1}\cdot\frac{1}{u^2}$

$= \frac{1}{\pi}\cdot\frac{1}{u^2+1} \qquad x_0 = 0 \quad \gamma = 1$

$\therefore f_U(U=u) \sim f(u; 0, 1)$ $\qquad \square$

(b). Show that $f\left(\frac{2ZY}{\sqrt{X^2+Y^2}}\right) = f(Z)$

$\because \frac{2ZY}{\sqrt{Z^2+Y^2}} = \frac{2R^2\cdot\cos\theta\cdot\sin\theta}{R}$

$= R\sin 2\theta$

$Z = R\cdot\cos\theta$

$\therefore f\left(\frac{2ZY}{\sqrt{X^2+Y^2}}\right) = f(R\cdot\sin 2\theta) = f(R)\cdot f(\sin 2\theta)$ ①

$f(Z) = f(R\cdot\cos\theta) = f(R)\cdot f(\cos\theta)$ ②

$\therefore$ To show ① = ② $\Rightarrow$ to show $f(\sin 2\theta) = f(\cos\theta)$

$f_{\sin 2\theta} = f_{\sin\theta}$

$2\theta = k$
$\theta = 0 \quad k = 0$
$\theta = \pi \quad \theta = \frac{1}{2}k$

$\left(f_{\sin\theta}\right)' \left(\int_0^{2\pi} f_{\sin} dk\right)'$

$f_{\sin 2\theta} = \frac{1}{2} f_{\sin\theta}$

Proof:

$f_{\sin 2\theta} = \frac{1}{2} f_{\sin\theta} \quad \theta\in(0,\pi)$

repeat $\left\{ \frac{1}{2} f_{\sin\theta} \quad \theta\in(\pi, 2\pi) \right.$

$\therefore f_{\sin 2\theta} = f_{\sin\theta}$

$\sin\theta \qquad \sin 2\theta$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

Proof:

Let $\arccos x = y$ $\qquad$ Aim $\frac{dy}{dx}$

$\cos y = x$

$(\cos y)' = (x)'$

$-\sin y\cdot\frac{dy}{dx} = 1$

$\therefore \frac{dy}{dx} = -\frac{1}{\sin y}$

$= -\frac{1}{\sqrt{1-\cos^2 y}}$

$= -\frac{1}{\sqrt{1-x^2}}$

$f_{\cos\theta}$?

Repeat

Let $U = \cos\theta \qquad \theta = \arccos U \qquad \theta' = -\frac{1}{\sqrt{1-U^2}}$

$f_U(U=u) = 2\frac{1}{2\pi}|\theta'| = \frac{1}{\pi}\cdot\frac{1}{\sqrt{1-u^2}}$

Let $V = \sin\theta \qquad \theta = \arcsin V \qquad \theta' = \frac{1}{\sqrt{1-V^2}}$

$f_U(V=v) = 2\frac{1}{2\pi}\cdot|\theta'| = \frac{1}{\pi}\cdot\frac{1}{\sqrt{1-v^2}}$

Repeat

$\therefore \frac{2ZY}{\sqrt{X^2+Y^2}}$ has the same dis as Z.

4.30 $f(y|x) \sim n(x, x^2)$, $f(x) \sim \text{uniform}(0,1)$

(a). Find EY, $Var\,Y$, $cov(X,Y)$

$E(Y) = E\{E(Y|X)\} = EX = \frac{1}{2}$

$Var(Y) = Var(E(Y|X)) + E(Var(Y|X))$

$= Var(X) + E(X^2)$

$= \frac{1}{12} + \frac{1}{3}$

$= \frac{5}{12}$

$\because f(y|x) \sim n(x, x^2)$

$\therefore E(Y|X) = X$

$Var(Y|X) = x^2$

$\because f(x) \sim \text{uniform}(0,1)$

$\therefore E(X) = \frac{1}{2}$

$Var(X) = E[x - E(x)]^2$

$= EX^2 - E^2(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$EX^2 = \int_0^1 x^2 f(x)\,dx$

$= \int_0^1 x^2\,dx$

$= \frac{1}{3} \cdot x^3 \Big|_0^1$

$= \frac{1}{3}$

$Cov(X,Y) = E[(X - EX)(Y - EY)]$

$= E[XY - EX \cdot Y - EY \cdot X + EX \cdot EY]$

$= E\,XY - EX \cdot EY$

$EX = \frac{1}{2}$ $\quad EY = \frac{1}{2}$ $\quad EX \cdot EY = \frac{1}{4}$

$EXY = E\{\underbrace{E(XY|X)}_{\text{cons}}\} = E\{X \cdot \underbrace{E(Y|X)}_{\text{cons } X}\}$

$= EX^2$

$= \frac{1}{3}$

$\therefore Cov(X,Y) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

(b) Prove Y/X & X are independent

$U = Y/X$ $\quad X = V$ $\quad J = \begin{vmatrix} 0 & 1 \\ V & U \end{vmatrix} = -V$

$V = X \in (0,1)$ $\quad Y = UV$

$f_{U,V}(U=u, V=v) = f_{X,Y}(X=v, Y=uv) \cdot |J| = \frac{1}{\sqrt{2\pi}\,v} \cdot e^{-\frac{(uv-v)^2}{2v^2}} \cdot v$

$f(y|x) = \frac{1}{\sqrt{2\pi} \cdot x} \cdot e^{-\frac{(y-x)^2}{2x^2}}$

$= \frac{1}{\sqrt{2\pi}\,\cancel{v}} \cdot e^{-\frac{(u-1)^2 \cdot v^2}{2v^2}} \cdot \cancel{v}$

$= \frac{1}{\sqrt{2\pi}\,\cancel{v}} \cdot e^{-\frac{(u-1)^2}{2}} \cdot \cancel{v}$

$f_U(U=u) = \int_0^1 f_{U,V}(u,v)\,dv$

$= \int_0^1 \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(u-1)^2}{2}}\,dv$

$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(u-1)^2}{2}}$

$f_X(X=x) = 1$

$\therefore$ Y/X and X are independent. $\square$

4.31 $Y \sim$ binomial dis n trails

$Z \sim$ uniform$(0,1)$ $\quad$ P(success)

(a) Find EY & $Var\,Y$

$EY = E\{E(Y|Z)\} = E[nZ] = nEZ \qquad \because EZ = \frac{1}{2}$

$\therefore EY = n \cdot \frac{1}{2} = \frac{n}{2}$

$Var\,Y = E\{(Y-EY)^2\} = E\{(\underbrace{Y - E(Y|Z)} + \underbrace{E(Y|Z) - EY})^2\}$

$= E\{(\underbrace{Y-E(Y|Z)})^2\} + E\{(\underbrace{E(Y|Z)-EY})^2\}$

$+ 2E\{(\underbrace{Y-E(Y|Z)})(\underbrace{E(Y|Z)-EY})\}$

$E\{(\underbrace{Y-E(Y|Z)})(\underbrace{E(Y|Z)-EY})\}$

$= E\{E[(\underbrace{Y-E(Y|Z)})(\overset{cons}{E(Y|Z)}-\overset{cons}{EY}) \mid Z]\}$

$= E\{(E(Y|Z)-EY)\cdot\underbrace{[E(Y|Z)-E(Y|Z)]}_{0}$

$= 0$

$\therefore Var\,Y = E\{(Y-E(Y|Z))^2\} + E\{(E(Y|Z)-EY)^2\}$

$= E\{E[(Y-E(Y|Z))^2 \mid Z]\} + Var[E(Y|Z)]$

$= E[Var(Y|Z)] + Var[E(Y|Z)]$

$= E[nZ(1-Z)] + Var(nZ)$

$= nE(Z-Z^2) + n^2 Var(Z)$

$= n\cdot(\frac{1}{2}-\frac{1}{3}) + n^2\cdot\frac{1}{12}$

$= n\cdot\frac{1}{6} + n^2\cdot\frac{1}{12}$

$E[E(Y|Z)] = EY$

$\overset{a}{Z} \sim \text{uniform}(\overset{b}{0},1)$

$Var(Z) = E(Z-EZ)^2$

$= EZ^2 - (EZ)^2 = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$

$EZ^2 = \int_a^b x^2 \frac{1}{b-a}\,dx$

$= \frac{1}{b-a}\cdot\frac{1}{3}x^3\Big|_a^b$

$= \frac{b^3-a^3}{3(b-a)}$

$= \frac{1}{3}$

$Var(Z) = \frac{b^3-a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2$

(b) Find joint distri of Z & Y

$P_{Y,Z}(Y=y, Z=x) = \binom{n}{y}\cdot x^y(1-x)^{n-y}$

(c) Find marginal dis of Y

$P_Y(Y=y) = \int_0^1 \binom{n}{y} x^y(1-x)^{n-y}\,dx$

$= \binom{n}{y}\underbrace{\int_0^1 x^y(1-x)^{n-y}\,dx}$

$= \binom{n}{y}\cdot\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$

Beta dis:

$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot x^{\alpha-1}(1-x)^{\beta-1}$

$f(x; y+1, n-y+1) = \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\,x^y(1-x)^{n-y}$

$\because \int_0^1 f(x; y+1, n-y+1)\,dx = 1$

$\therefore \int_0^1 x^y(1-x)^{n-y}\,dx = \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n-2)}$

4.32 (a) $Y \mid \Lambda \sim \text{Poisson}(\Lambda)$ $\quad \Lambda \sim \text{gamma}(\alpha, \beta)$

Find $f_Y(y)$, $E(Y)$, $Var(Y)$

$$f(y \mid \Lambda) = \frac{\Lambda^y e^{-\Lambda}}{y!} \qquad y = 0, 1, 2, \cdots$$

$$f_Y(y) = \int_0^\infty \frac{\Lambda^y e^{-\Lambda}}{y!} \, d\Lambda$$

$$= \frac{1}{y!} \cancel{\int_0^\infty \Lambda^y e^{-\Lambda} \, d\Lambda}$$

$$= \cancel{\frac{\Gamma(y+1)}{y!}}$$

let $\alpha = y+1$ $\quad \beta = 1$

$$f(\Lambda; y+1, 1) = \frac{\Lambda^y \cdot e^{-\Lambda}}{\Gamma(y+1)}$$

$$\because \int_0^\infty \frac{\Lambda^y e^{-\Lambda}}{\Gamma(y+1)} d\Lambda = 1$$

$$\therefore \int_0^\infty \Lambda^y \cdot e^{-\Lambda} d\Lambda = \Gamma(y+1)$$

$$E(Y) = E(E(Y \mid \Lambda))$$

$$= E(\Lambda)$$

$$= \alpha\beta$$

$$Var(Y) = Var[E(Y \mid \Lambda)] + E[Var(Y \mid \Lambda)]$$

$$= Var(\Lambda) + E(\Lambda)$$

$$= \beta^2 \alpha + \alpha\beta \qquad \square$$

$$f_Y(y) = \int_0^\infty f(y \mid \Lambda) \cdot f_\Lambda(\Lambda) \, d\Lambda$$

$$= \int_0^\infty \frac{\Lambda^y \cdot e^{-\lambda}}{y!} \cdot \frac{\Lambda^{\alpha-1} \cdot e^{-\Lambda/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)} \, d\Lambda$$

$$= \frac{1}{y! \cdot \beta^\alpha \cdot \Gamma(\alpha)} \underbrace{\int_0^\infty \Lambda^{y+\alpha-1} \cdot e^{-\Lambda(1+\frac{1}{\beta})} \, d\Lambda}$$

$$= \frac{\left(\frac{\beta}{1+\beta}\right)^{y+\alpha}}{y! \cdot \beta^\alpha \cdot \Gamma(\alpha)} \underbrace{\int_0^\infty \left[\Lambda(1+\tfrac{1}{\beta})\right]^{y+\alpha-1} \cdot e^{-\Lambda(1+\frac{1}{\beta})} \, d\Lambda}_{\Gamma(y+\alpha)}$$

$$= \frac{\beta^y \quad \Gamma(y+\alpha)}{y! \cdot \Gamma(\alpha) \cdot (1+\beta)^{y+\alpha}}$$

If α is an integer, $\Gamma(y+\alpha) = (y+\alpha-1)!$

$\Gamma(\alpha) = (\alpha-1)!$

$$f_Y(y) = \frac{(y+\alpha-1)!}{y! \cdot (\alpha-1)!} \cdot$$

$$= \binom{y+\alpha-1}{y} \left(\frac{\beta}{1+\beta}\right)^y \left(\frac{1}{1+\beta}\right)^\alpha$$

$Y \sim$ bionomial distribution $(\alpha, \frac{1}{1+\beta})$

Poisson

$$f(y \mid \Lambda) = \frac{\Lambda^y \cdot e^{-\Lambda}}{y!}$$

$$E(Y) = \sum_{y=1}^\infty y \cdot f(y) = \sum \frac{\Lambda^y e^{-\Lambda}}{(y-1)!}$$

$$= \Lambda \cdot e^{-\Lambda} \cdot \sum_{y=1}^\infty \frac{\Lambda^{y-1}}{(y-1)!} = \Lambda \cdot e^{-\Lambda} \underbrace{\sum_{k=0}^\infty \frac{\Lambda^k}{k!}}_{e^\Lambda}$$

$$= \Lambda$$

$$Var(Y) = E(Y - EY)^2$$

$$= EY^2 - (EY)^2$$

$$= \Lambda^2 + \Lambda - \Lambda^2 = \Lambda$$

$$EY^2 = \sum_{y=1}^\infty y^2 \cdot f(y)$$

$$= \sum_1^\infty y^2 \cdot \frac{\Lambda^y \cdot e^{-\Lambda}}{y!}$$

$$= \sum_1^\infty \frac{y \cdot \Lambda^y \cdot e^{-\Lambda}}{(y-1)!}$$

$$= e^{-\Lambda} \sum_1^\infty \frac{\Lambda^y}{(y-1)!} (y-1+1)$$

$$= e^{-\Lambda} \left[\sum_2^\infty \frac{\Lambda^y}{(y-2)!} + \sum_1^\infty \frac{\Lambda^y}{(y-1)!} \right]$$

$$= e^{-\Lambda} \{ \Lambda^2 \cdot e^\Lambda + \Lambda \cdot e^\Lambda \}$$

$$= \Lambda^2 + \Lambda$$

Gamma

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

$$f(\Lambda; \alpha, \beta) = \frac{\Lambda^{\alpha-1} e^{-\Lambda/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

$$E(\Lambda) = \int_0^\infty \Lambda \cdot f(\Lambda; \alpha, \beta) \, d\Lambda$$

$$= \int_0^\infty \frac{\Lambda^\alpha \cdot e^{-\Lambda/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)} \, d\Lambda$$

$$= \frac{\beta^\alpha \beta}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int_0^\infty \left(\frac{\Lambda}{\beta}\right)^\alpha e^{-\Lambda/\beta} \frac{1}{\beta} \, d\Lambda}_{\Gamma(\alpha+1)}$$

gamma function $\Gamma(z) = \int_0^\infty x^{z-1} \cdot e^{-x} dx$

$$\therefore E(\Lambda) = \frac{\Gamma(\alpha+1)\beta}{\Gamma(\alpha)}$$

$$= \alpha\beta$$

$$Var(\Lambda) = E\Lambda^2 - E^2(\Lambda)$$

$$E\Lambda^2 = \int_0^\infty \Lambda^2 \cdot f(\Lambda; \alpha, \beta) \, d\Lambda$$

$$= \int_0^\infty \frac{\Lambda^{\alpha+1} \cdot e^{-\Lambda/\beta}}{\beta^\alpha \quad \Gamma(\alpha)} \, d\Lambda$$

$$= \frac{\beta^{\alpha+1} \beta}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int \left(\frac{\Lambda}{\beta}\right)^{\alpha+1} \cdot e^{-\Lambda/\beta} \frac{1}{\beta} \, d\Lambda}_{\Gamma(\alpha+2)}$$

$$= \beta^2 \cdot \frac{(\alpha+1) \cdot \alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)} = \beta^2 \cdot (\alpha+1)\alpha$$

$$\therefore Var(\Lambda) = E\Lambda^2 - E^2(\Lambda)$$

$$= \beta^2(\alpha^2 + \alpha) - \alpha^2\beta^2$$

$$= \beta^2\alpha$$

(b). $Y|N \sim \text{binomial}(N, p)$ $\quad N|\Lambda \sim \text{Poisson}(\Lambda)$, $\Lambda \sim \text{gamma}(\alpha, \beta)$

Find $P(Y)$

$$P_Y(Y=y|\Lambda) = \sum_{n=y}^{\infty} P(Y=y|\overset{\Sigma}{N=n}, \Lambda=\lambda) \cdot P(N=n|\Lambda)$$

$$= \sum_{n=y}^{\infty} \binom{n}{y} \cdot p^y \cdot (1-p)^{n-y} \cdot \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= \sum_{n=y}^{\infty} \frac{\cancel{n!}}{y!(n-y)!} \cdot p^y (1-p)^{n-y} \cdot \frac{\lambda^n \cdot e^{-\lambda}}{\cancel{n!}}$$

Since need to ΣN & $\int_0^\infty \lambda$ $\therefore$ has to have two layers (steps)

$$= \sum_{n=y}^{\infty} \frac{1}{y!(n-y)!} \cdot \left(\frac{p}{1-p}\right)^y \cdot [(1-p)\cdot\lambda]^n \cdot e^{-\lambda} \qquad \text{let } m=n-y$$

$$= e^{-\lambda} \sum_{m=0}^{\infty} \frac{1}{y!\, m!} \cdot \left(\frac{p}{1-p}\right)^y [(1-p)\cdot\lambda]^{m+y}$$

$$= e^{-\lambda} \frac{1}{y!} \left(\frac{p}{1-p}\right)^y [(1-p)\lambda]^y \cdot \underbrace{\sum_{m=0}^{\infty} \frac{[(1-p)\lambda]^m}{m!}}_{} \rightarrow e^{(1-p)\lambda}$$

$$= e^{-\lambda} \frac{1}{y!} \left(\frac{p}{\cancel{1-p}}\right)^y [(\cancel{1-p})\lambda]^y \cdot e^{(1-p)\lambda}$$

$$= \frac{1}{y!} (p\lambda)^y \cdot e^{-p\lambda} \sim \text{Poisson distri}$$

$$f_Y(Y=y) = \int_0^\infty P_Y(Y=y|\lambda) \cdot f(\lambda)\, d\lambda \qquad \lambda \sim \text{gamma}(\alpha, \beta)$$

$$= \int_0^\infty \frac{1}{y!} (p\lambda)^y \cdot e^{-p\lambda} \cdot \frac{\lambda^{\alpha-1} e^{-\lambda/\beta}}{\beta^\alpha \Gamma(\alpha)}\, d\lambda$$

$$= \frac{1}{y!} p^y \frac{1}{\Gamma(\alpha)\beta^\alpha} \int \lambda^y \lambda^{\alpha-1} e^{-\left(p\lambda + \frac{\lambda}{\beta}\right)}$$

$$= \frac{p^y e^{p+\frac{1}{\beta}}}{y!\, \beta^\alpha \Gamma(\alpha)} \int_0^\infty \lambda^{y+\alpha-1} e^{-\lambda(p+\frac{1}{\beta})}\, d\lambda \qquad \text{gamma fun}$$

$$= \frac{p^y}{y! \cdot \beta^\alpha \Gamma(\alpha)(p+\frac{1}{\beta})^{y+\alpha+1}} \underbrace{\int [\lambda(p+\tfrac{1}{\beta})]^{y+\alpha-1} \cdot e^{-\lambda(p+\frac{1}{\beta})} (p+\tfrac{1}{\beta})\, d\lambda}_{\Gamma(y+\alpha)}$$

If α is an int

$$f(Y) = \frac{(y+\alpha)!}{(\alpha-1)! \cdot y!} \left(\frac{p\beta}{p\beta+1}\right)^y \left(1 - \frac{p\beta}{p\beta+1}\right)^\alpha$$

$\sim$ negative binomial dis $\left(\alpha, \frac{1}{p\beta+1}\right)$ $\square$

取得 α 次成功之前
需要 y 失敗

4.33. $N \sim \text{Poisson}(\lambda)$ $\quad H|N = Z_1 + Z_2 + \cdots + Z_N$, $\quad P(Z_i = t) = \frac{-1}{\log(p)} \frac{(1-p)^t}{t}$

Aim: $f(H) \sim$ negative binomial(r, p)

$$r = -\frac{\lambda}{\log p}$$

$y = \sum_{n=1}^{\infty} \frac{a^n}{n} \qquad \frac{dy}{da} = \sum \frac{1}{n} \cdot n a^{n-1} = \sum_{n=1}^{\infty} a^{n-1} = \sum_{k=0}^{\infty} a^k = \frac{a^{n+1}-1}{a-1}$

if $|a|<1$, $a^{n+1} \to 0$ $\quad \frac{dy}{da} = \frac{1}{1-a}$

$\therefore y = \int_0^a \frac{dy}{dx} dx = \int_0^a \frac{1}{1-x} dx = -\ln^{1-x}\Big|_0^a = -\ln^{1-a}$

$\because H = Z_1 + Z_2 + \cdots Z_N$ $\quad$ Hint: use mgf

$$E e^{Ht} = E\{E(e^{Ht}|N)\} = E\{E(e^{(Z_1+Z_2+\cdots+Z_N)t}|N)\}$$

$\because Z_1, Z_2, \cdots Z_N$ are identical independent distributed (i.i.d)

$$\therefore E e^{Ht} = E\{E(e^{N \cdot Z_i t}|N)\} \qquad E(e^{Z_i t}) = \sum_{x_i=0}^{\infty} e^{Z_i t} \cdot P(Z_i) = \sum_1^{\infty} e^{Z_i t} \frac{-1}{\log p} \cdot \frac{(1-p)^{Z_i}}{Z_i}$$

$$= E\{E(e^{Z_i t}|N)^N\} \qquad = -\frac{1}{\log p} \sum e^{Z_i t} \frac{(1-p)^{Z_i}}{Z_i}$$

$$= \sum_{n=0}^{\infty} \left[\frac{\log(1-e^t(1-p))}{\log p}\right]^n \cdot \frac{\lambda^n e^{-\lambda}}{n!} \qquad = -\frac{1}{\log p} \sum_1^{\infty} \frac{[e^t(1-p)]^{Z_i}}{Z_i} \qquad \text{if } |e^t(1-p)|<1$$

$$= e^{-\lambda} \cdot e^{\lambda \frac{\log[1-e^t(1-p)]}{\log p}} \sum \frac{1}{n!} \left(\lambda \frac{\log[1-e^t(1-p)]}{\log p}\right)^n \cdot e^{-\lambda \frac{\log[1-e^t(1-p)]}{\log p}} \qquad = -\frac{1}{\log p} \{-\log[1-e^t(1-p)]\}$$

$$= e^{-\lambda}\left(e^{\log[\quad]}\right)^{\frac{\lambda}{\log p}} \qquad = \frac{\log[1-e^t(1-p)]}{\log p}$$

$$= e^{-\lambda} [1-e^t(1-p)]^{\frac{\lambda}{\log p}} = \left(e^{\log p}\right)^{-\frac{\lambda}{\log p}} \cdot [\quad]^{\frac{\lambda}{\log p}}$$

$$= \left[\frac{p}{1-e^t(1-p)}\right]^{-\frac{\lambda}{\log p}}$$

$\sim$ the mgf of negative binomial dis $\square$

① The mgf of binomial dis (n, p)

$$M(t) = \sum_{x=0}^{n} e^{xt} \cdot f(x)$$
$$= \sum_{x=0}^{n} e^{xt} \cdot \binom{n}{x} \cdot p^x (1-p)^{n-x}$$
$$= \sum_{x=0}^{n} e^{xt} \cdot \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^{n-x}$$
$$= \sum \binom{n}{x} (e^t p)^x (1-p)^{n-x}$$
$$= \left[e^t p + (1-p)\right]^n$$

② The mgf of negative binomial dis $(x; r, p)$

Ps

the number of failures (x) before reaching r success

第 $r+x$ 次是成功的

在 $r+x-1$ 次中有 x 次失敗 $r-1$ 次成功

∴ 排列組合是 $\binom{r+x-1}{x}$ OR $\binom{r+x-1}{r-1}$

$$M(t) = \sum_{x=0}^{n} e^{xt} f(x)$$
$$= \sum_{x=0}^{n} e^{xt} \cdot \binom{r+x-1}{r-1} \cdot p^r (1-p)^x$$

$$\binom{r+x-1}{r-1} = \binom{r+x-1}{x} = \frac{(r+x-1)!}{x!(r-1)!}$$
$$= \frac{(r+x-1)(r+x-2)\cdots r}{x!} \quad ①$$

$$\binom{-r}{x} = \frac{(-r)!}{x!(-r-x)!} = \frac{(-r)(-r-1)\cdots(-r-x+1)}{x!} \quad ②$$

① = ② $\cdot (-1)^x$, that is $\binom{-r}{x}(-1)^x = \binom{r+x-1}{r-1}$

$$\therefore M(t) = \sum_{x=0}^{n} \left[e^t(1-p)\right]^x \cdot p^r \cdot (-1)^x \binom{-r}{x}$$
$$= \sum \binom{-r}{x} \left[-e^t(1-p)\right]^x p^r$$
$$= \left[1 - e^t(1-p)\right]^{-r} \cdot p^r$$
$$= \left[\frac{p}{1 - e^t(1-p)}\right]^r$$

$$\sum \binom{n}{x} \cdot a^x = (a+1)^n$$

4.34 (a) $X|P \sim$ binomial (P) $\quad i = 1, \cdots, n$

$P \sim$ beta (α, β)

Show $P(X = x) = \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}$ beta (α, β)

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot p^{\alpha-1}(1-p)^{\beta-1}$$

$$E(P) = \int_0^1 p \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot p^{\alpha-1}(1-p)^{\beta-1} dp$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} \int_0^1 \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} p^{\alpha}(1-p)^{\beta-1} dp$$
$$= \frac{\cancel{(\alpha+\beta-1)!}}{\cancel{(\alpha+1)!}(\beta-1)!} \cdot \frac{\alpha \cancel{\Gamma(\beta-1)!}}{(\alpha+\beta)\cancel{!}} = \frac{\alpha}{\alpha+\beta}$$

$$P(X = x) = \int_0^1 P(x|P) \cdot f(P)\, dp$$
$$= \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot p^{\alpha-1} \cdot (1-p)^{\beta-1} dp$$
$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{x+\alpha-1} (1-p)^{n+\beta-x-1} dp$$
$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+x)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(n-x+\beta)} \cdot p^{x+\alpha-1}(1-p)^{n+\beta-x-1} dp}_{1}$$
$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}$$

$$E\left(\frac{1}{p}\right) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha-1)\Gamma(\beta)}{\Gamma(\alpha-1+\beta)}$$
$$= \frac{\alpha+\beta-1}{\alpha-1} = 1 + \frac{\beta}{\alpha-1}$$

□

! $E\left(\frac{1}{p}\right) \neq \frac{1}{E(p)}$

(b) $X|P \sim$ negative binomial (r, p) $\quad P \sim$ beta (α, β)

Find $P(X = x)$, $E(X)$, and $Var(X)$

$$f(x|p) = \binom{r+x-1}{x} \cdot p^r (1-p)^x$$

$$E[f(x|p)] = \sum_{x=0}^{\infty} x \binom{r+x-1}{x} p^r (1-p)^x$$
$$= \sum x \cdot \frac{(r+x-1)!}{x!(r-1)!} \cdot p^r (1-p)^x$$
$$= \sum_{x=1}^{\infty} \frac{(r+x-1)!}{(x-1)!(r-1)!} p^r (1-p)^x$$

Let $k = x-1$

$$= \sum_{k=0}^{\infty} \frac{(k+r)!}{k!(r-1)!} p^r (1-p)^k (1-p)$$
$$= (1-p) \sum_{k=0}^{\infty} \frac{[k+(r+1)-1]!}{k! \cdot r!} r \cdot \frac{p^{r+1}}{p} (1-p)^k$$
$$= \frac{r(1-p)}{p} \underbrace{\sum_{k=0}^{\infty} \frac{[k+(r+1)-1]!}{k!\, r!} p^{r+1} (1-p)^k}_{\sim (r+1,\ p)\quad 1}$$

$$P(X = x) = \int_0^1 P(x|P) \cdot f(P)\, dp$$
$$= \int_0^1 \binom{r+x-1}{x} \cdot p^r (1-p)^x \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp$$
$$= \binom{r+x-1}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{r+\alpha-1} (1-p)^{x+\beta-1} dp$$
$$= \binom{r+x-1}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(r+\alpha)\Gamma(x+\beta)}{\Gamma(r+\alpha+x+\beta)} \underbrace{\int_0^1 \frac{\Gamma(r+\alpha+x+\beta)}{\Gamma(r+\alpha)\Gamma(x+\beta)} p^{r+\alpha-1} (1-p)^{x+\beta-1} dp}_{1}$$
$$= \binom{r+x-1}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(r+\alpha)\Gamma(x+\beta)}{\Gamma(r+\alpha+x+\beta)}$$

□

$$E(X) = E\{E(X|P)\} = E\{r(1-p)/p\} = r \cdot E\left(\frac{1}{p} - 1\right)$$
$$= r\left[E\left(\frac{1}{p}\right) - 1\right] = r \cdot \frac{\beta}{\alpha-1}$$

$$\boxed{E(P) = \alpha/\alpha+\beta \quad \therefore E(X) = r\left[1 + \frac{\beta}{\alpha} - 1\right] = \frac{r\beta}{\alpha}}$$

□

$$\mathrm{Var}(Z) = \mathrm{Var}\{E(Z|P)\} + E\{\mathrm{Var}(Z|P)\}$$

$$= \mathrm{Var}\left\{\frac{r(1-P)}{P}\right\} + E\left\{\frac{r(1-P)}{P^2}\right\}$$

$$= E\left\{\left[r\left(\tfrac{1}{P}-1\right) - r\cdot E\left(\tfrac{1}{P}-1\right)\right]^2\right\} + r\cdot\left[E\left(\tfrac{1}{P^2}\right) - E\left(\tfrac{1}{P}\right)\right]$$

$$= r^2\cdot E\left[\left(\tfrac{1}{P}-1\right)^2\right] - r^2\cdot E^2\left(\tfrac{1}{P}-1\right) + r\cdot E\left(\frac{1-P}{P^2}\right)$$

$$= r^2\left[\frac{(\beta+1)\cdot\beta}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2}\right] + r\cdot\frac{(\alpha+\beta-1)\cdot\beta}{(\alpha-1)(\alpha-2)}$$

$$= r^2\,\frac{\beta(\beta+\alpha-1)}{(\alpha-1)^2(\alpha-2)} + r\cdot\frac{(\alpha+\beta-1)\beta}{(\alpha-1)(\alpha-2)}$$

$$\mathrm{Var}(Z|P) = E(Z^2|P) - E^2(Z|P) = \frac{r^2(1-P)^2 + r(1-P)}{P^2} - \frac{r^2(1-P)^2}{P^2} = \frac{r(1-P)}{P^2}$$

$$E(Z^2|P) = \sum_{x=0}^{\infty} x^2\binom{r+x-1}{x}P^r(1-P)^x$$

$$= \sum \frac{(r+x-1)!}{x!\,(r-1)!}\,x^2\cdot P^r(1-P)^x$$

$$= \sum x\cdot\frac{(r+x-1)!}{(x-1)!\,(r-1)!}\,P^r(1-P)^x$$

$$= \sum (x-1+1)\,\frac{(r+x-1)!}{(x-1)!\,(r-1)!}\,P^r(1-P)^x$$

$$= \sum_{x=2}^{\infty}(x-1)\cdot\frac{(r+x-1)!}{(x-1)!\,(r-1)!}\,P^r(1-P)^x + \sum_{x=1}^{\infty}\frac{(r+x-1)!}{(x-1)!\,(r-1)!}\cdot P^r(1-P)^x$$

$$= \sum_{x=2}^{\infty}\frac{(r+x-1)!}{(x-2)!\,(r-1)!}\,P^r(1-P)^x + \sum_{k=0}^{\infty}\frac{(r+1+k-1)!}{k!\,r!}\,r\cdot\frac{P^{r+1}}{P}(1-P)^k(1-P)$$

$$= \sum_{s=0}^{\infty}\frac{(r+2+s-1)!}{s!\,(r+1)!}\,r(r+1)\cdot\frac{P^{r+2}}{P^2}(1-P)^s(1-P)^2$$

$$+ \frac{r(1-P)}{P}\cdot\underbrace{\sum_{k=0}^{\infty}\binom{r+1+k-1}{k}P^{r+1}(1-P)^k}_{1}$$

$$= \frac{r(r+1)\cdot(1-P)^2}{P^2}\cdot\underbrace{\sum_{s=0}^{\infty}\binom{r+2+s-1}{s}P^{r+2}(1-P)^s}_{1} + \frac{r(1-P)}{P}$$

$$= \frac{r(1-P)}{P}\left[1 + \frac{1-P}{P}(r+1)\right]$$

$$= \frac{r(1-P)}{P}\,\frac{[(1-P)r+1]}{P}$$

$$= \frac{r^2(1-P)^2 + r(1-P)}{P^2}$$

$$E(P) = \frac{\alpha}{\alpha+\beta}$$

$$E\left(\frac{1}{P}\right) = \frac{\alpha+\beta}{\alpha} = 1 + \frac{\beta}{\alpha} \qquad 1+\frac{\beta}{\alpha-1}$$

$$E\left(\frac{1}{P}-1\right) = \frac{\beta}{\alpha} \qquad \frac{\beta}{\alpha-1}$$

$$E^2\left(\frac{1}{P}-1\right) = \frac{\beta^2}{\alpha^2} \qquad \frac{\beta^2}{(\alpha-1)^2}$$

$$E\left(\frac{1}{P^2}\right) = \frac{(\alpha+\beta+1)(\alpha+\beta)}{(\alpha+1)\alpha}$$

$$E\left(\frac{1}{P^2}\right) - 2E\left(\frac{1}{P}\right) + 1$$

$$= \frac{(\alpha+\beta+1)(\alpha+\beta)}{(\alpha+1)\alpha} - 2\left(1+\frac{\beta}{\alpha}\right) + 1$$

$$= \frac{(\alpha+\beta+1)(\alpha+\beta)}{(\alpha+1)\alpha} - 2 + \frac{2\beta}{\alpha} + 1$$

$$= \frac{(\alpha+\beta+1)(\alpha+\beta) + (\alpha+1)(2\beta-1)}{\alpha(\alpha+1)}$$

$$E\left(\frac{1-P}{P^2}\right) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\int_0^1 P^{\alpha-3}(1-P)^{\beta}\,dp$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha-2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta-1)} = \frac{(\alpha+\beta-1)\cdot\beta}{(\alpha-1)(\alpha-2)}$$

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot P^{\alpha-1}(1-P)^{\beta-1}$$

$$E(P^2) = \int_0^1 P^2\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot P^{\alpha-1}\cdot(1-P)^{\beta-1}\,dp$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\int_0^1 P^{\alpha+1}(1-P)^{\beta-1}\,dp$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)}\underbrace{\int_0^1\frac{\Gamma(\alpha+2+\beta)}{\Gamma(\alpha+2)\Gamma(\beta)}P^{\alpha+1}(1-P)^{\beta-1}\,dp}_{1}$$

$$= \frac{(\alpha+\beta-1)!}{(\alpha-1)!\,(\beta-1)!}\,\frac{(\alpha+1)!\,(\beta-1)!}{(\alpha+\beta+1)!}$$

$$= \frac{(\alpha+1)\cdot\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$E\left[\left(\frac{1}{P}-1\right)^2\right]$$

$$= E\left[\frac{(1-P)^2}{P^2}\right]$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\int_0^1 P^{\alpha-3}(1-P)^{\beta+1}\,dp$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha-2)\cdot\Gamma(\beta+2)}{\cancel{\Gamma(\alpha+\beta)}}$$

$$= \frac{(\beta+1)\cdot\beta}{(\alpha-1)(\alpha-2)}$$

4.35 (a) $X|P \sim \text{binomial}(P)$ $\quad i = 1, \cdots, n$

$P \sim \text{beta}(\alpha, \beta)$

Show that $\mathrm{Var}X = nEP(1-EP) + n(n-1)\,\mathrm{Var}P$

$$\mathrm{Var}X = E\{\mathrm{Var}(X|P)\} + \mathrm{Var}\{E(X|P)\}$$

$$E(X|P) = nP$$

$$E(X^2|P) = n(n-1)\cdot P^2 + nP$$

$$\mathrm{Var}(X|P) = E[X - E(X|P)]^2$$

$$= E(X^2|P) - E^2(X|P) = nP(1-P)$$

$$P(X) = \binom{n}{x}P^x\cdot(1-P)^{n-x}$$

$$E(X^2) = \sum_{x=0}^{n} x^2\cdot P^x(1-P)^{n-x}\binom{n}{x} \qquad E(X) = \sum_{x=0}^{n} x\cdot P^x(1-P)^{n-x}\binom{n}{x}$$

$$= \sum\frac{n!\,(x-1+1)}{(x-1)!\,(n-x)!}\cdot P^x(1-P)^{n-x} \qquad = \sum_{x=1}^{\infty}\frac{n!}{(x-1)!\,(n-x)!}\,P^x(1-P)^{n-x}$$

$$= \underbrace{\sum\frac{n!}{(x-2)!\,(n-x)!}\cdot P^x(1-P)^{n-x}}_{n(n-1)P^2} \qquad = \sum_{x=1}^{\infty}\frac{n(n-1)!}{(x-1)!\,(n-1-(x-1))!}\,P^{x-1}\cdot P(1-P)^{n-1-(x-1)}$$

$$+ \underbrace{\sum\frac{n!}{(x-1)!\,(n-x)!}\cdot P^x(1-P)^{n-x}}_{nP} \qquad = n\cdot P\underbrace{\left(\frac{n-1}{x-1}\right)\cdot P^{x-1}(1-P)^{n-x}}_{1}$$

$$\because \operatorname{Var} Z = E\{np(1-p)\} + \operatorname{Var}\{np\}$$
$$= n\,E\{p(1-p)\} + E\,n^2p^2 - E^2(np)$$
$$= n\cdot E\{p-p^2\} + n^2Ep^2 - n^2E^2p$$
$$= n(Ep - Ep^2) + n^2Ep^2 - n^2E^2p$$
$$= \underline{n\cdot Ep} \underbrace{- nEp^2 + n^2Ep^2}_{n(n-1)Ep^2} \underbrace{- n^2E^2p + n\cdot E^2p}_{-n(n-1)E^2p} \underline{- nE^2p}$$
$$= n(Ep - E^2p) + n(n-1)(Ep^2 - E^2p)$$
$$= nEp(1-Ep) + n(n-1)\operatorname{Var}(p) \qquad \square$$

(b). $Y|\Lambda \sim \text{Poisson}(\Lambda)$ $\quad \Lambda \sim \text{gamma}(\alpha,\beta)$

$$\operatorname{Var} Y = E\Lambda + \operatorname{Var}\Lambda = \mu + \frac{1}{\alpha}\mu^2 \qquad \mu = E\Lambda$$

$$\because \operatorname{Var} Y = E\{\operatorname{Var}(Y|\Lambda)\} + \operatorname{Var}\{E(Y|\Lambda)\}$$

$$E(Y|\Lambda) = \sum_{y=0}^{\infty} y\cdot f(y) = \sum_{y=0}^{\infty} \frac{\Lambda^y e^{-\Lambda}}{y!} y = e^{-\Lambda}\cdot \sum_{y=0}^{\infty} \frac{\Lambda^y}{(y-1)!} = e^{-\Lambda}\Lambda\cdot \sum \frac{\Lambda^{y-1}}{(y-1)!} = e^{-\Lambda}\Lambda\cdot e^{\Lambda} = \Lambda$$

$$E(Y^2|\Lambda) = \sum y^2 f(y) = \sum_{y=0}^{\infty} \frac{\Lambda^y e^{-\Lambda}}{y!} y^2 = e^{-\Lambda}\left[\sum \frac{\Lambda^y}{(y-1)!}(y-1+1)\right] = e^{-\Lambda}\left[\sum \frac{\Lambda^y}{(y-2)!} + \sum \frac{\Lambda^y}{(y-1)!}\right]$$
$$= e^{-\Lambda}\left[\Lambda^2 e^{\Lambda} + \Lambda\cdot e^{\Lambda}\right] = \Lambda^2 + \Lambda$$

$$\therefore \operatorname{Var}(Y|\Lambda) = \Lambda$$

$$\operatorname{Var} Y = E\{\Lambda\} + \operatorname{Var}\{\Lambda\} = E\Lambda + \underline{E\Lambda^2 - E^2\Lambda}$$
$$= E\Lambda + \beta^2(\alpha+1)\alpha - \alpha^2\beta^2$$
$$= E\Lambda + \beta^2\alpha$$
$$= E\Lambda + \frac{1}{\alpha}\cdot\beta^2\alpha^2$$
$$= E\Lambda + \frac{1}{\alpha}\cdot(E\Lambda)^2$$

Let $\mu = E\Lambda$

$$\operatorname{Var} Y = \mu + \frac{1}{\alpha}\cdot\mu \qquad \square$$

$$f(\Lambda) = \frac{\Lambda^{\alpha-1}\cdot e^{-\Lambda/\beta}}{\beta^{\alpha}\cdot\Gamma(\alpha)}$$

$$\cdot\ E\Lambda^2 = \int_0^{\infty} \Lambda^2\cdot f(\Lambda)\,d\Lambda$$
$$= \int_0^{\infty} \Lambda^2 \frac{\Lambda^{\alpha-1}\cdot e^{-\Lambda/\beta}}{\beta^{\alpha}\cdot\Gamma(\alpha)}\,d\Lambda$$
$$= \int_0^{\infty} \frac{\Lambda^{\alpha+1}\cdot e^{-\Lambda/\beta}}{\beta^{\alpha}\cdot\Gamma(\alpha)}\,d\Lambda$$
$$= \beta^2\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\underbrace{\int_0^{\infty} \frac{\Lambda^{\alpha+1}\cdot e^{-\Lambda/\beta}}{\beta^{\alpha+2}\cdot\Gamma(\alpha+2)}\,d\Lambda}_{1}$$
$$= \beta^2\frac{(\alpha+1)!}{(\alpha-1)!}$$
$$= \beta^2(\alpha+1)\alpha$$

$$\cdot\ E\Lambda = \alpha\beta$$

4.36 $(Z_1, P_1), \cdots (X_n, P_n)$ independent

$Z_i|P_i \sim \text{Bernoulli}(P_i)$ $\ i=1,\cdots,n$ $\quad P_i \sim \text{beta}(\alpha,\beta)$ $\quad Y = \sum_{i=1}^{n} Z_i$

(a) $EY = n\alpha/(\alpha+\beta)$

$$Y = Z_1 + Z_2 + \cdots + Z_n$$
$$EY = E(Z_1 + Z_2 + \cdots + Z_n)$$
$$= E(Z_1) + E(Z_2) + \cdots + E(Z_n)$$
$$= n\cdot E(Z_i)$$
$$= n\cdot \underline{E\{E(Z_i|P_i)\}}$$

$$E(Z_i|P_i) = P_i$$
$$E(P_i) = \frac{\alpha}{\alpha+\beta}$$
$$\therefore EY = n\cdot\frac{\alpha}{\alpha+\beta} \qquad \square$$

Bernoulli dis:

$$f(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$$

$$E(Z) = 1\cdot p + 0(1-p)$$
$$= p$$

beta dis:

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\cdot p^{\alpha-1}(1-p)^{\beta-1}$$

$$E(P) = \int_0^1 P\cdot f(p)\,dp$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\cdot\int_0^1 p^{\alpha}(1-p)^{\beta-1}\,dp = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+1)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}$$

(b) $\mathrm{Var}Y = n\alpha\beta/(\alpha+\beta)^2$, find $P_Y(y)$

$$\mathrm{Var}Y = \mathrm{Var}(Z_1+Z_2+\cdots+Z_n)$$
$$= \mathrm{Var}(Z_1)+\mathrm{Var}(Z_2)+\cdots+\mathrm{Var}(Z_n)$$
$$= n\cdot\mathrm{Var}(Z_i)$$

Bernoulli dis:

$$f(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$$

$$E(Z) = 1\cdot p + 0(1-p) = p \qquad E(Z^2) = 1^2 p + 0^2(1-p) = p$$

$$\mathrm{Var}(Z_i) = E\{\mathrm{Var}(Z_i|P_i)\} + \mathrm{Var}\{E(Z_i|P_i)\}$$
$$E(Z_i|P_i) = P_i$$
$$E(Z_i^2|P_i) = P_i$$
$$\mathrm{Var}(Z_i|P_i) = E(Z_i^2|P_i) - E^2(Z_i|P_i) = P_i - P_i^2$$
$$\therefore \mathrm{Var}(Z_i) = E(P_i - P_i^2) + \mathrm{Var}(P_i)$$
$$= E(P_i) - E(P_i^2) + E(P_i^2) - E^2(P_i)$$
$$= E(P_i) - E^2(P_i)$$
$$= \frac{\alpha}{\alpha+\beta}\left(1-\frac{\alpha}{\alpha+\beta}\right)$$
$$= \frac{\alpha\beta}{(\alpha+\beta)^2}$$
$$\therefore \mathrm{Var}Y = n\cdot\frac{\alpha\beta}{(\alpha+\beta)^2}$$

$$P_Y(Y=y) = P(Z_1+Z_2+\cdots+Z_n = y)$$

n trails of B
success y times

$$= \int_0^1 \binom{n}{y} P_i^{\,y}(1-P_i)^{n-y}\cdot f(P_i)\,dP_i$$
$$= \int_0^1 \binom{n}{y} P_i^{\,y}(1-P_i)^{n-y}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot P_i^{\alpha-1}(1-P_i)^{\beta-1}\,dP_i$$
$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 P_i^{\alpha+y-1}(1-P_i)^{n-y+\beta-1}\,dP_i$$
$$= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+y)\Gamma(n-y+\beta)}{\Gamma(\alpha+\beta+n)}$$

(c) $Z_i|P_i \sim \text{binomial}(n_i, P_i)$ $\quad P_i \sim \text{beta}(\alpha,\beta)$, $Y = \sum Z_i$

Show that $EY = \frac{\alpha}{\alpha+\beta}\sum n_i \qquad \mathrm{Var}Y = \sum \mathrm{Var}Z_i$

$$\mathrm{Var}Z_i = n_i\cdot\frac{\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$Y = Z_1+Z_2+\cdots+Z_k$$
$$EY = EZ_1+EZ_2+\cdots+EZ_k \quad \because Z_i \text{ are i.i.d}$$

$$Z_i|P_i \sim \text{binomial}(n_i, P_i)$$
$$\therefore P(x_i|P_i) = \binom{n_i}{x_i}P_i^{x_i}(1-P_i)^{n_i-x_i}$$
$$EZ_i = E\{E(x_i|P_i)\}$$
$$= E\{n_i\cdot P_i\} \qquad \mathrm{Var}P_i$$
$$= n_i\cdot E(P_i) \qquad \underbrace{EP_i^2 - E^2P_i}$$

$$E(x_i|P_i)$$
$$= \sum_0^{n_i} x_i\cdot P(x_i)$$
$$= \sum\frac{n_i!}{x_i!(n_i-x_i)!}\cdot x_i\cdot P_i^{x_i}(1-P_i)^{n_i-x_i}$$
$$= \sum\frac{(n_i-1)!\cdot n_i}{(x_i-1)!(n_i-1-(x_i-1))!}P_i^{x_i-1}\cdot P_i(1-P_i)^{n_i-x_i}$$
$$= n_iP_i$$

$$E(Z_i^2|P_i)$$
$$= \sum\frac{n_i!}{x_i!(n_i-x_i)!}\cdot x_i^2 P_i^{x_i}(1-P_i)^{n_i-x_i}$$
$$= \sum\frac{n_i!}{(x_i-1)!(n_i-x_i)!}(x_i-1+1)\cdot P_i^{x_i}(1-P_i)^{n_i-x_i}$$
$$= \sum\frac{(n_i-2)!(n_i-1)\cdot n_i}{(x_i-2)!(n_i-2-x_i+2)!}P_i^{x_i-2}\cdot P_i^2(1-P_i)^{n_i-x_i} + n_iP_i$$
$$= n_i(n_i-1)P_i^2 + n_iP_i$$
$$\therefore \mathrm{Var}(Z_i|P_i) = \cancel{n_i^2P_i^2} - n_iP_i^2 + n_iP_i - \cancel{n_i^2P_i^2}$$
$$= n_iP_i(1-P_i)$$

$$\because P_i \sim \text{beta}(\alpha,\beta)$$
$$\therefore E(P_i) = \frac{\alpha}{\alpha+\beta}$$
$$\therefore EZ_i = n_i\frac{\alpha}{\alpha+\beta}$$
$$\therefore EY = \frac{\alpha}{\alpha+\beta}\sum_1^k n_i$$

$$f(P_i) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}P_i^{\alpha-1}(1-P_i)^{\beta-1}$$

$$E(P_i) = \int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}P_i^{\alpha}(1-P_i)^{\beta-1} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\cdot 1 = \frac{\alpha}{\alpha+\beta}$$

$$E(P_i^2) = \int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}P_i^{\alpha+1}(1-P_i)^{\beta-1} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+2)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}\cdot 1 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\mathrm{Var}(P_i)$$
$$= E(P_i^2) - E^2P_i$$
$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$
$$= \frac{(\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2}$$
$$= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2} \quad ②$$

$$\mathrm{Var}Y = \mathrm{Var}(Z_1+Z_2+\cdots+Z_k)$$
$$= \mathrm{Var}(Z_1)+\mathrm{Var}(Z_2)+\cdots+\mathrm{Var}(Z_k)$$
$$= \sum\mathrm{Var}(Z_i)$$

$$\mathrm{Var}Z_i = E\{\mathrm{Var}(Z_i|P_i)\} + \mathrm{Var}\{E(Z_i|P_i)\}$$
$$= E\{n_iP_i(1-P_i)\} + \mathrm{Var}\{n_i\cdot P_i\}$$
$$= n_i[EP_i - EP_i^2] + n_i^2\,\mathrm{Var}(P_i) = n_i\cdot ① + n_i^2\cdot ②$$

$$EP_i - EP_i^2 = \frac{\alpha}{\alpha+\beta} - \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} = \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} \quad ①$$

$$= \frac{n_i\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad \square$$

4.37. $C = \frac{1}{2}(1+p)$ $\quad X_1, X_2, \cdots X_n \sim \text{Bernoulli}(c)$ $\quad K = \sum X_i$

$P(K=k \mid c) = \binom{n}{k} c^k (1-c)^{n-k}$ $\quad p \sim \text{beta}(a,b)$

(a) Show that $\sum X_i$ is a mixture of beta-binomials

$$P(K \mid p) = \binom{n}{k}\left[\tfrac{1}{2}(1+p)\right]^k\left[\tfrac{1}{2}(1-p)\right]^{n-k} \qquad c = \tfrac{1}{2}(1+p)$$

$$= \left(\tfrac{1}{2}\right)^n \binom{n}{k}(1+p)^k(1-p)^{n-k} \qquad 1-c = \tfrac{1}{2}(1-p)$$

$$P(K) = \int_0^1 P(K \mid p)\cdot f(p)\,dp$$

$$= \int \left(\tfrac{1}{2}\right)^n \binom{n}{k}(1+p)^k(1-p)^{n-k}\cdot\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\cdot p^{a-1}(1-p)^{b-1}\,dp$$

$$= \left(\tfrac{1}{2}\right)^n \binom{n}{k}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 (1+p)^k p^{a-1}(1-p)^{n+b-k-1}\,dp$$

$$= \sum_{j=0}^{k}\binom{k}{j}\int_0^1 p^{j+a-1}(1-p)^{n-k+b-1}\,dp$$

$$\boxed{\because (1+p)^k = \sum_{j=0}^{k}\binom{k}{j}p^j} \quad = \sum_{j=0}^{k}\binom{k}{j}\frac{\Gamma(j+a+n-k+b)}{\Gamma(j+a)\,\Gamma(n-k+b)}$$

$$\therefore P(K) = \sum_{j=0}^{k}\binom{k}{j}\left(\tfrac{1}{2}\right)^n\binom{n}{k}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\cdot\frac{\Gamma(j+a+n-k+b)}{\Gamma(j+a)\,\Gamma(n-k+b)}$$

$$\boxed{(a+b)^n = \sum_{i=0}^{k}\binom{n}{i}a^i b^{n-i}}$$

(b) Find the mean & Var of K

$$E(K) = E(E(K \mid p)) = \tfrac{1}{2}n + \tfrac{n}{2}\cdot E(p)$$

$$E(K \mid c) = nc = n\,\tfrac{1}{2}(1+p)$$

$\because p \sim \text{beta}(a,b)$ $\quad \therefore E(p) = \frac{a}{a+b}$ $\qquad E(p^2) = \frac{(a+1)\cdot a}{(a+b+1)(a+b)}$

$$\therefore E(K) = \frac{n}{2}\left(1+\frac{a}{a+b}\right)$$

$$Var(K) = E\{Var(K \mid c)\} + Var\{E(K \mid c)\}$$

1° $Var(K \mid c) = nc(1-c) = n\cdot\frac{1}{2}(1+p)\cdot\frac{1}{2}(1-p) = \frac{n}{4}(1-p^2) = \frac{n}{4} - \frac{n}{4}p^2$

$$E\{Var(K \mid c)\} = \frac{n}{4} - \frac{n}{4}\cdot E(p^2) = \frac{n}{4}\left[1 - \frac{(a+1)a}{(a+b+1)(a+b)}\right]$$

2° $Var\{E(K \mid c)\} = Var\{\frac{n}{2}(1+p)\} = \frac{n^2}{4}\cdot Var(1+p)$

$$E(1+p) = 1 + E(p) = 1 + \frac{a}{a+b} \quad ①$$

$$E[(1+p)^2] = E(1+2p+p^2) = 1 + 2E(p) + E(p^2)$$

$$= 1 + 2\frac{a}{a+b} + \frac{(a+1)\cdot a}{(a+b+1)(a+b)} \quad ②$$

$$Var(1+p) = E[(1+p)^2] - E^2(1+p)$$

$$= ② - ①^2$$

$$= \cancel{1} + \frac{a}{a+b}\left(2 + \frac{a+1}{a+b+1}\right) - \cancel{1} - \frac{2a}{a+b} - \frac{a^2}{(a+b)^2}$$

$$= \frac{a}{a+b}\left(\cancel{2} + \frac{a+1}{a+b+1} - \cancel{2} - \frac{a}{a+b}\right)$$

$$= \frac{ab}{(a+b+1)(a+b)^2}$$

$$\therefore Var\{E(K \mid c)\} = \frac{n^2}{4}\cdot\frac{ab}{(a+b+1)(a+b)^2} \quad ✓$$

Combine 1° and 2°.

$$Var(K) = \frac{n}{4}\left[1 - \frac{(a+1)a}{(a+b+1)(a+b)} + \frac{nab}{(a+b+1)(a+b)^2}\right]$$

4.38. $X \sim \text{gamma}(r, \lambda)$ $f(x) = \dfrac{x^{r-1} \cdot e^{-x/\lambda}}{\lambda^r \, \Gamma(r)}$

(a) Show that if $r \le 1$, $f(x) = \int_0^\lambda \frac{1}{v} \cdot e^{-x/v} \cdot P_\lambda(v)\, dv$

$$P_\lambda(v) = \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \frac{v^{r-1}}{(\lambda - v)^r} \qquad 0 < v < \lambda$$

$v \to u$

$u = x/v - x/\lambda$

思路：还原回去

Let $u = \frac{x}{v} - \frac{x}{\lambda} = x\left(\frac{1}{v} - \frac{1}{\lambda}\right) = x \cdot \frac{\lambda - v}{v \cdot \lambda}$

$$P_\lambda(v) = \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \frac{v^{r-1}}{(\lambda - v)^r}$$

$$P_\lambda(u) = P_\lambda\left(\frac{x\lambda}{u\lambda + x}\right)$$

$$= \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \left(\frac{x}{u\lambda}\right)^r \cdot \frac{u\lambda + x}{x\lambda}$$

$$f(x) = \int_0^\infty \frac{1}{v} \cdot e^{-x/v} \cdot P_\lambda(v)\, dv$$

$$= \int_0^\infty \frac{u\lambda + x}{x\lambda} \cdot e^{-\frac{u\lambda + x}{\lambda}} \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \left(\frac{x}{u\lambda}\right)^r \cdot \frac{u\lambda + x}{x\lambda} \left| (-x) \cdot \left(\frac{x\lambda}{u\lambda + x}\right)^2 \right| du$$

$$= \frac{1}{\Gamma(r)\Gamma(1-r)} \int_0^\infty e^{-\frac{u\lambda + x}{\lambda}} \cdot \frac{x^{r-1}}{(u\lambda)^r}\, du$$

$$= \frac{x^{r-1} \cdot e^{-x/\lambda}}{\Gamma(r)\Gamma(1-r)\lambda^r} \underbrace{\int_0^\infty e^{-u} \cdot \frac{1}{u^r}\, du}_{\Gamma(1-r)}$$

$$= \frac{x^{r-1} \cdot e^{-x/\lambda}}{\Gamma(r) \cdot \lambda^r} \quad \square$$

$X \sim$ gamma

$\in (0,1) > 0$

Gamma function:

$$\Gamma(z) = \int_0^\infty x^{z-1} \cdot e^{-x}\, dx$$

$$\Gamma(1-r) = \int_0^\infty u^{-r} \cdot e^{-u}\, du$$

$$\frac{du}{dv} : -x \cdot v^{-2} = (-x)\left(\frac{x\lambda}{u\lambda + x}\right)^2$$

$v = 0 \quad u = \infty$

$v = \lambda \quad u = 0$

$$v = \left(\frac{u}{x} + \frac{1}{\lambda}\right)^{-1}$$

$$= \frac{x\lambda}{u\lambda + x}$$

$$\lambda - v = \lambda - \frac{x\lambda}{u\lambda + x}$$

$$= \frac{u\lambda^2}{u\lambda + x}$$

$$\left(\frac{v}{\lambda - v}\right)^r = \left[\frac{x\lambda}{u\lambda + x} \cdot \frac{u\lambda + x}{u\lambda^2}\right]^r$$

$$= \left(\frac{x}{u\lambda}\right)^r$$

$$\left(\frac{v}{\lambda - v}\right)^r \cdot \frac{1}{v} = \left(\frac{x}{u\lambda}\right)^r \cdot \frac{u\lambda + x}{x\lambda}$$

$$-\frac{x}{v} = -x \cdot \frac{u\lambda + x}{x\lambda}$$

$$= -\frac{u\lambda + x}{\lambda}$$

(b). Show that $\int_0^\lambda P_\lambda(v)\, dv = 1$

$$P_\lambda(v) = \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \frac{v^{r-1}}{(\lambda - v)^r}$$

$$\int_0^\lambda P_\lambda(v)\, dv = \int_0^\infty \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \left(\frac{x}{u\lambda}\right)^r \cdot \frac{u\lambda + x}{x\lambda} \cdot x \cdot \left(\frac{x\lambda}{u\lambda + x}\right)^2 du$$

$$= \frac{x^{r+2}\lambda^{1-r}}{\Gamma(r)\Gamma(1-r)} \int_0^\infty u^{-r} \cdot \frac{1}{u\lambda + x}\, du$$

$$\frac{1}{u^r(u\lambda + x)}\, du$$

$$\frac{1}{\lambda u^{r+1} + x \cdot u^r}\, du$$

× 思路断了 ⇒ 直接积分

$$\int_0^\lambda P_\lambda(v)\, dv = \frac{1}{\Gamma(r)\Gamma(1-r)} \int_0^\lambda \frac{v^{r-1}}{(\lambda - v)^r}\, dv$$

$$= \frac{1}{\Gamma(r)\Gamma(1-r)} \int_0^1 t^{r-1}(1-t)^{-r}\, dt$$

$$= \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \frac{\Gamma(r) \cdot \Gamma(1-r)}{\Gamma(1)}$$

$$= 1$$

$$v^{r-1} \cdot (\lambda - v)^{-r}\, dv$$

1° let $\frac{1}{v} = t$ $\quad v = 0 \quad t = \infty$, $\quad v = \lambda \quad t = 1$

$v = \lambda t^{-1}$

$dv = (-1)\lambda t^{-2}\, dt$

$$(\lambda t^{-1})^{r-1} \cdot (\lambda - \lambda t^{-1})^{-r} \left|(-1)\lambda t^{-2}\right| dt$$

$$= \lambda^{r-1} \cdot \left(\frac{1}{t}\right)^{r-1} \cdot \lambda^{-r}\left(1 - \frac{1}{t}\right)^{-r} \lambda \cdot t^{-2}\, dt$$

手算 again

$$\Gamma(1) = \int_0^\infty x^0 \cdot e^{-x}\, dx$$

$$= \int_0^\infty e^{-x}\, dx$$

$$= -e^{-x}\Big|_0^\infty$$

$$= -(0 - 1)$$

$$= 1$$

2° let $\frac{v}{\lambda} = t$ $\quad v = 0 \quad t = 0$, $\quad v = \lambda \quad t = 1$

$v = \lambda t$

$dv = \lambda\, dt$

$$\int_0^1 (\lambda t)^{r-1} (\lambda - \lambda t)^{-r} \lambda\, dt$$

$$\int_0^1 \lambda^{r-1} \cdot t^{r-1} \lambda^{-r} (1-t)^{-r} \lambda\, dt$$

$$= \int_0^1 t^{r-1} \cdot (1-t)^{-r}\, dt \quad \sim \text{beta}$$

(c). $f(x) = \int_0^\infty \frac{1}{v} \cdot e^{-x/v} g_\lambda(v)\,dv$ $\qquad f(x) \sim \text{gamma}(r, \lambda)$

If $r>1$, then $\frac{\partial}{\partial x}\log(f(x)) > 0$ but $\frac{\partial}{\partial x}\log\left(\int_0^\infty \frac{1}{v} \cdot e^{-x/v} \cdot g_\lambda(v)\,dv\right) < 0$

$\because f(x) \sim \text{gamma}(r, \lambda)$

$\therefore f(x) = \dfrac{x^{r-1} \cdot e^{-x/\lambda}}{\lambda^r \cdot \Gamma(r)}$ $\qquad \frac{\partial}{\partial x}\log(f(x)) = \dfrac{f'(x)}{f(x)}$

$$f'(x) = \frac{1}{\lambda^r \Gamma(r)} \cdot \left[x^{r-1} e^{-x/\lambda} \cdot \left(-\frac{1}{\lambda}\right) + (r-1) \cdot x^{r-2} \cdot e^{-x/\lambda}\right]$$

$$= \frac{1}{\lambda^r \cdot \Gamma(r)} \cdot x^{r-2} \cdot e^{-x/\lambda} \left(-\frac{x}{\lambda} + r - 1\right)$$

$$\frac{\partial}{\partial x}\log(f(x)) = \frac{1}{\cancel{\lambda^r \cdot \Gamma(r)}} \cdot x^{r-2} \cancel{e^{-x/\lambda}} \left(-\frac{x}{\lambda} + r - 1\right) \cdot \frac{\cancel{\lambda^r \cdot \Gamma(r)}}{x^{r-1} \cancel{e^{-x/\lambda}}}$$

$$= \left(-\frac{x}{\lambda} + r - 1\right) \cdot \frac{1}{x}$$

$$= -\frac{1}{\lambda} + \frac{r-1}{x} > 0 \text{ for some } x$$

$$\frac{\partial}{\partial x}\log\left(\int_0^\infty \frac{1}{v} \cdot e^{-x/v} \cdot g_\lambda(v)\,dv\right)$$

$$\log\left[e^{-x} \int_0^\infty \frac{1}{v} \cdot e^{\frac{1}{v}} \cdot g_\lambda(v)\,dv\right]$$

$\log ab = \log a + \log b$

$$\frac{\partial}{\partial x}\left(\log e^{-x} + \log\left(\int_0^\infty \frac{1}{v} e^{\frac{1}{v}} \cdot g_\lambda(v)\,dv\right)\right)$$

$$= -1 + \frac{\int_\infty^0 t e^t \cdot g_\lambda(\frac{1}{t}) \cdot t^2\,dt}{\int_0^\infty \frac{1}{v} e^{\frac{1}{v}} \cdot g_\lambda(v)\,dv} \qquad \left(\int_0^\infty \frac{1}{v} e^{\frac{1}{v}} \cdot g_\lambda(v)\,dv\right)'$$

< 0 definitely

let $\frac{1}{v} = t$ $\quad v=0 \quad t=\infty \quad v = t^{-1}$

$v=\infty \quad t=0 \quad dv = (-1) \cdot t^{-2}\,dt$

$$\frac{\int_\infty^0 t e^t g_\lambda(\frac{1}{t}) \cdot t^2\,dt}{+}$$

$\therefore$ conflict $\square$

4.39. $(Z_1, Z_2, \cdots, Z_n) \sim$ multinomial distribution with m trails $P_1, P_2, \cdots P_n$

Pf: $f(x_1, \cdots, x_n) = \dfrac{m!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n} = m! \prod_{i=1}^n \dfrac{p_i^{x_i}}{x_i!}$ $\qquad \sum_{i=1}^n x_i = m.$

Show that $Z_i | Z_j = x_j \sim \text{binomial}\left(m - x_j, \frac{P_i}{1-P_j}\right)$

$Z_j \sim \text{binomial}(m, P_j)$

$\text{Cov}(Z_i, Z_j) = -m P_i P_j$

$$f(x_1, \cdots x_{i-1}, x_{j+1}, \cdots x_n) = \frac{(m-x_j)!}{x_1! \cdots x_{j-1}! \cdot x_{j+1}! \cdots x_n!} \cdot \left(\frac{P_1}{1-P_j}\right)^{x_1} \cdots \left(\frac{P_{j-1}}{1-P_j}\right)^{x_{j-1}} \left(\frac{P_{j+1}}{1-P_j}\right)^{x_{j+1}} \cdots \left(\frac{P_n}{1-P_j}\right)^{x_n} \qquad 1-P_1-P_j$$

Then

$$f(x_i | x_j) = \sum_{(x_1, \cdots x_{i-1}, x_{i+1}, \cdots x_{j-1}, x_{j+1}, \cdots x_n)} f(x_1, \cdots x_{i-1}, x_{i+1}, \cdots x_{j-1}, x_{j+1}, \cdots x_n | x_j)$$

$$= \sum \frac{(m-x_j)!}{x_1, \cdots x_{i-1}, x_{i+1}, \cdots x_{j-1}, x_{j+1}, \cdots x_n} \left(\frac{P_1}{1-P_j}\right)^{x_1} \cdots \left(\frac{P_{i-1}}{1-P_{i-1}}\right)^{x_{i-1}} \left(\frac{P_{i+1}}{1-P_{i+1}}\right)^{x_{i+1}} \cdots \left(\frac{P_{j-1}}{1-P_j}\right)^{x_{j-1}} \left(\frac{P_{j+1}}{1-P_j}\right)^{x_{j+1}} \cdots \left(\frac{P_n}{1-P_j}\right)^{x_n} \cdot \frac{(m-x_i-x_j)! \left(1-\frac{P_i}{1-P_j}\right)^{m-x_i-x_j}}{(m-x_i-x_j)! \left(1-\frac{P_i}{1-P_j}\right)^{m-x_i-x_j}}$$

$$= \binom{m-x_j}{x_i} \left(\frac{P_i}{1-P_j}\right)^{x_i} \left(1 - \frac{P_i}{1-P_j}\right)^{m-x_i-x_j} \underbrace{\sum \frac{(m-x_i-x_j)!}{x_1, \cdots x_{i-1}, x_{i+1}, \cdots x_{j-1}, x_{j+1}, \cdots x_n} \left(\frac{P_1}{1-P_i-P_j}\right)^{x_1} \cdots \left(\frac{P_{i-1}}{1-P_i-P_j}\right)^{x_{i-1}} \left(\frac{P_{i+1}}{1-P_i-P_j}\right)^{x_{i+1}} \cdots \left(\frac{P_{j-1}}{1-P_i-P_j}\right)^{x_{j-1}} \left(\frac{P_{j+1}}{1-P_i-P_j}\right)^{x_{j+1}} \cdots \left(\frac{P_n}{1-P_i-P_j}\right)^{x_n}}_{1}$$

$\sim \text{binomial}\left(m-x_j, \frac{P_i}{1-P_j}\right)$

$f(x_i, x_j) = f(x_i)\, x_j) \cdot f(x_j)$

$$= \frac{m!}{x_i!\, x_j!\, (m-x_i-x_j)!} \cdot p_i^{x_i}\, p_j^{x_j} (1-p_i-p_j)^{m-x_i-x_j}$$

$\therefore x_i + x_j \sim \text{binomial}(m,\ p_i + p_j)$ binomial for multi-var

$Var(X_i + X_j) = m(p_i + p_j)(1 - p_i - p_j)$

$Var(x_i + x_j) = Var(x_i) + Var(x_j) + 2Cov(x_i, x_j)$

$\therefore Cov(x_i, x_j) = [Var(x_i + x_j) - Var(x_i) - Var(x_j)]/2$

$= [m(p_i + p_j)(1 - p_i - p_j) - m\, p_i(1-p_i) - m\, p_j(1-p_j)]/2$

$= \frac{m}{2}[p_i + p_j - (p_i + p_j)^2 - p_i + p_i^2 - p_j + p_j^2]$

$= \frac{m}{2}(\cancel{p_i} + \cancel{p_j} - \cancel{p_i^2} - 2p_ip_j - \cancel{p_j^2} - \cancel{p_i} + \cancel{p_i^2} - \cancel{p_j} + \cancel{p_j^2})$

$= -p_ip_jm$ □

$Cov(X, Y) = E(XY) - (EX)(EY)$

4.40. $f(x, y) = C \cdot x^{a-1} y^{b-1} (1-x-y)^{c-1}$

(a) Show that $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$

$\because f(x, y)$ is a pdf

$\therefore \int_0^1 \int_0^{1-x} f(x, y)\, dy\, dx = 1$

that is $\int_0^1 \int_0^{1-x} C \cdot x^{a-1} y^{b-1} (1-x-y)^{c-1}\, dy\, dx = 1$

Let $u = \frac{y}{1-x}$ $\quad y = u(1-x)$ $\quad 1-x = \frac{y}{u}$ $\quad 1-x-y = 1-x-u(1-x)$

$= (1-x)(1-u)$

$\cancel{x}\, dy = (1-x)\, du$ $\quad y=0 \quad u=0$

$y = 1-x \quad u = 1$

思路: $\because \int_0^{1-x} \cdots dy$

$\therefore$ 垂直条: $\int_0^1$

$0 \longrightarrow 0$

y map u

$1-x \longrightarrow 1$

$u = \frac{y}{1-x}$

$\therefore \int_0^1 \int_0^1 C \cdot x^{a-1} \cdot [u(1-x)]^{b-1} [(1-x)(1-u)]^{c-1} (1-x)\, dx\, du = 1$

$\int_0^1 \int_0^1 C \cdot u^{b-1} \cdot x^{a-1} \cdot (1-x)^{b+c-1} (1-u)^{c-1}\, dx\, du = 1$

$\int_0^1 C\, u^{b-1} (1-u)^{c-1} \underbrace{\int_0^1 x^{a-1} (1-x)^{b+c-1} dx}\, du = 1$

$C \frac{\Gamma(a)\Gamma(b+c)}{\Gamma(a+b+c)} \underbrace{\int_0^1 u^{b-1} (1-u)^{c-1}\, du} = 1$

$C \frac{\Gamma(a)\cancel{\Gamma(b+c)}}{\Gamma(a+b+c)} \frac{\Gamma(b)\Gamma(c)}{\cancel{\Gamma(b+c)}} = 1$

$\therefore C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$

(b) Show that $f(X)$ & $f(Y) \sim$ beta

$f(x) = \int_0^{1-x} f(x, y)\, dy$

$= \int_0^{1-x} C \cdot x^{a-1} y^{b-1} (1-x-y)^{c-1}\, dy$

$= C \cdot x^{a-1} \underbrace{\int_0^{1-x} y^{b-1} (1-x-y)^{c-1}\, dy}_{A}$

Let $u = \frac{y}{1-x}$ $\quad y=0 \quad u=0$

$y = 1-x \quad u = 1$

$dy = (1-x)\, du$

$A = \int_0^1 [(1-x) \cdot u]^{b-1} [(1-x)(1-u)]^{c-1} (1-x)\, du$

$= (1-x)^{b+c-1} \int_0^1 u^{b-1} (1-u)^{c-1}\, du$

$= (1-x)^{b+c-1} \cdot \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}$

$\therefore f(x) = \frac{\Gamma(a+b+c)}{\Gamma(a)\cancel{\Gamma(b)\Gamma(c)}} \cdot \frac{\cancel{\Gamma(b)\Gamma(c)}}{\Gamma(b+c)} \cdot x^{a-1} (1-x)^{b+c-1}$

$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1} (1-x)^{b+c-1} \sim \text{beta}(a, b+c)$

$f(y) = \int_0^{1-y} f(x, y)\, dx$

$= \int_0^{1-y} C \cdot x^{a-1} y^{b-1} (1-x-y)^{c-1}\, dx$

$= C \cdot y^{b-1} \underbrace{\int_0^{1-y} x^{a-1} (1-x-y)^{c-1}\, dx}_{B}$

Let $u = \frac{x}{1-y}$ $\quad x = u(1-y)$ $\quad x=0 \quad u=0$ $\quad dx = (1-y)\, du$

$x = 1-y \quad u = 1$

$B = \int_0^1 [u(1-y)]^{a-1} [(1-u)(1-y)]^{c-1} (1-y)\, du$

$= (1-y)^{a+c-1} \int_0^1 u^{a-1} (1-u)^{c-1}\, du$

$= (1-y)^{a+c-1} \frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}$

$\therefore f(y) = \frac{\Gamma(a+b+c)}{\cancel{\Gamma(a)\Gamma(b)\Gamma(c)}} \cdot y^{b-1} (1-y)^{a+c-1} \cdot \frac{\cancel{\Gamma(a)\Gamma(c)}}{\Gamma(a+c)}$

$= \frac{\Gamma(a+b+c)}{\Gamma(b)\Gamma(a+c)} \cdot y^{b-1} (1-y)^{a+c-1}$

$\sim \text{beta}(b, a+c)$

(c). Find $f(Y|Z=x)$, show that $Y/(1-x) \sim \text{beta}(b,c)$

$$f(Y|Z=x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{c\,\cancel{x^{a-1}}\,y^{b-1}(1-x-y)^{c-1}}{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)}\,\cancel{x^{a-1}}(1-x)^{b+c-1}}$$

$$= \frac{\cancel{\Gamma(a+b+c)}}{\cancel{\Gamma(a)}\Gamma(b)\Gamma(c)} \cdot \frac{\cancel{\Gamma(a)}\Gamma(b+c)}{\cancel{\Gamma(a+b+c)}} \cdot \frac{y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-1}}$$

$$= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \cdot \frac{y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-1}}$$

$$f\left(\frac{Y}{1-x} \middle| Z=x\right) = f(Y|Z=x) \cdot \frac{1}{1-x}$$ $\frac{1}{1-x}$ 是参数.

$$= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \cdot \frac{y^{b-1}(1-x-y)^{c-1}}{(1-x)^{b+c-2}}$$

$$= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \cdot \left(\frac{y}{1-x}\right)^{b-1}\left(1-\frac{y}{1-x}\right)^{c-1}$$

$\sim \text{beta}(b,c)$ □

(d) show that $E(ZY) = \dfrac{ab}{(a+b+c+1)(a+b+c)}$ and find $Cov(Z,Y)$

$Cov(Z,Y) = E(ZY) - E(Z)E(Y)$

$$E(ZY) = \int_0^1\int_0^{1-x} xy \cdot f(x,y)\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} xy \cdot c \cdot x^{a-1} y^{b-1}(1-x-y)^{c-1}\,dy\,dx$$

Let $u = \frac{y}{1-x}$ $\quad y = (1-x)u \quad dy = (1-x)du$

$$E(ZY) = c\int_0^1\int_0^1 x^a[(1-x)u]^b \cdot [(1-x)(1-u)]^{c-1}(1-x)\,du\,dx$$

$$= c \cdot \int_0^1 x^a(1-x)^{b+c} \frac{\int_0^1 u^b(1-u)^{c-1}\,du}{\int_0^1 x^a(1-x)^{b+c}\,dx}\,dx$$

$$= c \cdot \frac{\Gamma(b+1)\Gamma(c)}{\Gamma(b+c+1)}$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\cancel{\Gamma(b)\Gamma(c)}} \cdot \frac{\Gamma(b+1)\cancel{\Gamma(c)}}{\cancel{\Gamma(b+c+1)}} \cdot \frac{\Gamma(a+1)\cancel{\Gamma(b+c+1)}}{\Gamma(a+b+c+2)}$$

$$= \frac{(a+b+c-1)!}{(a-1)!(b-1)!} \cdot b! \cdot \frac{a!}{(a+b+c+1)!}$$

$$= \frac{ab}{(a+b+c+1)(a+b+c)}$$ □

$$\therefore f(x) = \frac{\Gamma(a+b+c)}{\Gamma(a)\cancel{\Gamma(b)\Gamma(c)}} \cdot \frac{\cancel{\Gamma(b)\Gamma(c)}}{\Gamma(b+c)} \cdot x^{a-1}(1-x)^{b+c-1}$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1}(1-x)^{b+c-1} \sim \text{beta}(a, b+c)$$

$$\therefore f_Y(y) = \frac{\Gamma(a+b+c)}{\cancel{\Gamma(a)\Gamma(b)\Gamma(c)}} \cdot y^{b-1}(1-y)^{a+c-1} \cdot \frac{\cancel{\Gamma(a)\Gamma(c)}}{\Gamma(a+c)}$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(b)\Gamma(a+c)} \cdot y^{b-1}(1-y)^{a+c-1}$$

$\sim \text{beta}(b, a+c)$

$\because Z \sim \text{beta}(a, b+c)$

$$EZ = \int_0^1 x f(x)\,dx$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} \int_0^1 x^a(1-x)^{b+c-1}\,dx$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\cancel{\Gamma(b+c)}} \cdot \frac{\Gamma(a+1)\cancel{\Gamma(b+c)}}{\Gamma(a+b+c+1)}$$

$$= \frac{a}{a+b+c}$$

$E = \dfrac{\alpha}{\alpha+\beta}$

$\because Y \sim \text{beta}(b, a+c)$

$$EY = \frac{b}{a+b+c}$$

$$\therefore Cov(Z,Y) = EZY - EZ \cdot EY$$

$$= \frac{ab}{a+b+c}\left(\frac{1}{a+b+c+1} - \frac{1}{a+b+c}\right)$$

$$= \frac{-ab}{(a+b+c+1)(a+b+c)^2}$$ □

4.41. Show that $Cov(X, c) = 0$ for any X with finite EX and any constant c

$$Cov(X,c) = E(Xc) - E(X)\cdot E(c)$$
$$= c\cdot E(X) - c\cdot E(X)$$
$$= 0$$

$\square$

4.42. X, Y are independent, $EX = \mu_X$ $EY = \mu_Y$, $Var X = \sigma_X^2$ $Var Y = \sigma_Y^2$. Find $corr(XY, Y)$

$$corr(XY, Y) = \frac{Cov(XY, Y)}{\sigma_{XY}\ \sigma_Y}$$

$$Cov(XY, Y) = E(XY\cdot Y) - E(XY)\cdot E(Y)$$
$$= E(XY^2) - E(XY)\cdot E(Y)$$

$\because X, Y$ are independent

$\therefore E(XY) = E(X)E(Y) = \mu_X \mu_Y$

$E(XY^2) = E(X)\cdot E(Y^2) = \mu_X(\sigma_Y^2 + \mu_Y^2)$

$Var Y = EY^2 - E^2Y$

$EY^2 = Var Y + E^2Y$

$= \sigma_Y^2 + \mu_Y^2$

$EX^2 = \sigma_X^2 + \mu_X^2$

$$Var XY = E(X^2Y^2) - E^2(XY)$$
$$= E(X^2)E(Y^2) - \mu_X^2\mu_Y^2$$
$$= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2$$

$$\sigma_{XY} = \sqrt{Var XY} = \left(\sigma_X^2\sigma_Y^2 + \mu_X^2\cdot\sigma_Y^2 + \mu_Y^2\cdot\sigma_X^2\right)^{\frac{1}{2}}$$

$$corr(XY, Y) = \frac{\mu_X(\sigma_Y^2 + \mu_Y^2) - \mu_X\mu_Y\cdot\mu_Y}{\left(\sigma_X^2\sigma_Y^2 + \mu_X^2\cdot\sigma_Y^2 + \mu_Y^2\cdot\sigma_X^2\right)^{\frac{1}{2}}\ \sigma_Y}$$

$$= \frac{\mu_X\cdot\sigma_Y}{\left(\sigma_X^2\sigma_Y^2 + \mu_X^2\cdot\sigma_Y^2 + \mu_Y^2\cdot\sigma_X^2\right)^{\frac{1}{2}}}$$

4.43. X_1, X_2, X_3 are uncorrelated, each has mean μ & var σ^2

Find $Cov(X_1 + X_2, X_2 + X_3)$ & $Cov(X_1 + X_2, X_1 - X_2)$

$Cov(X_1 + X_2, X_2 + X_3)$

$= E[(X_1 + X_2)(X_2 + X_3)] - \underline{E(X_1 + X_2)E(X_2 + X_3)}$

$(EX_1 + EX_2)(EX_2 + EX_3)$

$(\mu_1 + \mu_2)(\mu_2 + \mu_3)$

$E[(X_1 + X_2)(X_2 + X_3)]$

$= E[X_1X_2 + X_2^2 + X_1X_3 + X_2X_3]$

$= E(X_1X_2) + E(X_2^2) + E(X_1X_3) + E(X_2X_3)$

$= \mu_1\mu_2 + \sigma_2^2 + \mu_2^2 + \mu_1\mu_3 + \mu_2\mu_3$

$= \mu_2(\mu_1 + \mu_2) + \mu_3(\mu_1 + \mu_2) + \sigma_2^2$

$= (\mu_1 + \mu_2)(\mu_2 + \mu_3) + \sigma_2^2$

$Var X_2 = EX_2^2 - E^2X_2$

$\therefore EX^2 = \sigma_2^2 + \mu_2^2$

$\therefore Cov(X_1 + X_2, X_2 + X_3) = \sigma_2^2$ $\square$

$Cov(X_1 + X_2, X_1 - X_2)$

$= E[(X_1 + X_2)(X_1 - X_2)] - E(X_1 + X_2)E(X_1 - X_2)$

$= \sigma_1^2 + \cancel{\mu_1^2} - (\sigma_2^2 + \cancel{\mu_2^2}) - (\cancel{\mu_1} + \cancel{\mu_2})(\cancel{\mu_1} - \cancel{\mu_2})$

$= \sigma_1^2 - \sigma_2^2$ $\square$

- Uncorrelated : $\rho(X, Y) = 0 = \frac{Cov(X, Y)}{\sigma_X\ \sigma_Y}$

$Cov(X, Y) = E(XY) - E(X)E(Y) = 0$

$E(XY) = E(X)E(Y)$

Indepen

uncor

- Independent

$P_{X,Y}(x, y) = P_X(x)\ P_Y(y)$ $\Leftrightarrow$ $P_{Y|X}(y|x) = P_Y(y)$

$E(XY) = \iint xy\cdot P_{X,Y}(x, y)\,dx\,dy$

$= \iint \underline{x}\,\underline{y}\cdot \underline{P_X(x)}\cdot \underline{P_Y(y)}\,dx\,dy$

$= \underline{E(X)}\cdot \underline{E(Y)}$

$\therefore$ Independent $\Rightarrow$ Uncorrelated

$\not\Leftarrow$

4.44. $Var(\sum_{i=1}^{n} Z_i) = \sum_{i=1}^{n} Var Z_i + 2 \sum_{1\le i<j\le n} Cov(Z_i, Z_j)$

$$Var(\sum_{i=1}^{n} Z_i) = E[(\sum_{i=1}^{n} Z_i)^2] - E^2(\sum_{i=1}^{n} Z_i) \quad \text{展开} \qquad Var Z_i = E(Z_i^2) - E^2 Z_i$$

$$= E[(X_1+X_2+\cdots+X_n)^2] - E^2(X_1+X_2+\cdots+X_n)$$

$$= E(X_1^2+X_2^2+\cdots+X_n^2+2\sum_{1\le i<j\le n} X_iX_j) - E^2X_1 - E^2X_2 - \cdots - E^2X_n - 2\sum_{1\le i<j\le n} EX_iEX_j$$

$$= \underbrace{EX_1^2 - E^2X_1}_{Var Z_1} + EX_2^2 - E^2X_2 + \cdots + EX_n^2 - E^2X_n + 2\sum_{1\le i<j\le n} \underbrace{E(X_iX_j) - EX_iEX_j}_{Cov(X_i, Z_j)}$$

$$= \sum_{i=1}^{n} Var Z_i + 2\sum_{1\le i<j\le n} Cov(Z_i, Z_j) \quad \square$$

4.45 If $(X, Y) \sim$ bivariate normal $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$

(a) $f_X(x) \sim n(\mu_X, \sigma_X^2)$ $\quad f_Y(y) \sim n(\mu_Y, \sigma_Y^2)$

$\because (X, Y) \sim$ bivariate normal $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$

$$\therefore f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]}$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\underbrace{\left(\frac{y-\mu_Y}{\sigma_Y} - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right)^2}_{} - \rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]}$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\cdot\left[\frac{y-[\mu_Y+\frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X)]}{\sigma_Y\sqrt{1-\rho^2}}\right]^2} e^{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} e^{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\cdot\left[\frac{y-[\mu_Y+\frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X)]}{\sigma_Y\sqrt{1-\rho^2}}\right]^2}}_{} \cdot A$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} \underbrace{e^{-\frac{1}{2}\left[\quad\right]^2}}_{A} = 1$$

$$A = \sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}$$

$$= \frac{\sqrt{2\pi}\,\cancel{\sigma_Y}\cancel{\sqrt{1-\rho^2}}}{2\pi\sigma_X\cancel{\sigma_Y}\cancel{\sqrt{1-\rho^2}}} e^{-\frac{1-\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \sim n(\mu_X, \sigma_X) \quad \square$$

$\because X$ and Y are equivalent

$\therefore f_Y(y) \sim n(\mu_Y, \sigma_Y)$

(b) $f_Y(y|X=x) \sim n(\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_X),\ \sigma_Y^2(1-\rho^2))$

$$f_Y(y|X=x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

$$= \frac{1}{2\pi\cancel{\sigma_X}\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]}$$

$$\cdot \sqrt{2\pi}\,\cancel{\sigma_X} \cdot e^{\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \qquad 1-1+\rho^2$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right] + \frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

$$= -\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) - (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]$$

$$= -\frac{1}{2(1-\rho^2)}\left[\rho\left(\frac{x-\mu_X}{\sigma_X}\right) - \frac{y-\mu_Y}{\sigma_Y}\right]^2 = -\frac{1}{2(1-\rho^2)\sigma_Y^2}\left[y - \left[\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right]\right]^2$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X))}{\sqrt{1-\rho^2}\,\sigma_Y}\right)^2} \sim n\left(\sigma_Y\sqrt{1-\rho^2}, \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right) \quad \square$$

(c) $f(aX+bY) \sim n(a\mu_X+b\mu_Y,\ a^2\sigma_X^2+b^2\sigma_Y^2+2ab\rho\sigma_X\sigma_Y)$

$\because (X,Y) \sim$ bivariate normal $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$

$\therefore f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2-2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]}$

Let $u = ax+by$ $\quad x = \frac{u-bv}{a}$ $\quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{vmatrix} = \frac{1}{a}$

$v = y$ $\quad y = v$

$f(u,v) = f_{X,Y}\left(x=\frac{u-bv}{a},\ y=v\right)\cdot|J|$

$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\cdot\frac{1}{a}\cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u-bv-a\mu_X}{a\sigma_X}\right)^2+\left(\frac{v-\mu_Y}{\sigma_Y}\right)^2-2\rho\left(\frac{u-bv-a\mu_X}{a\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)\right]}$

O, 卡了3个番茄钟, 计算过于复杂, 考虑化, 将 X, Y 视作 $n(0,1)$ 来做.

$f(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}}\cdot\frac{1}{a}\cdot e^{\underbrace{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u-bv}{a}\right)^2+v^2-2\rho\cdot\frac{u-bv}{a}\cdot v\right]}_{A}}$

$A = -\frac{1}{2(1-\rho^2)}\left[\frac{1}{a^2}(u-bv)^2+v^2-2\rho\frac{1}{a}(u-bv)v\right]$

$= -\frac{1}{2(1-\rho^2)}\cdot\left[\frac{1}{a^2}u^2-\frac{2}{a^2}b\,uv+\frac{b^2}{a^2}v^2+v^2-2\rho\frac{1}{a}uv+2\rho\frac{b}{a}v^2\right]$

$= -\frac{1}{2(1-\rho^2)}\frac{1}{a^2}\left[u^2-2(b+\rho a)uv+(b^2+2\rho ab+a^2)v^2\right]$

$\because$ Aim is to proof $u \sim$ norm dis

$\therefore$ Reform $f(u,v) \sim$ bivariable norm dis

$A = -\frac{1}{2(1-\rho^2)}\frac{a^2+2\rho ab+b^2}{a^2}\left(\frac{u^2}{a^2+2\rho ab+b^2}-2\frac{b+\rho a}{a^2+2\rho ab+b^2}uv+v^2\right)$

$U = ax+by$: $EX = EY = 0$ $\quad \therefore EU = 0 = EV$

$\sigma_u^2 = Var(aX+bY) = E[(aX+bY)^2] - E^2(aX+bY)$

$\sigma_v^2 = 1$ $\qquad = E(a^2X^2+2abXY+b^2Y^2) - 0$

$= a^2\underbrace{\sigma_X^2}_{1}+b^2\underbrace{\sigma_Y^2}_{1}+2ab\underbrace{E(XY)}_{\rho}$ $\qquad \rho = corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{E(XY)-\overset{0}{E(X)}E(Y)}{1\ 1}$

$= a^2+b^2+2ab\rho$ $\qquad = E(XY)$

$\rho_2 = \frac{Cov(U,V)}{\sigma_U\sigma_V}$ $\qquad Cov(U,V) = E(UV) - E(U)\cdot\underbrace{E(V)}_{0}$

$= \frac{a\rho+b}{\sqrt{a^2+b^2+2ab\rho}}$ $\qquad = E[(aX+bY)Y]$

$= aE(XY)+b\underbrace{E(Y^2)}_{1}$

$= a\rho+b$

$1-\rho_2^2 = 1-\frac{a^2\rho^2+2ab\rho+b^2}{a^2+b^2+2ab\rho} = \frac{a^2(1-\rho^2)}{a^2+b^2+2ab\rho} = \frac{a^2(1-\rho^2)}{\sigma_u^2}$

$\therefore 1-\rho^2 = \frac{\sigma_u^2(1-\rho_2^2)}{a^2}$ $\qquad \sqrt{1-\rho^2} = \frac{\sigma_u}{a}\sqrt{1-\rho_2^2}$

$A = -\frac{\cancel{a^2}}{2\cancel{\sigma_u^2}(1-\rho_2^2)}\cdot\frac{\cancel{\sigma_u^2}}{\cancel{a^2}}\left(\frac{u^2}{\sigma_u^2}-2\frac{\rho_2\sigma_u}{\sigma_u^2}uv+v^2\right)$

$= -\frac{1}{2(1-\rho_2^2)}\left(\frac{u^2}{\sigma_u^2}-2\frac{\rho_2}{\sigma_u}\cdot uv+v^2\right)$

$\therefore f_{UV}(u,v) = \frac{\cancel{a}}{2\pi\sigma_u\sqrt{1-\rho_2^2}}\cdot\frac{1}{\cancel{a}}e^{-\frac{1}{2(1-\rho_2^2)}\left(\frac{u^2}{\sigma_u^2}-2\cdot\frac{\rho_2}{\sigma_u}\cdot uv+v^2\right)}$

$= \frac{1}{2\pi\sigma_u\sigma_v\sqrt{1-\rho_2^2}}\cdot e^{-\frac{1}{2(1-\rho_2^2)}\left(\frac{u^2}{\sigma_u^2}-2\rho_2\cdot\frac{uv}{\sigma_u\sigma_v}+\frac{v^2}{\sigma_v^2}\right)} \sim$ bivariable normal dis $(\underset{0}{\mu_u}, \underset{0}{\mu_v}, \sigma_u, \underset{1}{\sigma_v}, \rho_2)$

$(\sigma_v = 1)$

$\because U \sim$ norm distribution (μ_u, σ_u^2) $\qquad \mu_u = a\mu_X+b\mu_Y$ $\qquad \sigma_u^2 = a^2\sigma_X^2+b^2\sigma_Y^2+2ab\,E(XY)$ $\qquad \square$

446 $Z_1, Z_2 \sim n(0,1)$, $X = a_X Z_1 + b_X Z_2 + C_X$

$Y = a_Y Z_1 + b_Y Z_2 + C_Y$

(a). Show that $EX = C_X$ $\quad Var\,X = a_X^2 + b_X^2$ $\quad Cov(X,Y) = a_X a_Y + b_X b_Y$

$EY = C_Y$ $\quad Var\,Y = a_Y^2 + b_Y^2$

$$f_{Z_1}(z_1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{z_1-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z_1^2}{2}}$$

$E(Z_1) = E(Z_2) = 0$

$E(X) = E(a_X Z_1 + b_X Z_2 + C_X) = a_X \underbrace{E(Z_1)}_{0} + b_X \underbrace{E(Z_2)}_{0} + C_X = C_X$

$E(Y) = C_Y$

$$\begin{aligned} Var\,X &= EX^2 - E^2X \\ &= E(a_X Z_1 + b_X Z_2 + C_X)^2 - C_X^2 \\ &= E\left(a_X^2 Z_1^2 + 2a_X b_X Z_1 Z_2 + b_X^2 Z_2^2 + 2a_X C_X Z_1 + 2b_X C_X Z_2\right) \\ &= a_X^2 + b_X^2 + 2\left(a_X b_X \underbrace{Z_1 Z_2}_{0} + \underbrace{a_X C_X Z_1}_{0} + \underbrace{b_X C_X Z_2}_{0}\right) \end{aligned}$$

$\because Z_1, Z_2$ are i.i.d $\therefore E(Z_1 Z_2) = E(Z_1)E(Z_2) = 0$

$\therefore Var\,X = a_X^2 + b_X^2$

Analogously $\therefore Var\,Y = a_Y^2 + b_Y^2$

$$\begin{aligned} Cov\,XY &= E(XY) - EX \cdot EY \\ &= E[(a_X Z_1 + b_X Z_2 + C_X)(a_Y Z_1 + b_Y Z_2 + C_Y)] - C_X C_Y \\ &= a_X a_Y E(Z_1^2) + b_X a_Y E(Z_2 Z_1) + C_X a_Y E(Z_1) \\ &\quad + a_X b_Y E(Z_1 Z_2) + b_X b_Y E(Z_2^2) + C_X b_Y E(Z_2) \\ &\quad + C_Y a_X E(Z_1) + C_Y b_X E(Z_2) + \cancel{C_X C_Y} - \cancel{C_X C_Y} \\ &= a_X a_Y + b_X b_Y \end{aligned}$$

$\square$

(b). Define $a_X = \sqrt{\frac{1+\rho}{2}}\,\sigma_X$ $\quad b_X = \sqrt{\frac{1-\rho}{2}}\,\sigma_X$ $\quad C_X = \mu_X$ $\quad \rho \in [0,1]$

$a_Y = \sqrt{\frac{1+\rho}{2}}\,\sigma_Y$ $\quad b_Y = \sqrt{\frac{1-\rho}{2}}\,\sigma_Y$ $\quad C_Y = \mu_Y$

Show $EX = \mu_X$ $\quad Var\,X = \sigma_X^2$ $\quad \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \rho$

$EY = \mu_Y$ $\quad Var\,Y = \sigma_Y^2$

$E(X) = C_X = \mu_X$ $\quad Var\,X = a_X^2 + b_X^2 = \frac{1+\rho}{2}\sigma_X + \frac{1-\rho}{2}\sigma_X = \sigma_X$

$E(Y) = C_Y = \mu_Y$ $\quad Var\,Y = a_Y^2 + b_Y^2 = \frac{1+\rho}{2}\sigma_Y + \frac{1-\rho}{2}\sigma_Y = \sigma_Y$

$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$ $\quad Cov(X,Y) = a_X a_Y + b_X b_Y = \frac{1+\rho}{2}\cdot\sigma_X\sigma_Y - \frac{1-\rho}{2}\sigma_X\sigma_Y$

$= \rho$ $\quad = \rho\sigma_X\sigma_Y$

(c) Show that (X,Y) has the bivariate normal pdf with $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$

$$f_{Z_1 Z_2}(z_1, z_2) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(z_1^2 + z_2^2)}$$

$\rho_{Z_1 Z_2} = \frac{Cov(Z_1, Z_2)}{\sigma_{Z_1}\sigma_{Z_2}} = E(Z_1 Z_2) - E Z_1 E Z_2 = 0$

$$|J| = \begin{vmatrix} \frac{\partial Z_1}{\partial X} & \frac{\partial Z_1}{\partial Y} \\ \frac{\partial Z_2}{\partial X} & \frac{\partial Z_2}{\partial Y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2(1+\rho)}\,\sigma_X} & \frac{1}{\sqrt{2(1+\rho)}\,\sigma_Y} \\ \frac{1}{\sqrt{2(1-\rho)}\,\sigma_X} & -\frac{1}{\sqrt{2(1-\rho)}\,\sigma_Y} \end{vmatrix} = \frac{-1}{\sqrt{1-\rho^2}\,\sigma_X\sigma_Y}$$

$X = a_X Z_1 + b_X Z_2 + C_X$ $\quad b_Y(X - C_X) = a_X Z_1 + b_X a_X Z_1 + \underbrace{b_X b_Y Z_2}$

$Y = a_Y Z_1 + b_Y Z_2 + C_Y$ $\quad b_X(Y - C_Y) = a_Y Z_1 + b_Y Z_2 = b_X a_Y Z_1 + \underbrace{b_X b_Y Z_2}$

let $D = a_X b_Y - b_X a_Y = -\sqrt{1-\rho^2}\,\sigma_X\sigma_Y$

$$\therefore Z_1 = \frac{b_Y(X - C_X) - b_X(Y - C_Y)}{D} = \frac{-\sqrt{\frac{1-\rho}{2}}\,\sigma_Y(X-\mu_X) - \sqrt{\frac{1+\rho}{2}}\,\sigma_X(Y-\mu_Y)}{-\sqrt{1-\rho^2}\,\sigma_X\sigma_Y} = \frac{\sigma_Y(X-\mu_X) + \sigma_X(Y-\mu_Y)}{\sqrt{2(1+\rho)}\,\sigma_X\sigma_Y}$$

$$Z_2 = \frac{b_Y(X - C_X) + b_X(Y - C_Y) - Z_1(a_X b_Y + b_X a_Y)}{2 b_X b_Y} = \frac{b_Y(X - C_X) + b_X(Y - C_Y)}{2 b_X b_Y} = \frac{\sigma_Y(X-\mu_X) - \sigma_X(Y-\mu_Y)}{\sqrt{2(1-\rho)}\,\sigma_X\sigma_Y}$$

$$f_{X,Y}(x,y) = f_{Z_1,Z_2}(z_1, z_2)\cdot|J| = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{1-\rho^2}\,\sigma_X\sigma_Y}\cdot e^{-\frac{1}{2}\left\{\frac{[\sigma_Y(x-\mu_X)+\sigma_X(y-\mu_Y)]^2}{2(1+\rho)\sigma_X^2\sigma_Y^2}+\frac{[\sigma_Y(x-\mu_X)-\sigma_X(y-\mu_Y)]^2}{2(1-\rho)\sigma_X^2\sigma_Y^2}\right\}}$$

let $A = \sigma_Y(x-\mu_X)$ $\quad e^{M}$

$B = \sigma_X(y-\mu_Y)$

$$M = -\frac{1}{2}\cdot\frac{1}{(2\sigma_X\sigma_Y)^2}\left[\frac{(A+B)^2}{1+\rho}+\frac{(A-B)^2}{1-\rho}\right]$$

$$= -\frac{1}{2}\cdot\frac{1}{2(\sigma_X\sigma_Y)^2}\cdot\frac{2(A^2+B^2-2\rho AB)}{(1+\rho)(1-\rho)}$$

$$= -\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2-\frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]$$

$\therefore f_{X,Y}(x,y) \sim$ bivariate $n(\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho)$ $\square$

(d) $X, Y \sim$ bivariate normal $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$

$\mu_X = c_X \qquad \sigma_X^2 = a_X^2 + b_X^2 \qquad \rho = \dfrac{a_X a_Y + b_X b_Y}{\sigma_X\sigma_Y}$

$\mu_Y = c_Y \qquad \sigma_Y^2 = a_Y^2 + b_Y^2$

Show (b) is not unique, show another selection.

$\because \sigma_X^2 = a_X^2 + b_X^2 \quad \therefore b_X = \pm\sqrt{\sigma_X^2 - a_X^2} \quad \therefore b_X b_Y = \pm\sqrt{\sigma_X^2 - a_X^2}\sqrt{\sigma_Y^2 - a_Y^2}$

$\sigma_Y^2 = a_Y^2 + b_Y^2 \quad \therefore b_Y = \pm\sqrt{\sigma_Y^2 - a_Y^2}$

$\rho = \dfrac{a_X a_Y + b_X b_Y}{\sigma_X\sigma_Y} \qquad a_X a_Y \pm\sqrt{\sigma_X^2 - a_X^2}\cdot\sqrt{\sigma_Y^2 - a_Y^2} = \rho\sigma_X\sigma_Y$ ①

Let $a_X = \sqrt{1-\rho^2}\,\sigma_X$ $\quad$ Satisfy equation ①

$a_Y = 0$

$\therefore$ (b) is not unique, there are infinite selections for ① $\square$

4.47. X, Y are i.i.d $\sim n(0,1)$, define $Z = \begin{cases} X & XY > 0 \\ -X & XY < 0 \end{cases}$

(a) Show that Z has normal distribution

$f_X(x) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{y^2}{2}}$

$f(Y>0) = f(Y<0) = \frac{1}{2}$

$f_Z(z=z \mid Y>0) = \dfrac{f_Z(z, Y>0)}{f(Y>0)} \qquad f_Z(z=z \mid Y<0) = \dfrac{f_Z(z, Y<0)}{Y(Y<0)}$

$f_Z(z, Y>0) = f_Z(|Z|, Y>0)\cdot 1$

$= f_Z(|Z|)\cdot f(Y>0)$

$= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot\frac{1}{2}$

$f_Z(z, Y<0) = f_Z(-|Z|, Y<0)\cdot|(-1)|$

$= \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}\cdot\frac{1}{2}$

$\therefore f_Z(z=z \mid Y<0) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}$

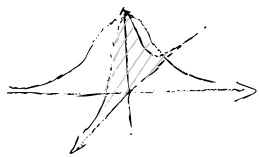
$\therefore f_Z(z=z \mid Y>0) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}$

$f_Z(z) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}} \sim n(0,1)$

(b). Show that $f_{Z,Y}(z,y)$ is not bivariate normal (Z, Y always have the same sign)

From (a), when $Y>0$ $\quad Z = |Z| > 0$, $\therefore Z, Y$ always have the same sign.

when $Y<0$ $\quad Z = -|Z| < 0$ $\quad \therefore$ cannot be bivariate normal



4.48. X, Y, $f(x,y) \propto \exp\{-\frac{1}{2}[Ax^2y^2 + x^2 + y^2 - 2Bxy - 2Cx - 2Dy]\}$

(a). Show that $f(X|Y=y) \sim n\left(\frac{By+C}{Ay^2+1}, \sqrt{\frac{1}{Ay^2+1}}\right)$

$$f_X(x|Y=y) = \frac{f(x,y)}{f_Y(y)}$$

$\because f(x,y) \propto \exp\{\cdots\}$

$\therefore$ without the loss of generality, assume

$$f(x,y) = \alpha \cdot e^{-\frac{1}{2}[(Ay^2+1)x^2 - 2(By+C)x + y^2 - 2Dy]}$$

$$= \alpha \cdot e^{-\frac{1}{2}(Ay^2+1)\left(x^2 - 2\frac{By+C}{Ay^2+1}x + \frac{y^2-2Dy}{Ay^2+1}\right)}$$

$$= \alpha \cdot e^{-\frac{1}{2}(Ay^2+1)\left[\left(x - \frac{By+C}{Ay^2+1}\right)^2 - \left(\frac{By+C}{Ay^2+1}\right)^2 + \frac{y^2-2Dy}{Ay^2+1}\right]}$$

$$= \alpha \cdot e^{-\frac{1}{2}(Ay^2+1)\cdot\left(x - \frac{By+C}{Ay^2+1}\right)^2} \cdot e^{\frac{1}{2}\left[\frac{(By+C)^2}{Ay^2+1} - y^2 + 2Dy\right]}$$

$$= \alpha \cdot e^{\frac{1}{2}\left[\frac{(By+C)^2}{Ay^2+1} - y^2 + 2Dy\right]} \cdot e^{-\frac{1}{2\left(\sqrt{\frac{1}{Ay^2+1}}\right)^2}\left(x - \frac{By+C}{Ay^2+1}\right)^2}$$

$f_Y(y) = \beta$

$$\therefore f_X(x|Y=y) = \frac{\alpha}{\beta} \cdot e^{\frac{1}{2}\left[\frac{(By+C)^2}{Ay^2+1} - y^2 + 2Dy\right]} \cdot e^{-\frac{1}{2\left(\sqrt{\frac{1}{Ay^2+1}}\right)^2}\left(x - \frac{By+C}{Ay^2+1}\right)^2}$$

mean $= \frac{By+C}{Ay^2+1}$ Var $= \frac{1}{Ay^2+1}$

$$f_X(x|Y=y) \sim n\left(\frac{By+C}{Ay^2+1}, \sqrt{\frac{1}{Ay^2+1}}\right)$$

$$f_Y(y|X=x) = \frac{f(x,y)}{f(x)}$$

$$f(x,y) = \alpha' \cdot e^{-\frac{1}{2}[(Ax^2+1)y^2 - 2(Bx+D)y + x^2 - 2Cx]}$$

$$= \alpha' \cdot e^{-\frac{1}{2}(Ax^2+1)\left[y^2 - 2\frac{Bx+D}{Ax^2+1} + \frac{x^2-2Cx}{Ax^2+1}\right]}$$

$$= \alpha' \cdot e^{-\frac{1}{2\left(\sqrt{\frac{1}{Ax^2+1}}\right)^2}\left[\left(y - \frac{Bx+D}{Ax^2+1}\right)^2 - \left(\frac{Bx+D}{Ax^2+1}\right)^2 + \frac{x^2-2Cx}{Ax^2+1}\right]}$$

$$= \alpha' \cdot e^{\frac{1}{2}\left[\frac{(Bx+D)^2}{Ax^2+1} - x^2 + 2Cx\right]} \cdot e^{-\frac{1}{2\left(\sqrt{\frac{1}{Ax^2+1}}\right)^2}\left(y - \frac{Bx+D}{Ax^2+1}\right)^2}$$

$f_X(x) = \beta'$

$$\therefore f_Y(y|X=x) = \frac{\alpha'}{\beta'} e^{\frac{1}{2}\left[\frac{(Bx+D)^2}{Ax^2+1} - x^2 + 2Cx\right]} \cdot e^{-\frac{1}{2\left(\sqrt{\frac{1}{Ax^2+1}}\right)^2}\left(y - \frac{Bx+D}{Ax^2+1}\right)^2}$$

mean $= \frac{Bx+D}{Ax^2+1}$ Var $= \frac{1}{Ax^2+1}$

$$f_Y(y|X=x) \sim n\left(\frac{Bx+D}{Ax^2+1}, \sqrt{\frac{1}{Ax^2+1}}\right)$$

(b) $A=1$, $B=0$, $C=D=8$. Show that this joint distribution is bimodal.

$$f(x,y) \propto \exp\{-\frac{1}{2}[Ax^2y^2 + x^2 + y^2 - 2Bxy - 2Cx - 2Dy]\}$$

$$f(x,y) \propto \exp\{-\frac{1}{2}[(x+y-8)^2 + (xy-2)^2 - 68]\}$$

when $\begin{cases} x+y=8 \\ xy=2 \end{cases}$ $f(x,y)$ have the max value

$(8-y)y = 2$

$y^2 - 8y + 2 = 0$

$(y-4)^2 - 16 + 2 = 0$

$(y-4)^2 = 14$

$y = \pm\sqrt{14} + 4$

$x = 8 - y$

$= 4 \pm \sqrt{14}$

$\therefore \begin{cases} x_1 = 4+\sqrt{14} \\ y_1 = 4-\sqrt{14} \end{cases}$ or $\begin{cases} x_2 = 4-\sqrt{14} \\ y_2 = 4+\sqrt{14} \end{cases}$ $\therefore f(x,y)$ is bimodal $\square$

4.49. f_1, f_2, g_1, g_2 are univariate densities with $\mu_1, \mu_2, \xi_1, \xi_2$

$$f(X,Y) = a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)$$

(a) Show $f_X(x) = a f_1(x) + (1-a) f_2(x)$

$f_Y(y) = a g_1(y) + (1-a) g_2(y)$

$$f_X(x) = \int_{-\infty}^{\infty} f(X,Y)\, dy$$

$$= \int_{-\infty}^{\infty} a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)\, dy$$

$$= a f_1(x) \underbrace{\int_{-\infty}^{\infty} g_1(y)\, dy}_{1} + (1-a) f_2(x) \underbrace{\int_{-\infty}^{\infty} g_2(y)\, dy}_{1}$$

$$= a f_1(x) + (1-a) f_2(x)$$

Analogously $f_Y(y) = a g_1(y) + (1-a) g_2(y)$

(b) Show that X, Y are independent if & only if

$$[f_1(x) - f_2(x)][g_1(y) - g_2(y)] = 0$$

1° If $[f_1(x) - f_2(x)][g_1(y) - g_2(y)] = 0$, X, Y are independent.

Proof:

$\because [f_1(x) - f_2(x)][g_1(y) - g_2(y)] = 0$

$\therefore$ (1) $f_1(x) - f_2(x) = 0$ OR (2) $g_1(y) - g_2(y) = 0$

$f_1(x) = f_2(x)$ $\qquad$ $g_1(y) = g_2(y)$

$f(x,y) = a f_1(x) g_1(y) + (1-a) f_1(x) g_2(y)$ $\qquad$ $f(x,y) = a f_1(x) g_1(y) + (1-a) f_2(x) g_1(y)$

$= f_1(x)[a g_1(y) + (1-a) g_2(y)]$ $\qquad$ $= g_1(y)[a f_1(x) + (1-a) f_2(x)]$

$f(x,y)$ can be factorized

$\therefore X, Y$ are independent

(3) $f_1(x) = f_2(x)$ meanwhile $g_1(y) = g_2(y)$

$f(x,y) = a f_1(x) g_1(y) + (1-a) f_1(x) g_1(y)$

$= f_1(x) g_1(y)$

$\therefore$ 1° $\square$

2° Only if $\underbrace{[f_1(x) - f_2(x)][g_1(y) - g_2(y)]}_{A} = 0$, X, Y are independent.

Proof: Assume X, Y are independent, but $A \neq 0$

$\therefore X, Y$ are independent

$\therefore f(x,y) = f(x) \cdot f(y)$ that is

$$[a f_1(x) + (1-a) f_2(x)][a g_1(y) + (1-a) g_2(y)] = a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)$$

$$\begin{cases} [a f_1(x) + (1-a) f_2(x)]\, a\, g_1(y) = a f_1(x) g_1(y) \\ [a f_1(x) + (1-a) f_2(x)](1-a) g_2(y) = (1-a) f_2(x) g_2(y) \end{cases}$$

$$\begin{cases} a f_1(x)[a g_1(y) + (1-a) g_2(y)] = a f_1(x) g_1(y) \\ (1-a) f_2(x)[a g_1(y) + (1-a) g_2(y)] = (1-a) f_2(x) g_2(y) \end{cases}$$

$$\therefore \begin{cases} a f_1(x) + (1-a) f_2(x) = f_1(x) \\ a f_1(x) + (1-a) f_2(x) = f_2(x) \end{cases}$$

$(1-a) f_2(x) = (1-a) f_1(x)$

$\therefore f_1(x) = f_2(x)$

$$\begin{cases} a g_1(y) + (1-a) g_2(y) = g_1(y) \\ a g_1(y) + (1-a) g_2(y) = g_2(y) \end{cases} \quad \therefore g_1(y) = g_2(y)$$

$\therefore A = 0$ contradicts with $A \neq 0$

$\therefore$ If X, Y are independent $A = 0$

(c) Show $Cov(Z,Y) = a(1-a)[\mu_1-\mu_2][\xi_1-\xi_2]$

$Cov(Z,Y) = E(ZY) - E(Z)E(Y)$

$E(ZY) = \int_Z \int_Y xy\, f(x,y)\, dx\, dy$

$= \iint xy\,[a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)]\, dx\, dy$

$= \int a x f_1(x) \underbrace{\int y\, g_1(y)\, dy}_{\xi_1}\, dx$

$+ \int (1-a) x f_2(x) \underbrace{\int y\, g_2(y)\, dy}_{\xi_2}\, dx$

$= a\mu_1\xi_1 + (1-a)\mu_2\xi_2$

$E(Z) = \int x \cdot [a f_1(x) + (1-a) f_2(x)]\, dx = a\cdot\mu_1 + (1-a)\mu_2$

$E(Y) = \int y\, a g_1(y) + (1-a) g_2(y)\, dy = a\cdot\xi_1 + (1-a)\xi_2$

$\therefore E(Z)E(Y) = [a\mu_1 + (1-a)\mu_2][a\xi_1 + (1-a)\xi_2] = [a(\mu_1-\mu_2)+\mu_2][a(\xi_1-\xi_2)+\xi_2]$

$\therefore Cov(Z,Y) = a\mu_1\xi_1 + (1-a)\mu_2\xi_2 - a^2(\mu_1-\mu_2)(\xi_1-\xi_2) - a\mu_2(\xi_1-\xi_2) - a(\mu_1-\mu_2)\xi_2 - \mu_2\xi_2$

$= a\mu_1\xi_1 + \cancel{\mu_2\xi_2} - a\mu_2\xi_2 - a^2(\mu_1-\mu_2)(\xi_1-\xi_2) - a\mu_2(\xi_1-\xi_2)$

$- a(\mu_1-\mu_2)\xi_2 - \cancel{\mu_2\xi_2}$

$= a\mu_1\xi_1 - \cancel{a\mu_2\xi_2} - a\mu_2\xi_1 + \cancel{a\mu_2\xi_2} - a\mu_1\xi_2 + a\mu_2\xi_2$

$- a^2(\mu_1-\mu_2)(\xi_1-\xi_2)$

$= a(\mu_1-\mu_2)(\xi_1-\xi_2) - a^2(\mu_1-\mu_2)(\xi_1-\xi_2)$

$= a(1-a)(\mu_1-\mu_2)(\xi_1-\xi_2)$ $\square$

If uncorrelated, $Cov(Z,Y) = 0 \Rightarrow \mu_1 = \mu_2$ or $\xi_1 = \xi_2$

dependent, $[f_1(x) - f_2(x)][g_1(y) - g_2(y)] \neq 0 \Rightarrow f_1(x) \neq f_2(x)$ AND $g_1(y) \neq g_2(y)$

(d). Independent (Z,Y) pairs binomial pmf (n,p)

$f_1(x) \sim$ binomial (n,p) $g_1(y) \sim$ binomial (n,p) $E = np$

$f_2(x) \sim$ binomial (n,p) $g_2(y) \sim$ ... (n,p)

$[f_1(x) - f_2(x)][g_1(y) - g_2(y)] = 0$

Correlated (Z,Y)

$f_1(x) \sim$ binomial (n_1, p) $(n_1 p \neq n_3 p)$ $g_1(y) \sim$ binomial (n_3, p) $(n_3 p \neq n_4 p)$

$f_2(x) \sim$ binomial (n_2, p) $g_2(y) \sim$ ... (n_4, p)

$\mu_1 \neq \mu_2$ and $\xi_1 \neq \xi_2$

Uncorrelated but dependent

$f_1(x) \sim$ binomial (n, p) $g_1(y) \sim$ binomial (n_2, p_2)

$f(x) \sim$ binomial $(2n, \frac{p}{2})$ $g_2(y) \sim$ binomial $(2n_2, \frac{p_2}{2})$

uncorrelated $\mu_1 = \mu_2$ or $\xi_1 = \xi_2$ $\square$

dependent $f_1(x) \neq f_2(x)$ and $g_1(x) \neq g_2(x)$

4.50. $f(X,Y) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \cdot e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}$ (Hint use conditional E

Show that $Corr(X,Y)=\rho$ and $Corr(X^2,Y^2)=\rho^2$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$\sigma_X^2 = Var X = EX^2 - E^2X$

$\sigma_Y^2 = Var Y = EY^2 - E^2Y$

$E(XY) = \iint xy\, f(x,y)\, dx\, dy$

$= \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \underbrace{\iint xy \cdot e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}\, dx\, dy}_{A}$

$A = \int x \cdot e^{-\frac{x^2-\rho^2x^2}{2(1-\rho^2)}} \int y \cdot e^{-\frac{1}{2(1-\rho^2)}(y^2-2\rho xy+\rho^2x^2)}\, dy\, dx$

$= \int x \cdot e^{-\frac{x^2}{2}} \int y \cdot e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2}\, dy\, dx$

$= \rho \int x^2 e^{-\frac{x^2}{2}}\, dx \cdot \sqrt{2\pi(1-\rho^2)} = \rho \cdot \int y^2 e^{-\frac{y^2}{2}}\, dy \cdot \sqrt{2\pi(1-\rho^2)}$

$\int y \cdot e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2}\, dy$

$= \int k \cdot e^{-\frac{1}{2(1-\rho^2)}k^2}\, dk + \rho x \cdot \int e^{-\frac{1}{2(1-\rho^2)}k^2}\, dk$

$= 0 + \rho x \cdot \sqrt{2\pi(1-\rho^2)}$

$\therefore E(XY) = \rho \int x^2 e^{-\frac{x^2}{2}}\, dx \cdot \sqrt{2\pi} = \frac{\rho \cdot \int y^2 e^{-\frac{y^2}{2}}\, dy}{\sqrt{2\pi}}$

$E(X) = \int x \cdot f(x,y)\, dy\, dx$

$= \int \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \int x \cdot e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}\, dy\, dx$

$= \int \frac{x}{2\pi(1-\rho^2)^{\frac{1}{2}}} \cdot e^{-\frac{x^2}{2}} \underbrace{\int e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2}\, dy}\, dx$

$= \int \frac{x}{2\pi(1-\rho^2)^{\frac{1}{2}}} \cdot e^{-\frac{x^2}{2}} \cdot \sqrt{2\pi(1-\rho^2)}\, dx$

$= \int \frac{x}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx$

$= 0$

Analogously

$E(Y) = \int \frac{y}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}\, dy = 0$

$Cov: EXY - EXEY = \frac{\rho(\int x^2 e^{-\frac{x^2}{2}}\, dx)^{\frac{1}{2}} (\int y^2 e^{-\frac{y^2}{2}}\, dy)^{\frac{1}{2}}}{2\pi}$

$EX^2 = \int \frac{x^2}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx \quad EY^2 = \int \frac{y^2}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}\, dy$

$\therefore \sigma_X^2 = EX^2 - (EX)^2 = \int \frac{x^2}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx - 0$

$\sigma_Y^2 = \int \frac{y^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{\rho^{\frac{1}{2}}(\int x^2 e^{-\frac{x^2}{2}}\, dx)^{\frac{1}{2}} \rho^{\frac{1}{2}}(\int y^2 e^{-\frac{y^2}{2}}\, dy)^{\frac{1}{2}}}{\sqrt{2\pi}\left(\int \frac{y^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy\right)^{\frac{1}{2}}\left(\int \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx\right)^{\frac{1}{2}}} = \rho$ □

$R = \int e^{-ax^2}\, dx$

$R^2 = \iint e^{-ax^2-ay^2}\, dx\, dy$

$= \iint e^{-a(x^2+y^2)}\, dx\, dy$

$x^2+y^2 = r^2$

$R^2 = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2} \cdot r\, dr\, d\theta$

$= \int_0^{\infty} e^{-ar^2} \cdot r \int_0^{2\pi} 1\, d\theta\, dr$

$= 2\pi\left(-\frac{1}{2a} e^{-ar^2}\Big|_0^{\infty}\right)$

$= \frac{\pi}{a}$

$R = \sqrt{\frac{\pi}{a}}$

$a = \frac{1}{2(1-\rho^2)}$

$R = \sqrt{2\pi(1-\rho^2)}$

$E(X^2) = \iint x^2 \cdot f(x,y)\, dy\, dx$

$= \int \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \int x^2 \cdot e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}\, dy\, dx$

$= \int \frac{x^2}{2\pi(1-\rho^2)^{\frac{1}{2}}} \cdot e^{-\frac{x^2}{2}} \int e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2}\, dy\, dx$

$= \int \frac{x^2}{2\pi(1-\rho^2)^{\frac{1}{2}}} \cdot e^{-\frac{x^2}{2}} \cdot \sqrt{2\pi(1-\rho^2)}\, dx$

$= \int \frac{x^2}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx$

$E(Y^2) = \int \frac{y^2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$E(X^4) = \int \frac{x^4}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx$

$E(Y^4) = \int \frac{y^4}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$

$E(X^4) - E^2(X^2) = \frac{1}{\sqrt{2\pi}} \int x^4 e^{-\frac{x^2}{2}}\, dx - \frac{1}{2\pi} \int x^4 \cdot e^{-x^2}\, dx$

$E(Y^4) - E^2(Y^2) = \frac{1}{\sqrt{2\pi}} \int y^4 e^{-\frac{y^2}{2}}\, dy - \frac{1}{2\pi} \int y^4 \cdot e^{-y^2}\, dy$

$\frac{x^2}{2} = s \qquad x^2 = 2s \qquad x^4 = 4s^2$

$x = \sqrt{2}\, s^{\frac{1}{2}}$

$dx = \sqrt{2} \cdot \frac{1}{2} s^{-\frac{1}{2}} ds$

$= \frac{\sqrt{2}}{2} s^{-\frac{1}{2}} ds$

$Corr(X^2,Y^2) = \frac{Cov(X^2,Y^2)}{\sigma_{X^2} \cdot \sigma_{Y^2}} = \frac{E(X^2Y^2) - E(X^2) \cdot E(Y^2)}{[E(X^4) - (E(X^2))^2]^{\frac{1}{2}} [E(Y^4) - (E(Y^2))^2]^{\frac{1}{2}}} = \frac{\frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \int x^4 \cdot e^{-\frac{x^2}{2}} \left[\int k^2 e^{-\frac{1}{2(1-\rho^2)}k^2} + \rho^2 x^2 \sqrt{2\pi(1-\rho^2)}\right] - \frac{1}{2\pi} \int x^4 y^2 \cdot e^{-\frac{x^2+y^2}{2}}\, dx\, dy}{\left(\frac{1}{\sqrt{2\pi}} \int x^4 e^{-\frac{x^2}{2}}\, dx - \frac{1}{2\pi} \int x^4 \cdot e^{-x^2}\, dx\right)^{\frac{1}{2}} \left(\frac{1}{\sqrt{2\pi}} \int y^4 e^{-\frac{y^2}{2}}\, dy - \frac{1}{2\pi} \int y^4 \cdot e^{-y^2}\, dy\right)^{\frac{1}{2}}}$

$E(X^2Y^2) = \iint x^2y^2 \cdot f(x,y)\, dx\, dy$

$= \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \underbrace{\iint x^2y^2 \cdot e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}\, dx\, dy}_{A}$

$A = \int x^2 \cdot e^{-\frac{x^2-\rho^2x^2}{2(1-\rho^2)}} \int y^2 e^{-\frac{1}{2(1-\rho^2)}(y^2-2\rho xy+\rho^2x^2)}\, dy\, dx$

$= \int x^2 e^{-\frac{x^2}{2}} \underbrace{\int y^2 e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2}\, dy}_{A}\, dx$

$= \int x^2 \cdot e^{-\frac{x^2}{2}} \left[\int k^2 e^{-\frac{1}{2(1-\rho^2)}k^2} + \rho^2 x^2 \sqrt{2\pi(1-\rho^2)}\right] dx$

$\int y^2 e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2}\, dy$

Let $k = y - \rho x \qquad y = k + \rho x$

$\int (k+\rho x)^2 \cdot e^{-\frac{1}{2(1-\rho^2)}k^2}\, dk$

$= \int k^2 e^{-\frac{1}{2(1-\rho^2)}k^2} + 2\rho x \underbrace{\int k \cdot e^{-\frac{1}{2(1-\rho^2)}k^2}\, dk}_{0} + \rho^2x^2 \int \cdot\, dk$

$= \int k^2 e^{-\frac{1}{2(1-\rho^2)}k^2} + \rho^2 x^2 \sqrt{2\pi(1-\rho^2)}$

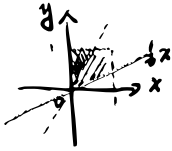
$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1-\rho^2)^{\frac{1}{2}}} \int x^4 e^{-\frac{x^2}{(1-\rho^2)}}\, dx + \sqrt{2\pi}(\rho^2-1) \int x^4 e^{-\frac{x^2}{2}}\, dx - \int x^4 e^{-x^2}\, dx\right]$

$= \frac{1}{\sqrt{2\pi}} \left[\sqrt{2\pi}(\rho^2-1)\right] + \frac{1}{\sqrt{2\pi}}$

$= \rho^2$ □

4.51. $X, Y, Z \sim$ independent uniform$(0,1)$

(a) Find $P(X/Y \le t)$ and $P(XY \le t)$



$\frac{x}{y} \le t \qquad y \ge \frac{1}{t}x \qquad t>0$

$\because x \in (0,1),\ y \in (0,1)$

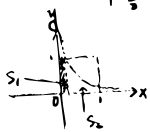
$\therefore \frac{x}{y} \le (0, +\infty) \qquad \frac{1}{t} \in (0, +\infty)$

$P(X/Y \le t) = 1 - S_\Delta = 1 - \frac{1}{2}\cdot 1 \cdot \frac{1}{t}$

$= \frac{2t-1}{2t} \quad (t \ge 1)$

$P(X/Y \le t) = S_\Delta = 1 \cdot t \cdot \frac{1}{2} = \frac{t}{2} \ (0<t<1)$

$\therefore P(X/Y \le t) = \begin{cases} 1 - \frac{1}{2t} & (t \ge 1) \\ \frac{t}{2} & (0<t<1) \end{cases}$ $\quad 0$



$xy = t \qquad y = t \cdot \frac{1}{x} \qquad \frac{t}{x} = 1 \qquad x = t$

$S_1 = t$

$S_2 = \int_t^1 t \cdot \frac{1}{x}\, dx$

$= t \cdot \ln^x \Big|_t^1$

$= t(0 - \ln^t)$

$\therefore P(XY \le t) = S_1 + S_2$

$= t - t\ln^t \qquad = -t \cdot \ln^t$

(b) Find $P(XY/Z \le t)$ $\qquad z \in (0,1)$

$P(XY/Z \le t) = \int_0^1 P(XY \le zt)\, dz$

$xy \le zt$

$y \le \frac{zt}{x}$

$= \int_0^1 zt(1 - \ln^{zt})\, dz \quad t \in (0,1)$

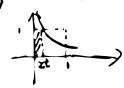
$\frac{zt}{x} = 1 \qquad x = zt$

$= \int_0^{\frac{1}{t}} zt(1 - \ln^{zt})\, dz$

$+ \int_{\frac{1}{t}}^1 1\, dz$

$t \in (1, \infty)$

$1^\circ\ t \in (0,1) \quad zt \in (0,1)$



$S_1 = zt$

$S_2 = \int_{zt}^1 zt \frac{1}{x}\, dx$

$= zt \ln^{zt}$

$S_1 + S_2 = zt(1 - \ln^{zt})$

$\therefore P(XY/Z \le t) = \begin{cases} t - \frac{t}{2}\ln^t + \frac{1}{4t} & t \in (0,1) \\ 1 + \frac{1}{4t^2} + 1 - \frac{1}{t} & t \in (1, \infty) \end{cases}$

$\left(= 2 + \frac{1}{4t^2} - \frac{1}{t}\right)$

$2^\circ\ t > 1 \qquad x = zt$

$t - \int_0^1 tz \cdot \ln^{tz}\, dz$

$u' = tz\, dz \qquad v = \ln^{tz}$

$u = \frac{t}{2}z^2 \qquad v' = \frac{1}{tz} t\, dz \qquad zt \le 1 \qquad zt > 1$

$z \le \frac{1}{t} \qquad 1 > z > \frac{1}{t}$

$(uv)' = \left(\frac{t}{2}z^2 \cdot \ln^{tz}\right)' = tz \cdot \ln^{tz} dz + \frac{t}{2}z^2 \cdot \frac{1}{tz} \cdot \frac{1}{t}\, dz$

$\int_0^1 tz \cdot \ln^{tz} dz = \frac{t}{2}z^2 \ln^{tz}\Big|_0^1 - \int_0^1 \frac{t}{2}\, dz = \frac{t}{2}\ln^t - \frac{1}{4t}$

$\int_0^{\frac{1}{t}} \cdots\cdots = \frac{t}{2} \cdot z^2 \cdot \ln^{tz}\Big|_0^{\frac{1}{t}} - \frac{1}{2t}\int_0^{\frac{1}{t}} z\, dz$

$= 0 - \frac{1}{2t} \cdot \frac{1}{2t^2} \qquad \frac{1}{2}z^2\Big|_0^{\frac{1}{t}} = \frac{1}{2t^2}$

$= -\frac{1}{4t^2}$

4.52. $X, Y \sim n(0,1)$ X, Y are independent

$A(x_1, y_1)$ $B(x_2, y_2)$

$d = AB = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$, find $f(D)$

$\because Z_1, Z_2 \sim n(0,1)$

$\therefore Z_2 - Z_1 \sim n(0,2)$

$Y_2 - Y_1 \sim n(0,2)$

$\left(\frac{X_2-X_1}{\sigma_1}\right)^2 + \left(\frac{Y_2-Y_1}{\sigma_2}\right)^2 \sim \chi_2^2$

$\left(\frac{X_2-X_1}{\sqrt{2}}\right)^2 + \left(\frac{Y_2-Y_1}{\sqrt{2}}\right)^2 \sim \chi_2^2$

that is $\frac{(x_2-x_1)^2}{2} + \frac{(y_2-y_1)^2}{2} \sim \chi_2^2$ $\quad k=2 \quad f(x,2) = \frac{e^{-\frac{x}{2}}}{2}$

Chi-square dis

k degree of freedom

$\sum_k X_i^2$

$$f(x;k) = \begin{cases} \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} & x>0 \\ 0 & x \le 0 \end{cases}$$

Let $U = \frac{(Z_2-Z_1)^2 + (Y_2-Y_1)^2}{2}$ $\quad f_U(u) = \frac{e^{-\frac{u}{2}}}{2}$

$d = (2U)^{\frac{1}{2}}$ $\quad U = \frac{d^2}{2}$ $\quad du = \frac{1}{2} 2d = d.$

$\therefore f_D(d) = f_U(\frac{d^2}{2}) \cdot d = \frac{e^{-\frac{d^2}{4}}}{2} \cdot d = \frac{d}{2} e^{-\frac{d^2}{4}}$ $d>0$ $\square$

4.53. $A, B, C \sim$ uniform $(0,1)$ independent

$P(Ax^2+Bx+C)$ has real roots.

Hint: If $X \sim$ uniform$(0,1)$. $-\log X \sim$ exponential.

Sum (two exponential) $\sim$ gamma.

Ax^2+Bx+C has real roots $\Rightarrow B^2 - 4AC \ge 0$

$B^2 \ge 4AC$

$\frac{B^2}{AC} \ge 4$

$\frac{B}{A} \cdot \frac{B}{C} \ge 4$

$\int P(Y < \log 4 + x) f_X(x) dx$

$P(X > \log 4 + Y)$

$\log A + \log B = \log AB$

$\log A - \log B = \log A/B$

$\log \frac{B}{A} + \log \frac{B}{C} \ge \log 4$

$\log B - \log A + \log B - \log C \ge \log 4$

$\underbrace{-\log A + (-\log C) \ge \log 4 - 2\log B}$

$-\log B \sim$ exponential(1) $\quad -2\log B \sim$ gamma$(k, c\theta)$ $\quad k=1 \quad (-4\frac{1}{2}\frac{1}{2}) \quad c\theta = 2$

$A \sim$ uniform $(0,1)$ $\quad -\log A \sim$ exponential(1) $\quad -\log A + (-\log C) \sim$ gamma(k, θ)

$C \sim$ uniform $(0,1)$ $\quad -\log C \sim$ exponential(1) $\quad k=2$ (形状参数) $\quad \theta = 1$

Let $Z = -\log A + (-\log C)$

$f_Z(x) = \frac{x^{k-1} e^{-x/\theta}}{\theta^k \cdot \Gamma(k)} = \frac{x \cdot e^{-x}}{(2-1)!} = x \cdot e^{-x}$

let $Y = -2\log B$

$f_Y(y) = \frac{1}{2} e^{-\frac{y}{2}}$?

$\because B \in (0,1)$

$\therefore \log B \in (-\infty, 0)$

$-\log B \in (0, +\infty)$ $\quad Y \in (0, +\infty)$

$P(Z > \log 4 + Y) = \int_0^\infty P(Z > \log 4 + y) \cdot f_Y(y) dy$

$= \int_0^\infty \int_{\log 4 + y}^\infty x \cdot e^{-x} \frac{1}{2} e^{-\frac{y}{2}} dy\, dx$

?

$\int_0^\infty e^{-\frac{2}{3}y} dy$

$= -\frac{2}{3} e^{-\frac{2}{3}y} \Big|_0^\infty$

$= -\frac{2}{3}(0-1)$

$= \frac{2}{3}$

$\int y \cdot e^{-\frac{2}{3}y} dy$

let $u = \frac{2}{3} y$ $\quad v' = -\frac{3}{2} e^{-\frac{2}{3}y} dy$

$u' = \frac{2}{3} dy$ $\quad v = e^{-\frac{2}{3}y}$

$(-\frac{2}{3} y \cdot e^{-\frac{2}{3}y})' = y \cdot e^{-\frac{2}{3}y} dy - \frac{2}{3} e^{-\frac{2}{3}y} dy$

$\int y \cdot e^{-\frac{2}{3}y} dy = \underbrace{-\frac{2}{3} y e^{-\frac{2}{3}y} \Big|_0^\infty}_{0} + \int \frac{2}{3} e^{-\frac{2}{3}y} dy$

$= \frac{2}{3}\left(-\frac{2}{3}\right) e^{-\frac{2}{3}y} \Big|_0^\infty$

$= -\frac{4}{9}(0-1)$

$= \frac{4}{9}$

$\int x \cdot e^{-x} \cdot \int_0^\infty e^{-2y} dy\, dx$

$\frac{1}{2} \int_0^\infty \int_{\log 4 + y}^\infty x \cdot e^{-x} dx \cdot e^{-\frac{y}{2}} dy$

$\frac{1}{2} \int_0^\infty (\log 4 + y + 1) \cdot \frac{1}{4} e^{-\frac{y}{2}} dy$

$= \frac{1}{8}(\log 4 + 1) \int_0^\infty e^{-\frac{2}{3}y} dy + \frac{1}{8} \int_0^\infty y e^{-\frac{2}{3}y} dy$

$= \frac{1}{8}(\log 4 + 1) \cdot \frac{2}{3} + \frac{1}{8} \cdot \frac{4}{9} = \frac{1}{12} \log 4 + \frac{5}{36}$ $\square$

let $u = -x$ $\quad v' = -e^{-x} dx$

$u' = -1 dx$ $\quad v = e^{-x}$

$\therefore (-x \cdot e^{-x})' = x \cdot e^{-x} dx - e^{-x} dx$

$\therefore \int_a^b x \cdot e^{-x} dx = -x \cdot e^{-x} \Big|_a^b + \int_a^b e^{-x} dx$

$= -x \cdot e^{-x} \Big|_{\log 4 + y}^\infty + (-e^{-x}) \Big|_{\log 4 + y}^\infty$

$= (\log 4 + y + 1) \frac{1}{4} e^{-y}$ ✓

4.54. Z_i are independent uniform $(0,1)$. find the pdf of $\prod_{i=1}^{n} Z_i$

Hint: If $Z \sim$ uniform $(0,1)$, $-\log Z \sim$ exponential

Calculate the cdf first.

$\because X_1, X_2, \cdots X_n \sim n(0,1)$

$\therefore -\log Z_1, -\log Z_2, \cdots, -\log Z_n \sim$ exponential (1)

$Y: (-\log Z_1) + (-\log Z_2) + \cdots + (-\log Z_n) \sim$ Gamma(k, θ)

$k=n \quad \theta=1$

$\log a + \log b = \log ab$

$\log a - \log b = \log a/b$

$-\log a = \log^{-} a$

$Y = \log^{-} Z_1 + \log^{-} Z_2 + \cdots + \log^{-} Z_n$

$= \log^{-} (Z_1 Z_2 \cdots Z_n)^{-1} \sim$ gamma $(n, 1)$ $\quad f_Y(y) = \frac{x^{k-1} \cdot e^{-x/\theta}}{\theta^k \cdot \Gamma(k)}$

$= \frac{y^{n-1} \cdot e^{-y}}{(n-1)!}$

let $Z = \prod_{i=1}^{n} Z_i$

$Y = \log Z^{-1} \quad Z = (e^{Y})^{-1} \quad dY = (-\log Z)'$

$= (-1) \cdot \frac{1}{Z} \quad Z \geq 0$

$f_Z(z) = f_Y(z^{-1}) \cdot \left|\frac{dy}{dz}\right|$

$= \frac{\cancel{z} \cdot (\log z)^{n-1}}{(n-1)!} \cdot \frac{1}{\cancel{z}}$

$= -\frac{(\log z)^{n-1}}{(n-1)!}$

4.55. Z_1, Z_2, Z_3 are independent $\sim$ exponential (λ)

$f_{Z_i}(x_i) = \lambda \cdot e^{-\lambda x_i} \quad x_i \geq 0$.

$Y = \max(Z_1, Z_2, Z_3)$, find $f_Y(y)$

先求 cdf Y，再求 pdf Y

$P_Y(y) = P(x_1 \leq y \,\&\, x_2 \leq y \,\&\, x_3 \leq y)$

$\because Z_1, Z_2, Z_3$ are independent

$\therefore P(x_1 \leq y \,\&\, x_2 \leq y \,\&\, x_3 \leq y) = P(x_1 \leq y) \cdot P(x_2 \leq y) \cdot P(x_3 \leq y)$

$f_Z(x) = \lambda \cdot e^{-\lambda x}$

$P(x \leq y) = \int_0^y \lambda \cdot e^{-\lambda x} dx$

$= -e^{-\lambda x} \Big|_0^y$

$= -(e^{-\lambda y} - 1)$

$= 1 - e^{-\lambda y}$

$\therefore P_Y(y) = (1 - e^{-\lambda y})^3$

$f_Y(y) = \frac{d P_Y(y)}{dy}$

$= 3(1 - e^{-\lambda y})^2 \cdot (-1) \cdot e^{-\lambda y} \cdot (-\lambda)$

$= 3\lambda (1 - e^{-\lambda y})^2 \cdot e^{-\lambda y}$ $y \geq 0$

0 $\quad y < 0$

4.56. Pool sample $N = mk$

negative $\rightarrow$ one test suffices for k people

positive $\rightarrow$ each test separately

(a). What is the probability that test for a pooled k people be positive. one is negative : $1-p$

$P = 1 - P(\text{all negative}) = 1 - (1-p)^k$

(b) X = number of blood tests necessary under pool sample, Find EX

$N = mk$ if positive, $(k+1)$ are requested for these k

$P_{k+} = 1-(1-p)^k$ ..... (a) $\sim k+1$ times

$P_{k-} = (1-p)^k$ ~ 1 times

In total there are m groups

Suppose there are n groups are positive among m, $n=0,1,\cdots m$

$f(n) = C_m^n \cdot (P_{k+})^n (1-P_{k+})^{m-n}$

$X = n(k+1) + (m-n)$

$EX = E[n(k+1)+(m-n)]$

$= (k+1)E(n) + m - E(n)$

$= k \cdot E(n) + m$ ②

$E(n) = \sum_{n=0}^{m} n \cdot f(n)$

$= \sum_{n=0}^{m} n \cdot \frac{m!}{n!(m-n)!} (P_{k+})^n (1-P_{k+})^{m-n}$

$= \sum_{n=1}^{m} \frac{m(m-1)!}{(n-1)!(m-1-n+1)!} \cdot (P_{k+})^{n-1}(P_{k+})(1-P_{k+})^{m-1-(n-1)}$

$= m \cdot P_{k+} \cdot \underbrace{\sum_{n=1}^{m} C_{m-1}^{n-1} (P_{k+})^{n-1} (1-P_{k+})^{(m-1)-(n-1)}}_{1}$

$= m \cdot P_{k+}$

$= m \cdot [1-(1-p)^k]$ ①

Substitute $E(n)$ in ② with ①:

$EX = k \cdot m[1-(1-p)^k] + m$ □

(c) Minimizing EX, if $p \to 0$, should (i) or (ii) be preferred

If $p \to 0$

with (ii) $EX \to m$ $\because$ $m < N$ $\therefore$ (ii) should be preferred

思考: 如果已知 p, 甚至可求最优 k (m) 来让检测成本降低.

4.57. Miscellanea 4.9.2

$$A_r = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^r\right]^{1/r} \qquad \left[\frac{1}{n}\sum_{i=1}^{n} x_i^r\right]^{1/r} \le \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{r'}\right]^{1/r'} \quad r \le r'$$

(a) Show that A_1 is the arithmetic mean,

A_{-1} is the harmonic mean,

$A_0 = \lim_{r\to 0} A_r$ is the geometric mean.

When $r=1$

$A_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$ by definition, it is the arithmetic mean.

When $r=-1$

$A_{-1} = \left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i}\right)^{-1} = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}$ ..... harmonic mean

when $r \to 0$

$$A_0 = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^r\right]^{1/r}$$

$\because \lim_{r\to 0} \frac{\log x^r}{x^r - 1} = 1 \quad \therefore x^r - 1 = \log x^r$

$x^r = \log x^r + 1 = \log e \cdot x^r$

$$A_0 = \left[\frac{1}{n}\left(\log x_1^r + \log x_2^r + \cdots + \log x_n^r + n\right)\right]^{1/r}$$

$$= \left[\frac{1}{n} \cdot \log e^n \cdot (x_1 x_2 \cdots x_n)^r\right]^{1/r}$$

$$= \left[\log e \cdot (x_1 x_2 \cdots x_n)^{r/n}\right]^{1/r}$$

$$= \left[\underbrace{\log (x_1 x_2 \cdots x_n)^{r/n} + 1}\right]^{1/r}$$

$$= \left[(x_1 x_2 \cdots x_n)^{r/n}\right]^{1/r}$$

$$= (x_1 x_2 \cdots x_n)^{\frac{1}{n}} \quad \cdots \text{Geometric mean} \quad \square$$

(b). (i) Varify that if $\log A_r$ is nondecreasing in r then it will follow that A_r is non-decreasing in r

If $\log A_r$ is nondecreasing

then $\log A_{r_1} \geq \log A_{r_2}$ if $r_1 \geq r_2$

that is $A_{r_1} \geq A_{r_2}$ if $r_1 \geq r_2$

which indicate A_r is nondecreasing. $\square$

(ii) Show that

$$\frac{d}{dr} \log A_r = \frac{1}{r^2}\left\{\frac{r \sum_i x_i^r \log x_i}{\sum_i x_i^r} - \log\left(\frac{1}{n}\sum_i x_i^r\right)\right\}$$

$$A_r = \left(\frac{1}{n}\sum x_i^r\right)^{1/r}$$

$$\log A_r = \frac{1}{r}\log\left(\frac{1}{n}\sum x_i^r\right)$$

$$\frac{d \log A_r}{dr} = \frac{1}{r}\frac{\cancel{\frac{1}{n}}\left(\sum \log x_i \cdot x_i^r\right)}{\cancel{\frac{1}{n}}\sum x_i^r} - \frac{1}{r^2}\log\left(\frac{1}{n}\sum_i x_i^r\right)$$

$$= \frac{1}{r^2}\left[\frac{r \sum \log x_i \, x_i^r}{\sum x_i^r} - \log\left(\frac{1}{n}\sum x_i^r\right)\right] \quad \square$$

(iii) Define $a_i = x_i^r / \sum x_i^r$ write the quantity in brevecas

$$\log(n) - \sum a_i \log(1/a_i)$$

where $\sum a_i = 1$.

Prove this quantity is nonnegative, establishing the monotonicity of A_r.

$$\frac{r \sum \log x_i \, x_i^r}{\sum x_i^r} - \log\left(\frac{1}{n}\sum x_i^r\right)$$

$$= r \cdot \left(\frac{x_1^r \log x_1 + x_2^r \log x_2 + \cdots + x_n^r \log x_n}{x_1^r + x_2^r + \cdots + x_n^r}\right) - \left(\log\frac{1}{n} + \log \sum x_i^r\right)$$

$$= r \cdot \left(a_1 \log x_1 + a_2 \log x_2 + \cdots + a_n \log x_n\right) + \log(n) - \log\left(x_1^r + x_2^r + \cdots + x_n^r\right)$$

$$= \log(n) + a_1 \log x_1^r + a_2 \log x_2^r + \cdots + a_n \log x_n^r - \log\left(x_1^r + x_2^r + \cdots + x_n^r\right)$$

$$= \log(n) + \log \frac{x_1^{ra_1} \cdot x_2^{ra_2} \cdot \cdots \cdot x_n^{ra_n}}{x_1^r + x_2^r + \cdots + x_n^r} = \log(n) + \sum a_i \log a_i = \log(n) - \sum a_i \log\left(\frac{1}{a_i}\right) \quad \square$$

Let $f(r) = \log(n) + \sum a_i \log a_i$

$= \log(n) + a_1 \log a_1 + a_2 \log a_2 + \cdots + a_n \log a_n$

$f'(r) = (\sum a_i \log a_i)' = 0$

$\therefore$ this quantity is nonnegative.

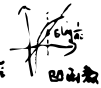
$\because r \to 0$

$f(r) = \log(n) + \sum \frac{x_i^r}{\sum x_i^r} \log\left(\frac{x_i^r}{\sum x_i^r}\right)$

$= \log(n) - \sum a_i \log \frac{1}{a_i}$

$\because \sum a_i = 1 \quad (f)$

$\therefore E \log \frac{1}{a_i} = \sum a_i \log \frac{1}{a_i}$



$\because E \log \frac{1}{a_i} \le \log E \frac{1}{a_i} = \log \sum a_i \cdot \frac{1}{a_i} = \log(n)$

$\therefore \sum a_i \log \frac{1}{a_i} \le \log n$

$\lim_{r \to 0} f(r) \ge 0 \quad \because f'(r) = 0$

$\therefore f(r) \ge 0 \quad (r \ge 0)$

$\therefore (\log A_r)' \ge 0$ that is A_r is nondecreasing $\square$

$a_i = x_i^r / \sum x_i^r$

$(a_i)' = \frac{r \cdot x_i^{r-1}}{\sum x_i^r} - \frac{r \cdot x_i^r \sum x_i^{r-1}}{(\sum x_i^r)^2}$

$(x \cdot \log x)'$

$= x \cdot \frac{1}{x} \cdot (x)' + (x') \cdot \log x$

$= (x)'(1 + \log x)$

$\sum (a_i \log a_i)'$

$\sum = (a_i)'(1 + \log a_i)$

$\sum = \left(\frac{r \cdot x_i^{r-1}}{\sum x_i^r} - \frac{r \cdot x_i^r \sum x_i^{r-1}}{(\sum x_i^r)^2}\right)(1 + \log x_i^r - \log \sum x_i^r)$

$= \left(\frac{r \sum x_i^{r-1} \sum x_i^r - r \sum x_i^r \sum x_i^{r-1}}{(\sum x_i^r)^2}\right)(1 + \log x_i^r - \log \sum x_i^r)$

$\to 0$

4.58 X, Y with finite variance, prove that

(a) $Cov(X, Y) = Cov(X, E(Y|X))$

$Cov(X, Y) = E((X - \mu_X)(Y - \mu_Y))$

$= E(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y)$

$= E(XY) - \mu_X \mu_Y$

$Cov(X, E(Y|X)) = E[(X - \mu_X)(E(Y|X) - \mu_Y)]$

$= E[X \cdot E(Y|X) - \mu_X \cdot E(Y|X) - X\mu_Y + \mu_X \mu_Y]$

$= E[X E(Y|X)] - \mu_X \cdot E[E(Y|X)] - \cancel{\mu_X \mu_Y} + \cancel{\mu_X \mu_Y}$

$= EXY - \mu_X \cdot EY = EXY - \mu_X \mu_Y = Cov(X, Y)$ $\square$

$E[E(Y|X)] = E_X(E_{Y|X}(Y|X))$

$= E_X(\sum_y y \cdot P(y|x))$

$= \sum_x (\sum_y y \cdot P(y|x)) \cdot P(x)$

$= \sum_x \sum_y y \cdot P(y|x) \cdot P(x)$

$= \sum_x \sum_y y \cdot P(x, y)$

$= \sum_y y \sum_x P(x, y)$

$= \sum_y y P(y)$

$= E(Y)$

$E[X E(Y|X)] = E_X[x \cdot E_{Y|X}(Y|X)]$

$= E_X[\sum_y xy P(y|x)]$

$= \sum_x [\sum_y xy P(y|x)] P(x)$

$= \sum_x \sum_y xy P(y|x) \cdot P(x)$

$= \sum_x \sum_y xy \cdot P(x, y)$

$= E(XY)$

(b) X & $Y - E(Y|X)$ are uncorrelated.

Uncorrelated $\Leftrightarrow Cov[X, Y - E(Y|X)] = 0$

$Cov(X, Y - E(Y|X)) = E[(X - \mu_X)(Y - E(Y|X) - 0)]$

$= E[(X - \mu_X)(Y - E(Y|X))]$

$= E[XY - \mu_X \cdot Y - X \cdot E(Y|X) + \mu_X \cdot E(Y|X)]$

$= E(XY) - \mu_X \mu_Y - E[X E(Y|X)] + \mu_X \mu_Y$

$= \cancel{E(XY)} - \cancel{\mu_X \mu_Y} - \cancel{E(XY)} + \cancel{\mu_X \mu_Y}$

$= 0$ $\square$

$E[Y - E(Y|X)] = EY - E[E(Y|X)]$

$= EY - EY$

$= 0$

$E[E^2(Y|X)]$

$E[=[E(Y|X)]^2]$

$E[=(\sum_y y \cdot P(y|x))^2]$

$(y_1 \cdot P(y_1|x) + y_2 \cdots]^2$

$E[E^2(Y|X)]$

$= E[E(Y|X)]^2$

$= E^2 Y$

(c) $Var(Y - E(Y|X)) = E(Var(Y|X))$

$Var(S) = ES^2 - E^2 S$

$\because Var(Y - E(Y|X)) = E[Y - E(Y|X)]^2 - E^2[Y - E(Y|X)]$

$= E[Y^2 - 2Y \cdot E(Y|X) + E^2(Y|X)] - 0$

$= EY^2 - 2E[Y E(Y|X)] + E^2[E(Y|X)]$

$= EY^2 - E^2 Y$

$Var(Y|X) = E(Y^2|X) - E^2(Y|X)$

$= \sum_y y^2 \cdot P(y|x) - E^2 Y$

$E[Var(Y|X)] = \sum_x \sum_y y^2 P(y|x) \cdot P(x) - E^2 Y$

$= \sum_x \sum_y y^2 P(x, y)$

$= EY^2 - E^2 Y$ $\square$

4.59. X, Y, Z with finite variances, prove

$$Cov(X,Y) = E(Cov(X,Y|Z)) + Cov(E(X|Z), E(Y|Z)),$$

$Cov(X,Y|Z)$ is the covariance of X, Y under $f(x,y|z)$.

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - \mu_X\mu_Y$$

$$Cov(X,Y|Z) = \iint (x-\mu_X)(y-\mu_Y)\cdot f(x,y|z)\,dx\,dy$$

$$E[Cov(X,Y|Z)] = \iiint (x-\mu_X)(y-\mu_Y)\cdot \underline{f(x,y|z)\cdot f(z)}\,dx\,dy\,\underline{dz}$$

$$= \iint (x-\mu_X)(y-\mu_Y)\cdot f(x,y)\,dx\,dy$$

$$= \iint (xy - \mu_X y - \mu_Y x + \mu_X\mu_Y)\, f(x,y)\,dx\,dy.$$

$$= E(XY) - \mu_X\mu_Y$$

$$E(X|Z) = \int_x x\cdot f(x|z)\,dx$$

$$E[E(X|Z)] = \int_z\int_x x\cdot f(x|z)f(z)\,dx\,dz$$

$$= \int_z\int_x x\cdot f(x,z)\,dx\,dz$$

$$= \int_x x\,\underline{\int_z f(x,z)\,dz}\,dx$$

$$= \int_x x\cdot f(x)\,dx$$

$$= \mu_X$$

$$E[E(Y|Z)] = \mu_Y$$

$$Cov[E(X|Z), E(Y|Z)] = E[(E(X|Z)-\mu_X)(E(Y|Z)-\mu_Y)]$$

$$= E[E(X|Z)E(Y|Z) - \mu_X\cdot E(Y|Z) - \mu_Y\cdot E(X|Z) + \mu_X\mu_Y]$$

$$= E\left[\int_x x\cdot f(x|z)dx\cdot\int_y y\cdot f(y|z)dy\right] - \mu_X\cdot\mu_Y$$

$$= \int_z\left(\int_x x\cdot f(x|z)dx\int_y y\,f(y|z)dy\right)dz - \mu_X\mu_Y$$

$$= \mu_X\mu_Y - \mu_X\mu_Y$$

$$= 0$$

$$\therefore Cov(X,Y) = E[Cov(X,Y|Z)] + Cov[E(X|Z), E(Y|Z)] \quad \square$$

4.60. Miscellanea 4.9.3 X, Y are independent $n(0,1)$

$f_Y(y \mid y=x)$ $\qquad f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

(1) $Z_1 = 0$; $Z_1 = Y - X$ $\qquad P(XY)$

(2) $Z_2 = 1$; $Z_2 = Y/X$ $\qquad P(X|Y) = \frac{P(XY)}{P(Y)}$

(3) $Z_3 = 1$; $Z_3 = I(Y=X)$

(1): Y, Z_1 $\quad \sigma_Y^2 = 1$ $\quad \sigma_{Z_1}^2 = 1+1 = 2$ $\quad \mu_Y = 0$ $\quad \mu_{Z_1} = 0$

$$\rho = Corr(Y, Z_1) = \frac{Cov(Y, Z_1)}{\sigma_Y\,\sigma_{Z_1}} = \frac{E[(Y-\mu_Y)(Z_1-\mu_{Z_1})]}{\sigma_Y\,\sigma_{Z_1}}$$

$$E(YZ_1) = E[Y(Y-X)] \qquad = \frac{E(Y^2)}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \frac{P(AB)}{P(B)}$$

$$= EY^2 - EXY$$

$$= EY^2 = 1$$

$$\therefore f(Y, Z_1) = \frac{1}{2\pi\sqrt{1-\frac{1}{2}}}\cdot e^{-\frac{1}{2(1-\frac{1}{2})}\left[Y^2 + Z_1^2 - 2\cdot\frac{1}{\sqrt{2}}YZ_1\right]}$$

$$\therefore f(y|z_1=0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2(1-\frac{1}{2})}} = \frac{1}{\sqrt{2\pi}}e^{-y^2}$$

$\therefore$ the conditional distribution of Y given Z_1 is

$$f(Y|Z_1=0) \sim N\left(\mu_Y + \frac{\sigma_Y}{\sigma_{Z_1}}\rho(0-\mu_{Z_1}),\ (1-\rho^2)\sigma_Y^2\right) = N(0, \tfrac{1}{2})$$

(2) $Z_2 = 1$; $Z_2 = Y/X$

$Z_2 \sim$ Cauchy distribution

$\because X, Y$ are independent $\sim N(0,1)$

$Z_2 \sim$ standard Cauchy distribution $(x_0 = 0, \gamma = 1)$

$P(AB)$

$P(A) = \frac{P(AB)}{P(B|B=1)}$

$= \int_B P(AB)/B$

$$f(z_2; x_0, \gamma) = \frac{1}{\pi\gamma} \cdot \frac{\gamma^2}{(z_2 - x_0)^2 + \gamma^2}$$

$$= \frac{1}{\pi} \cdot \frac{1}{z_2^2 + 1}$$

$$\frac{P(AB \mid B=1)}{P(B=1)}$$

$$f(z_2 = 1; x_0, \gamma) = \frac{1}{\pi} \cdot \frac{1}{2}$$

Let $s = y$, $z_2 = \frac{y}{x}$ $\begin{cases} x = \frac{s}{z_2} \\ y = s \end{cases}$ $|J| = \begin{vmatrix} \frac{1}{z_2} & -\frac{s}{z_2^2} \\ 1 & 0 \end{vmatrix} = \frac{s}{z_2^2}$

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} + \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]}$$

$$\rho = Corr(Y, Z) = \frac{Cov(Y,Z)}{\sigma_Z \sigma_Y} = \frac{E[(Y-\mu_Y)(Z-\mu_Z)]}{1} = \frac{E(ZY)}{1} = 0$$

$$\therefore f(x,y) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x^2+y^2)}$$

$$f(s, z_2) = f\left(\frac{s}{z_2}, s\right) \cdot |J|$$

$$= \frac{1}{2\pi} \cdot e^{-\frac{s^2(1+z_2^2)}{2z_2^2}} \frac{|s|}{z_2^2}$$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$f(s \mid z_2 = 1) = \frac{P(s, z_2 \mid z_2 = 1)}{P(z_2 = 1)}$$

$$= \frac{\cancel{\frac{1}{2\pi}} \cdot e^{-\frac{s^2 \cdot 2}{2}} \cdot \frac{|s|}{1}}{\cancel{\frac{1}{2\pi}}}$$

$$= |s| \cdot e^{-s^2} \quad \square$$

(3) $Z_3 = 1$; $Z_3 = I(Y = Z)$

I is the indicator function, $I(Y=Z) \begin{cases} 1 & \text{if } Y = Z \\ 0 & \text{otherwise} \end{cases}$

that is Y & Z_3 are independent

$P(Z_3 = 1) = 0$ $\underline{P(Z_3 = 0) = 1}$ (A constant is independent of any random variables)

$\therefore P(Y \mid Z_3 = 1) = P(Y)$

$\therefore f(Y \mid Z_3 = 1) = f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad \square$

4.61. Z_1, Z_2 are i.i.d $\sim$ exponential(1), $Z = (Z_2 - 1)/Z_1$

$P(Z = 0) \neq P(Z_2 = 1)$ but seems give the same info

(a) Find $f(Z_1 \mid Z = 0)$ and $f(Z_1 \mid Z_2 = 1)$

$\because Z_1, Z_2 \sim$ exponential(1)

$\therefore f_{Z_1}(x_1) = \lambda \cdot e^{-\lambda x_1}$ $x_1 > 0$ $\quad f_{Z_2}(x_2) = e^{-x_2}$

$= e^{-x_1}$

$f(A|B) = \frac{f(AB)}{f(B)}$

$$f(Z_1 \mid Z_2 = 1) = \frac{f(x_1, x_2 = 1)}{f(x_2 = 1)}$$

$$= \frac{f(x_1) \cdot f(x_2 = 1)}{f(x_2 = 1)}$$

$$= f(x_1)$$

$$= e^{-x_1}$$

$f(Z_1 \mid Z = 0)$

$f(Z_1, Z)$

$f(x_1, x_2) = f(x_1) \cdot f(x_2) = e^{-(x_1 + x_2)}$

Let $U = Z_1$ $\quad \therefore Z_1 = U$

$V = \frac{Z_2 - 1}{Z_1} = Z$ $\quad Z_2 = UV + 1$

$|J| = \begin{vmatrix} 1 & 0 \\ V & U \end{vmatrix} = U$

$\therefore f(U, V) = f(U, UV+1) \cdot |J|$

$= e^{-(U + UV + 1)} \cdot U$

$f(U, V=0) = e^{-(U+1)} \cdot U$

$f(V=0) = f(Z_2 = 1) = e^{-1}$

$\therefore f(U \mid V=0) = \frac{f(U, V=0)}{f(V=0)}$

$= \frac{e^{-(U+1)} \cdot U}{e^{-1}}$

$= e^{-U} \cdot U$

$= e^{-x_1} \cdot x_1$ ✓

(b) for a small $\varepsilon > 0$, $x_1 > 0$, $x_2 > 0$. Consider

$$B_1 = \left\{ (x_1, x_2) : -\varepsilon < \frac{x_2 - 1}{x_1} < \varepsilon \right\} \qquad B_2 = \{ (x_1, x_2) : 1-\varepsilon < x_2 < 1+\varepsilon \}$$

Visualize:

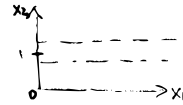
$$-\varepsilon x_1 < x_2 - 1 < \varepsilon x_1$$

$$\underline{-\varepsilon x_1 + 1 < x_2 < \varepsilon x_1 + 1}$$

B_1 x_2: 在这两条线之间

B_2: $\underline{1-\varepsilon < x_2 < 1+\varepsilon}$

x_2 在这两条线(常数函数)之间.



$\therefore$ B_1 is more informative as the range of x_2 changes.

(c). Calculate $P(Z_1 \le x \mid B_1)$ and $P(Z_1 \le x \mid B_2)$, Show that their limits ($\varepsilon \to 0$) agree with (a)

1° $P(Z_1 \le x \mid B_1) = P(U \le x_1 \mid -\varepsilon < V < \varepsilon)$

$$= \frac{P(U \le x_1, -\varepsilon < V < \varepsilon)}{P(-\varepsilon < V < \varepsilon)}$$

$$= \frac{\int_{-\varepsilon}^{\varepsilon} \int_0^{x_1} f(u,v)\, du\, dv}{\int_{-\varepsilon}^{\varepsilon} \int_0^{\infty} f(u,v)\, du\, dv}$$

$f(Z_1 \mid Z = 0)$

$f(Z_1, Z)$

$f(x_1 x_2) = f(x_1) \cdot f(x_2) = e^{-(x_1 + x_2)}$

Let $U = Z_1$ $\quad \therefore Z_1 = U$

$V = \frac{Z_2 - 1}{Z_1} = Z$ $\quad Z_2 = UV + 1$ $\quad |J| = \begin{vmatrix} 1 & 0 \\ V & U \end{vmatrix} = U$

$\therefore f(U,V) = f(U, UV+1) \cdot |J|$

$= e^{-(U + UV + 1)} \cdot U$

$$\int_{-\varepsilon}^{\varepsilon} -e^{-1}[(x_1+1)e^{-x_1} - 1](V+1)^{-2} dv$$

$$= -e^{-1}[(x_1+1) \cdot e^{-x_1} - 1] \cdot (-1)(V+1)^{-1} \Big|_{-\varepsilon}^{\varepsilon}$$

$$= e^{-1}[(x_1+1) \cdot e^{-x_1} - 1] \frac{-2\varepsilon}{(\varepsilon+1)(-\varepsilon+1)} \quad ①$$

$$\int_{-\varepsilon}^{\varepsilon} e^{-1}(V+1)^{-2} dv$$

$$= e^{-1} \cdot (-1) \cdot (V+1)^{-1} \Big|_{-\varepsilon}^{\varepsilon}$$

$$= e^{-1} \cdot (-1) \cdot \frac{-2\varepsilon}{(\varepsilon+1)(-\varepsilon+1)} \quad ②$$

$$\frac{①}{②} = \frac{(x_1+1) \cdot e^{-x_1} - 1}{(-1)} = \underline{1 - (x_1+1) \cdot e^{-x_1}}$$

$$\underline{= \int_0^{x_1} x \cdot e^{x} dx}$$

agree with (a)

$\int_0^{x_1} f(u,v)\, du$

$= \int_0^{x_1} e^{-(U + UV + 1)} \cdot U\, dU$

$= \int_0^{x_1} e^{-(V+1) \cdot U} \cdot U \cdot e^{-1}\, dU$

$= e^{-1} \int_0^{x_1} e^{-(V+1) \cdot U}\, U\, dU$

$= \frac{e^{-1}}{V+1} \underbrace{\int_0^{x_1} (V+1)U \cdot e^{-(V+1)U} dU}_{G}$

$G_1 = -\frac{1}{V+1}(s+1)e^{-s} \Big|_0^{x_1}$

$= -\frac{1}{V+1} \cdot [(x_1+1)e^{-x_1} - 1]$

$G_2 = -\frac{1}{V+1}(s+1) \cdot e^{-s} \Big|_0^{\infty}$

$= -\frac{1}{V+1}[0 - 1]$

$= \frac{1}{V+1}$

Let $(V+1)U = S$ $\quad U = 0$ $\quad S = 0$ $\quad U = \frac{S}{V+1}$

$U = x_1$ $\quad S = (V+1)x_1$ $\quad dU = \frac{1}{V+1} ds$

$$\frac{1}{V+1} \int_a^b S \cdot e^{-S} ds = -\frac{S+1}{V+1} e^{-S} \Big|_a^b$$

Let $m = S$ $\quad n' = e^{-S} ds$

$m' = 1 \cdot ds$ $\quad n = -e^{-S}$

$(-S \cdot e^{-S})' = S \cdot e^{-S} ds - e^{-S} ds$

$\int_a^b S \cdot e^{-S} ds = -S \cdot e^{-S} \Big|_a^b - e^{-S} \Big|_a^b$

$= (-S-1) \cdot e^{-S} \Big|_a^b$

2° $P(Z_1 \le x \mid B_2)$

B_2: (x_1, x_2): $-\varepsilon < x_2 - 1 < \varepsilon$

$\lim_{\varepsilon \to 0} B_2 \to (x_1, x_2)$: $x_2 = 1$ $\quad P(X_2 = 1)$ is a constant

$\because$ X_1 is independent with B_2

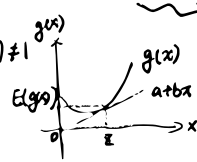
$\therefore P(Z_1 \le x \mid B_2) = P(Z_1 \le x) = \underline{\int_0^{x_1} e^{x} dx_1}$ $\quad \square$

4.62. Proof Jensen's Inequality

If $g(x)$ is a convex function, $Eg(X) \geq g(EX)$

Equality holds If and Only If $P(X = EX) = 1$

Assume $P(X = EX) \neq 1$



$Eg(X) = g(EX)$

$Eg(x) \geq E(a+bx)$

$= a + bE(X)$

If $Eg(x) = a + bE(X) = g(EX)$

$\therefore Eg(X) = \underbrace{g(X) = a+bX}_{\text{矛盾}} = g(EX)$

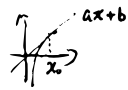
$X = EX$ which is conflict with assumption $P(X = EX) \neq 1$

$\therefore$ Equality holds Only If $P(X = EX) = 1$ □

4.63. X : $Z = \log X$, $EZ = 0$. EX ? 1

$\because Z = \log X$

$EZ = E(\log X) = 0$



Assume $E(\log X) = \log(x_0)$

$\because E(\log X) = 0$

$\therefore x_0 = 1$

$\because \log x$ is concave

$\therefore \log x \leq ax + b$

$\therefore E(\log X) \leq E(aX + b)$

$= aEX + b$

$Z = \log X$

$Z' = \frac{1}{X}$

$Z'(x_0) = 1 = a$

$E(x_0) = 1$

$aE(x_0) + b = 0$

$1 + b = 0$

$b = -1$

$\therefore E(\log X) \leq E(X - 1)$

$0 \leq E(X) - 1$

$\therefore E(X) \geq 1$ □

4.64. (a) Prove $|a+b| \leq |a| + |b|$ for any numbers a and b

1° $ab \geq 0$ $\quad (a+b)^2 = a^2 + 2ab + b^2 = (|a| + |b|)^2$

$\therefore \sqrt{(a+b)^2} = \sqrt{(|a|+|b|)^2}$ that is $|a+b| = |a| + |b|$

2° $ab < 0$ $\quad (a+b)^2 = a^2 + 2ab + b^2 < a^2 + 2 \cdot |a| \cdot |b| + b^2 = (|a| + |b|)^2$

$\therefore \sqrt{(a+b)^2} < \sqrt{(|a|+|b|)^2}$ that is $|a+b| < |a| + |b|$

Combine 1° and 2°, $|a+b| \leq |a| + |b|$ □

(b). Use (a) to establish that for any random variables X and Y with finite expectations

$$E|X+Y| \leq E|X| + E|Y|$$

$E|X+Y| = \int_X \int_Y |x+y| \cdot f(x,y)\, dx\, dy$ ①

$E|X| + E|Y| = E(|X| + |Y|) = \int_X \int_Y (|x| + |y|) \cdot f(x,y)\, dx\, dy$ ②

$\because |x+y| \leq |x| + |y|$

$\therefore$ ① $\leq$ ②

that is $E|X+Y| \leq E|X| + E|Y|$

□

4.65. Prove the Covariance Inequality-II by generalizing the argument given in the text immediately preceding the inequality.

4.7.9 Pg2 (220)

a. If $g(x)$ is a nondecreasing function and $h(x)$ is a nonincreasing function, then

$$E(g(Z)h(Z)) \le (Eg(Z))(Eh(Z))$$

b. If $g(x)$ and $h(x)$ are both nondecreasing or both nonincreasing, then

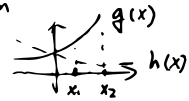
$$E(g(Z)h(Z)) \ge (Eg(Z))(Eh(Z))$$

$Cov(X,Y) = E[(X-EX)(Y-EY)]$

$= E(XY) - EXEY$

$\because$ $g(x)$ is nondecreasing function

$\therefore g(x_1) \le g(x_2)$ if $x_1 \le x_2$



$\because$ $h(x)$ is nonincreasing function

$\therefore h(x_1) \ge h(x_2)$

assume $E(g(Z)) = g(x_g)$

$E(h(Z)) = h(x_h)$

$[g(x) - E(g(Z))][h(x) - E(h(Z))]$

$= [g(x) - g(x_g)][h(x) - h(x_h)]$ ①

$x_g \le x_h$

without loss of generality

$x \le x_g \le x_h$

Let $x_g \le x_h$

1° when $x \le x_g \le x_h$

$g(x) \le g(x_g)$ $\quad h(x) \ge h(x_h)$

$g(x) - g(x_g) \le 0$ $\quad h(x) - h(x_h) \ge 0$

$\therefore$ ① ≤ 0

2° when $x_g \le x \le x_h$

$g(x) \ge g(x_g)$ $\quad h(x) \ge h(x_h)$

$g(x) - g(x_g) \ge 0$ $\quad h(x) - h(x_h) \ge 0$

① ≥ 0

3° when $x_g \le x_h \le x$

$g(x) \ge g(x_g)$ $\quad h(x) \le h(x_h)$

$g(x) - g(x_g) \ge 0$ $\quad h(x) - h(x_h) \le 0$

① ≤ 0

$E[(g(Z) - Eg(Z))(h(Z) - Eh(Z))] = E(g(Z)h(Z)) - (Eg(Z))(Eh(Z))$

$= \int_{-\infty}^{x_g}$ ① $f(x)dx + \int_{x_g}^{x_h}$ ① $f(x)dx + \int_{x_h}^{\infty}$ ① $f(x)dx \le 0$

$- \quad + \quad -$

$\therefore E(g(Z)h(Z)) \le (Eg(Z))(Eh(Z))$ □

If $g(x)$ and $h(Z)$ are both nondecreasing / nonincreasing

1° when $x \le x_g \le x_h$

$g(x) \le g(x_g)$ $\quad h(x) \le h(x_h)$

$g(x) - g(x_g) \le 0$ $\quad h(x) - h(x_h) \le 0$

$\therefore$ ① ≥ 0

2° when $x_g \le x \le x_h$

$g(x) \ge g(x_g)$ $\quad h(x) \le h(x_h)$

$g(x) - g(x_g) \ge 0$ $\quad h(x) - h(x_h) \le 0$

① ≤ 0

3° when $x_g \le x_h \le x$

$g(x) \ge g(x_g)$ $\quad h(x) \ge h(x_h)$

$g(x) - g(x_g) \ge 0$ $\quad h(x) - h(x_h) \ge 0$

① ≥ 0

$E[(g(Z) - Eg(Z))(h(Z) - Eh(Z))] = E(g(Z)h(Z)) - (Eg(Z))(Eh(Z))$

$= \int_{-\infty}^{x_g}$ ① $f(x)dx + \int_{x_g}^{x_h}$ ① $f(x)dx + \int_{x_h}^{\infty}$ ① $f(x)dx \ge 0$

$+ \quad - \quad +$

$\therefore E(g(Z)h(Z)) \ge (Eg(Z))(Eh(Z))$ □