Sec 6

6.1. Let X be one observation from $n(0, \sigma^2)$,

Is $|X|$ a sufficient static?

Definition of sufficient static:

$f(x|\theta) = g(T(x)|\theta) \cdot \underbrace{h(x)}$ independent with θ

$f(x|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{|x|^2}{2\sigma^2}}$

Let $T(x) = |x|$

$g(T(x)|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{|x|^2}{2\sigma^2}}$.

$h(x) = 1$.

then $f(x|\theta) = g(T(x)|\theta) \cdot h(x)$

$\therefore T(x) = |x|$ is a sufficient static for σ □.

6.2. Let $X_1, \cdots, X_n$ be independent random variables

$$f_{X_i}(x|\theta) = \begin{cases} e^{i\theta - x} & x > i\theta \\ 0 & x < i\theta \end{cases}$$

Prove $T(X) = \min_i (X_i / i)$ is a sufficient statistic for θ

$X = (X_1, X_2, \cdots, X_n)$

$f(x|\theta) = f_{X_1}(x_1|\theta) \cdot f_{X_2}(x_2|\theta) \cdots f_{X_n}(x_n|\theta)$

$= \prod_{i=1}^{n} e^{i\theta - x_i} \; I_{(i\theta, +\infty)} \cdot x_i$

$= \begin{cases} e^{\frac{n(n+1)}{2}\theta} \cdot e^{-\sum_{i=1}^{n} x_i} \; I_{(\theta, +\infty)} T(x) \\ 0 \quad I_{(0,\theta)} T(x) \end{cases}$

$= \underbrace{e^{\frac{n(n+1)}{2}\theta} I_{(\theta, +\infty)} T(x)}_{g(x)} \cdot \underbrace{e^{-\sum_{i=1}^{n} x_i}}_{h(x)}$ $\qquad T(X) = 1$

$\therefore T(x)$ is a sufficient static for θ. □

6.3. Let $X_1, \cdots, X_n$ be a random sample from

$$f(x|\mu, \sigma) = \frac{1}{\sigma} \cdot e^{-(x-\mu)/\sigma} \quad \mu < x < \infty, \; 0 < \sigma < \infty$$

Find a two-dimensional sufficient statistic for (μ, σ)

$T(X) = \min_i (X_i) \quad n\bar{X} = \sum X_i$

$f(X|\mu, \sigma) = f(x_1|\mu, \sigma) \cdot f(x_2|\mu, \sigma) \cdots$

$= \frac{1}{\sigma} \cdot e^{-(x_1-\mu)/\sigma} \cdot \frac{1}{\sigma} \cdot e^{-(x_2-\mu)/\sigma} \cdots I_{(\mu, +\infty)}(x_i)$

$= \left(\frac{1}{\sigma}\right)^n e^{-\frac{\sum_i x_i - n\mu}{\sigma}} \cdot I_{(\mu, +\infty)}(x_i)$

$= \left(\frac{1}{\sigma}\right)^n e^{-\frac{\sum x_i - n\mu}{\sigma}} \; I_{(\mu, +\infty)}(T(X))$

$= \underbrace{\left(\frac{1}{\sigma}\right)^n \cdot e^{-\frac{n\bar{X} - n\mu}{\sigma}} I_{(\mu, +\infty)}(T(X))}_{g(T(X), \bar{X})} \cdot \underbrace{1}_{h(X)}$

$\therefore (T(X), \bar{X})$ is sufficient statistic for (μ, σ) □

6.4 Prove 6.2.10. Let $X_1, \cdots, X_n$ iid, from

$$f(x|\theta) = h(x)\cdot c(\theta)\cdot \exp\left(\sum_{i=1}^{k} w_i(\theta)\cdot t_i(x)\right),$$

$\theta = (\theta_1, \theta_2, \cdots, \theta_d)$ $d \le k$,

$T(X) \left(\sum_{j=1}^{n} t_1(X_j), \cdots, \sum_{j=1}^{n} t_k(X_j)\right)$ is sufficient statistic for θ

$$\begin{aligned}
f(\underline{x}|\theta) &= f(x_1, \cdots, x_n|\theta) \\
&= \prod_{j=1}^{n} h(x_j)\cdot c(\theta)\cdot \exp\left(\sum_{i=1}^{k} w_i(\theta)\cdot t_i(x_j)\right) \\
&= [c(\theta)]^n \left[\prod_{j=1}^{n} h(x_j)\right] e^{\sum_{j=1}^{n}\sum_{i=1}^{k} w_i(\theta)\cdot t_i(x_j)} \\
&= [c(\theta)]^n \left[\prod_{j=1}^{n} h(x_j)\right] e^{\sum_{i=1}^{k} w_i(\theta)\cdot \sum_{j=1}^{n} t_i(x_j)} \\
&= \underbrace{[c(\theta)]^n \cdot e^{\sum_{i=1}^{k} w_i(\theta)\cdot \sum_{j=1}^{n} t_i(x_j)}}_{g(T(\underline{x}),\,\theta)} \cdot \underbrace{\prod_{j=1}^{n} h(x_j)}_{h(\underline{x})}
\end{aligned}$$

$\therefore T(\underline{x})$ is sufficient statistic for θ $\square$

6.5 Let $X_1, \cdots, X_n$ be iid.

$$f_{X_i}(x_i|\theta) = \begin{cases} \frac{1}{2i\theta} & -i(\theta-1) < x_i < i(\theta+1) \\ 0 & \text{otherwise} \end{cases} \qquad \theta > 0.$$

Find a two-dimensional sufficient statistic for θ.

$T_1(\underline{x}) = \min_i \left(\frac{x_i}{i}\right)$ $\qquad 1+\theta > \frac{x_i}{i} > 1-\theta$

$T_2(\underline{x}) = \max_i \left(\frac{x_i}{i}\right)$

$$f(\underline{x}|\theta) = \prod_{i=1}^{n} f_{X_i}(x_i|\theta) = (2\theta)^{-n}\cdot \prod_{i=1}^{n}\frac{1}{i}\cdot I_{(1-\theta,+\infty)}(T_1(\underline{x}))\cdot I_{(-\infty,1+\theta)}(T_2(\underline{x}))$$

Let $g([T_1(\underline{x}), T_2(\underline{x})]|\theta) = (2\theta)^{-n}\cdot \prod_{i=1}^{n}\frac{1}{i}\cdot I_{(1-\theta,+\infty)}(T_1(\underline{x}))\cdot I_{(-\infty,1+\theta)}(T_2(\underline{x}))$

$h(\underline{x}) = 1$

$f(\underline{x}|\theta) = g([T_1(\underline{x}), T_2(\underline{x})]|\theta)\cdot h(\underline{x})$

$\therefore [T_1(\underline{x}), T_2(\underline{x})]$ is a two-dimentional sufficient statistic for θ. $\square$

6.6. Let $X_1, \cdots, X_n$ be a random sample from gamma (α, β). Find two dimentional sufficient statistic for (α, β).

$$f_{X_i}(x_i|\alpha,\beta) = \frac{x_i^{\alpha-1}e^{-\frac{x_i}{\beta}}}{\beta^\alpha\,\Gamma(\alpha)}$$

$$\begin{aligned}
f(\underline{x}|\alpha,\beta) &= \prod_{i=1}^{n} f_{X_i}(x_i|\alpha,\beta) \\
&= [\beta^\alpha\Gamma(\alpha)]^{-n}\cdot e^{-\frac{n}{\beta}}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}\cdot e^{-\frac{1}{\beta}\sum x_i}
\end{aligned}$$

$T_1(\underline{x}) = \sum_{i=1}^{n} x_i$

$T_2(\underline{x}) = \prod_{i=1}^{n} x_i$

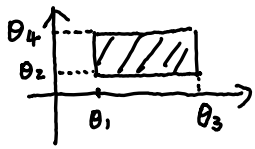
$f(\underline{x}|\alpha,\beta) = [\beta^\alpha\Gamma(\alpha)]^{-n}\cdot e^{-\frac{n}{\beta}}\cdot T_2(\underline{x})^{\alpha-1}\cdot e^{-T_1(\underline{x})}$

let $g(T_1(\underline{x}), T_2(\underline{x})|\alpha,\beta) = [\beta^\alpha\Gamma(\alpha)]^{-n}\cdot e^{-\frac{n}{\beta}}\cdot T_2(\underline{x})^{\alpha-1}\cdot e^{-T_1(\underline{x})}$

$h(\underline{x}) = 1$ $\qquad \therefore [T_1(\underline{x}), T_2(\underline{x})]$ is a 2D sufficient statistic for (α,β) $\square$

$f(\underline{x}|\alpha,\beta) = g([T_1(\underline{x}), T_2(\underline{x})]|\alpha,\beta)\cdot h(\underline{x})$

6.7. Let $f(x, y \mid \theta_1, \theta_2, \theta_3, \theta_4)$ be the bivariate pdf for a uniform distribution on the



Find a four-dimentted sufficient static for $(\theta_1, \theta_2, \theta_3, \theta_4)$

$$f_{(X,Y)}(x, y \mid \theta_1, \theta_2, \theta_3, \theta_4) = \begin{cases} \frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)} & x \in (\theta_1, \theta_3) \text{ and } y \in (\theta_2, \theta_4) \\ 0 & \end{cases}$$

$$f(\underline{X}, \underline{Y} \mid \theta_1, \theta_2, \theta_3, \theta_4) = \prod_{i=1}^{n} (x_i, y_i \mid \theta_1, \theta_2, \theta_3, \theta_4)\, I_{(\theta_1,\theta_3)}(x_i) \cdot I_{(\theta_2,\theta_4)}(y_i)$$

Let $T_1(\underline{X}) = \min_i x_i$ $\quad T_3(\underline{Y}) = \min_i y_i$

$T_2(\underline{X}) = \max_i x_i$ $\quad T_4(\underline{Y}) = \max_i y_i$

$$f(\underline{X}, \underline{Y} \mid \theta_1, \theta_2, \theta_3, \theta_4) = (\theta_3-\theta_1)^n(\theta_4-\theta_2)^n\, I_{(\theta_1,\infty)}(T_1(\underline{X}))\, I_{(-\infty,\theta_3)}(T_2(\underline{X}))\, I_{(\theta_2,\infty)}(T_3(\underline{Y}))\, I_{(-\infty,\theta_4)}(T_4(\underline{Y}))$$

Let $g(T_1(\underline{X}), T_2(\underline{X}), T_3(\underline{Y}), T_4(\underline{Y}) \mid \theta_1, \theta_2, \theta_3, \theta_4)$

$$= (\theta_3-\theta_1)^n(\theta_4-\theta_2)^n\, I_{(\theta_1,\infty)}(T_1(\underline{X}))\, I_{(-\infty,\theta_3)}(T_2(\underline{X}))\, I_{(\theta_2,\infty)}(T_3(\underline{Y}))\, I_{(-\infty,\theta_4)}(T_4(\underline{Y}))$$

$h(\underline{X}, \underline{Y}) = 1$.

then $f(\underline{X}, \underline{Y} \mid \theta_1, \theta_2, \theta_3, \theta_4) = g(T_1(\underline{X}), T_2(\underline{X}), T_3(\underline{Y}), T_4(\underline{Y}) \mid \theta_1, \theta_2, \theta_3, \theta_4) \cdot h(\underline{X}, \underline{Y})$

$\therefore$ $(T_1(\underline{X}), T_2(\underline{X}), T_3(\underline{Y}), T_4(\underline{Y}))$ is a 4-D sufficient statistic for $(\theta_1, \theta_2, \theta_3, \theta_4)$ ○

6.8. Let $X_1, \cdots, X_n$ be a random sample from a population with location pdf $f(x-\theta)$.
Show that the order statistics, $T(X_1, \cdots, X_n) = (X_{(1)}, \cdots, X_{(n)})$ are a sufficient static for θ and no further reduction is possible. P281 [31b]

1° $f_{X_i}(x_i \mid \theta) = f(x_i - \theta)$

$$f_{\underline{X}}(\underline{X} \mid \theta) = \prod_{i=1}^{n} f(x_i - \theta) = \prod_{j=1}^{n} f(x_{(j)} - \theta)$$

Let $T(X_1, X_2, \cdots X_n) = (X_{(1)}, X_{(2)}, \cdots X_{(n)})$

$$g(T(X_1, X_2, \cdots X_n) \mid \theta) = \prod_{j=1}^{n} f(x_{(j)} - \theta)$$

$h(\underline{X}) = 1$

then $f_{\underline{X}}(\underline{X} \mid \theta) = g(T(X_1, X_2, \cdots X_n) \mid \theta) \cdot h(\underline{X})$

$\therefore T(X_1, X_2, \cdots X_n)$ is a sufficient statistic for θ.

2° Proof that $T(X_1, X_2, \cdots, X_n)$ is the minimal sufficient sta.

Find Y, which make $\underset{②}{f_{\underline{X}}(\underline{Y} \mid \theta)} = \underset{①}{f_{\underline{X}}(\underline{X} \mid \theta)}$

namely: $\prod_{i=1}^{n} f(y_i - \theta) = \prod_{i=1}^{n} f(x_i - \theta)$

$$\frac{①}{②} = \frac{f_{\underline{X}}(\underline{X} \mid \theta)}{f_{\underline{X}}(\underline{Y} \mid \theta)} = \frac{\prod_{i=1}^{n} f(x_i-\theta)}{\prod_{i=1}^{n} f(y_i-\theta)} = \frac{\prod_{j=1}^{n} f(x_{(j)}-\theta)}{\prod_{j=1}^{n} f(y_{(j)}-\theta)} = \frac{g(T(X_1, X_2, \cdots X_n))}{g(T(Y_1, Y_2, \cdots Y_n))}$$

Assume $T(X_1, X_2 \cdots X_n) \neq T(Y_1, Y_2 \cdots Y_n)$

It means at least exists k, $Y_{(k)} \neq X_{(k)}$

$f(\underline{X} \mid \theta) / f(\underline{Y} \mid \theta)$ is constant only when $T(\underline{X}) = T(\underline{Y})$

$$\frac{①}{②} = \frac{f(X_{(k)} - \theta)}{f(Y_{(k)} - \theta)} \cdot \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{f(X_{(i)} - \theta)}{f(Y_{(i)} - \theta)}$$

which is not independent of θ

Thus if $\frac{①}{②}$ is independent of θ,

$T(X_1, X_2, \cdots X_n) = T(Y_1, Y_2, \cdots Y_n)$

According to Theorem 6.2.13,

$T(X_1, X_2, \cdots X_n) = (X_{(1)}, X_{(2)}, \cdots X_{(n)})$

is the minimal sufficient statistic of θ. ○

6.9. For each of the following distributions, Let $X_1, \cdots, X_n$ be a random sample. Find a minimal sufficient statistic for θ.

(a) $f(x|\theta) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2}$ $\quad x \in (-\infty, +\infty) \quad \theta \in (-\infty, \infty)$

Let X, Y be two sample points, $(\bar{x}, S_X^2)$ and $(\bar{y}, S_Y^2)$ be the sample means and var.

$f(X|\mu, \theta) = \left(\frac{1}{\sqrt{2\pi}\theta}\right)^n \cdot e^{-\frac{\sum(x_i-\mu)^2}{2\theta^2}}$

① $f(T(X)|\mu, \theta) = (\sqrt{2\pi}\theta)^{-n} \cdot e^{-\sum(x_i-\mu)^2} \cdot e^{-(2\theta^2)^{-1}}$

$= (\sqrt{2\pi}\theta)^{-n} \cdot e^{-(2\theta^2)^{-1}} \cdot e^{-[n(\bar{x}-\mu)^2 + (n-1)S_X^2]}$

Similarly

② $f(T(Y)|\mu, \theta) = (\sqrt{2\pi}\theta)^{-n} \cdot e^{-(2\theta^2)^{-1}} \cdot e^{-[n(\bar{y}-\mu)^2 + (n-1)\cdot S_Y^2]}$

$\frac{①}{②} = e^{-[n((\bar{x}-\mu)^2 - (\bar{y}-\mu)^2) + (n-1)(S_X^2 - S_Y^2)]}$

is independent with μ if & only if

$\bar{x} = \bar{y}$

$S_X^2 = S_Y^2$ $\quad \left(\frac{①}{②} = e^{-0} = 1\right)$

that is $\frac{f(T(X)|\mu, \theta)}{f(T(Y)|\mu, \theta)}$ is constant only

when $T(X) = T(Y)$ $\therefore T(X)$ is the minimal sufficient statistic for (μ, θ) □

(b) $f(x|\theta) = e^{-(x-\theta)}$, $\quad \theta < x < \infty \quad -\infty < \theta < \infty$ (location exponential)

$X = x_1, x_2, \cdots x_n$

$T_1(X) = \min_i x_i \qquad T_2(X) = \bar{X}$

$f_X(X|\theta) = \prod_{i=1}^n e^{-(x_i-\theta)} I_{(\theta,\infty)} x_i = e^{-\sum_{i=1}^n x_i + n\theta} \cdot I_{(\theta,\infty)}(T_1(X))$

$= e^{-n(T_2(X)-\theta)} \cdot I_{(\theta,\infty)}(T_1(X))$

Let $Y = y_1, y_2, \cdots y_n$ be another sample.

$f_Y(y|\theta) = e^{-n(T_2(Y)-\theta)} \cdot I_{(\theta,\infty)}(T_1(Y))$

$\frac{f_X(X|\theta)}{f_Y(y|\theta)} = e^{-n(T_2(X) - T_2(Y))} \cdot \frac{I_{(\theta,\infty)}(T_1(X))}{I_{(\theta,\infty)}(T_1(Y))}$

is a constant as of θ if and only if $T_1(X) = T_1(Y)$

$\therefore T(X) = \min_i x_i$ is the minimal sufficient statistic of θ. □

(c) $f(x|\theta) = \frac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}$ $\quad -\infty < x < \infty, \ -\infty < \theta < \infty$ (logistic)

$X = x_1, x_2, \cdots, x_n$

$f_X(X|\theta) = \prod_{i=1}^n f(x_i|\theta) = \frac{e^{-\sum x_i + n\theta}}{\prod_{i=1}^n (1+e^{-(x_i-\theta)})^2}$ 不清楚.

Let $Y = y_1, y_2, \cdots y_n$

Aim: $\prod_{i=1}^n \frac{1+e^{(y_i-\theta)}}{1+e^{-(x_i-\theta)}}$ ①

$= \prod_{i=1}^n \frac{1+e^{-y_i} \cdot e^{\theta}}{1+e^{-x_i} \cdot e^{\theta}}$ be independent with θ

② $\frac{f_X(X|\theta)}{f_Y(Y|\theta)} = \frac{e^{-n(\bar{x}-\theta)}}{\prod_{i=1}^n (1+e^{-(x_i-\theta)})^2} \cdot \frac{\prod_{i=1}^n (1+e^{-(y_i-\theta)})^2}{e^{-n(\bar{y}-\theta)}}$

$= e^{-n(\bar{x}-\bar{y})} \prod_{i=1}^n \left(\frac{1+e^{-(y_i-\theta)}}{1+e^{-(x_i-\theta)}}\right)^2$

② : $e^{-n(\bar{x}-\bar{y})}$ be independent with θ. thus order statistic $T(X)$ is the minimal sufficient statistic of θ □

Let $T(X) = x_{(1)}, x_{(2)}, \cdots x_{(n)}$ (order statistic)

If and only if $T(X) = T(Y)$

namely, $x_{(i)} = y_{(i)}$, can $e^{-x_{(i)}} = e^{-y_{(i)}}$, ① $= 1$

(d). $f(x|\theta) = \dfrac{1}{\pi[1+(x-\theta)^2]}$ $-\infty < x < \infty$ $-\infty < \theta < \infty$ (Cauchy)

Let $Z = x_1, x_2, \cdots x_n$

$f_Z(Z|\theta) = \pi^{-n} \cdot \prod_{i=1}^{n} (1+(x_i-\theta)^2)$

$Y = y_1, y_2, \cdots y_n$

$f_Y(Y|\theta) = \pi^{-n} \cdot \prod_{i=1}^{n} (1+(y_i-\theta)^2)$

手绘.

$\therefore \dfrac{f_Z(Z|\theta)}{f_Y(Y|\theta)} = \prod_{i=1}^{n} \dfrac{1+(x_i-\theta)^2}{1+(y_i-\theta)^2}$ $x_i + y_i = 2\theta$

Aim: make $\dfrac{f_Z(T(Z)|\theta)}{f_Y(T(Y)|\theta)}$ be independent with θ

Let $T(Z) = x_{(1)}, x_{(2)} \cdots x_{(n)}$

namely: $f_Z(T(Z)|\theta) \propto f_Y(T(Y)|\theta)$

when $\theta=0$ $f_Z(T(Z)|0) \propto f_Y(T(Y)|0)$ $f_Z(T(Z)|0) = \prod_{i=1}^{n} (1+x_{(i)})^2$

$\therefore \dfrac{f_Z(T(Z)|\theta)}{f_Z(T(Z)|0)} = \dfrac{f_Y(T(Y)|\theta)}{f_Y(T(Y)|0)}$

that is $\underbrace{\prod_{i=1}^{n} \dfrac{1+(x_{(i)}-\theta)^2}{1+x_{(i)}^2}}_{①} = \underbrace{\prod_{i=1}^{n} \dfrac{1+(y_{(i)}-\theta)^2}{1+y_{(i)}^2}}_{②}$

$\therefore$ polynomial ① = ②

$\therefore$ the roots of ① = the roots of ②, the number of roots $\le 2n$

$x^*_{(j)}$ $j = 1, 2, \cdots, 2n$

$x^*_{(j)} = x^*_{(k)} \pm i$ $i = 1, 2, \cdots n$.

$y^*_{(j)} = y^*_{(k)} \pm i$ $k = 1, 2, \cdots n$

$x^*_{(j)}$ and $y^*_{(j)}$ have the same order statistics

$\therefore$ Z, Y are permutations of one another.

$\therefore$ $T(Z)$ is a minimal sufficient statistics of θ □

(e). $f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}$ $-\infty < x < \infty$, $-\infty < \theta < \infty$ (double exponential)

$Z = x_1, x_2, \cdots, x_n$ $Y = y_1, y_2, \cdots y_n$

$f_Z(Z|\theta) = 2^{-n} \cdot \prod_{i=1}^{n} e^{-|x_i-\theta|}$

$f_Y(Y|\theta) = 2^{-n} \prod_{i=1}^{n} e^{-|y_i-\theta|}$

$\dfrac{f_Z(Z|\theta)}{f_Y(Y|\theta)} = \prod_{i=1}^{n} \dfrac{e^{-|x_i-\theta|}}{e^{-|y_i-\theta|}} = \prod_{i=1}^{n} e^{|y_i-\theta|-|x_i-\theta|} = e^{\sum_i^n |y_i-\theta|-|x_i-\theta|}$

$\dfrac{f(T(Z)|\theta)}{f(T(Y)|\theta)}$ is a costant as for θ if and only if $T(Z) = T(Y)$

$\therefore$ $T(Z)$ is a minimal sufficient statistic for θ □

Aim: $\sum_{i=1}^{n} |y_i-\theta| - |x_i-\theta|$ is independent with θ

Let $T(Z) = x_{(1)}, x_{(2)}, \cdots x_{(n)}$ $T(Y) = y_{(1)}, y_{(2)}, \cdots y_{(n)}$

If $T(Z) \ne T(Y)$ then at least $x_{(k)} \ne y_{(k)}$, assum $x_{(i)} = y_{(i)}$ $i \ne k$

$\sum_{i=1}^{n} |y_i-\theta| - |x_i-\theta| = |y_{(k)}-\theta| - |x_{(k)}-\theta|$ is not a costant for θ

6.10. Show that minimal sufficient statistics $T(\underline{Z}) = (\min_i Z_i, \max_i Z_i)$ for uniform $(\theta, \theta+1)$ is not complete

$\Leftarrow$ $T(\underline{Z}) = (\min_i x_{(i)}, \max_j x_{(j)})$ is not complete.

Aim: find $g(T(\underline{Z}))$ which satisfies ① $E_\theta g(T(\underline{Z})) = 0$

② $g(T(\underline{Z})|\theta) \neq 0$ 不恒为.

Assume $\underset{\text{range}}{R} = \max_j x_{(j)} - \min_i x_{(i)}$ $\quad \underbrace{\quad}_{0 \quad 1}$

$ER = \int_0^1 r \cdot f(r)\, dr$ $\quad f(r) = n(n-1) \cdot r^{n-2}(1-r)$ $\quad$ beta $(n-1, 2)$ $\quad f(x, \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$

$= \frac{n-1}{n+1}$ $\quad Ef(x, \alpha, \beta) = \frac{\alpha}{\alpha+\beta}$

Let $g(T(\underline{Z})) = R - ER$

$E\, g(T(\underline{Z})) = E(R - \underset{\text{constant}}{ER}) = ER - ER = 0$

$g(T(\underline{Z})|\theta) = R - ER = \max_j x_{(j)} - \min_i x_{(i)} - \underbrace{\frac{n-1}{n+1}}_{\text{constant}}$

$= R - ER$ $\quad \because ER = 0$ $\quad \therefore R - ER \neq 0$

$\therefore T(\underline{Z})$ is not a complete statistic to θ. □ $\quad f(\underline{z}|\theta) = g(T(\underline{z}), \theta) h(\underline{z})$

$\Rightarrow$ lemma: If a function of sufficient statistic is ancillary, $g(T(\underline{Z}))$ not inde. then $T(\underline{Z})$ is not complete.

$\because E(g(T(\underline{Z})) = 0$ doesn't depend on θ $\therefore g(T(\underline{Z})|\theta)$ can $\neq 0$

In complete $\because$ $T(\underline{Z})$ is independent with ancellary statistics

6.11. Refer to pdfs in 6.9. For each, Let $Z_{(1)} < \cdots < Z_{(n)}$ be the ordered sample, and define $Y_i = Z_{(n)} - Z_{(i)}$, $i = 1, \cdots, n-1$.

(1). For each of the pdfs, verify that the set $(Y_1, \cdots, Y_{n-1})$ is ancillary for θ.

$\because$ All pdfs in 6.9 belong to location families. $\quad$ P283 [3.9]

Let $Z_{(1)}, \cdots, Z_{(n)}$ be the order of sample $Z_1, Z_2, \cdots Z_n$

of $f(z|\theta=0) = f(z)$

Let $Z_{(1)} = Z_{(1)} + \theta$ $\quad Y_{(1)} = Z_{(n)} - Z_{(1)} = Z_{(n)} + \theta - (Z_{(1)} + \theta) = Z_{(n)} - Z_{(1)}$

$Z_{(2)} = Z_{(2)} + \theta$ $\quad Y_{(2)} = Z_{(n)} - Z_{(2)} = Z_{(n)} + \theta - (Z_{(2)} + \theta) = Z_{(n)} - Z_{(2)}$

$\vdots$

$Z_{(n)} = Z_{(n)} + \theta$ $\quad Y_{(n)} = Z_{(n)} - Z_{(n)} = Z_{(n)} + \theta - (Z_{(n)} - \theta) = Z_{(n)} - Z_{(n)}$

$f(Z_{(1)} + \theta, Z_{(2)} + \theta, \cdots Z_{(n)} + \theta) = f_{\underline{Z}}(\underline{z}|\theta)$

$\because$ Sample $(Z_{(n)} - Z_{(1)}, Z_{(n)} - Z_{(2)}, \cdots Z_{(n)} - Z_{(n-1)})$

depends only on $(Z_{(1)}, Z_{(2)}, \cdots Z_{(n)})$

$f_Y(Y|\theta) = f(Z_{(1)}, Z_{(2)}, \cdots Z_{(n)})$ which doesn't depends on θ. $\therefore (Y_{(1)}, Y_{(2)}, \cdots Y_{(n)})$ is ancillary.

(b) In each case determine whether the set $(Y_1, \cdots, Y_{n-1})$ is independent of the minimal statistic.

(a) $f(x|\theta) = \frac{1}{\sqrt{2\pi}} \cdot e^{-(x-\theta)^2/2}$ $\quad -\infty < x < \infty \quad -\infty < \theta < \infty$

(b) $f(x|\theta) = e^{-(x-\theta)}$ $\quad \theta < x < \infty \quad -\infty < \theta < \infty$

(c) $f(x|\theta) = \frac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}$ $\quad -\infty < x < \infty \quad -\infty < \theta < \infty$

(d) $f(x|\theta) = \frac{1}{\pi[1+(x-\theta)^2]}$ $\quad -\infty < x < \infty \quad -\infty < \theta < \infty$

(e) $f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}$ $\quad -\infty < x < \infty \quad -\infty < \theta < \infty$

minimal sufficient statistics

(a) $T(\underline{X}) = (\bar{X}, S_X^2)$

(b) $T(\underline{X}) = \min_i (X_i)$

(c) $T(\underline{X}) = (X_{(1)}, X_{(2)}, \cdots X_{(n)})$

(d) $T(\underline{X}) = (X_{(1)}, X_{(2)}, \cdots X_{(n)})$

(e) $T(\underline{X}) = (X_{(1)}, X_{(2)}, \cdots X_{(n)})$

$(Y_1, Y_2, \cdots, Y_{n-1}) = (X_{(n)} - X_{(1)}, X_{(n)} - X_{(2)}, \cdots, X_{(n)} - X_{(n-1)})$

To see if $(Y_1, Y_2, \cdots, Y_n)$ is independent of $T(\underline{X})$, need to check whether can find $g(T(\underline{X}))$ • $(Y_1, Y_2, \cdots, Y_{n-1})$

$Y_i = X_{(n)} - X_{(i)}, \quad i = 1, \cdots, n-1.$

(b): If independent

Yes For (b) $\quad f(x|\theta) = e^{-(x-\theta)}$

Yes Let $T(\underline{X}) = X_{(1)}$, $Y_{(n)} = X_{(n)}$ $\quad f(X_{(1)}, X_{(2)} \cdots X_{(n)}) = e^{-\sum_{i=1}^{n}(x_{(i)} - \theta)}$

No Then $f(Y|\theta) = \Pi$

No $f(y_n) = f(X_{(1)})$

No $f(y_1, y_2, \cdots y_n)$

$= f(y_1, y_2, \cdots y_{n-1}) \cdot f(y_n)$

$= f(y_1) \cdot f(y_2) \cdots f(y_{n-1}) \cdot f(y_n)$

$f(y_1, y_2 \cdots y_{n-1}|\theta) = e^{-[(X_{(n)} - X_{(1)} - \theta) + (X_{(n)} - X_{(2)} - \theta) + \cdots + (X_{(n)} - X_{(n-1)} - \theta)]}$

$= e^{-(n-1)X_{(n)}} \cdot e^{\sum_{i=1}^{n-1} X_{(i)}} \cdot e^{n\theta}$

$f(y_1, y_2, \cdots y_n) = e^{-(n-1)X_{(n)}} \cdot e^{\sum_{i=1}^{n-1} X_{(i)}} \cdot e^{n\theta} \cdot e^{-(X_{(1)} - \theta)}$

$Y_{(n)} = T(\underline{X}) \quad = e^{-(n-1)X_{(n)}} \cdot e^{\sum_{i=2}^{n-1} X_{(i)}} \cdot e^{(n+1)\theta}$

$\because X_{(1)}$ is independent with $(y_1, y_2, \cdots y_n)$

that is $(y_1, y_2, \cdots y_n)$ is independent with the minimal statistic $T(\underline{X}) = \min_i X_i$ D

6.12. N is a random variable taking values $1, 2, \cdots$

$P(N=1) = P_1 \quad P(N=2) = P_2, \quad \cdots \quad \sum P_i = 1$

Having observed $N=n$, perform n. Bernoulli (θ) getting X success.

(a). Prove that (X, N) is minimal sufficient and N is ancillary for θ.

minimal sufficient of θ

$\Leftarrow \quad P_X(x|\theta, n) = C_n^x \cdot \theta^x \cdot (1-\theta)^{n-x}$

$P_{\underline{X}}(x, n|\theta) = P_X(x|\theta, n) \cdot P(n)$

$= P_X(x|\theta, n) \cdot P_n$

$= C_n^x \cdot \theta^x \cdot (1-\theta)^{n-x} \cdot P_n$

$= g((X, N)|\theta) \cdot \underbrace{h((X, N))}_{1}$

$\therefore (X, N)$ is a sufficient static for θ.

To prove N is ancellary to θ

need to prove $P_N(n|\theta)$ is independent with θ

Namely, the value of N contains no info about θ

$\because P_N(n|\theta) = P_N(n) = P_n$

$\therefore N$ is cancellary to θ.

A·B is independent with θ only when A is independent with θ, it requires $x = y$, $n = n'$

Method 2:

$$\frac{P(x, n|\theta)}{P(y, n'|\theta)} = \frac{P_X(x|\theta, n) \cdot P(n)}{P_X(y|\theta, n') \cdot P(n')} = \frac{C_n^x \theta^x (1-\theta)^{n-x} \cdot P_n}{C_{n'}^y \theta^y (1-\theta)^{n'-y} \cdot P_n} = \underbrace{\frac{\theta^x (1-\theta)^{n-x}}{\theta^y (1-\theta)^{n'-y}}}_{\text{Contains } \theta} \cdot \underbrace{\frac{C_n^x \cdot P_n}{C_{n'}^y \cdot P_n}}_{\text{independ } \theta}$$

A B

$\therefore (X, N)$ is the minimal statistic for θ D

(b). Prove that the estimator X/N is unbiased for θ and has variance $\theta(1-\theta)\,E(1/N)$

1° Unbias estimator of θ

$E(X/N) = \theta$

$E(X/N) = E_N\left(E(X/n) \mid N=n\right) = E_N\left(\frac{1}{n} E(X) \mid N=n\right)$ $\quad \because E(X) = n\cdot\theta$

$\therefore = E_N\left(\frac{1}{n}\cdot n\cdot\theta \mid N=n\right) = E_N(\theta) = \theta$ □

2° Var $\theta(1-\theta)\cdot E(1/N)$

$Var(X/N) = E(X/N)^2 - \underbrace{E^2(X/N)}_{\theta^2}$

$E(X^2/N^2) = E_N\left(E(X^2/n^2) \mid N=n\right)$

$= E_N\left(\frac{1}{n^2}\cdot E(X^2) \mid N=n\right)$ ①

$E(X^2) = \sum x^2 \cdot f(x)$

$= \sum x^2 \binom{n}{x}\cdot\theta^x\cdot(1-\theta)^{n-x}$

$= \sum x^2 \cdot \frac{n!}{(n-x)!\,x!}\cdot\theta^x(1-\theta)^{n-x}$ 代入

$Var(X/N) = E\left(Var(X/N \mid N)\right) + Var\left(\underbrace{E(X/N \mid N)}_{\theta}\right)$

$= E\left(\frac{1}{N^2} Var(X \mid N)\right) + \underbrace{Var(\theta)}_{0}$

$= E\left(\frac{1}{N^2}\cdot Var(\theta)\right)$

$= E\left(\frac{1}{N^2}\cdot N\theta(1-\theta)\right) = E\left(\frac{1}{N}\right)\theta(1-\theta)$

$E\left(Var(\frac{X}{N} \mid N)\right) = E\left(\frac{1}{N^2} Var(X \mid N)\right)$

The law of variance

$Var(X) = \underbrace{E(Var(X \mid Y))}_{①} + \underbrace{Var(E(X \mid Y))}_{②}$

$Var(X) = E(X^2) - E^2(X)$

$Var(X \mid Y) = E(X^2 \mid Y) - E^2(X \mid Y)$

① : $E(Var(X \mid Y)) = E\left(E(X^2 \mid Y)\right) - E\left(E^2(X \mid Y)\right)$

$= E(X^2) - E\left(E^2(X \mid Y)\right)$

② : $E\left(E^2(X \mid Y)\right) - E^2\left(E(X \mid Y)\right)$

$= E\left(E^2(X \mid Y)\right) - (EX)^2$

$\therefore$ ① + ② $= E(X^2) - (EX)^2 = Var(X)$

$Var(AB \mid B) = B^2\cdot Var(A \mid B)$

$Var(aX) = a^2 Var(X)$

$= E(aX - E(aX))^2$

$= E(aX - aE(X))^2$

$= a^2 E(X - E(X))^2$

$= a^2\cdot Var(X)$

$\frac{1}{X} = \log X$

$X = \left(e^{\frac{1}{X}}\right)$

6.B. X_1, X_2 are iid observations from $f(x \mid \alpha) = \alpha\cdot x^{\alpha-1}\cdot e^{-x^\alpha}$, $x>0$, $\alpha>0$

Show that $(\log X_1)/(\log X_2)$ is an ancillary statistic.

To prove $\frac{\log X_1}{\log X_2}$ is ancillary statistic for α.

need to show $\frac{\log X_1}{\log X_2}$ doesn't depends on α.

let $Y \overset{g(X)}{=} \log X$ $\mapsto$

$f(Y_1 \mid \alpha) = f(\log X_1 \mid \alpha)$

$f(x \mid \alpha) = \alpha\cdot x^{\alpha-1}\cdot e^{-x^\alpha}$

$f(Y \mid \alpha) = \alpha\cdot e^{\alpha y} - e^{\alpha y}$

Let $\begin{cases} Y_1 = \frac{1}{\alpha} Z_1 = \log Z_1 \\ Y_2 = \frac{1}{\alpha} Z_2 = \log Z_2 \end{cases}$ $\quad y = \frac{1}{\alpha} z_i$

$Z \sim f(z \mid \alpha, 1) = 1\cdot z^0\cdot e^{-z} = e^{-z}$ ✓

$F_Y(y) = P(Y \le y)$

$= P(\log Z \le y)$

$= P(Z < g^{-1}(y))$

$= F_Z(g^{-1}(y))$

$f_Y(y) = \frac{d}{dy} F_Z(x_1)$

$= f_Z(x_1)\cdot\left|\frac{dx_1}{dy}\right|$

$= f_Z(x_1)\cdot\left[g^{-1}(y)\right]'$

$= f_Z(g^{-1}(y))\cdot e^y$

$= \alpha\cdot (e^y)^{\alpha-1}\cdot e^{-(e^y)^\alpha}\cdot e^y$

$= \alpha\cdot e^{\alpha y}\cdot e^{-e^{\alpha y}}$ $\quad = \alpha\cdot e^{\alpha y - e^{\alpha y}}$

$\because Y = g(Z)$

assume

$y = g(x_1)$

$g^{-1}(y) = x_1$

$y = \log x$

$x = e^y$

$f(x \mid \alpha) = \alpha\cdot x^{\alpha-1}\cdot e^{x^\alpha}$

$f(y \mid \alpha) = \alpha\cdot e^{(\alpha y - e^{\alpha y})}$

if $y = \frac{1}{\alpha} z \Rightarrow z = \alpha y$

$f(z) = \alpha\cdot e^{(z - e^z)}$ $\quad \frac{1}{\alpha} = e^{z - e^z}$ ..... □

$\therefore \frac{\log Z_1}{\log Z_2} = \frac{Z_1}{Z_2}$ which is independent with α

Idea: $Y = \log X$ is a scale variable

By constructing $Z = \alpha Y$

Z absorb α

$\therefore \frac{Y_1}{Y_2} = \frac{\log Z_1}{\log Z_2} = \frac{Z_1}{Z_2}$ independent of α

$\frac{\alpha\cdot y_1}{\alpha\, y_2}$, the α will cancle out.

For scale variable X

$\frac{X_i}{X_j}$ is ancillary

6.14. $Z_1, \cdots Z_n$ is random sample from a location family. Show that $M - \bar{Z}$ is an ancillary statistic, where M is the sample median

To show $M - \bar{Z}$ is an ancellary statistic

$\Leftarrow$ $M - \bar{Z}$ doesn't depends on μ

Assume that $Z_1, \cdots, Z_n$ are iid from $f(z)$

Let $Y = Z - \mu$, μ is the location parameter

$y_1 = z_1 - \mu$, $y_2 = z_2 - \mu$, $\cdots$ $y_n = z_n - \mu$

$\bar{Y} = \bar{Z} - \mu$

Assume $M(Z)$ is the median of $z_1, z_2, \cdots z_n$

$M(Y)$ $\cdots$ $y_1, y_2, \cdots y_n$

$M(Y) = M(Z) - \mu$ ①

$\because$ $\bar{Y} = \bar{Z} - \mu$ ②

①-②: $M(Y) - \bar{Y} = M(Z) - \bar{Z}$

$\therefore$ Statistic $M(Y) - \bar{Y}$ is independent to the location parameter μ.

※

Idea: Construct a new random var that $(Z - \mu)$

Every sample move μ

Thus mean & median move μ

median - mean ①

(μ cancle out)

$\therefore$ ① contains no info about μ

For location variable Z, $Z_i - Z_j$ is ancillary.

6.15. Let $Z_1, \cdots, Z_n$ be iid $n(\theta, a\theta^2)$ where a is known constant and $\theta > 0$

(a) Show that the parameter space doesn't contain a two-dimensional open set.

What is parameter space?

what is open set?

The parameter space is composed by points (θ, σ^2)

$\therefore$ let $x = \theta$ $y = a\theta^2 = ax^2$ is a line (boundry)

$\therefore$ The parameter space doesn't contain two-dimensional open set.

Open Set — red

Boundary — blue

Closed set — red + blue

with every point P contain all points that are sufficiently near to P

The intersection of unlimited open sets might not be an open set.

Example: $\bigcap (-\frac{1}{n}, 1+\frac{1}{n})$ $n \in R$

$= [0, 1]$ (closed set)

(b) Show that statistic $T = (\bar{Z}, S^2)$ is a sufficient statistic for θ, but the family of distributions is not complete.

1° Proof is sufficient:

$Z = Z_1, Z_2, \cdots Z_n$ $\bar{Z} = \frac{1}{n}(Z_1 + Z_2 + \cdots + Z_n)$

$$P_{Z_i}(x_i | \theta) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{a} \cdot \theta} \cdot e^{-\frac{(x-\theta)^2}{2a\theta^2}}$$

$$P_Z(\bar{Z} | \theta) = \prod_{i=1}^{n} P_{Z_i}(x_i | \theta)$$

$$= (\sqrt{2\pi a}\ \theta)^{-n} \cdot e^{-\frac{1}{2a\theta^2} \cdot \sum (x_i - \theta)^2}$$

$$= (\sqrt{2\pi a}\ \theta)^{-n} \cdot e^{-\frac{1}{2a\theta^2}[n \cdot (\bar{Z} - \theta)^2 + (n-1) \cdot S^2]}$$

$$= \underbrace{(\sqrt{2\pi a})^{-n} \cdot (\theta)^{-n} \cdot e^{-\frac{1}{2a\theta^2}[n \cdot (\bar{Z} - \theta)^2 + (n-1) \cdot S^2]}}$$

$= h(\bar{Z}) \cdot f((\bar{Z}, S^2) | \theta)$ $\therefore$ $T = (\bar{Z}, S^2)$ is sufficient static for θ.

a

$\therefore$ family of $T(\bar{Z}, S^2)$ is not complete

$\Leftarrow$ Find $g((\bar{Z}, S^2))$ that satisfy $\begin{cases} ① \ E\, g((\bar{Z}, S^2)) = 0 \\ ② \ g((\bar{Z}, S^2) \mid \theta) \neq 0 \end{cases}$

$E(S^2) = n\theta^2$

$$E(\bar{Z}^2) = Var(\bar{Z}) + E^2\bar{Z}$$
$$= Var(\tfrac{1}{n}(X_1 + \cdots + X_n)) + \theta^2$$
$$= \tfrac{1}{n^2} \cdot \underbrace{Var(X_1 + \cdots + X_n)} + \theta^2$$
$$= \tfrac{1}{n^2} \cdot n \cdot Var(X_1) + \theta^2$$
$$= \tfrac{1}{n} \cdot a \cdot \theta^2 + \theta^2$$
$$= (\tfrac{a}{n} + 1)\theta^2$$

$Var\bar{Z} = E\bar{Z}^2 - E^2\bar{Z}$

Let $g((\bar{Z}, S^2)) = \dfrac{(\frac{a}{n}+1)\theta^2}{(\frac{a}{n}+1)} - \dfrac{n\theta^2}{n} = \dfrac{\bar{Z}^2}{(\frac{a}{n}+1)} - \dfrac{S^2}{n}$

$$E[g((\bar{Z}, S^2))] = E\left[(\tfrac{a}{n}+1)^{-1} \cdot \bar{Z}^2 - (n)^{-1} S^2\right] \quad \text{for all } \theta$$
$$= (\tfrac{a}{n}+1)^{-1} \cdot E(\bar{Z}^2) - n^{-1} \cdot E(S^2)$$
$$= \theta^2 - \theta^2$$
$$= 0$$

$$g((\bar{Z}, S^2) \mid \theta) = (\tfrac{a}{n}+1)^{-1} \cdot \bar{Z}^2 - n^{-1} \cdot S^2$$
$$P(g((\bar{Z}, S^2) \mid \theta) = 0) = P\left((\tfrac{a}{n}+1)^{-1} \cdot \bar{Z}^2 - n^{-1} \cdot S^2 = 0\right)$$
$$= P\left(\bar{Z}^2 = \tfrac{S^2}{n^2}(a+1)\right)$$

Intuition about complete statistics

$T(Z)$ is sufficient if $Z \mid T$ X depends on θ

T provide all infor about θ

$V(Z)$ is ancillary if $f(V)$ x depends on θ

V provide no info about θ

$U(Z)$ is an unbiased estimator of 0 if $E(U) = 0$

$S(Z)$ is a complete static if Any $U = g(S)$ is 0

$E[g(S)] = 0 \quad g(S) = 0$ for all θ

Intu: eg $T_1(Z), T_2(Z)$ are two sufficient stat of θ

$\therefore E\, f(T_1) = \theta \qquad P(f(T_1) \neq f(T_2)) > 0$

$E\, f(T_2) = \theta$

$\therefore E(f(T_1) - f(T_2)) = 0$

$\therefore U = f(T_1) - f(T_2)$ but $P(f(T_1) \neq f(T_2)) > 0$

$\therefore T_1$ or T_2 is not complete

$T = (Z_{(1)}, Z_{(n)})$ is sufficient $\quad Z_i \sim (\theta, \theta+1)$

$V = Z_{(n)} - Z_{(1)} = g(T)$ is ancellary

Assume that $E(V) = E(g(T)) = c$

Then $V - c$ is an unbiased estimator of 0

$U = V - c$ but $P(g_2(T) \neq 0) > 0$

$= g(T) - c$

$= g_2(T) \Rightarrow$ provides no info about θ

$\therefore T$ is not complete

complete T $\Rightarrow$ no function of it is uninfo $U = g(T)$

every function of it $\cdots$ info $U = g(T)$

6.16. Genetic modeling

observe (x_1, x_2, x_3, x_4) with cell probabilites given by

$(\frac{1}{2} + \frac{\theta}{4}, \frac{1}{4}(1-\theta), \frac{1}{4}(1-\theta), \frac{\theta}{4})$

(a) Show that this is a curved exponential family.

Denote the cell probability P_1, P_2, P_3, P_4

x_1: the times for event 1 happen

x_2: the times for event 2 happen

x_3: " events "

x_4: " event 4 "

$P(\text{event } i) = P_i \qquad$ Assume $\underbrace{x_1 + x_2 + x_3 + x_4 = n}_{③}$

$P_1 = \frac{1}{2} + \frac{\theta}{4} \qquad \underbrace{P_2 = \frac{1}{4}(1-\theta) = P_3}_{①} \qquad P_4 = \frac{\theta}{4}$

$$P(x_1, x_2, x_3, x_4 \mid \theta) = \frac{n!}{x_1!\, x_2!\, x_3!\, x_4!} \cdot P_1^{x_1} P_2^{x_2} P_3^{x_3} P_4^{x_4}$$
$$= \frac{n!}{x_1!\, x_2!\, x_3!\, x_4!} \cdot P_1^{x_1} P_2^{x_2 + x_3} P_4^{n - x_1 - (x_2 + x_3)}$$
$$= \frac{n!}{x_1!\, x_2!\, x_3!\, x_4!} \cdot \left(\frac{P_1}{P_4}\right)^{x_1} \cdot \overset{(①)}{\left(\frac{P_2}{P_4}\right)^{x_2 + x_3}} \overset{(②)}{P_4}$$
$$= \frac{n!}{x_1!\, x_2!\, x_3!\, x_4!} \cdot \left(\frac{P_1}{P_4}\right)^{x_1} \cdot \left(\frac{P_2}{P_4}\right)^{x_2 + x_3} \cdot P_4^{n}$$
$$= \underbrace{\frac{n!}{x_1!\, x_2!\, x_3!\, x_4!}} \cdot \left(\frac{2}{\theta} + 1\right)^{x_1} \cdot \left(\frac{1}{\theta} - 1\right)^{x_2 + x_3} \cdot \left(\frac{\theta}{4}\right)^{n}$$

let $Z = (x_1, x_2, x_3, x_4)$

$h(Z) = \dfrac{n!}{x_1!\, x_2!\, x_3!\, x_4!}$

$c(\theta) = \left(\frac{\theta}{4}\right)^n$

$t(Z) = (x_1, x_2 + x_3)$

$w(\theta) = (\log \frac{2}{\theta} + 1, \log \frac{1}{\theta} - 1)$

$f(Z \mid \theta) = h(Z) \cdot c(\theta) \cdot e^{(w_1(\theta) \cdot t_1(Z) + w_2(\theta) \cdot t_2(Z))}$

Based on definition, $Z \mid \theta$ is exponential distribution of

$\because w(\theta)$ is two demension $k=2$ that is $w(\theta)$ is a curve in 2D space.

θ is one demension $d=1$

(b) Find a sufficient statistic for θ

Let $T(\underline{X}) = (\underset{T_1}{x_1}, \underset{T_2}{x_2+x_3})$

$\Leftarrow$ T is sufficient

$\Leftarrow$ $\underline{X} | T$ x depends on θ :

$\because$ when x_1, x_2+x_3 known

the the cell is $(x_1, \frac{x_2+x_3}{2}, \frac{x_2+x_3}{2}, n-x_1-(x_2+x_3))$

is independent to θ

$\therefore$ (x_1, x_2+x_3) is a sufficient statistic to θ.

(c) Find a minimal sufficient sta for θ

$\Leftarrow$ T is minimal sufficient

$\Leftarrow$ $\frac{P(x|\theta)}{P(y|\theta)}$ is independent to θ if and only if $T(x) = T(y)$

$P(\underline{X}|\theta)$

$= h(\underline{X}) \cdot c(\theta) \cdot e^{\omega_1(\theta)\cdot T_1 + \omega_2(\theta)\cdot T_2}$

$h(\underline{X}) = \frac{n!}{x_1! \cdot x_2! \cdot x_3! \cdot x_4!}$

$c(\theta) = \left(\frac{\theta}{4}\right)^n$

$\omega(\theta) = (\log \frac{1}{\theta}+1, \log \frac{1}{\theta}-1)$

$\frac{P(\underline{X}=x|\theta)}{P(Y=y|\theta)} = \frac{h(x)}{h(y)} e^{\omega_1(\theta)(x_1-y_1) + \omega_2(\theta)\cdot(x_2+x_3-y_2-y_3)}$

Only when $\begin{cases} x_1 = y_1 \\ x_2+x_3 = y_2+y_3 \end{cases}$ can $\frac{P(\underline{X}=x|\theta)}{P(Y=y|\theta)}$ be independent to θ.

$\therefore$ $T(x_1, x_2+x_3)$ is a minimal sufficient sta to θ.

6.17. $X_1, X_2, \cdots, X_n$ are iid with geometric distribution

$P_\theta(X=x) = \theta(1-\theta)^{x-1}$ $x=1,2,\ldots$ $\theta \in (0,1)$

Show the $\sum X_i$ is sufficient for θ, and find the family of distributions of $\sum X_i$. Is the family complete?

$P(X=x|\theta) = \theta(1-\theta)^{x-1}$

Let $\underline{X} = X_1, X_2, \cdots, X_n$

$T(\underline{X}) = \sum X_i$

$P(\underline{X}=x|\theta) = P(x_1+x_2+\cdots+x_n|\theta) = \frac{x!}{x_1!x_2!\cdots x_n!} \cdot P(x_1)P(x_2)\cdots P(x_n)$

$= \frac{(x_1+x_2+\cdots+x_n)!}{x_1!x_2!\cdots x_n!} \cdot \theta^n(1-\theta)^{x-n}$

$= \frac{x!}{x_1!x_2!\cdots x_n!}\left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^x$

Let $H(\underline{X}) = \frac{x!}{x_1!x_2!\cdots x_n!}$

$f(T(\underline{X})) = \left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^x$

$P(\underline{X}=x|\theta) = H(\underline{X}) \cdot f(T(\underline{X})|\theta)$

$\therefore$ $T(\underline{X})$ is sufficient.

$\frac{P(\underline{X}=a)}{P(\underline{X}=b)} = \frac{a!}{x_1!x_2!\cdots x_n!} \cdot \frac{x_1!x_2!\cdots x_n!}{b!} \cdot \frac{}{(1-\theta)^{a-b}}$

Only when $a=b$

can $\frac{P(\underline{X}=a)}{P(\underline{X}=b)}$ be independent to θ

$\therefore$ $T(\underline{X})$ $\underline{X} = \sum x_i$ is a minimal sufficient statistic for θ.

$\Leftarrow$ Complete

$E(g(\underline{X})) = 0$ $P(g(\underline{X}|\theta)=0) = 1$

Let $g(\underline{X}) = \underline{X} - n = \sum_i x_i - n$

$E(g(\underline{X})) = E(\underline{X}-n) = \int_\theta \int_x [(\underline{X}-n)\cdot P(\underline{X}-n|\theta)]\, dx\, d\theta$

$= \int \int (\underline{X}-n)\cdot P(\underline{X}|\theta)\, dx\, d\theta$

6.20. Let $X_1, \cdots, X_n$ be iid observations. Find a complete sufficient statistic, or show that one doesn't exist.

(a). $f(x|\theta) = \frac{2x}{\theta^2}$ $\quad 0<x<\theta$, $\theta>0$

(b) $f(x|\theta) = \frac{\theta}{(1+x)^{1+\theta}}$ $\quad 0<x<\infty$ $\quad \theta>0$

(c) $f(x|\theta) = \frac{(\log\theta)\theta^x}{\theta-1}$ $\quad 0<x<1$ $\quad \theta>1$

(d) $f(x|\theta) = e^{-(x-\theta)} \exp(-e^{-(x-\theta)})$, $\quad -\infty<x<\infty$ $\quad -\infty<\theta<\infty$

(e) $f(x|\theta) = \binom{2}{x}\theta^x (1-\theta)^{2-x}$, $\quad x=0,1,2,$ $\quad 0\le\theta\le1$

For (b), (c) and (e), they are exponential family.

(b) $f(x|\theta) = \theta\cdot(1+x)^{-(1+\theta)} = \theta\cdot e^{-(1+\theta)\cdot\log(1+x)}$

$h(x)=1$ $\quad c(\theta)=\theta$ $\quad w(\theta)=-(1+\theta)$ $\quad t(x)=\log(1+x)$

Complete sufficient statistic:

$T(\underline{X}) = \sum_{i=1}^n t(x_i) = \sum_{i=1}^n \log(1+x_i)$

(c) $f(x|\theta) = \frac{\log\theta}{\theta-1}\cdot e^{x\cdot\log\theta}$

$h(x)=1$ $\quad c(\theta)=\frac{\log\theta}{\theta-1}$ $\quad w(\theta)=\log\theta$ $\quad t(x)=x$

$T(\underline{X}) = \sum_{i=1}^n t(x) = \sum_{i=1}^n x_i$

(e) $f(x|\theta) = \binom{2}{x}\cdot e^{\log\theta^x(1-\theta)^{2-x}}$

$= \binom{2}{x}\cdot e^{\log\theta^x + \log(1-\theta)^{2-x}}$

$= \binom{2}{x}\cdot e^{x\log\theta + (2-x)\log(1-\theta)}$

$T(\underline{X}) = \left[\sum_{i=1}^n t_1(x_i), \sum_{i=1}^n t_2(x_i)\right]$

$= \left[\sum_{i=1}^n x_i, \sum_{i=1}^n 2-x_i\right]$

$h(x) = \binom{2}{x}$ $\quad c(\theta)=1$ $\quad w_1(\theta)=\log\theta$ $\quad t_1(x)=x$ $\quad w_2(\theta)=\log(1-\theta)$ $\quad t_2(x)=2-x$ $\quad$ (6.2.25)

(a). $T(\underline{X}) = \max x_i$ is a sufficient statistic for θ

$f(x|\theta) = \frac{2x}{\theta^2}$ $\quad x\in(0,\theta)$

$f(\underline{x}|\theta) = \prod_{i=1}^n f(x_i|\theta) = (2\theta^{-2})^n\cdot(x_1x_2\cdots x_n)$ ①

$f(y|\theta) = f(\max\{x_i\}=y|\theta)$ $\quad y\in(0,\theta)$ $\quad \frac{1}{\theta^2}y^2$

$g(\max_j x_j = y|\theta) = n\cdot\frac{2y}{\theta^2}\prod_{j\neq i}\int_0^y f(x_i)\,dx_i$

$j=1,2,\cdots,n$

$= 2yn\cdot\theta^{-2}\cdot\theta^{-2(n-1)}\cdot y^{2(n-1)}$

$= 2n\cdot\theta^{-2n}\cdot y^{2n-1}$ ②

$E[g(y)] = \int_0^\theta g(y)\cdot P(y|\theta)\,dy$

$= \int_0^\theta g(y)\cdot \underbrace{2n\cdot\theta^{-2n}\cdot y^{2n-1}}_{>0}\,dy$

$\therefore$ If $E[g(y)]=0$

$g(y)$ has to be 0 for any θ,

that is $P(g(y|\theta)=0)=1$ $\quad\therefore$ $\max x_i$ is a complete sufficient sta for θ. $\square$

(e) is a location family distribution

order $X_{(n)}-X_{(1)}$ is ancillary, $E(R)$ is independent to θ

$\therefore$ sufficient sta is not complete.

6.

$\frac{①}{②} = \frac{P(\underline{X}|\theta)}{P(Y|\theta)} = \frac{x_1x_2\cdots x_n}{2n\cdot y^{2n-1}}$ $\quad$ (6.2.2 [3097])

which is constant to θ

$\therefore$ Y is a sufficient statistic

6.21. Let X be one observation from

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}, \quad x = -1, 0, 1 \quad \theta \in (0,1)$$

(a) Is X a complete sufficient statistic?

X is sufficient

Complete $\Leftarrow$

$$E[g(X)] = (-1)\cdot f(-1|\theta) + 0\cdot f(0|\theta) + 1\cdot f(1|\theta)$$

$$= (-1)\cdot\left(\frac{\theta}{2}\right)(1-\theta)^{1-1} + 0 + \left(\frac{\theta}{2}\right)(1-\theta)^{1-1}$$

$$= -\frac{\theta}{2} + \frac{\theta}{2}$$

$$= 0$$

$\because P[g(X|\theta) = 0] \neq 1$ $\therefore$ the family of distribution of X is not a complete sufficient statistic for θ ◻

(b) Is $|X|$ a complete sufficient statistic?

$\because f(x|\theta)$ depends on $|x|$

$\therefore$ If $|X|$ already known

$x \mid |X|$ doesn't depend on θ

$\therefore |X|$ is sufficient.

Complete $E[g(|X|)] = 1\cdot|-1|\cdot f(|-1||\theta) + 0 + 1\cdot|1|\cdot f(|1||\theta)$

$$= \theta$$

If $E[g(|X|)] = 0$ then $\theta = 0$

$P(g(|X|)|0) = 0) = 1$ always hold $\therefore$ the family of distribution of X is a complete sufficient statistic for θ ◻

(c) Does $f(x|\theta)$ belong to the exponential class?

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}$$

$$= \left(\frac{\theta}{2}\right)^{|x|}\cdot(1-\theta)\cdot(1-\theta)^{-|x|}$$

$$= \left(\frac{\frac{\theta}{2}}{1-\theta}\right)^{|x|}(1-\theta)$$

$\log A + \log B = \log AB$

$$= (1-\theta)\left(\frac{\theta}{2(1-\theta)}\right)^{|x|}$$

$$= e^{\log(1-\theta)\cdot\left(\frac{\theta}{2(1-\theta)}\right)^{|x|}}$$

$$= e^{\log(1-\theta) + \log\left(\frac{\theta}{2(1-\theta)}\right)^{|x|}}$$

$$= e^{\log(1-\theta) + |x|\cdot\log\frac{\theta}{2(1-\theta)}} \quad ①$$

$\because$ ① is of the formation of exponential distribution

$\therefore f(x|\theta)$ belongs to the exponential class.

6.22. Let $X_1, \cdots, X_n$ be a random sample from a population

$$f(x|\theta) = \theta\cdot x^{\theta-1}$$

(a) Is ΣX_i sufficient for θ?

Let $X = X_1, X_2, \cdots X_n$

$$f(X|\theta) = \prod_{i=1}^{n}\theta\cdot x_i^{\theta-1} = \theta^n\cdot(x_1\cdot x_2\cdots x_n)^{\theta-1}$$

ΠX_i is sufficient for θ

ΣX_i is not a sufficient statistic for θ ◻

(b) Find a complete sufficient statistic for θ

Proof $\prod x_i$ is complete:

$f(x|\theta) = \theta \cdot x^{\theta-1}$

$= e^{\log\theta + (\theta-1)\log x}$

$f(\underline{x}|\theta) = \prod_{i=1}^{n} \theta \cdot x_i^{\theta-1} = \theta^n \cdot (x_1 \cdot x_2 \cdots x_n)^{\theta-1}$

$= e^{n\log\theta + (\theta-1)\log\prod x_i}$

From 6.2.25: $\log\prod x_i$ is a complete sufficient statistic for θ

$\because \prod x_i = g(\log\prod x_i)$

$\therefore$ the family of distribution of $\prod x_i$ is a complete sufficient statistic for θ $\square$

6.23. Let $X_1, \dots, X_n$ be a random sample from $U(\theta, 2\theta)$, $\theta > 0$. Find a minimal sufficient statistic for θ. Is the statistic complete?

Let $\underline{X} = X_1, \cdots, X_n$ $\qquad Y = Y_1, \cdots, Y_n$

$P(\underline{X}|\theta) = \theta^{-n} \cdot I_{(X_{(n)}/2,\, X_{(1)})}(\theta)$

$T = (X_{(1)}, X_{(n)})$

$\Leftarrow$ minimal sufficient statistic

$\dfrac{f(\underline{X}|\theta)}{f(\underline{Y}|\theta)}$ is independent with θ if and only if

$T(\underline{X}) = T(\underline{Y})$

$\because \dfrac{f(\underline{X}|\theta)}{f(\underline{Y}|\theta)} = \dfrac{\theta^{-n} I_{(X_{(n)}/2,\, X_{(1)})}(\theta)}{\theta^{-n} I_{(Y_{(n)}/2,\, Y_{(1)})}(\theta)}$

Only when $X_{(n)} = Y_{(n)}$ and $X_{(1)} = Y_{(1)}$ can $\dfrac{f(\underline{X}|\theta)}{f(\underline{Y}|\theta)}$ be independent to θ

$\therefore T = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.

$\Leftarrow$ Complete

If $E[g(X_{(1)}, X_{(n)})] = 0$, $P(g(X_{(1)}, X_{(n)}|\theta) = 0) = 1$

Let $g(X_{(1)}, X_{(n)})$ be $\dfrac{X_{(n)}}{X_{(1)}}$, $Z \sim U(1, 2)$

$\therefore X_i = \theta \cdot Z_i$

$X_{(1)} = \theta \cdot Z_{(1)}$

$X_{(n)} = \theta \cdot Z_{(n)}$

$E\left(\dfrac{X_{(n)}}{X_{(1)}}\right) = E\left(\dfrac{\theta Z_{(n)}}{\theta Z_{(1)}}\right) = E\left(\dfrac{Z_{(n)}}{Z_{(1)}}\right) = 2 \neq 0$

不能用这种方法证 complete.

If $g(X_{(1)}, X_{(n)})$ is ancillary, then $X_{(1)}, X_{(n)}$ is not complete

$\Leftarrow$ If $g(X_{(1)}, X_{(n)})$ is ancillary to θ

$\Leftarrow$ $P(g(X_{(1)}, X_{(n)}))$? depends on θ

$\because P(g(X_{(1)}, X_{(n)})) = P\left(\dfrac{X_{(n)}}{X_{(1)}}\right) = P\left(\dfrac{\theta X_{(n)}}{\theta X_{(1)}}\right) = P\left(\dfrac{Z_{(n)}}{Z_{(1)}}\right)$ independent to θ

$\therefore g(X_{(1)}, X_{(n)})$ is an ancillary sta to θ

$\therefore X_{(1)}, X_{(n)}$ is not complete

6.24. Consider:

$P = \{ P_\lambda(X=x) : P_\lambda(X=x) = \lambda^x \cdot e^{-\lambda} / x! \ ; \ x = 0, 1, 2, \cdots ; \ \lambda = 0 \text{ or } 1 \}$.

Show that the family P is not complete.

Demonstrate the completeness depends on the range of parameter.

$\Leftarrow$ P is not complete

$\Leftarrow$ $E(g(P)) = 0$ but $P(g(P(\theta)) = 0) \neq 0$

$P_1(x) = \frac{1}{e} \cdot \frac{1}{x!}$ Let $g(P_\lambda(x_1), \cdots) = g(X)$

$P_0(x) = 1$

$$E[g(P_0(X))] = \underbrace{\sum_{x=0}^{n} g(P_0(x))}_{A} \cdot \qquad (\lambda=0)$$

$$E[g(P_1(X))] = \sum_{x=0}^{n} g(P_\lambda(x)) \cdot e^{-1} \cdot (x!)^{-1} \qquad (\lambda=1)$$

$$= e^{-1} \cdot \underbrace{\sum_{x=0}^{n} g(P_1(x)) \cdot (x!)^{-1}}_{B}$$

$A = 0$

$B = 0$

then $E[g(P_\lambda(X))] = 0$

$\because (x!)^{-1} > 0$ $\therefore g(P_\lambda(x))$ is not necessary $= 0$

that is

$P(g(P_\lambda(X) \mid \lambda = 1) = 0) \neq 0$

6.25. By applying Theorem 6.2.13 to $T(X)$ of 6.2.10 establish

6.2.13

minimal sufficient

$\Leftarrow$ $\frac{f(X|\theta)}{f(y|\theta)}$ is $\times$ θ if and only if $T(X) = T(y)$

6.2.10

If $f(x|\theta) = h(x) c(\theta) \cdot \exp\left(\sum_{i=1}^{k} w_i(\theta) \cdot t_i(x)\right)$

$T(X) = \left(\sum_{j=1}^{n} t_1(X_j), \ \ldots, \ \sum_{j=1}^{n} t_k(X_j)\right)$ is sufficient

(a) $\left(\sum X_i, \sum X_i^2\right)$ is sufficient, but not minimal sufficient in $n(\mu, \mu)$

1° $f(x|\mu) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad n(\mu, \mu)$

$= \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)}$

$= \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}} \cdot e^{\frac{\mu}{\sigma^2} x} \cdot e^{-\frac{\mu^2}{2\sigma^2}}$

$= (\sqrt{2\pi}\mu)^{-1} \cdot e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2} x^2 + \frac{\mu}{\sigma^2} x}$ $\quad$ Let $c(\theta) = (\sqrt{2\pi}\mu)^{-1} \cdot e^{-\frac{\mu^2}{2\sigma^2}}$ $\quad w_1(\mu) = -\frac{1}{2\sigma^2}$ $\quad w_2(\mu) = \frac{\mu}{\sigma^2}$

$T_1(x) = x^2$ $\quad T_2(x) = x$

$\therefore f(x|\theta) \sim$ exponential family

Then $T(X) = \left(\sum_{i=1}^{n} x_i^2, \ \sum_{i=1}^{n} x_i\right)$ is sufficient statistic for μ

(a): $n(\mu, \mu)$

$\sigma^2 = \mu$

$\frac{f(X|\theta)}{f(Y|\theta)} = e^{-\frac{1}{2\sigma^2}(\sum x_i^2 - \sum x_j^2) + \frac{\mu}{\sigma^2}(\sum x_i - \sum x_j)}$ ①

if ① $\times$ θ, $\sum x_i \equiv \sum y_i$ is not necessary

$\therefore T(\sum x_i^2, \sum x_i)$ is not minimal

2° Not minimal

$\Leftarrow$ Let $X = X_1, X_2, \cdots X_n$

$f(X|\mu) = \prod_{i=1}^{n} f(x_i|\mu) = [c(\theta)]^n \cdot e^{-\frac{1}{2\sigma^2}\sum_{i}^{n} x_i^2} \cdot e^{\frac{\mu}{\sigma^2}\sum_{i}^{n} x_i}$

$$\therefore \frac{f(X|\mu)}{f(Y|\mu)} = \frac{\cancel{[c(\theta)]^n} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i}^{n} x_i^2} \cdot e^{\frac{\mu}{\sigma^2}\sum_{i}^{n} x_i}}{\cancel{[c(\theta)]^n} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i}^{n} y_i^2} \cdot e^{\frac{\mu}{\sigma^2}\sum_{i}^{n} y_j}} = e^{-\frac{1}{2\sigma^2}(\sum x_i^2 - \sum x_j^2) + \frac{\mu}{\sigma^2}(\sum x_i - \sum x_j)}$$

(b) The statistic $\sum x_i^2$ is minimal sufficient in the $n(\mu, \mu)$

when $\sigma^2 = \mu$ $\quad \sigma = \sqrt{\mu}$

1° $f(x|\mu,\sigma) = (\sqrt{2\pi}\sigma)^{-1} \cdot e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} = (\sqrt{2\pi\mu})^{-1} \cdot e^{-\mu} \cdot e^{-\frac{1}{2\mu}x^2 + x}$

Let $c(\mu) = (\sqrt{2\pi\mu})^{-1} e^{-\mu}$

$h(x) = e^{x}$

$w(\mu) = -\frac{1}{2\mu}$

$T(x) = x^2$

$\therefore \sum X_i^2$ is sufficient statistic for μ

2° minimal

$$\frac{f(X|\mu)}{f(Y|\mu)} = \frac{e^{-\frac{1}{2\sigma^2}\sum_1^n x_i^2} e^{\frac{\mu}{\sigma^2}\sum x_i}}{e^{-\frac{1}{2\sigma^2}\sum y_i^2} e^{\frac{\mu}{\sigma^2}\sum y_i}} = \frac{e^{-\frac{1}{2\mu}\sum_1^n x_i^2} e^{\sum x_i}}{e^{-\frac{1}{2\mu}\sum y_i^2} e^{\sum y_i}}$$

$\frac{f(X|\mu)}{f(Y|\mu)}$ is independent to μ if and only if $\sum_1^n x_i^2 = \sum_1^n y_i^2$

$\therefore \sum X_i^2$ is a minimal sufficient statistic for μ. $\square$

(c) The statistic $(\sum X_i, \sum X_i^2)$ is minimal sufficient in the $n(\mu, \mu^2)$ family

when $\sigma^2 = \mu^2$ $\quad \sigma = \mu$

1° $f(x|\mu,\sigma) = (\sqrt{2\pi}\sigma)^{-1} \cdot e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} = (\sqrt{2\pi}\mu)^{-1} \cdot e^{-1} \cdot e^{-\frac{1}{2\mu^2}x^2 + \frac{1}{\mu}x}$

Let $c(\mu) = (\sqrt{2\pi}\mu)^{-1} e^{-1}$

$h(x) = 1$

$w_1(\mu) = -\frac{1}{2\mu^2}$ $\qquad w_2(\mu) = \frac{1}{\mu}$

$T_1(x) = x^2$ $\qquad T_2(x) = x$

$\therefore \left(\sum_1^n X_i, \sum_1^n X_i^2\right)$ is sufficient statistic for μ.

2° minimal

$$\frac{f(X|\mu)}{f(Y|\mu)} = \frac{e^{-\frac{1}{2\sigma^2}\sum_1^n x_i^2} e^{\frac{\mu}{\sigma^2}\sum x_i}}{e^{-\frac{1}{2\sigma^2}\sum y_i^2} e^{\frac{\mu}{\sigma^2}\sum y_i}} = \frac{e^{-\frac{1}{2\mu^2}\sum_1^n x_i^2} e^{\frac{1}{\mu}\sum_1^n x_i}}{e^{-\frac{1}{2\mu^2}\sum y_i^2} e^{\frac{1}{\mu}\sum y_i}}$$

$\frac{f(X|\mu)}{f(Y|\mu)}$ is independent to μ if and only if $\begin{cases} \sum x_i^2 = \sum y_i^2 \\ \sum x_i = \sum y_i \end{cases}$

$\therefore (\sum X_i^2, \sum X_i)$ is a minimal sufficient statistic for μ. $\square$

(d). The statistic $(\sum X_i, \sum X_i^2)$ is minimal sufficient in $n(\mu, \sigma^2)$

1° $f(x|\mu,\sigma) = (\sqrt{2\pi}\sigma)^{-1} \cdot e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x}$

$c(\mu,\sigma) = (\sqrt{2\pi}\sigma)^{-1} \cdot e^{-\frac{\mu^2}{2\sigma^2}}$

$h(x) = 1$

$w_1(\mu,\sigma) = -\frac{1}{2\sigma^2}$ $\qquad w_2(\mu,\sigma) = \frac{\mu}{\sigma^2}$

$T_1(x) = x^2$ $\qquad T_2(x) = x$

$\therefore (\sum X_i, \sum X_i^2)$ is sufficient

2° minimal

$$\frac{f(X|\mu)}{f(Y|\mu)} = \frac{e^{-\frac{1}{2\sigma^2}\sum x_i^2} \cdot e^{-\frac{\mu}{\sigma^2}\sum x_i}}{e^{-\frac{1}{2\sigma^2}\sum y_i^2} \cdot e^{-\frac{\mu}{\sigma^2}\sum y_i}}$$

is independent to (μ, σ) if and only if $\begin{cases} \sum x_i^2 = \sum y_i^2 \\ \sum x_i = \sum y_i \end{cases}$

$\therefore (\sum X_i^2, \sum X_i)$ is a minimal sufficient sta for (μ, σ)

6.2b. Use Theorem 6.6.5 to establish that, given a sample $X_1, \cdots, X_n$ the following statics are minimal sufficient.

	sta	Dis
(a)	$\bar{X}$	$n(\theta, 1)$
(b)	ΣX_i	gamma (α, β), α known
(c)	$\max X_i$	uniform $(0, \theta)$
(d)	$X_{(1)}, \cdots, X_{(n)}$	Cauchy $(\theta, 1)$
(e)	$X_{(1)}, \cdots, X_{(n)}$	logistic (μ, β)

Theorem 6.6.5 Suppose that the family of density $\{f_0(x), \cdots, f_k(x)\}$ all have common support. Then

a. The statistic

$$T(X) = \left(\frac{f_1(X)}{f_0(X)}, \frac{f_2(X)}{f_0(X)}, \cdots \frac{f_k(X)}{f_0(X)} \right)$$

is minimal sufficient for the family $\{f_0(x), \cdots, f_k(x)\}$.

(a). $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad \mu = \theta$, $\sigma \cdot \sigma^2 = 1$

Let $X = X_1, X_2, \cdots, X_n$

$$f_\theta(X) = \prod_{i=1}^{n} (\sqrt{2\pi})^{-n} e^{-\frac{\Sigma x_i^2}{2}} \cdot e^{\theta \Sigma x_i} \cdot e^{-\frac{n}{2}\theta^2}$$

$$= \underbrace{(\sqrt{2\pi})^{-n}} \cdot e^{-\frac{n}{2}\theta^2} \cdot \underbrace{e^{-\frac{\Sigma x_i^2}{2}}} \cdot e^{\theta \Sigma x_i} \quad ①$$

$$f_0(X) = \underbrace{(\sqrt{2\pi})^{-n}} \cdot 1 \cdot \underbrace{e^{-\frac{\Sigma x_i^2}{2}}} \cdot 1 \quad ② \quad (\theta = 0)$$

$$T(X) = \frac{①}{②} = e^{-\frac{n}{2}\theta^2} \cdot e^{\theta \cdot \Sigma x_i}$$

$$= e^{-\frac{n}{2}\theta^2} \cdot e^{\theta n \cdot \bar{X}} \quad \text{is a function of } \bar{X}$$

$\therefore \bar{X}$ is minimal sufficient for θ $\square$

(b) $f(x) = \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\beta^\alpha \cdot \Gamma(\alpha)}$ $\quad \alpha$ known

Let $X = X_1, X_2, \cdots, X_n$

$$f_\beta(X) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1} \cdot e^{-\frac{x_i}{\beta}}}{\beta^\alpha \cdot \Gamma(\alpha)} = \frac{(\prod_{i=1}^{n} x_i)^{\alpha-1} \cdot e^{-\frac{1}{\beta}(\Sigma x_i)}}{(\beta^\alpha \cdot \Gamma(\alpha))^n}$$

$$f_0(X) = f_{\beta=1}(X) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1} e^{-x_i}}{\Gamma(\alpha)} = \frac{(\prod_{i=1}^{n} x_i)^{\alpha-1} \cdot e^{-\Sigma x_i}}{(\Gamma(\alpha))^n}$$

$$\frac{f_\theta(X)}{f_0(X)} = \frac{e^{-\Sigma x_i(\frac{1}{\beta}-1)}}{\beta^{n\alpha}} = e^{(1-\frac{1}{\beta})n\alpha} \cdot e^{-\Sigma x_i} \quad \text{is a function of } \Sigma x_i$$

$\therefore \Sigma x_i$ is minimal sufficient for β $\square$

(c) $f_\theta(X) = \left(\frac{1}{\theta}\right)^n \cdot I(\max x_i < \theta)$

$f_1(X) = I(\max x_i < 1)$

$\frac{f_\theta(X)}{f_1(X)} = \left(\frac{1}{\theta}\right)^n \frac{I(\max x_i < \theta)}{I(\max x_i < 1)}$ is a fun of $\max x_i$

$\therefore \max x_i$ is a sufficient sta for θ. $\square$

(d) Cauchy $(\theta, 1)$

$$f(x) = \frac{1}{\pi \gamma}\left[\frac{\gamma^2}{(x-\theta)^2+\gamma^2}\right] \quad \gamma = 1 \quad \frac{X}{Y} \sim \text{cauchy}(0,1)$$

$$= \frac{1}{\pi}\left[\frac{1}{(x-\theta)^2+1}\right] \quad X, Y \text{ iid} \sim n(0,1)$$

$$f(X) = \pi^{-n} \cdot \prod_{i=1}^{n}\left[(x_i-\theta)^2+1\right]^{-1}$$

$$f_{\theta=0}(X) = \pi^{-n} \cdot \prod_{i=1}^{n}(x_i^2+1)^{-1}$$

$\frac{f(X)}{f_{\theta=0}(X)} = \frac{x_i^2+1}{(x_i-\theta)^2+1}$ is a function of $X_{(1)}, \cdots X_{(n)}$

$\therefore X_{(1)}, \cdots X_{(n)}$ is a minimal sufficient sta for θ

(e) Similar

$$f(x, \mu, \beta) = \frac{e^{-(x-\mu)/\beta}}{\beta(1+e^{-(x-\mu)/\beta})^2}$$

$$f(x, 0, 1) = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$f(X, \mu, \beta) = \frac{\beta^{-n} \cdot e^{-\frac{1}{\beta}(\Sigma x - n\mu)}}{\prod_{i=1}^{n}(1+e^{-\frac{1}{\beta}(x_i-\mu)})^2}$$

$$f(X, 0, 1) = \frac{e^{-\Sigma x_i}}{\prod_{i=1}^{n}(1+e^{-x_i})^2}$$

$\frac{f(X, \mu, \beta)}{f(X, 0, 1)}$ is a function of $X_{(1)}, \cdots X_{(n)}$

$\therefore \cdots \cdots$ $\square$

6.27. $X_1, \dots X_n$ iid, $\sim f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \cdot e^{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}}$ $x \in (0,\infty)$

(a) Show that $\bar{X} = \frac{1}{n}\sum X_i$ and $T = \frac{n}{\sum_{i=1}^n \frac{1}{X_i} - \frac{1}{\bar{X}}}$ are sufficient and complete.

Sufficient <= $X \mid T$ $\times$ depends on μ, λ

$$f(\underline{x}|\mu,\lambda) = e^{\frac{n\lambda}{\mu}} \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^n x_i^{-\frac{3}{2}} \cdot e^{-\frac{\lambda}{2\mu^2}\sum x_i} \cdot e^{-\frac{\lambda}{2}\sum \frac{1}{x_i}}$$

$$= e^{\frac{n\lambda}{\mu}} \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \left(\prod x_i\right)^{-\frac{3}{2}} \cdot e^{-\frac{\lambda}{2\mu^2}\sum x_i} \cdot e^{-\frac{\lambda}{2}\sum (x_i)^{-1}}$$

is a exponential distribution.

Theorem 6.2.25 : $T(\underline{x}) = \left(\sum x_i, \sum (x_i)^{-1}\right)$ is a complete sufficient static for (μ,λ)

$\because T = \frac{n}{\sum \frac{1}{x_i} - \frac{1}{\bar{x}}} = g\left(\sum (x_i)^{-1}, \sum x_i\right)$

if we know $\sum x_i$, then $T = \frac{n}{\sum \frac{1}{x_i} - \frac{1}{\bar{x}}}$ is a function of $\sum (x_i)^{-1}$

$\therefore$ T can be regarded as a function of $T(\underline{x})$

$\therefore$ T is a complete and sufficient statistic for μ, λ $\square$

(b) For $n=2$, show that $\bar{X}$ has an inverse Gaussian dis $\frac{n\lambda}{T}$ has a χ^2_{n-1} distribution, and they are independent.

$n=2$

$$f(\underline{x}|\mu,\lambda) = e^{\frac{2\lambda}{\mu}} \left(\frac{\lambda}{2\pi}\right) (x_1+x_2)^{-\frac{3}{2}} \cdot e^{-\frac{\lambda}{2\mu^2}(x_1+x_2)} \cdot e^{-\frac{\lambda}{2}\left(\frac{1}{x_1}+\frac{1}{x_2}\right)}$$

$$= e^{\frac{2\lambda}{\mu}} \left(\frac{\lambda}{2\pi}\right) (x_1+x_2)^{-\frac{3}{2}} \cdot e^{-\frac{\lambda}{2}(x_1+x_2)\left(\frac{1}{\mu^2}+\frac{1}{x_1x_2}\right)}$$

$\frac{\partial x_1}{\partial u} = \frac{1}{2}\left(u^2 + \frac{uv}{\lambda+v}\right)^{-\frac{1}{2}} \cdot \frac{u\lambda}{(\lambda+v)^2}$

Let $U = \frac{X_1+X_2}{2}$ $\quad x_1+x_2 = 2u$ ① $\quad u>0$ $\quad x_1 = 2u - x_2$ $\quad \frac{x_1+x_2}{x_1x_2}$

$$V = \frac{n\lambda}{T} = \frac{2\lambda}{T} = \lambda \cdot \frac{2}{\frac{1}{x_1}+\frac{1}{x_2} - \frac{2}{x_1+x_2}}$$

$$= 2\lambda \cdot \frac{1}{\frac{x_1+x_2}{x_1x_2} - \frac{2}{x_1+x_2}}$$

$$= 2\lambda\left[\frac{(x_1+x_2)^2 - 2x_1x_2}{x_1x_2(x_1+x_2)}\right]^{-1}$$

$$= 2\lambda \cdot \frac{x_1x_2(x_1+x_2)}{x_1^2+x_2^2}$$

$$= 2\lambda \cdot \frac{x_1x_2 \cdot 2u}{(2u)^2 - 2x_1x_2} = v$$

$$2\lambda \frac{x_1x_2}{2u - 2x_1x_2} = v$$

$$\frac{\lambda x_1x_2}{u - x_1x_2} = v$$

$$\lambda x_1x_2 = uv - v(x_1x_2)$$

$$(\lambda+v)x_1x_2 = uv$$

$$x_1x_2 = \frac{uv}{\lambda+v} \quad ②$$

$$(2u - x_2)x_2 = \frac{uv}{\lambda+v}$$

$$2ux_2 - x_2^2 = \frac{uv}{\lambda+v}$$

$$x_2^2 - 2ux_2 + \frac{uv}{\lambda+v} = 0$$

$$(x_2-u)^2 = u^2 - \frac{uv}{\lambda+v}$$

$$x_2 = \pm\left(u^2 - \frac{uv}{\lambda+v}\right)^{\frac{1}{2}} + u$$

$\because x_2 > 0$ $\quad x_2 = \left(u^2 - \frac{uv}{\lambda+v}\right)^{\frac{1}{2}} + u$

or $x_2 = -\left(u^2 - \frac{uv}{\lambda+v}\right)^{\frac{1}{2}} + u$

$\because x_1+x_2 = 2u$

$\therefore x_1 = 2u - x_2$

$\therefore \begin{cases} x_1 = \left(u^2 - \frac{uv}{\lambda+v}\right)^{\frac{1}{2}} + u \\ x_2 = -\left(u^2 - \frac{uv}{\lambda+v}\right)^{\frac{1}{2}} + u \end{cases}$

or $\begin{cases} x_1 = -\left(u^2 - \frac{uv}{\lambda+v}\right)^{\frac{1}{2}} + u \\ x_2 = \left(u^2 - \frac{uv}{\lambda+v}\right)^{\frac{1}{2}} + u \end{cases}$

$$f(u,v|\mu,\lambda) = e^{\frac{2\lambda}{\mu}} \left(\frac{\lambda}{2\pi}\right) (2u)^{-\frac{3}{2}} \cdot e^{-\frac{\lambda}{2}2u \cdot \left(\frac{1}{u^2} + \frac{\lambda+v}{uv}\right)} \cdot \left(u^2 + \frac{uv}{\lambda+v}\right)^{-\frac{1}{2}} \frac{u\lambda}{(\lambda+v)^2}$$

把 U, V 分开

$$= e^{\frac{2\lambda}{\mu}} \frac{\lambda}{2\pi} 2^{-\frac{3}{2}} u^{-\frac{3}{2}} \cdot e^{-\lambda u/\mu^2 - \lambda u\left(\frac{\lambda+v}{uv}\right)}$$

$$= e^{-\frac{\lambda u}{\mu^2}} \cdot e^{-\lambda\frac{\lambda+v}{v}}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \underbrace{\frac{1}{2}\left(u^2 - \frac{uv}{\lambda+v}\right)^{-\frac{1}{2}}\left(2u - \frac{v}{\lambda+v}\right) + 1}_{A} & \underbrace{\frac{1}{2}\left(u^2 + \frac{uv}{\lambda+v}\right)^{-\frac{1}{2}} \frac{u\lambda}{(\lambda+v)^2}}_{B} \\ -\frac{1}{2}\left(u^2 - \frac{uv}{\lambda+v}\right)^{-\frac{1}{2}}\left(2u - \frac{v}{\lambda+v}\right) + 1 & -\frac{1}{2}\left(u^2 + \frac{uv}{\lambda+v}\right)^{-\frac{1}{2}} \frac{u\lambda}{(\lambda+v)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2}A + 1 & \frac{1}{2}B \\ -\frac{1}{2}A + 1 & -\frac{1}{2}B \end{vmatrix} = \left[-\frac{1}{4}AB - \frac{1}{2}B + \frac{1}{4}AB - \frac{1}{2}B(-\frac{1}{2}A+1)\right]$$

$$= \left|-\frac{1}{4}AB - \frac{1}{2}B + \frac{1}{4}AB - \frac{1}{2}B\right|$$

$= |-B|$

$(2u)^{-\frac{3}{2}} \left(u^2+\frac{uv}{\lambda+v}\right)^{-\frac{3}{2}} \frac{u\lambda}{(\lambda+v)^2}$

$= 2^{-\frac{3}{2}} u^{-\frac{3}{2}} \cdot u \frac{\lambda}{(\lambda+v)^2} \cdot \left(u^2+\frac{uv}{\lambda+v}\right)^{-\frac{3}{2}}$

$= 2^{-\frac{3}{2}} \cdot u^{-\frac{1}{2}} \frac{\lambda}{(\lambda+v)^2} \left(u^2+\frac{uv}{\lambda+v}\right)^{-\frac{3}{2}}$

$= 2^{-\frac{3}{2}} \frac{\lambda}{(\lambda+v)^2} \left(u^3+\frac{u^2v}{\lambda+v}\right)^{-\frac{1}{2}}$ 手抖，做不下去

6.28. Prove Theorem 6.6.5

① The minimal sufficiency of $T(\underline{X})$ in the family $\{f_0(\underline{x}), \cdots, f_k(\underline{x})\}$ follows from 6.2.13.

Theorem 6.6.5 Suppose that the family of density $\{f_0(\underline{x}), \cdots, f_k(\underline{x})\}$ all have common support. Then

a. The statistic

$$T(\underline{X}) = \left(\frac{f_1(\underline{X})}{f_0(\underline{X})}, \frac{f_2(\underline{X})}{f_0(\underline{X})}, \cdots \frac{f_k(\underline{X})}{f_0(\underline{X})}\right)$$

is minimal sufficient for the family $\{f_0(\underline{x}), \cdots, f_k(\underline{x})\}$.

6.2.13 minimal sufficient

$\in \frac{f(\underline{x}|\theta)}{f(y|\theta)}$ is $\times\ \theta$ if and only if $T(\underline{x}) = T(y)$

Assume $T(\underline{X}) = \left\{\frac{f_1(\underline{X})}{f_0(\underline{X})}, \frac{f_2(\underline{X})}{f_0(\underline{X})}, \cdots \frac{f_k(\underline{X})}{f_0(\underline{X})}\right\}$ when & only when $T(\underline{X}) = T(Y)$

$T(Y) = \left\{\frac{f_1(Y)}{f_0(Y)}, \frac{f_2(Y)}{f_0(Y)}, \cdots \frac{f_k(Y)}{f_0(Y)}\right\}$ $\frac{f_i(\underline{X})}{f_0(\underline{X})} = \frac{f_i(Y)}{f_0(Y)} = S$

$\frac{f_i(\underline{X}|\ i\ )}{f_i(Y|\ i\ )} = \frac{f_0(\underline{X})\cdot S}{f_0(Y)\cdot S} = \frac{f_0(\underline{X})}{f_0(Y)}$ which doesn't depends on i

$\therefore T(\underline{X})$ is minimal sufficient according to Theorem 6.2.13 □

② Any sta that is sufficient in $\mathcal{F}$ must be a function of $T(\underline{X})$

Assume $T'(\underline{X})$ is sufficient for $f_i(\underline{X})$

Then $f_i(\underline{X}) | T'(\underline{X})$ doesn't depends on i

$\therefore P(f_1(\underline{X}) | T'(\underline{X})) = P(f_2(\underline{X}) | T'(\underline{X})) = \cdots = P(f_k(\underline{X}) | T'(\underline{X}))$

$\therefore T'(\underline{X})$ is a function of $(f_1(\underline{X}), f_2(\underline{X}), \cdots f_k(\underline{X}))$

also is a function of $\left(\frac{f_1(\underline{X})}{f_0(\underline{X})}, \frac{f_2(\underline{X})}{f_0(\underline{X})} \cdots \frac{f_k(\underline{X})}{f_0(\underline{X})}\right)$, namely $T(\underline{X})$ □

6.29. Show that if $X_1, \ldots, X_n$ are random sample from f, then the order statistics are minimal sufficient.

Hint: Use 6.6.5 taking the family $\{f_0(\underline{x}), \cdots, f_k(x)\}$ to be logistic densities.

b. If $\mathcal{F}$ is a family of densities with common support

(i) $f_i(x) \in \mathcal{F}$, $i = 0, 1, \cdots, k$,

(ii) $T(\underline{X})$ is sufficient for $\mathcal{F}$

then $T(\underline{X})$ is minimal sufficient for $\mathcal{F}$.

Let $f_i(x) = \text{logistic}(\alpha_i, \beta_i) = \frac{e^{-(x-\alpha_i)/\beta_i}}{\beta_i(1+e^{-(x-\alpha_i)/\beta_i})^2}$

$T(\underline{X}) = \left(\frac{\prod_{i=1}^n f_1(x_i)}{\prod_{i=1}^n f_0(x_i)}, \frac{\prod_{i=1}^n f_2(x_i)}{\prod_{i=1}^n f_0(x_i)}, \cdots, \frac{\prod_{i=1}^n f_k(x_i)}{\prod_{i=1}^n f_0(x_i)}\right) = \left(\frac{\prod_{i=1}^n f_1(x_{(i)})}{\prod_{i=1}^n f_0(x_{(i)})}, \frac{\prod_{i=1}^n f_2(x_{(i)})}{\prod_{i=1}^n f_0(x_{(i)})}, \cdots, \frac{\prod_{i=1}^n f_k(x_{(i)})}{\prod_{i=1}^n f_0(x_{(i)})}\right)$

$= f(x_{(1)}, x_{(2)}, \cdots x_{(n)})$ $[x_{(1)}, x_{(2)}, \cdots x_{(n)}]$ is a minimal sufficient sta for $\{f_0, f_1, \cdots, f_k\}$

6.30. Let $X_1, \cdots, X_n$ be a random sample $f(x|\mu) = e^{-(x-\mu)}$

$-\infty < \mu < x < \infty$

(a). Show that $X_{(1)} = \min_i X_i$ is a complete sufficient sta.

(b) Use Basu's Theorem show $X_{(1)}$ and S^2 are independent

(a) $\Leftarrow$ Complete sufficient

If $E_\mu(g(X_{(1)})) = 0 \Rightarrow P(g(X_{(1)}|\theta) = 0) = 1$

$\underline{X} = X_1, X_2, \cdots X_n$

$\because f(\underline{X}|\mu) = \prod_{i=1}^{n} e^{-(x_i-\mu)} I_{(\mu,\infty)}(\min x_i)$

$= e^{-\sum x_i + n\mu} I_{(\mu,\infty)} \cdot x_{(1)}$

$= e^{n\mu} \cdot e^{-\sum x_i} I_{(\mu,\infty)} \cdot x_{(1)}$

$= e^{-\sum x_i} \cdot e^{n\mu} I_{(\mu,\infty)} \cdot x_{(1)}$

$= f(\underline{x}) \cdot f(\mu, x_{(1)})$

$\therefore X_{(1)}$ is a sufficient sta for μ

$E(g(X_{(1)}))$

$\Leftarrow g(X_{(1)})$ Let $Y = X_{(1)}$

$F(y) = \int_\mu^y e^{-(x-\mu)} dx \quad y \geq \mu$

$= -e^{-(x-\mu)}\Big|_\mu^y = -e^{-(y-\mu)} + 1$

$f_Y(y) = f(y) \cdot \underbrace{(1-F(y))^{n-1}}_{\text{the rest is bigger than } y} C_n^1$

$= e^{-(y-\mu)} \cdot [e^{-(y-\mu)}]^{n-1} n$

$= n e^{-n(y-\mu)} \quad y > \mu$

$E_\mu(g(Y)) = \int_\mu^\infty g(y) \cdot f_Y(y)\, dy$

$= \int_\mu^\infty g(y) \cdot n \cdot e^{-n(y-\mu)} dy \quad \odot$

$= \int_\mu^\infty g(y) e^{-ny} \cdot \boxed{n \cdot e^{n\mu}} dy$

If $\odot = 0$, then $\int_\mu^\infty g(y) \cdot e^{-ny} dy = 0$ for all μ

$\int_\mu^\infty g(y) \cdot e^{-ny} dy \longrightarrow$ Assume $k'(y) = g(y) e^{-ny}$

$= k(y)\Big|_\mu^\infty$

$= k(\infty) - k(\mu) = 0$ for all μ

$\Rightarrow k$ is a constant

$\therefore k'(y) = 0, \quad g(y) \cdot e^{-ny} = 0 \quad \because e^{-ny} > 0$

$\therefore g(y) \equiv 0$

$\therefore P(g(y)|\theta) = 0) = 1$

$\therefore X_{(1)}$ is a complete sufficient sta for μ ∎

(b) Basu's theorem: If $X_{(1)}$ is complete sufficient sta for μ, then $X_{(1)}$ is independent of any ancillary statistic.

$\Leftarrow S^2$ is ancillary (is independent of μ)

$S^2 = \sum (x_i - \bar{x})^2 / (n-1)$ 0 info

$\because f(x|\mu)$ is a location family

$\therefore X_i = Z_i + \mu$

$Z_i \sim f(x|0) \qquad x_i = z_i + \mu$

$\bar{x} = \bar{z} + \mu$

$S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$

$= \frac{1}{n-1} \sum [(z_i + \mu) - (\bar{z} + \mu)]^2$

$= \frac{1}{n-1} \sum (z_i - \bar{z})^2$

$\therefore S^2$ is independent with μ

$\therefore S^2$ is an ancillary statistic for μ

According to Basu's theorem, if $X_{(1)}$ is complete sufficient, then it's independent of any ancillary sta.

$\therefore X_{(1)}$ is independent to S^2

6.31. Basu's Theorem can simplify calculations.

(a). Let $X_1, \cdots, X_n$ be iid $n(\mu, \sigma^2)$, where σ^2 is known.

(i) Show that $\bar{X}$ is complete sufficient for μ, and S^2 is ancillary. Hence by Basu's Theorem, $\bar{X}$ and S^2 are independent.

1° $\Leftarrow$ $\bar{X}$ is complete and sufficient

If $E_\mu(g(\bar{X})) = 0$, then $P(g(\bar{X}) = 0 \mid \mu) = 1$

$\because X \sim n(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}} \cdot e^{x^2} \cdot e^{-2\mu x} \cdot e^{\mu^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2\sigma^2}} e^{\mu^2} \cdot e^{x^2} \cdot e^{-2\mu x}$$

let $w(\theta) = -2\mu$, $t(x) = x$

the $T(\bar{X}) = \sum_1^n x_i$ is a complete sufficient sta for μ.

2° $\Leftarrow$ S^2 is independent to μ

$\because X \sim n(\mu, \sigma^2)$ is a position statistic

let $Z : X - \mu \sim f(0, \sigma^2)$ $\quad x_i = z_i - \mu$, $\bar{x} = \bar{z} - \mu$

$$S^2 = \frac{1}{n-1} \sum_1^n (x_i - \bar{x})^2$$

$$= \frac{1}{n-1} \sum_1^n \left[(z_i - \mu) - (\bar{z} - \mu)\right]^2$$

$$= \frac{1}{n-1} \sum_{i=1}^n (z_i - \bar{z})^2 \quad \text{is independent to } \mu$$

$\therefore S^2$ is an ancellary sta of μ.

$\therefore$ Based on Basu's theorem, $\bar{X}$ is independent to S^2 $\square$

6.2.25. Complete sta for exponential family.

$f(x|\theta) = h(x) \cdot c(\theta) \cdot \exp\left(\sum_{j=1}^k w_j(\theta) t_j(x)\right)$, then

$T(\bar{X}) = \left(\sum_{i=1}^n t_1(X_i), \sum_1^n t_2(X_i), \cdots \sum_1^n t_k(X_i)\right)$ is complete sta

open set in R^k

for $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots \theta_k)$

(ii) Show that this independent carries even if σ^2 is unknown. as knowledge of σ^2 has no bearing on distributions.

$\because$ the distribution of $\bar{X}$ completely decided by μ

& $\because$ the distribution of S^2 completely independent to μ but depends on σ^2

$\therefore P(\bar{X}) \cdot P(\sigma^2) = f_1(\mu) \cdot f_2(\sigma^2)$ for each (μ, σ^2)

$\therefore$ This independence carries even σ^2 is unknown. $\square$

(b). A Monte Carlo swindle is a technique for improving variance estimates.

Suppose that $X_1, \cdots, X_n$ are iid $n(\mu, \sigma^2)$, we compute the variance of the median M.

(i). Apply Basu's Theorem to show that

$Var(M) = Var(M-\bar{X}) + Var(\bar{X})$, thus we only have to simulate the $Var(M-\bar{X})$ piece of $Var(M)$

$\Leftarrow$ $Var(M) = Var(M-\bar{X}) + Var(\bar{X})$

$M-\bar{X}$ and $\bar{X}$ are independent

If A, B are independent

$Var(A+B)$

$= Var(A) + Var(B)$

From (a), we have $\bar{X}$ is a complete sufficient statistic for μ

$\Leftarrow$ $M-\bar{X}$ is an ancellary $\quad$ cal$(M-\bar{X})$ doesn't depends on μ

$\because$ X is a position statistic

let M be the median of $X_1, X_2, \cdots X_n$

Y be the median of $Y_1, Y_2, \cdots Y_n$

$Y_i = X_i - \mu \sim n(0, \sigma^2)$

$\therefore$ $Y = M - \mu$ ①

$\bar{Y} = \bar{X} - \mu$ ②

$\therefore$ ①-②:

$Y - \bar{Y} = M - \mu - (\bar{X} - \mu)$

$= M - \mu - \bar{X} + \mu$

$= M - \bar{X}$

$\therefore$ $M-\bar{X}$ doesn't depends on μ

$\therefore$ $M-\bar{X}$ is an ancellary statistic to μ.

Based on Basu's Theorem, $M-\bar{X}$ is independent with $\bar{X}$

$\therefore$ $Var(M-\bar{X}+\bar{X}) = Var(M) = Var(M-\bar{X}) + Var(\bar{X})$ ☐

(ii) Show that $Var(M)$ is approximately $2[Var(M)]^2/(N-1)$ and that of $M-\bar{X}$ is approximately $2[Var(M-\bar{X})]^2/(N-1)$

$\left.\begin{array}{l} M_1, M_2, \cdots, M_n \\ M_1-\bar{X}, M_2-\bar{X}, \cdots M_n-\bar{X} \end{array}\right\}$ are asymptotically normal,

Assume S^2 is a sample variance of N X, $X \sim n(\mu, \sigma^2)$

then $\frac{S^2(N-1)}{\sigma^2} \sim \chi^2_{N-1}$

$Var\left(\frac{S^2(N-1)}{\sigma^2}\right) = 2(N-1)$

that is $\left(\frac{N-1}{\sigma^2}\right)^2 Var(S^2) = 2(N-1)$

$\therefore$ $Var(S^2) = \frac{2(N-1)\sigma^2}{(N-1)^2}$

$= \frac{2\sigma^2}{N-1}$

$S^2 = \frac{1}{n}\sum(X_i - \mu)^2$

$\frac{S^2}{\sigma^2} = \frac{1}{n}\cdot\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$

$n\frac{S^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$ ☐

Let $\sigma^2 = Var(M)$, $Var(M) = 2[Var(M)]^2/N-1$

$\sigma^2 = Var(M-\bar{X})$ $Var(M-\bar{X}) = 2[Var(M-\bar{X})]^2/(N-1)$

☐

(c). (i) If X/Y and Y are independent random variables, show that

$$E\left(\frac{X}{Y}\right)^k = \frac{E(X^k)}{E(Y^k)}$$

$\because$ $\frac{X}{Y}$ and Y are independent

$\therefore$ $E(X) = E\left(\frac{X}{Y}\cdot Y\right) = E\left(\frac{X}{Y}\right)\cdot E(Y)$

$E(X^k) = E\left(\frac{X^k}{Y^k}\cdot Y^k\right) = E\left(\frac{X}{Y}\right)^k \cdot E(Y^k)$

$\therefore$ $E\left(\frac{X}{Y}\right)^k = \frac{E(X^k)}{E(Y^k)}$ ☐

(ii) Using this result and Basu's Theorem to show that if $X_1, \dots, X_n$ are iid gamma(α, β), where α is known, then for $T = \sum_i X_i$

$$E\left(\frac{X}{T}\right)^k = \frac{E(X^k)}{E(T^k)}$$

$$E(X_{(i)} \mid T) = E\left(\frac{X_{(i)}}{T} T \mid T\right) = T \frac{E(X_{(i)})}{ET}$$

From (i) $\quad T \dfrac{E(X_{(i)})}{ET} = T \cdot E\left(\dfrac{X_{(i)}}{T}\right)$

$\Leftarrow \quad E\left(\dfrac{X_{(i)}}{T} T \mid T\right) = T \cdot E\left(\dfrac{X_{(i)}}{T}\right)$

$\Leftarrow$ T is independent with $\dfrac{X_{(i)}}{T}$

$\Leftarrow$ T is complete sufficient statistic for β

$\dfrac{X_{(i)}}{T}$ is ancellary, then by Basu's Theorem, ...

6.2.25. Complete sta for exponential family.

$f(x|\theta) = h(x) \cdot c(\theta) \cdot \exp\left(\sum_{j=1}^{k} w_j(\theta)\, t_j(x)\right)$, then $\quad$ open set in R^k

$T(X) = \left(\sum_{i=1}^{n} t_1(X_i), \sum_{i=1}^{n} t_2(X_i), \dots \sum_{i=1}^{n} t_k(X_i)\right)$ is complete sta

for $\theta = (\theta_1, \theta_2, \cdots \theta_k)$

$\because$ $X_1, \cdots, X_n$ are iid gamma(α, β)

$\therefore$ $f_{X_i}(x_i) = \dfrac{x_i^{\alpha-1} e^{-x_i/\beta}}{\beta^\alpha \, \Gamma(\alpha)}$ is exponential distribution

$= \underbrace{\dfrac{1}{\Gamma(\alpha)} \cdot \dfrac{1}{\beta^\alpha}}_{c(\beta)} \cdot \underbrace{x_i^{\alpha-1}}_{h(x)} \cdot e^{-\frac{1}{\beta} x}$ $\quad$ Let $t(X) = X$, $\quad w(\beta) = -\dfrac{1}{\beta}$

$T(X) = \sum_{i=1}^{n} t(X_i) = \sum_{i=1}^{n} X_i = T$ is a complete sufficient statistic for β. ①

X_i is a scale statistic $\quad E(X_i) = \alpha\beta$

let $Z_i = \dfrac{X_i}{\beta}$, $\quad Z_1, Z_2, \cdots Z_n$ are iid. gamma$(\alpha, 1)$

$X_i = \beta \cdot Z_i$

$\dfrac{X_i}{T} = \dfrac{X_i}{\sum_{j=1}^{n} X_j} = \dfrac{\beta \cdot Z_i}{\beta(Z_1 + Z_2 + \cdots + Z_n)} = \dfrac{Z_i}{\sum_{j=1}^{n} Z_j}$ which is independent to β

$\therefore$ $\dfrac{X_i}{T}$ is an ancellary statistic for β ②

Combine ① and ②, according to Basu's Theorem, T and $\dfrac{X_i}{T}$ are independent

$\therefore \quad E(X_i \mid T) = E\left(T \cdot \dfrac{X_i}{T} \mid T\right) = E(T) \cdot E\left(\dfrac{X_i}{T}\right)$

$= T \cdot E\left(\dfrac{X_i}{T}\right)$ $\quad \square$

6.32. Prove the Likelihood Principle Corollary.

Assuming both the Formal Sufficient Principle & Conditionality Principle, prove that if $E = (X, \theta, \{f(x|\theta)\})$ is an experiment, then $\underbrace{Ev(E, x)}_{\text{evidence about } \theta \text{ raising from } E \,\&\, x}$ should depend on E & x only through $L(\theta|x)$.

X, Y contain same info about θ

Let $E_1 = E_2$, then $Ev(E_1, x_1) = Ev(E_2, x_2)$

the evidence got from $E_1(x_1)$ or $E_2(x_2)$ are the same.

From formal Likelyhood Principle,

If $Ev(E_1, x_1) = Ev(E_2, x_2)$, if

$L(\theta, x_1) = C L(\theta, x_2)$

That is the evidence is equal when Likelihood function is equal $\quad \square$

Likelihood Principle: $L(\theta|x) = c(x, y) \cdot L(\theta|y)$

Formal Sufficiency Principle: $E = (X, \theta, \{f(x|\theta)\})$

If $T(X)$ is sufficient sta for θ,

and $T(X) = T(Y)$, then $Ev(E, x) = Ev(E, y)$

Conditional Principle: $X^* = (j, x_j)$

$E^* = (X^*, \theta, \{f^*(x^*|\theta)\})$

$= f^*((j, x_j)|\theta)$

Experiment Form

then $Ev(E^*, (j, x_j)) = Ev(E_j, x_j)$

Formal Likelihood Principle: $E_1 = (X_1, \theta, \{f_1(x_1|\theta)\})$ $\quad x_1^* \in X_1$

$E_2 = (X_2, \theta, \{f_2(x_2|\theta)\})$ $\quad x_2^* \in X_2$

If $L(\theta|x_1^*) = C L(\theta|x_2^*)$

Then $Ev(E_1, x_1^*) = Ev(E_2, x_2^*)$

Likelyhood Principle Corollary: If $E = (X, \theta, \{f(x|\theta)\})$, then $Ev(E, x)$ should depend on E & x only through $L(\theta|x)$

6.33. (a). Proof Theorem 6.3.6 Birnbaum's Theorem

Define $g(t|\theta) = g((j, x_j)|\theta) = f^*((j, x_j)|\theta)$

$$h(j, x_j) = \begin{cases} c & \text{if } (j, x_j) = (2, x_2^*) \\ 1 & \text{otherwise} \end{cases}$$

Show that $T(j, x_j)$ is a sufficient sta in E^* experiment by varifying that

$$g(T(j, x_j)|\theta) \cdot h(j, x_j) = f^*((j, x_j)|\theta) \text{ for all } (j, x_j)$$

When $j \neq 2$ $T(j, x_j) = (j, x_j)$

when $j = 2$

$T(j, x_j) = \begin{cases} (1, x_1^*) & j=1 \ x_1 = x_1^* \text{ or if } j=2 \text{ and } x_2 = x_2^* \\ (j, x_j) & \text{otherwise} \end{cases}$

$$\underbrace{g(T(2, x_2^*)|\theta)}_{} \cdot \underbrace{h(2, x_2^*)}_{c} = g((1, x_1^*)|\theta) \cdot c = f^*((1, x_1^*)|\theta) \cdot c \quad \because j \text{ and } x_j \text{ are independent}$$

$$\overset{j=2}{=} \frac{1}{2} \cdot c \cdot f_1(x_1^*|\theta) \quad \therefore f^*((1, x_1^*)|\theta)$$

$$= c \cdot \frac{1}{2} \cdot L(\theta|x_1^*) \quad = f_1(x_1^*|\theta) \cdot \frac{1}{2}$$

$T(1, x_1^*) = T(2, x_2^*) = (1, x_1^*)$

$$\overset{\text{why?}}{=} \frac{1}{2} \cdot L(\theta|x_2^*) \quad = f_2(x_2^*|\theta) \cdot \frac{1}{2}$$

$$= \frac{1}{2} \cdot f_2(x_2|\theta)$$

$$= f^*((2, x_2^*)|\theta)$$

?

$\therefore$ By Factorization Theorem, $T(j, x_j)$ is sufficient.

6.34. Consider Exercise 6.12

Show that the Formal Likelihood Principle implies any conclusion about θ x depend on sample size n.

The Likelyhood for (n, x) is proportional to likelihood x from binomial $n(n, \theta)$ experiment.

X, Y contain same info about θ

Likelihood Principle: $L(\theta|X) = c(X, Y) \cdot L(\theta|Y)$ $\quad L(\theta_1|X) = 2L(\theta_2|X)$ if $c(X, Y) = 1$

$L(\theta_1|Y) = 2L(\theta_2|Y)$ doesn't matter if we know X, or Y

Formal Sufficiency Principle: $E = (X, \theta, \{f(x|\theta)\})$

If $T(X)$ is sufficient sta for θ,

and $T(X) = T(Y)$, then $Ev(E, x) = Ev(E, Y)$

Conditional Principle: $X^* = (j, x_j)$

$E^* = (X^*, \theta, \{f^*(x^*|\theta)\})$

$= f^*((j, x_j)|\theta)$

then $Ev(E^*, (j, x_j)) = Ev(E_j, x_j)$

Experiment Form

Formal Likelihood Principle: $E_1 = (X_1, \theta, \{f_1(x_1|\theta)\})$ $\quad x_1^* \in X_1$

$E_2 = (X_2, \theta, \{f_2(x_2|\theta)\})$ $\quad x_2^* \in X_2$

If $L(\theta|x_1^*) = c\,L(\theta|x_2^*)$

Then $Ev(E_1, x_1^*) = Ev(E_2, x_2^*)$

Likelyhood Principle Corollary: If $E = (X, \theta, \{f(x|\theta)\})$, then $Ev(E, x)$ should depend on E & x only through $L(\theta|x)$

$L(\theta|(n, x)) = P((n, x)|\theta) = C_n^x \cdot \theta^x \cdot (1-\theta)^{n-x} \cdot P_n$ ①

$L(\theta, n|x) = P(x|\theta, n) = C_n^x \cdot \theta^x \cdot (1-\theta)^{n-x}$ ②

$\because$ ① $= P_n \cdot$ ②

$= c \cdot$ ②

$\therefore$ The likelihood for (n, x) is proportional to the likelihood for x ($x \sim$ binomial(n, θ)) ▷

6.35. Outcome for patients $X \sim \text{Bernoulli}(P)$

X_1, X_2, X_3 are iid.

$$X_i = \begin{cases} 1 & P \\ 0 & 1-P \end{cases}$$

Sample Points : $\{0, 10, 110, 111\}$

$P(0) = 1-P$

$P(10) = P(1-P)$

$P(110) = P^2(1-P)$

$P(111) = P^3$

From likelihood principle, $\mathcal{L}(P|x)$ infor about P depends on x.

$\mathcal{L}(P|0) = P(0|P) = 1-P$

$\mathcal{L}(P|10) = P(10|P) = P(1-P)$

$\mathcal{L}(P|110) = P(110|P) = P^2(1-P)$

$\mathcal{L}(P|111) = P(111|P) = P^3$

(Sample size = 1)

are independent to the sample size

6.36 T_1 is sufficient

T_2 is minimal sufficient

U is an unbiased estimator of θ. $E(U) = \theta$

$U_1 = E(U|T_1)$ and $U_2 = E(U|T_2)$

(a) Show that $U_2 = E(U_1|T_2)$

T_1 is sufficient

T_2 is minimal sufficient $T_2 = f(T_1)$

$U_1 = E(U|T_1)$

$U_2 = E(U|T_2)$

If T_2 is minimal sufficient

T_1 is sufficient,

$T_2 = f(T_1)$

6.2.13

minimal sufficient

$\Leftarrow \frac{f(x|\theta)}{f(y|\theta)}$ is $\times\ \theta$ if and only if $T(x) = T(y)$

$E(U_1|T_2)$

$= E(E(U|T_1)|T_2)$

$= E(U|T_1)$

$\Leftarrow U_2 = E(U|T_1)$

$\Leftarrow E(U|T_2) = E(U|T_1)$

$E(U|T_2)$ ①

$= \int_U \frac{P(U|T_2)}{P(T_2)} \cdot P(u)$

$E(U|T_1)$ ②

$= \int_U \frac{P(u|T_1)}{P(T_1)} \cdot P(u)$

$\because T_2(x) = f(T_1(x))$ f is the map of the value

$P(T_2(x)) = P(f(T_1(x))) = P(T_1(x)) = P(x)$

that is $P(T_2) = P(T_1)$

$\therefore P(U|T_2) = P(U|T_1)$

$\therefore ① = ②$

Intuition about complete statistics

$T(X)$ is sufficient if $X|T$ X depends on θ

T provide all infor about θ

$V(X)$ is ancillary if $f(V)$ $\times$ depends on θ

V provide no info about θ

$U(X)$ is an unbiased estimator of 0 if $E(U) = 0$

$S(X)$ is a complete static if Any $U = g(S)$ is 0

(b) Use the conditional var formula 4.4.7

Show $\text{Var}\, U_2 \le \text{Var}\, U_1$

$\text{Var}\, X = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$

$E(U_1) = E(E(U|T_1)) = U$

$E(U_2) = E(E(U|T_2)) = U$

$\therefore U_1, U_2$ are unbiased estimator for U

$\because U_2 = E(U_1|T_2)$ (a)

后问用前问结论

$\therefore \text{Var}(U_1) = E(\text{Var}(U_1|T_2)) + \text{Var}(E(U_1|T_2))$

$= E(\text{Var}(U_1|T_2)) + \text{Var}(U_2)$

4.4.7

$\therefore \text{Var}(U_1) - \text{Var}(U_2) = E(\text{Var}(U_1|T_2)) \ge 0$

12

6.37. $(X, Y) \sim f(x, y \mid \theta) = \exp\{-(\theta x + y/\theta)\]$ $\quad x>0,\ y>0$

(a) For an iid sample of size n.

Show that the Fisher info is $I(\theta) = 2n/\theta^2$

$\because f(x, y \mid \theta) = e^{-(\theta x + y/\theta)}$

$\log(f(x, y \mid \theta)) = \underbrace{-(\theta x + y/\theta)} = f$

$f' = -x - y \cdot (-1) \cdot \theta^{(-2)} = y/\theta^2 - x$

$f'' = (y \cdot \theta^{-2})' = -2y \cdot \theta^{-3}$

that is $\log(f(x, y \mid \theta))$ is twice differentiable.

$\therefore$ the Fisher info is

$$I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \log(f(x, y \mid \theta))\right]$$

$$= -E\left[-2y \cdot \theta^{-3}\right]$$

$$= 2E(y/\theta^3)$$

$$= 2Ey/\theta^3 \quad = 2 \cdot \theta/\theta^3 = \frac{2}{\theta^2} \quad \text{for } n=1, \quad n=n \quad I(\theta) = \frac{2n}{\theta^2} \quad \text{IZ}$$

$$Ey \Leftarrow EY = \int_0^\infty y \cdot f(y)\, dy. = \int_0^\infty y \cdot \frac{1}{\theta} \cdot e^{-y/\theta}\, dy. = \frac{1}{\theta} \cdot \int_0^\infty y \cdot e^{-y/\theta}\, dy. = \frac{1}{\theta} \cdot \theta^2 = \theta$$

$$\Leftarrow f_Y(y) = \int_0^\infty f(x, y \mid \theta)\, dx$$

$u = y \qquad v = -\theta e^{-y/\theta}$

$u' = 1 \qquad v' = e^{-y/\theta}$

$$= \int_0^\infty e^{-(\theta x + y/\theta)}\, dx$$

$$= e^{-y/\theta} \cdot \int_0^\infty e^{-\theta x}\, dx$$

$$= e^{-y/\theta} \cdot \left(-\frac{1}{\theta}\right) e^{-\theta x} \Big|_0^\infty$$

$$= \frac{1}{\theta} \cdot e^{-y/\theta} \qquad (\theta > 0)$$

$$\int (-y\theta \cdot e^{-y/\theta})' = \int y \cdot e^{-y/\theta} - \int \theta \cdot e^{-y/\theta}$$

$$\int y \cdot e^{-y/\theta}\, dy = -y\theta \cdot e^{-y/\theta}\Big|_0^\infty + \int \theta \cdot e^{-y/\theta}\, dy$$

$$= 0 + \left(-\theta^2 e^{-y/\theta}\right)\Big|_0^\infty$$

$$= \theta^2$$

(b) For the estimator

$$T = \sqrt{\sum Y_i / \sum X_i} \quad \text{and} \quad U = \sqrt{\sum X_i \sum Y_i}$$

Show that

(i) the info in T alone is $[2n/(2n+1)]I(\theta)$

$\Leftarrow$ Info of T $\quad -E\left[\frac{\partial^2}{\partial \theta^2} \underbrace{\log f(t \mid \theta)}\right] = -E\left[\frac{2n}{\theta^2}\left[\frac{2}{(\frac{t^2}{\theta^2}+1)^2} - 1\right]\right] = \frac{2n}{\theta^2}\left[1 - 2E\left(\frac{1}{\frac{t^2}{\theta^2}+1}\right)^2\right]$ see next page

$\Leftarrow f(t \mid \theta) = F'(t \mid \theta)$

$\Leftarrow F(t \mid \theta) = P(T \le t)$

$\because (2Y_i/\theta,\ 2X_i \cdot \theta)$

$= P(\sqrt{\sum Y_i / \sum X_i} \le t)$

$= P(\sum Y_i / \sum X_i \le t^2)$

$= P\left(\frac{2}{\theta} \frac{\sum Y_i/\theta}{\sum X_i} \le \frac{t^2}{\theta^2}\right)$

$= P\left(\frac{\sum 2Y_i/\theta}{\sum 2X_i \theta} \le \frac{t^2}{\theta^2}\right)$

$\because f(x, y \mid \theta) = e^{-(\theta x + y/\theta)}$

let $x = 2Y_i/\theta$

$y = 2X_i \cdot \theta$

$f(2Y_i/\theta,\ 2X_i\theta \mid \theta) = e^{-\left(\theta \cdot \frac{2Y_i}{\theta} + \frac{2X_i\theta}{\theta}\right)}$

$= e^{-2(Y_i + X_i)}$

$= \underbrace{e^{-2Y_i}} \cdot e^{-2X_i}$

$\therefore 2Y_i/\theta$ and $2X_i\theta \sim$ exponential (1)

$f_{2Y_i}(2y_i) = 1 \cdot e^{-2y_i}$

$\therefore \sum 2Y_i/\theta$ and $\sum 2X_i\theta \sim$ gamma $(1, n)$

$\frac{\sum 2Y_i/\theta}{\sum 2X_i\theta} \sim F(2n, 2n) \quad \sim \chi^2_{2n}$

$\left(\frac{t^2}{\theta^2}\right)^{n-1} \left(\frac{\theta^2}{t^2+\theta^2}\right)^{n-1} \frac{2t\theta^2}{(t^2+\theta^2)^2}$

$\frac{2}{\theta}\left(\frac{t^2}{t^2+\theta^2}\right)^n \left(\frac{\theta^2}{t^2+\theta^2}\right)^n$

$= I_{\frac{t^2}{\theta^2}}\left(\frac{2n}{2}, \frac{2n}{2}\right)$ $\quad 2n \cdot \frac{t^2}{\theta^2} + 2n$

$= I_{\frac{t^2}{t^2+\theta^2}}(n, n)$

$= \frac{B\left(\frac{t^2}{t^2+\theta^2}; n, n\right)}{B(n, n)}$

$= \frac{\int_0^{\frac{t^2}{t^2+\theta^2}} x^{n-1}(1-x)^{n-1}\, dx}{\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)}}$

$\therefore f(t \mid \theta) = \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)} \cdot \frac{2}{\theta}\left(\frac{t^2}{t^2+\theta^2}\right)^n \left(\frac{\theta^2}{t^2+\theta^2}\right)^n$

$$\frac{\partial^2}{\partial\theta^2}\log f(t|\theta) \quad \therefore f(t|\theta) = \underbrace{\frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}}_{A}\cdot\frac{2}{\theta}\left(\frac{t^2}{t^2+\theta^2}\right)^n\left(\frac{\theta^2}{t^2+\theta^2}\right)^n \quad ②$$

$$= \left[\frac{f'}{f}\right]'$$

$$f' = \frac{\partial f(t|\theta)}{\partial\theta} = \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\frac{2}{\theta}\left[\frac{(t^2\theta^2)^n}{(t^2+\theta^2)^{2n}}\right]'$$

$$= \left[2n\cdot\theta^{-1}\cdot\frac{(t^2-\theta^2)}{t^2+\theta^2}\right]'$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\frac{2}{\theta}\cdot t^{2n}\cdot 2n\cdot\frac{\theta^{2n-1}(t^2-\theta^2)}{(t^2+\theta^2)^{2n+1}} \quad ①$$

$$= 2n\cdot B'$$

$$\frac{f'}{f} = \frac{①}{②} = 4n\cdot t^{2n-1}\cdot\theta^{2n-1}\cdot\frac{t^2-\theta^2}{(t^2+\theta^2)^{2n+1}}\times\frac{t}{2}\cdot\frac{(t^2+\theta^2)^{2n}}{t^{2n}\theta^{2n}}$$

$$= 2n\cdot\theta^{-1}\cdot\underbrace{\frac{(t^2-\theta^2)}{t^2+\theta^2}}_{B} \quad \checkmark$$

$$\cancel{t^{2n}}\cdot 2n\cdot\frac{\cancel{\theta^{2n-1}}(t^2-\theta^2)}{(t^2+\theta^2)^{2n+1}}\times\frac{(t^2+\theta^2)^{2n}}{\cancel{t^{2n}}\cancel{\theta^{2n}}}$$

$$= 2n\cdot\frac{t^2-\theta^2}{\theta(t^2+\theta^2)}$$

$$A = \frac{t^{2n}\theta^{2n}}{(t^2+\theta^2)^{2n}} \qquad A'_\theta$$

$$A = t^{2n}\theta^{2n}(t^2+\theta^2)^{-2n}$$

$$A' = \underbrace{t^{2n}\theta^{2n}}_{m}\underbrace{(-2n)(t^2+\theta^2)^{-2n-1}}_{n}\cdot 2\theta + t^{2n}\cdot 2n\cdot\theta^{2n-1}(t^2+\theta^2)^{-2n}$$

$$= t^{2n}\cdot 2n\left(\frac{\theta^{2n-1}}{(t^2+\theta^2)^{2n}} - \frac{2\theta^{2n+1}}{(t^2+\theta^2)^{2n+1}}\right)$$

$$= t^{2n}\cdot 2n\cdot\frac{\theta^{2n-1}t^2+\theta^{2n+1}-2\theta^{2n+1}}{(t^2+\theta^2)^{2n+1}}$$

$$= t^{2n}\cdot 2n\cdot\frac{\theta^{2n-1}t^2-\theta^{2n+1}}{(t^2+\theta^2)^{2n+1}}$$

$$= t^{2n}\cdot 2n\cdot\frac{\theta^{2n-1}(t^2-\theta^2)}{(t^2+\theta^2)^{2n+1}}$$

$$B = \left(\frac{t^2}{\theta}-\theta\right)(t^2+\theta^2)^{-1}$$

$$B' = \left(\frac{t^2}{\theta}-\theta\right)\cdot(-1)(t^2+\theta^2)^{-2}\cdot 2\theta + \left(t^2(-1)\cdot\theta^{-2}-1\right)(t^2+\theta^2)^{-1}$$

$$= \frac{2\theta^2-2t^2}{(t^2+\theta^2)^2} - \frac{1}{\theta^2}$$

$$= \frac{2(1-t^2/\theta^2)}{(t^2+\theta^2)^2}$$

$$= \frac{2\theta^4-2t^2\theta^2-(t^2+\theta^2)^2}{\theta^2(t^2+\theta^2)^2}$$

$$B' = \frac{2\theta^4-2t^2\theta^2-t^4-2t^2\theta^2-\theta^4}{\theta^2(t^2+\theta^2)^2}$$

$$= \frac{1}{\theta^2}\cdot\frac{(2\theta^4-2t^2\theta^2-t^4-2t^2\theta^2-\theta^4)/\theta^4}{(t^2+\theta^2)^2/\theta^4}$$

$$= \frac{1}{\theta^2}\cdot\frac{1-2\frac{t^2}{\theta^2}-\frac{t^4}{\theta^4}-2\frac{t^2}{\theta^2}-1}{(\frac{t^2}{\theta^2}+1)^2}$$

$$= \frac{1}{\theta^2}\cdot\frac{-(\frac{t^4}{\theta^4}+4\frac{t^2}{\theta^2}+2)+2}{(\frac{t^2}{\theta^2}+1)^2}$$

$$= \frac{1}{\theta^2}\cdot\frac{2-(\frac{t^2}{\theta^2}+1)^2}{(\frac{t^2}{\theta^2}+1)^2}$$

$$= \frac{1}{\theta^2}\left[\frac{2}{(\frac{t^2}{\theta^2}+1)^2}-1\right]$$

$$E\left(\frac{1}{t^2/\theta^2+1}\right)^2 = E\left(\frac{1}{F^2_{2n,2n}+1}\right)^2 = E(Y) = \frac{n+1}{2(2n+1)}$$

$\therefore$ Info in T:

$$\frac{2n}{\theta^2}\left[1-2E\left(\frac{1}{t^2/\theta^2+1}\right)^2\right]$$

$$= \frac{2n}{\theta^2}\left[1-2\cdot\frac{n+1}{2(2n+1)}\right]$$

$$= \frac{2n}{\theta^2}\cdot\frac{n}{2n+1}$$

$$= \frac{2n^2}{\theta^2}\cdot\frac{1}{2n+1} \quad \square$$

Let $t^2/\theta = x \qquad x \sim F^2_{2n,2n}$

$$f(x) = \frac{1}{B(n,n)}\left(\frac{2n}{2n}\right)^{2n/2}x^{2n/2-1}\left(1+\frac{2n}{2n}x\right)^{-(2n+2n)/2}$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot x^{n-1}(1+x)^{-2n}$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\frac{x^{n-1}}{(1+x)^{2n}}$$

$x\in(0,+\infty) \quad Y\in(0,1)$

Let $Y = \left(\frac{1}{x+1}\right)^2 \qquad x = \frac{1}{\sqrt{Y}}-1 \qquad x' = (Y^{-\frac{1}{2}})' = (-\frac{1}{2})\cdot Y^{-\frac{3}{2}}$

$$f_Y(y) = f_X\left(\frac{1}{\sqrt{y}}-1\right)\cdot\left(\frac{1}{2}\right)y^{-\frac{3}{2}}$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\frac{(\frac{1}{\sqrt{y}}-1)^{n-1}}{(\frac{1}{\sqrt{y}})^{2n}}\left(\frac{1}{2}\right)y^{-\frac{3}{2}}$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{y}}-1\right)^{n-1}\cdot y^{n-\frac{5}{2}}$$

$$E(Y) = \int_0^1 y\cdot f_Y(y)\,dy$$

$$= \int_0^1\frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\frac{1}{2}\left(\frac{1}{\sqrt{y}}-1\right)^{n-1}\cdot y^{n-\frac{1}{2}}\,dy$$

$$= \frac{\Gamma(2n)}{(\Gamma(n))^2}\cdot\frac{1}{2}\int_0^1\left(\frac{1-\sqrt{y}}{\sqrt{y}}\right)^{n-1}\cdot y^n(\sqrt{y})^{-1}\,dy$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\frac{1}{2}\underbrace{\int_0^1(1-\sqrt{y})^{n-1}(\sqrt{y})^{1-n}y^n(\sqrt{y})^{-1}\,dy}$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)}\cdot\frac{\Gamma(n)\Gamma(n+2)}{\Gamma(2n+2)}\,\alpha$$

$$= \frac{\Gamma(2n)\Gamma(n+2)}{\Gamma(2n+2)\cdot\Gamma(n)} = \frac{(2n-1)!\,(n+1)!}{(2n+1)!\,(n-1)!} = \frac{(n+1)n}{(2n+1)2n} = \frac{n+1}{2(2n+1)}$$

$$\alpha = \int_0^1(1-\sqrt{y})^{n-1}(\sqrt{y})^{-n}(\sqrt{y})^{2n}\,dy$$

$$= \int_0^1(1-\sqrt{y})^{n-1}(\sqrt{y})^n\,dy$$

Let $k = \sqrt{y} \qquad y = k^2 \qquad dy = 2k\,dk$

$\because y\in[0,1]$

$\therefore k\in[0,1]$

$$\therefore \alpha = \int_0^1(1-k)^{n-1}\cdot k^n\cdot 2k\,dk$$

$$= 2\int_0^1(1-k)^{n-1}k^{n+1}\,dk$$

$$= 2\cdot B(n,n+2)$$

$$= 2\cdot\frac{\Gamma(n)\Gamma(n+2)}{\Gamma(n+n+2)}$$

(ii) the information in (T, U) is $I(\theta)$

Info of (T,U) $\quad -E\left[\frac{\partial^2}{\partial\theta^2}\log f(t,u|\theta)\right]$

$T = \sqrt{\Sigma Y_i / \Sigma Z_i}$

$U = \sqrt{\Sigma Z_i \Sigma Y_i}$

$f(t,u|\theta) = F'(t,u|\theta)$

$\Leftarrow F(t,u|\theta) = P(T \le t,\ U \le u)$

$= P\left(\sqrt{\Sigma Y_i/\Sigma Z_i} \le t,\ \sqrt{\Sigma Z_i \Sigma Y_i} \le u\right)$

$= P\left(\Sigma Y_i/\Sigma Z_i \le t^2,\ \Sigma Z_i \Sigma Y_i \le u^2\right)$

卡住

Since T, U both contain ΣZ_i and ΣY_i

can thus first get $f(\Sigma Z_i, \Sigma Y_i) \Rightarrow f(t,u|\theta)$

Let $S = \Sigma Z_i$ $\quad K = \Sigma Y_i$

$\because Z_i \sim$ exponential $(\frac{1}{\theta})$ $\quad \because Y_i \sim$ exponential (θ)

$S = \Sigma Z_i$ $\quad K = \Sigma Y_i$

$\therefore S \sim$ gamma $(n, \frac{1}{\theta})$ $\quad K \sim$ gamma (n, θ)

$f_S(s) = \dfrac{s^{n-1} e^{-s\theta}}{(\frac{1}{\theta})^n \Gamma(n)}$ $\quad f_K(k) = \dfrac{k^{n-1} e^{-\frac{k}{\theta}}}{(\theta)^n \Gamma(n)}$

$f_{S,K}(s,k|\theta) = f_S(s) \cdot f_K(k)$

$= \dfrac{1}{[\Gamma(n)]^2} (s \cdot k)^{n-1} e^{\left(-s\theta - \frac{k}{\theta}\right)}$

$T = \sqrt{k/s}$ $\quad T^2 = k/s$ ①

$U = \sqrt{sk}$ $\quad U^2 = sk$ ②

$\therefore k = \sqrt{① \cdot ②} = TU$

$s = \sqrt{②/①} = \frac{U}{T}$

$|J| = \begin{vmatrix} \frac{\partial k}{\partial T} & \frac{\partial k}{\partial U} \\ \frac{\partial s}{\partial T} & \frac{\partial s}{\partial U} \end{vmatrix} = \begin{vmatrix} U & T \\ \frac{U}{T^2} & \frac{1}{T} \end{vmatrix}$

$= 2\frac{U}{T}$

$\therefore f_{T,U}(t,u|\theta) = f_{S,K}\left(\frac{u}{t}, tu|\theta\right) \cdot |J|$

$= \dfrac{1}{[\Gamma(n)]^2} \cdot 2 \cdot \frac{u}{t} \cdot (u^2)^{n-1} e^{\left(-\frac{u}{t}\theta - \frac{tu}{\theta}\right)}$

$\therefore \log f_{T,U}(t,u|\theta) = \log \dfrac{2}{[\Gamma(n)]^2} \frac{1}{t} \cdot u^{2n-1} e^{\left(-\frac{u}{t}\theta - \frac{tu}{\theta}\right)}$

$= \log 2 \cdot u^{2n-1} - \log [\Gamma(n)]^2 t + \left(-\frac{u}{t}\theta - \frac{tu}{\theta}\right)$

$\frac{\partial}{\partial\theta} \log f_{T,U}(t,u|\theta) = -\frac{u}{t} - tu(-1)\cdot\theta^{(-2)}$

$\frac{\partial^2}{\partial^2\theta} \log f_{T,U}(t,u|\theta) = tu(-2)\cdot\theta^{(-3)}$

$-E\left(\frac{\partial^2}{\partial^2\theta} \log f_{T,U}(t,u|\theta)\right) = 2E\left(\frac{tu}{\theta^3}\right) = 2E\left(\frac{k}{\theta^2}\right)$ $\quad K \sim$ gamma (n, θ)

$= 2n\theta/\theta^3 = \frac{2n}{\theta^2} = I(\theta)$ $\square$

$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$

$f_K(k) = \dfrac{k^{n-1} e^{-\frac{k}{\theta}}}{(\theta)^n \Gamma(n)}$

$E(k) = \int_0^\infty k \cdot f_K(k)\, dk = \frac{1}{\Gamma(n)} \frac{1}{\theta^n} \int_0^\infty k^n e^{-\frac{k}{\theta}}\, dk$

$= \frac{1}{(n-1)!} \frac{1}{\theta^n} \int_0^\infty \left(\frac{k}{\theta}\theta\right)^n e^{-\frac{k}{\theta}}\, dk$

$= \frac{\theta^{n+1}}{(n-1)!} \frac{1}{\theta^n} \int_0^\infty \left(\frac{k}{\theta}\right)^n e^{-\frac{k}{\theta}} \theta\, d\frac{k}{\theta}$

$= \frac{\theta}{(n-1)!} \Gamma(n+1)$

$= n\theta$

(iii) (T,U) is jointly sufficient but not complete (for θ)

1° $\because f(\underline{z}, \underline{y}|\theta) = e^{-\theta \Sigma z_i - \Sigma y_i/\theta}$ $\quad \underline{z} = z_1, z_2, \cdots, z_n$ $\quad \underline{y} = y_1, y_2, \cdots, y_n$

$\therefore (S,K)$ is sufficient for θ

$\because (T,U)$ and (S,K) are one-to-one map

$\therefore (T,U)$ is sufficient for θ

6.2.25. Complete stat for exponential family.

$f(x|\theta) = h(x) \cdot c(\theta) \cdot \exp\left(\sum_{j=1}^{k} q_j(\theta) t_j(x)\right)$, then $\xrightarrow{\text{open set in } R^k}$

$T(\underline{z}) = \left(\sum_{i=1}^{n} t_1(z_i), \sum_{i=1}^{n} t_2(z_i), \cdots \sum_{i=1}^{n} t_k(z_i)\right)$ is complete stat

for $\theta = (\theta_1, \theta_2, \cdots \theta_k)$

Summary

Possion dis, Exponential dis & Gamma

$X \sim$ Possion (λt)

$X :=$ # email arrive with in $(\alpha, \alpha+t]$ $\alpha \in (0, \infty)$

$W \sim$ Exponential (λw)

$W^i :=$ the time between receiving email $(i-1, i)$

$P(W_i \le w) = 1 - e^{-\lambda w}$

didn't receive email within time w

$f(w) = F'(W \le w_i) = -e^{-\lambda w_i} \cdot (-\lambda) = \lambda \cdot e^{-\lambda w_i}$

$Y \sim$ Gamma

$Y_j :=$ the time that will take when receive the jth email

$Y_j = \sum_{i=1}^{j} w^i$ $\quad \therefore Y_j$ is the samup of w^i

$Y \sim$ gamma $(j, \frac{1}{\theta})$

$\Leftarrow$ (T,U) is complete $\begin{cases} E[g(T,U)] = 0 \\ g[(T,U)|\theta] = 0 \end{cases}$

$T = \sqrt{k/s}$ $\quad T^2 = k/s$ ① $\quad$ $k \sim gamma(n,\theta)$ $\quad$ $S \sim gamma(n, \frac{1}{\theta})$

$U = \sqrt{s \cdot k}$ $\quad U^2 = s \cdot k$ ② $\quad$ $Ek = n\theta$ $\quad$ $ES = \frac{n}{\theta}$

$T^2 = k/s$ $\quad E(U^2) = E(SK) = E(S) \cdot E(K) = n^2$

$U^2 = S \cdot K$

Assume $g(T,U) = U^2 - n^2$

$E[g(T,U)] = E(g(T,U)] - n^2 = 0$

but $g[(T,U)|\theta]$ doesn't depend on θ, and doesn't equal to 0

$\therefore$ is not complete. $\square$

6.38. In Definition 6.4.2, show that (iii) is implied by (i) and (ii).

(b) For the estimator

$T = \sqrt{\sum Y_i / \sum X_i}$ and $U = \sqrt{\sum X_i \sum Y_i}$

Show that

(i) the info in T alone is $[2n/(2n+1)]I(\theta)$

(ii) the information in (T,U) is $I(\theta)$

(iii) (T,U) is jointly sufficient but not complete (for θ)

Def 6.4.2 $\{g(x) : g \in G\}$ is called a group of transformations

If (i) $g'(g(x)) = x$

(ii) exist $g'' \in G$ $\quad g'(g(x)) = g''(x)$

From (ii) and (i), we have

$g_1(T,U) = \theta$ $\qquad g_2(T) = \theta$

$\therefore$ Exists $g'(g(T,U)) = (T,U)$ $\qquad$ Assume $E[g(T,U)] = 0$

Exists $g_3(T,U) = g_2(g_1(T,U))$ $\qquad$ $g_1(T,U) + g_2(T,U) + g_3(T,U) = 0$

Assume that $g_2(\quad) = g_1'$ $\qquad$ $g_1(T,U) + (T,U) + (T,U) = 0$

$g_2(g_1(T,U)) = (T,U) = g_3(T,U)$

$g_1(T,U) = -2(T,U) \neq 0$

$\therefore$ (T,U) is not complete. $\square$

6.39. To estimate θ by $T(x) = 0.5x + 0.5(100)$. Celsius

X is the single observation.

$T^*(y) = 0.5y + 0.5(212)$ Fahrenheit

(a) Can convert Fahrenheit to Celsius by $\frac{5}{9}(T^*(y) - 32)$.

Show that $\frac{5}{9}(T^*(y) - 32) = T(x)$

$\frac{5}{9}(T^*(y) - 32) = \frac{5}{9}(0.5y + 106 - 32)$

$= \frac{5}{9}(\frac{1}{2}y + 74)$ ②

$\because \frac{5}{9}(y - 32) = x$ $\quad \therefore y = \frac{9}{5}x + 32$ ①

① → ②:

$\frac{5}{9}(\frac{1}{2}(\frac{9}{5}x + 32) + 74)$

$= \frac{5}{9}(\frac{9}{10}x + 16 + 74)$

$= \frac{1}{2}x + 50 = T(x)$

(b) Formal invariance require $T(x) = T^*(x)$ for all x.
Show that the estimator we have defined do not satisfy this.

$T(x) = 0.5x + 50$ $\quad \therefore T(x) \neq T^*(x)$

$T^*(x) = 0.5x + 106$

✓ Measurement equivariant 测量等变.
Same data $\Rightarrow$ same answer (estimate θ)

X Formal invariance. 形式不变.
Same value $(T(x) = T^*(x))$

6.40. Let $Z_1, \cdots, Z_n$ be iid observations from a location-scale family.
Let $T_1(Z_1, \cdots, Z_n)$ and $T_2(Z_1, \cdots, Z_n)$ be two statistics that satisfy:

$$T_i(ax_1+b, \cdots, ax_n+b) = a \cdot T_i(x_1, \cdots, x_n)$$

for all values of $x_1, \cdots, x_n$ and b and for any $a>0$.

(a) Show that T_1/T_2 is an ancillary statistic

T_1/T_2 is an ancillary statistic.

$\Leftarrow$ $f(T_1/T_2)$ doesn't depends on θ

Let $T_1(a_1x_1+b_1, \cdots a_1x_n+b_1) = a_1 \cdot T_1(x_1, \cdots x_n)$

$T_2(a_2x_1+b_2, \cdots a_2x_n+b_2) = a_2 \cdot T_2(x_1, \cdots x_n)$

$$\frac{T_1(x_1, \cdots x_n)}{T_2(x_1, \cdots x_n)} = \frac{\frac{1}{a_1} \cdot T_1(a_1x_1+b_1, \cdots, a_1x_n+b_1)}{\frac{1}{a_2} \cdot T_2(a_2x_1+b_2, \cdots, a_2x_n+b_2)}$$

$$= \frac{a_2}{a_1} \cdot \frac{T_1(a_1x_1+b_1, \cdots, a_1x_n+b_1)}{T_2(a_2x_1+b_2, \cdots a_2x_n+b_2)}$$

$\because x_1, x_2, \cdots x_n$ are location scale family

$\therefore$ Let $z_1 = x_1 - \theta, \cdots z_n = x_n - \theta$

$a_1 = 1$, let

$T_1(z_1, \cdots, z_n) = 1\, T_1(z_1, \cdots z_n)$ ①

$T_2(2z_1, \cdots 2z_n) = 2\, T_2(z_1, \cdots z_n)$ ②

$\frac{①}{②} = \frac{1}{2}$ doesn't depends on θ $\therefore$ T_1/T_2 is an ancillary statistic.

Let $z_1, \cdots z_n \sim$ standard pdf $f(z)$

$Z_1 = \sigma z_1 + \mu \cdots Z_n = \sigma z_n + \mu$

$$\frac{T(Z_1, \cdots, Z_n)}{T_2(Z_1, \cdots, Z_n)} = \frac{T_1(\sigma z_1+\mu, \cdots, \sigma z_n+\mu)}{T_2(\sigma z_1+\mu, \cdots, \sigma z_n+\mu)}$$

$$= \frac{\sigma\, T_1(z_1, z_2, \cdots z_n)}{\sigma\, T_2(z_1, z_2, \cdots z_n)}$$

$$= \frac{T_1(z_1, z_2, \cdots, z_n)}{T_2(z_1, z_2, \cdots, z_n)}$$

which is independs with σ or μ.
$\therefore$ is an ancillary statistic. a

(b) Let R be the sample range and S be the sample standard deviation. Verify that R and S satisfy the above condition so that R/S is an ancillary statistic.

1° $R(x_1, \cdots, x_n) = x_{(n)} - x_{(1)}$

$\because a>0$

$\therefore \max(ax_1+b, \cdots, ax_n+b) = ax_{(n)}+b$

$\min(ax_1+b, \cdots, ax_n+b) = ax_{(1)}+b$

$\therefore R(ax_1+b, \cdots, ax_n+b) = ax_{(n)}+b - ax_{(1)}+b = a(x_{(n)} - x_{(1)}) = a\,R(x_1, \cdots, x_n)$

$\therefore$ R satisfy the condition

2° For S

Let $m = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$

$S(x_1, \cdots, x_n) = \sqrt{\frac{\sum (x_i - m)^2}{n}}$

Let $n = \frac{1}{n}(ax_1+b + \cdots + ax_n+b) = am+b$

$S(ax_1+b, \cdots ax_n+b) = \sqrt{\frac{\sum (ax_i+b-m)^2}{n}} = a\sqrt{\frac{\sum (x_i-m)^2}{n}} = a \cdot S(x_1, x_2, \cdots x_n)$

$\therefore$ S satisfy the condition

$\therefore$ R/S is an ancillary sta. □

6.41. To estimate μ, let $T(\underline{x})$ be the estimate used if $\underline{X} = \underline{x}$.
If $g_a(\underline{X}) = Y = y$ is observed, then let $T^*(y)$ be the estimate of $\mu + a$, the mean of each Y_i.
If $\mu + a$ is estimated by $T^*(y)$, then μ would be estimated by $T^*(y) - a$.

(a) Show that measurement equivariance requires that $T(\underline{x}) = T^*(y) - a$ for all $\underline{x} = (x_1, x_2, \cdots, x_n)$ and all a.

思路: $g_a(\underline{x})$ is a function of $\underline{x}$.

$T^*(y)$ is to estimate $\mu + a$.

measurement equivariance: $\mu | y = \mu | \underline{x}$

that is $T^*(x_1+a, x_2+a, \cdots, x_n+a) = T^*(y)$

$T^*(x_1+a, x_2+a, \cdots, x_n+a) - a = T^*(y) - a = T(\underline{x})$

(b) Show that formal invariance requires that $T(\underline{x}) = T^*(\underline{x})$ and hence the Equivariance Principle requires that $T(x_1, \cdots x_n) + a = T(x_1+a, \cdots x_n+a)$ for all $(x_1, \cdots, x_n)$ and a

思路. Formal invariance requires: $T(\underline{x}) = T^*(\underline{x})$

from (a): $T(\underline{x}) = T^*(x_1+a, x_2+a, \cdots x_n+a) - a$

$\therefore T(\underline{x}) + a = T^*(x_1+a, x_2+a, \cdots, x_n+a)$

又 $\because T(\underline{x}) = T^*(\underline{x})$

$\therefore T^*(x_1+a, \cdots x_n+a) = T(x_1+a, \cdots x_n+a)$

$\therefore T(\underline{x}) + a = T(x_1+a, \cdots x_n+a)$ □

(c) If $Z_1, \cdots, Z_n$ are iid $f(x-\theta)$, show that, as long as $E_0 Z_1 = 0$, the estimator $W(Z_1, \cdots, Z_n) = \bar{Z}$ is equivariant for θ and satisfies $E_\theta W = \theta$

思路: (a) $T^*(x_1+a, \cdots, x_n+a) = T^*(y) = T(z)$

(b) $T(x_1+a, \cdots, x_n+a) - a = T(x_1, x_2, \cdots, x_n)$

$\because W(Z_1, \cdots, Z_n) = \bar{Z}$

$\therefore W(Z_1, \cdots, Z_n) = \frac{1}{n}(Z_1 + Z_2 + \cdots + Z_n)$

$W(Z_1-\theta, \cdots, Z_n-\theta) = \frac{1}{n}(Z_1-\theta + \cdots + Z_n-\theta)$

From (b)

$W(Z_1-\theta, \cdots, Z_n-\theta) + \theta = W(Z_1, Z_2, \cdots, Z_n) = W(Z)$

$E_\theta(W) = E_\theta(W(Z)|\theta)$

$= E_\theta(W(Z-\theta)+\theta|\theta) = E_\theta(W(Y)+\theta|\theta)$ ①

$\because E_0(Z_1) = 0$

that is $E(Z_1|\theta=0) = x_1 \cdot f(x_1) = 0$

Assume $Y_i = Z_i - \theta$ $\because Z_i \sim f(x_i-\theta)$

$Z_i = Y_i + \theta$ $\therefore Y_i \sim f(y_i)$ $E(Y_i) = 0$

Let $Y = (Y_1, Y_2, \cdots Y_n) = (Z_1-\theta, Z_2-\theta, \cdots, Z_n-\theta)$

$Z = (Z_1, Z_2, \cdots Z_n)$

① $= E_\theta(W(Y)+\theta|\theta)$

$= E(Y) + E(\theta)$

$= E(\theta)$ □

6.42. Random sample $Z_1, \cdots, Z_n \sim \frac{1}{\sigma} f((x-\theta)/\sigma)$, a location-scale pdf. We want to estimate θ.

$G_1 = \{g_{a,c}(z) : -\infty < a < \infty, c > 0\}$. $G_2 = \{g_a(z) : -\infty < a < \infty\}$

$g_{a,c}(z) = (cx_1+a, \cdots, cx_n+a)$ $g_a(z) = (x_1+a, \cdots x_n+a)$

(a) Show that estimators of $W(z) = \bar{x}+k$ $(k \neq 0, \text{cons})$ is equivariant to G_2, but not G_1.

思路: $c_\theta|W(z) = \theta|G_2(z)$ ①

$\theta|W(z) \neq \theta|G_1(z)$ ②

① : $W(z) = \bar{z}+k = \frac{1}{n}(x_1+x_2+\cdots+x_n)+k$

$= \frac{1}{n}(x_1+k, x_2+k, \cdots, x_n+k)$ $k \neq 0$

$G_2(z) = g_a(x_1+a, \cdots, x_n+a)$

assume $a=k$ $g_k(z) = (x_1+k, \cdots x_n+k)$

$G_2 = \{g_k(z), k \neq 0\}$ $\therefore W(z) = \frac{1}{n} G_2(z)$

$\therefore W(z)$ is equivariant to G_2 □

② $G_1(z) = g_{ac}(cx_1+a, \cdots cx_n+a)$

$= g_{ac}(c(x_1+\frac{a}{c}, \cdots x_n+\frac{a}{c}))$

Let $\frac{a}{c} = k$

$g_{ac}(z) = c(x_1+k, \cdots x_n+k)$

$\therefore W(z) = \frac{c}{n}(x_1+k, \cdots x_n+k) = \frac{c}{n} G_1(z)$

depends on c, $\therefore W(z)$ is not equivariant to G_1 □

(b) For each group, under what conditions does an equivarant estimator W satisfy $E_\theta W = \theta$, that is unbiased for estimating θ.

$W(Z) = \bar{Z} + k = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) + k$

$= \frac{1}{n}(x_1 + k + x_2 + k, \cdots + x_n + k) \quad k \neq 0$

Assume $Y_i = Z_i - \theta \quad \sim \frac{1}{\sigma} f(y_i / \sigma) \quad E(Y) = 0$

$Y = (Y_1, \cdots, Y_n)$

$W(Z - \theta) = W(Z) - \theta$

$W(Z - \theta) + \theta = W(Z)$

$W(Y) + \theta = W(Z)$

$\therefore E_\theta[W(Z)] = E_\theta[W(Y) + \theta]$

$= E_\theta[W(Y)] + \theta$

If $E_\theta W = \theta$, then $E_\theta[W(Y)]$ has to be 0

$E_\theta(W(Y)) = E\left(\frac{1}{n}(y_1 + k + y_2 + k + \cdots + y_n + k)\right) = 0$

$E(Y) + E(k) = 0 \quad \because E(Y) = 0$

$\therefore$ As long as $k = 0$ $E_\theta W$ is an unbiased estimator for θ.

(b) Show that the larger class of estimators of σ^2

$$W(X_1, \cdots, X_n) = \phi\left(\frac{\bar{X}}{S}\right) S^2$$

$\phi(x)$ is a function, are equivariant w.r.t G_3 but not G_1, G_2 unless $\phi(x)$ is a constant.

证明: $g_c(\underline{x}) = (cx_1, \cdots cx_n)$ $\quad \bar{g}_c(\underline{x}) = c^2\sigma^2$

$$\phi\left(\frac{\bar{x}}{s}\right) S^2_{(\underline{x})} = \phi\left(\bar{x} \Big/ \sqrt{\tfrac{1}{n-1}\Sigma(x_i-\bar{x})^2}\right) \cdot \frac{1}{n-1}\cdot\Sigma(x_i-\bar{x})^2$$

$$S = \sqrt{\tfrac{1}{n-1}\Sigma(x_i-\bar{x})^2}$$

$$\phi\left(\frac{\overline{cx}}{s}\right) S^2_{(c\underline{x})} \Leftarrow$$

$$cx_1 + cx_2 + \cdots + cx_n = c(x_1 + x_2 + \cdots + x_n)$$

$$\overline{cx} = c\cdot\bar{x}$$

$$S^2_{(c\underline{x})} = \frac{1}{n-1}\sum_{i=1}^{n}(cx_i - c\cdot\bar{x})^2$$

$$= \frac{1}{n-1} c^2 \cdot \Sigma(x_i-\bar{x})^2$$

$$= c^2 \cdot S^2_{(\underline{x})}$$

$$\sqrt{S^2_{(c\underline{x})}} = S_{(c\underline{x})} = c\cdot S_{(\underline{x})}$$

$$\therefore \phi\left(\frac{\overline{cx}}{s}\right) S^2_{(c\underline{x})} = \phi\left(c\cdot\bar{x}/c\cdot S_{(\underline{x})}\right)\cdot c^2 S^2_{(\underline{x})}$$

$$\hat{W}(c\underline{x}) = \qquad = c^2\,\phi\left(\frac{\bar{x}}{s}\right)\cdot S^2_{(\underline{x})}$$

$\therefore$ W is equivariant w.r.t G_3 $\square$

For G_1

$$\bar{g}_{ac}(\underline{x}) = c^2\sigma^2$$

$$\phi\left(\frac{\overline{c\underline{x}+a}}{S_{(c\underline{x}+a)}}\right)\cdot S^2_{(c\underline{x}+a)} \Leftarrow$$

$$\overline{c\underline{x}+a} = c\cdot\bar{x}+a$$

$$S^2_{(c\underline{x}+a)} = \frac{1}{n-1}\cdot\sum_{i=1}^{n}\left[cx_i + a - (c\bar{x}+a)\right]^2$$

$$= \frac{1}{n-1}\cdot c^2\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$= c^2\cdot S^2_{(\underline{x})}$$

$$\therefore S_{(c\underline{x}+a)} = c\cdot S_{(\underline{x})}$$

$$\phi\left(\frac{\overline{c\underline{x}+a}}{S_{(c\underline{x}+a)}}\right)\cdot S^2_{(c\underline{x}+a)}$$

$$= \phi\left(\frac{c\bar{x}+a}{c\cdot S_{(\underline{x})}}\right)\cdot c^2 S^2_{(\underline{x})}$$

$\therefore$ is not equivariant, unless

$$\phi\left(\frac{c\bar{x}+a}{c\cdot S_{(\underline{x})}}\right) = \phi\left(\frac{\bar{x}}{S_{(\underline{x})}}\right)$$

which means $\phi(\alpha)$ is a constant.

For G_2, similar:

$$W(\underline{x}+a) = \phi\left(\frac{\overline{\underline{x}+a}}{S_{(\underline{x})}}\right)\cdot S^2_{(\underline{x}+a)} \Leftarrow$$

$$\overline{\underline{x}+a} = \bar{x}+a$$

$$S^2_{(\underline{x}+a)} = S^2_{(\underline{x})}$$

$$\therefore W(\underline{x}+a) = \phi\left(\frac{\bar{x}+a}{S_{(\underline{x})}}\right)\cdot S^2_{(\underline{x})}$$

Only when $\phi(\alpha)$ is constant, can $\phi\left(\frac{\bar{x}+a}{S_{(\underline{x})}}\right) = \phi\left(\frac{\bar{x}}{S_{(\underline{x})}}\right)$

$\therefore$ W is equivariant w.r.t G_2. $\square$