Sec 5

?

5.8 (b) (c)
5.19 (c)

5.1. $P = 1\%$ N? $P(\text{Containing a color-blind}) \geq 0.95$

$P(\text{Contain}) \geq 0.95$ equals to $P(\text{Doesn't contain}) < 0.05$

$P(\text{Doesn't contain}) = \underset{0.99}{(1-P)}^N < 0.05$ $N \geq 299$

5.2. X_i : annual rainfall at a given location $X_i \sim f(x)$

(a) Find the distribution of number of years until the first year's rainfall X_1 is exceeded for the first time.

X_1 : rainfall of the first year

X_2 : rainfall of the second year

Let $1-p = P(X_i \leq x_1)_{i \neq 1} = \int_0^{x_1} f(x)dx = F(x_1)$

Let the number of years that X_1 is exceeded be n

$N \sim$ negative binomial (N, P)

$p = 1 - F(x_1)$ $n-2$ times failure last time success

$f_N(n) = \int_0^\infty F(x_1)^{n-2} \cdot (1 - F(x_1)) \cdot \underbrace{f(x_1)}_{\text{first time is } x_1} dx_1$ $\square$

$\int_0^\infty F(x_1)^{n-2}(1-F(x_1))f(x)\,dx_1$

$= \int_0^\infty F(x_1)^{n-2} f(x_1)dx_1 - \int_0^\infty F(x_1)^{n-1} f(x_1)dx_1$

$= \frac{1}{n-1}F(x_1)^{n-1}\Big|_0^\infty - \frac{1}{n}F(x_1)^n\Big|_0^\infty$

$= \frac{1}{n-1} - \frac{1}{n}$

$= \frac{1}{n(n-1)}$

(b) Show that the mean number of years until X_1 is exceeded for the first time is infinite.

$$EN = \sum_{n=2}^{\infty} n \cdot f_N(n)$$

$$= \sum_{n=2}^{\infty} n \cdot \frac{1}{n(n-1)}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n-1}$$

$$= \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}} + \cdots$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$= 1 + 1 + 1 + \cdots$$

$\therefore EN$ is infinite $\square$

5.3. $X_1, X_2 \cdots X_n$ are iid. F_X, $EX_i = \mu$

Define $Y_1, Y_2, \cdots, Y_n$ by

$$Y_i = \begin{cases} 1 & \text{if } X_i > \mu \\ 0 & \text{if } X_i \leq \mu \end{cases}$$

Find the distribution of $\sum_{i=1}^n Y_i$

$1 - P = P(X_i > \mu) = F(\mu)$

Let $k = \sum_{i=1}^n Y_i \in [0, n]$ $\sim$ bionomial dis

$f(k, n, p) = \binom{n}{k} p^k (1-p)^{n-k}$

$= \binom{n}{k} [1 - F(\mu)]^k F(\mu)^{n-k}$ $\square$

5.4. $Z_i | p \sim$ iid Bernoulli(p) $\quad p \sim U(0,1)$ $\quad Z_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$

(a) $P(Z_1 = x_1, \cdots, Z_k = x_k) = \int_0^1 p^t (1-p)^{k-t} dp = \frac{t!\,(k-t)!}{(k+1)!}$ $\quad t = \sum_1^k x_i$

$\because$ Z_i are independent

$\therefore P(Z_1 = x_1, \cdots, Z_k = x_k) = \int_0^1 p^t (1-p)^{k-t} dp$ $\quad t = \sum_1^k x_i$

Beta dis

$f(p, t+1, k-t+1) = \frac{\Gamma(k+2)}{\Gamma(t)\Gamma(k-t)} p^t (1-p)^{k-t}$

$\therefore \int_0^1 \frac{\Gamma(k+1)\Gamma(k-t+1)}{\Gamma(k+2)} f(p, t+1, k-t+1)\, dp$ $\quad \int_0^1 \left(\frac{p}{1-p}\right)^t (1-p)^k dp$

$= \frac{\Gamma(k+1)\Gamma(k-t+1)}{\Gamma(k+2)}$ $\quad 1$ $\quad$ Beta dis

$= \frac{k!\,(k-t)!}{(k+1)!}$

(b). Show that $P(Z_1 = x_1, \cdots Z_n = x_n) \neq \prod_{i=1}^n P(Z_i = x_i)$

$P(Z_i = x_i) = \int_0^1 p^{x_i} (1-p)^{1-x_i} dp$

$f(p, x_i+1, 1-x_i+1) = \frac{\Gamma(3)}{\Gamma(x_i+1)\Gamma(1-x_i+1)} p^{x_i} (1-p)^{1-x_i}$

$\therefore P(Z_i = x_i) = \frac{x_i!\,(1-x_i)!}{3!}$

$x_i = 0 \quad P(Z_i = 0) = \frac{0!\,(1)!}{3!} = \frac{1}{6}$

$x_i = 1 \quad P(Z_i = 1) = \frac{1!\,0!}{3!} = \frac{1}{6}$

$\prod_{i=1}^n P(Z_i = x_i) = \left(\frac{1}{6}\right)^k \neq P(Z_1 = x_1, \cdots Z_n = x_n)$ $\quad \square$

5.5. $Z_1, \cdots, Z_n$ be iid $\sim f_Z(x)$ $\quad \bar{X} = \bar{Z}_1 + \bar{Z}_2 + \cdots \bar{Z}_n)^{\frac{1}{n}}$

Show that $f_{\bar{Z}}(x) = n f_{Z_1 + \cdots + Z_n}(nx)$

$\because$ $Z_1, \cdots, Z_n$ are iid $\sim f_Z(x)$

$\therefore P(Z_1 = x_1, Z_2 = x_2 \cdots Z_n = x_n) = f_Z(x_1) \cdot f_Z(x_2) \cdots f_Z(x_n)$

Let $x = (x_1 + x_2 + \cdots + x_n)^{\frac{1}{n}}$

$nx = x_1 + x_2 + \cdots + x_n$

Assume $x_1 + \cdots + x_n = S \sim f_{Z_1 + \cdots + Z_n}(S)$ $\quad S = nx$

Then $f_{\bar{Z}}(x) = f_{Z_1 + \cdots + Z_n}(S) \frac{dS}{dx}$

$= n f_{Z_1 + \cdots + Z_n}(nx)$ $\quad \square$

5.6. $Z \sim f_Z(x)$ $\quad Y \sim f_Y(y)$ $\quad Z, Y$ are independent

(a) $Z = X - Y$ $\quad f_{Z,Z}(z, x) = f_{X,Y}(x, x-z)|J| = \int_{-\infty}^{\infty} f_Z(x) \cdot f_Y(x-z)\, dx$ $\quad \square$

$\because Z = X - Y$

$\therefore Y = X - Z$ $\quad |J| = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = |-1| = 1$

(b) $Z = XY$ $\quad f_{Z,Z}(z, x) = f_{X,Y}(x, \frac{z}{x}) \cdot |J| = \int_{-\infty}^{\infty} f_Z(x) f_Y(\frac{z}{x}) \frac{1}{|x|} dx$ $\quad \square$

$\because Z = XY$

$\therefore Y = \frac{z}{x}$ $\quad |J| = \begin{vmatrix} 1 & 0 \\ -\frac{z}{x^2} & \frac{1}{x} \end{vmatrix} = \left|\frac{1}{x}\right|$ $\quad |J| = \begin{vmatrix} y & z \\ 0 & 1 \end{vmatrix} = |y|$

(c) $Z = X/Y$ $\quad f_{Z,Y}(z, y) = f_{X,Y}(zy, y) \cdot |J| = \int_{-\infty}^{\infty} f_Z(zy) f_Y(y) |y|\, dy$ $\quad \square$

$Z = XY$

5.7. (a) Find the constants A, B, C and D that satisfy

$$\frac{1}{1+(\omega/\sigma)^2}\,\frac{1}{1+((z-\omega)/\tau)^2}$$

$$= \frac{A\omega}{1+(\omega/\sigma)^2} + \frac{B}{1+(\omega/\sigma)^2} - \frac{C\omega}{1+((z-\omega)/\tau)^2} - \frac{D}{1+((z-\omega)/\tau)^2}$$

$$\frac{A\omega+B}{1+(\omega/\sigma)^2} - \frac{C\omega+D}{1+((z-\omega)/\tau)^2}$$

令: $A\omega+B-C\omega-D + (A\omega+B)\cdot\frac{(z-\omega)^2}{\tau^2} - (C\omega+D)\cdot\frac{\omega^2}{\sigma^2}$

$$(A-C)\omega+B-D + \frac{(A\omega+B)(z-2z\omega+\omega^2)}{\tau^2} - \frac{(C\omega+D)\omega^2}{\sigma^2}$$

$$= \frac{1}{\tau^2\sigma^2}\Big[\tau^2\sigma^2(A-C)\cdot\omega + \tau^2\sigma^2(B-D) + \sigma^2 Az\omega - 2\sigma^2 Az\omega^2 + \sigma^2 A\omega^3 + \sigma^2 Bz - 2\sigma^2 Bz\omega + \sigma^2 B\omega^2 - \tau^2 C\omega^3 - \tau^2 D\omega^2\Big]$$

$\sigma^2 A\omega + \sigma^2 B$

$\tau^2\omega^2(C\omega+D)$

ω^3: $\sigma^2 A = \tau^2 C$ ①

ω^2: $-2\sigma^2 Az + \sigma^2 B - \tau^2 D = 0$ ②

ω: $\tau^2\sigma^2(A-C) + \sigma^2 Az - 2\sigma^2 Bz = 0$ ③

ω^0: $\tau^2\sigma^2(B-D) + \sigma^2 Bz = \tau^2\sigma^2$ ④

From ①: $C = \frac{\sigma^2 A}{\tau^2}$

From ④: $B = \frac{\tau^2\sigma^2(1+D)}{\tau^2\sigma^2+z\sigma^2}$

$\therefore A-C = A\left(1-\frac{\sigma^2}{\tau^2}\right) = A\cdot\frac{\tau^2-\sigma^2}{\tau^2}$

$1+D = \frac{\tau^2\cancel{\sigma^2}+z\cancel{\sigma^2}}{\tau^2\cancel{\sigma^2}}B$

$D = \frac{\tau^2+z}{\tau^2}B - 1$

$\therefore$ ③: $\cancel{\tau^2}\cancel{\sigma^2}\cdot A\cdot\frac{\tau^2-\sigma^2}{\cancel{\tau^2}} + \cancel{\sigma^2}Az - 2\cancel{\sigma^2}Bz = 0$

$A(\tau^2-\sigma^2+z) = 2Bz$

$B = \frac{\tau^2-\sigma^2+z}{2z}A$ ⑥

Let $A = 2z$ $B = \tau^2-\sigma^2+z$

$C = \frac{2\sigma^2 z}{\tau^2}$

$D = \frac{(\tau^2+z)(\tau^2+z-\sigma^2)}{\tau^2} - 1$

(b) Using the facts that $\int \frac{t}{1+t^2}\,dt = \frac{1}{2}\log(1+t^2)+c$

and $\int \frac{1}{1+t^2}\,dt = \arctan(t) + C$

evaluate

(5.2.4) P216 (245) $\bar{Z} = \frac{1}{n}(Z_1 + \cdots + Z_n)$ $Z_i \sim \text{Cauchy}(0,1)$

let $U \sim \text{Cauchy}(0,\sigma)$ $V \sim \text{Cauchy}(0,\tau)$

$\bar{Z} \sim \text{Cauchy}(0,1)$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\pi\sigma}\cdot\frac{1}{1+(w/\sigma)^2}\,\frac{1}{\pi\tau}\,\frac{1}{1+(z-w)/\tau)^2}\,dw$$

(5.2.5) $$f_Z(z) = \frac{1}{\pi(\sigma+\tau)}\,\frac{1}{1+(z/(\sigma+\tau))^2}$$

Aim : proof1 $f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\pi\sigma}\cdot\frac{1}{1+(w/\sigma)^2}\,\frac{1}{\pi\tau}\,\frac{1}{1+((z-w)/\tau)^2}\,dw$

Proof2 $\frac{1}{\pi(\sigma+\tau)}\,\frac{1}{1+(z/(\sigma+\tau))^2} = \int \frac{1}{\pi\sigma}\,\frac{1}{1+(w/\sigma)^2}\,\frac{1}{\pi\tau}\,\frac{1}{1+((z-w)/\tau)^2}\,dw$

Proof 2: 结合 (a)

$$\int_{-\infty}^{\infty} \frac{B}{1+(\frac{w}{\sigma})^2}\,dw = B\cdot\arctan\left(\frac{w}{\sigma}\right)\Big|_{-\infty}^{\infty} = B\left[\frac{\pi\sigma}{2} - \left(-\frac{\pi\sigma}{2}\right)\right] = B\pi\sigma$$

Similar: $\int_{-\infty}^{\infty} \frac{D}{1+(\frac{w-z}{\tau})}\,dw = D\pi\tau$

$d\frac{w}{\sigma} = \frac{1}{\sigma}\,dw$ $\quad \sigma\, d\frac{w}{\sigma} = dw$

$d\frac{w-z}{\tau} = \frac{1}{\tau}\,dw$ $\quad \tau\, d\frac{w-z}{\tau} = dw$

$$\int \frac{Aw}{1+(\frac{w}{\sigma})^2} - \underbrace{\frac{Cw}{1+(\frac{w-z}{\tau})^2}}\,dw \to ③$$

$$= \int \frac{\sigma^2 A\,(\frac{w}{\sigma})}{1+(\frac{w}{\sigma})^2}\,d\frac{w}{\sigma} - \int \frac{\tau^2 C\,(\frac{w-z}{\tau})}{1+(\frac{w-z}{\tau})^2}\,d\frac{w-z}{\tau} - \int \frac{Cz}{1+(\frac{w-z}{\tau})^2}\,dw \quad ③$$

$$= \underbrace{\sigma^2 A\cdot\frac{1}{2}\log\left[1+\left(\frac{w}{\sigma}\right)^2\right]\Big|_{-\infty}^{\infty}}_{①} - \underbrace{\tau^2 C\cdot\frac{1}{2}\log\left[1+\left(\frac{w-z}{\tau}\right)^2\right]\Big|_{-\infty}^{\infty}}_{②} - Cz\arctan\left(\frac{w-z}{\tau}\right)\Big|_{-\infty}^{\infty}$$

If $A = \frac{2}{\sigma^2}M_0$ $\quad C = \frac{2}{\tau^2}M_1$ $\quad$ ① − ② = 0 $\quad$ ③ = $-Cz\cdot\tau\pi$

$$f_Z(z) = \int \frac{Aw}{1+(\frac{w}{\sigma})^2} + \frac{B}{1+(\frac{w}{\sigma})^2} - \frac{Cw}{1+(\frac{w-z}{\tau})^2} - \frac{D}{1+(\frac{w-z}{\tau})^2}\,dw\cdot\frac{1}{\pi\sigma}\cdot\frac{1}{\pi\tau}$$

$$= \frac{1}{\pi^2\sigma\tau}\left[\sigma\pi B - \tau\pi D - \frac{2}{\tau^2}z\cancel{\tau}\pi M_1\right]$$

$$\frac{1}{\pi(\sigma+\tau)}\cdot\frac{(\sigma+\tau)^2}{(\sigma+\tau)^2+z^2}\quad\frac{1}{\pi\sigma\tau}\cdot\frac{\tau\sigma B - \tau^2 D - 2zM_1}{\tau}$$

$$④ = \frac{1}{\pi^2\sigma\tau}\left(\sigma\pi B - \tau\pi D - \frac{2\pi z M_1}{\tau}\right)$$

$$⑤ = \frac{1}{\pi(\sigma+\tau)}\cdot\frac{1}{1+(\frac{z}{\sigma+\tau})^2}$$

$$\frac{(\sigma+\tau)^2}{(\sigma+\tau)[(\sigma+\tau)^2+z^2]}\quad\frac{\frac{1}{\sigma\tau^2}\cdot\tau\sigma B - \frac{1}{\sigma\tau^2}\cdot\tau^2 D - \frac{1}{\sigma\tau^2}\cdot 2zM_1}{1}$$

$$= \frac{\frac{B}{\tau} - \frac{D}{\sigma} - \frac{2z}{\sigma\tau^2}M_1}{1}$$

when

$B = \frac{\tau}{\sigma+\tau}$ $\quad D = \frac{\sigma}{\sigma+\tau}$ $\quad M = \frac{\sigma\tau^2(-1)}{2z(\sigma+\tau)}\,\frac{(\sigma+\tau)^2}{(\sigma+\tau)[(\sigma+\tau)^2+z^2]}$

④ = ⑤ $\square$

Proof 1:

$f_U(u) = \frac{1}{\pi\sigma}\,\frac{1}{1+(u/\sigma)^2}$ $\quad Z = U+V$

$U = w$ $\quad V = z-w$

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$f_V(v) = \frac{1}{\pi\tau}\cdot\frac{1}{1+(v/\tau)^2}$ $\quad \therefore f_Z(z) = \int f_{U,V}(w, z-w)\,dw$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi\sigma}\,\frac{1}{1+(w/\sigma)^2}\,\frac{1}{\pi\tau}\,\frac{1}{1+((z-w)/\tau)^2}\,dw$$

$\square$

5.8. $X_1, \cdots, X_n$ $\quad \bar{X} = \frac{1}{n}\sum X_i \quad S^2 = \frac{1}{n-1}(X_i - \bar{X})^2$

(a) Show that

$$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - X_j)^2$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X}^2)$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - \underbrace{2\sum_{i=1}^{n} X_i\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)}_{①} + \underbrace{n\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)^2}_{②}\right]$$

$X_i^2 - 2X_iX_j + X_j^2$

$\sum\sum X_j^2 \qquad \sum\sum 2X_iX_j$

$\underbrace{n\sum_{i=1}^{n} X_i^2 + n\sum X_j^2}$ $\quad 4x_1 4x_1x_3 \; x_1x_3 \cdots$

$2n\sum X_j^2$

$$① = \left(-\frac{2}{n}\right)\begin{bmatrix} X_1(X_1+X_2+\cdots+X_n) \\ X_2(X_1+X_2+\cdots+X_n) \\ \vdots \\ X_n(X_1+X_2+\cdots+X_n)\end{bmatrix} = \left(-\frac{2}{n}\right)\begin{bmatrix} X_1^2 + X_1X_2 + \cdots X_1X_n \\ X_2X_1 + X_2^2 + \cdots X_2X_n \\ X_3X_1 \\ \vdots \\ X_nX_1 + \cdots + X_n^2 \end{bmatrix}$$

$$= \frac{1}{n} 2\sum\sum X_iX_j$$

$$② = \frac{1}{n}\left(\sum X_j\right)^2 = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)^2 \qquad (X_1 + \cdots + X_n)(X_1 + \cdots + X_n)$$

$$= \frac{1}{n}\begin{bmatrix} X_1(X_1 + \cdots + X_n) \\ X_2(X_1 + \cdots + X_n) \\ \vdots \\ X_n(X_1 + \cdots + X_n)\end{bmatrix} = \frac{1}{n}\sum\sum X_iX_j$$

$$\therefore \sum_{i=1}^{n} X_i^2 + ① + ②$$

$$= \sum_{i=1}^{n} X_i^2 - \frac{2}{n}\sum\sum X_iX_j + \frac{1}{n}\sum\sum X_iX_j$$

$$= \frac{1}{2n}\left(2n\cdot\sum_{i=1}^{n} X_i^2 - \frac{2n}{n}\sum\sum X_iX_j\right)$$

$$= \frac{1}{2n}\left(\sum_{i}^{n}\sum_{j}^{n} X_i^2 - 2\sum\sum X_iX_j + \sum_{i}^{n}\sum_{j}^{n} X_j^2\right)$$

$$= \frac{1}{2n}\sum\sum(X_i - X_j)^2$$

$$\therefore S^2 = \frac{1}{n-1}\cdot\frac{1}{2n}\sum\sum(X_i - X_j)^2 \quad \square$$

(b) Assume $\theta_1 = EX_i \quad \theta_j = E(X_i - \theta_1)^j, \; j = 2, 3, 4$ 用数学归纳法思路记啊

Show that $\operatorname{Var} S^2 = \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\theta_2^2\right)$

Hint: mgf

$$M_{X_i}(t) = E[e^{tX_i}] = \int_{-\infty}^{\infty} e^{tx_i} f_{X_i}(x_i)\,dx_i$$

$$\operatorname{Var} S^2 = E(S^2 - ES^2)^2$$

$$= ES^4 - (ES^2)^2$$

$$ES^2 = E\left(\frac{1}{n-1}\cdot\sum(X_i - \bar{X})^2\right)$$

$$= E\left(\frac{1}{n-1}\left[\sum X_i^2 - n\bar{X}^2\right]\right)$$

$$= \frac{1}{n-1}(nEX_1^2 - nE\bar{X}^2)$$

$$= \frac{n}{n-1}\underbrace{(EX_1^2 - E\bar{X}^2)}_{①}$$

$$= \frac{n}{n-1}\cdot\theta_2$$

$$\theta_2 = E(X_i - \theta_1)^2$$

$$= E(X_i - \bar{X})^2$$

$$= E(X_i^2 - 2X_i\bar{X} + \bar{X}^2)$$

$$= EX_i^2 - 2\bar{X}^2 + \bar{X}^2$$

$$= \underbrace{EX_1^2 - \bar{X}^2}_{①}$$

$EX_i = \bar{X}$

$$S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$$

$$= \frac{1}{n-1}\cdot n\theta_2$$

$$\theta_4 = E(X_i - \theta_1)^4$$

$$= E(X_i^4 - 4X_i^3\bar{X} + 6X_i^2\bar{X}^2 - 4X_i\bar{X}^3 + \bar{X}^4)$$

$$= EX_i^4 - 4\bar{X}\cdot EX_i^3 + 6\bar{X}^2\cdot EX_i^2 - 4\bar{X}^3\cdot EX_i + \bar{X}^4$$

$$\theta_2^2 =$$

$$ES^4 = E\left(\frac{1}{n-3}\sum(X_i - \bar{X})^4\right)$$

$$= \frac{1}{n-3}\cdot E\sum(X_i - \bar{X})^4 = \frac{n}{n-3}\theta_4$$

Assume: we don't know

without loss of generality, we can assume $\theta_1 = 0$, $E\bar{Z}_i = 0$

(1) when $n = 4$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{2n(n-1)}\sum_i\sum_j (X_i - X_j)^2 \quad \text{from a.}$$

$$= \frac{1}{24}\sum_{i=1}^{4}\sum_{j=1}^{4}(X_i - X_j)^2$$

$$ES^2 = E\left(\frac{1}{24}\sum_{i=1}^{4}\sum_{j=1}^{4}(X_i - X_j)^2\right) \qquad \theta_2 = E(\bar{Z}_i - \theta_1)^2 = E(\bar{Z}_i - 0)^2 = E\bar{Z}_i^2$$

$$= \frac{1}{24}E\left[\sum\sum(X_i - X_j)^2\right]$$

$$= \frac{1}{24}E\Big[(X_1-X_1)^2 + (X_1-X_2)^2 + \cdots + (X_1-X_n)^2$$
$$+ (X_2-X_1)^2 + (X_2-X_2)^2 + \cdots (X_2-X_n)^2$$
$$\vdots$$
$$+ (X_n-X_1)^2 + \cdots + (X_n-X_n)^2\Big]$$

$$4X_1^2 + \underbrace{(X_1^2 + \cdots X_4^2)}_{4\theta_2} = 4(X_1^2 + \theta_2)$$
$$+ \quad 4X_2^2 + (\qquad)$$
$$\vdots$$
$$4X_4^2 + (\qquad)$$

$$= \frac{1}{24}E\Big[X_1^2 - 2X_1^2 + X_1^2 + X_1^2 - 2X_1X_2 + X_2^2 + \cdots + X_1^2 - 2X_1X_n + X_n^2$$
$$\vdots$$
$$X_n^2 - 2X_nX_1 + X_1^2 + \cdots + X_n^2 - 2X_nX_n + X_n^2\Big]$$

$\cancel{3\times4\times X}$

$\underbrace{6\times4}_{\theta}\times E(X_1^2)$

θ_2

$$24\,\frac{E(X_i^2)}{\theta_2}$$

$$= \frac{1}{24}\Big[E(2n^2X_i^2) - 2E(X_1X_1 + X_1X_2 + \cdots + X_1X_n \to X_1\underbrace{(X_1 + X_2 + \cdots + X_n)}_{n\theta_1 = 0} = 0$$
$$+ X_2X_1 + \cdots + X_2X_n \to 0$$
$$+ \vdots$$
$$+ X_nX_1 + \cdots + X_nX_n)\Big] \to 0$$

$$= \frac{1}{24}\cdot 2(X_1^2 + X_2^2 + \cdots + X_4^2)\times 4$$

$$= \frac{1}{24}\cdot 2\cdot 4\theta_2\times 4 = \frac{4}{3}\theta_2 \quad \checkmark \quad Var($$

$$(ES^2)^2 = \frac{16}{9}\theta_2^2$$

$$Var\,A = E(A - \bar{A})^2$$
$$= EA^2$$

$$EA = E\sum\sum(X_i - X_j)^2$$
$$= 24\,ES^2$$
$$= 24\cdot\frac{4}{3}\theta_2$$
$$= 32\theta_2$$

$$Var\,S^2 = Var\left[\frac{1}{24}\sum\sum(X_i - X_j)^2\right]$$

$$= \frac{1}{24^2}\cdot Var\sum\sum(X_i - X_j)^2$$

$$= \frac{1}{24^2}E\left(\sum\sum(X_i - X_j)^2 - E\left[\sum\sum(\quad)^2\right]\right)^2$$

$$= \frac{1}{24^2}\left\{E\left[\sum\sum(X_i - X_j)^2\right]^2 - 2E(\sum\sum(\quad)^2) + E^2(\quad)\right\}^2 = \frac{16}{9}\theta_2^2$$

$$= \frac{1}{24^2}\Big[\underbrace{E\left[\sum\sum(X_i - X_j)^2\right]^2}_{A} - E^2(\sum\sum(\quad)^2)\Big]$$

$$(X_1 - X_2)^2$$

$$A = E\Big[(X_1-X_1)^2 + (X_1-X_2)^2 + \cdots + (X_1-X_n)^2$$
$$+ (X_2-X_1)^2 + (X_2-X_2)^2 + \cdots (X_2-X_n)^2$$
$$\vdots$$
$$+ (X_n-X_1)^2 + \cdots + (X_n-X_n)^2\Big]^2$$

4

$$= E\,(X_1-X_1)^2\begin{bmatrix}(X_1-X_1)^2 + (X_1-X_2)^2 + \cdots + (X_1-X_n)^2\\ + (X_2-X_1)^2 + (X_2-X_2)^2 + \cdots (X_2-X_n)^2\\ \vdots\\ + (X_n-X_1)^2 + \cdots + (X_n-X_n)^2\end{bmatrix} + \cdots + (X_1-X_n)^2\begin{bmatrix}(X_1-X_1)^2 + (X_1-X_2)^2 + \cdots + (X_1-X_n)^2\\ + (X_2-X_1)^2 + (X_2-X_2)^2 + \cdots (X_2-X_n)^2\\ \vdots\\ + (X_n-X_1)^2 + \cdots + (X_n-X_n)^2\end{bmatrix}$$

$$+ (X_n-X_1)^2\begin{bmatrix}(X_1-X_1)^2 + (X_1-X_2)^2 + \cdots + (X_1-X_n)^2\\ + (X_2-X_1)^2 + (X_2-X_2)^2 + \cdots (X_2-X_n)^2\\ \vdots\\ + (X_n-X_1)^2 + \cdots + (X_n-X_n)^2\end{bmatrix} + \cdots + (X_n-X_n)^2\begin{bmatrix}(X_1-X_1)^2 + (X_1-X_2)^2 + \cdots + (X_1-X_n)^2\\ + (X_2-X_1)^2 + (X_2-X_2)^2 + \cdots (X_2-X_n)^2\\ \vdots\\ + (X_n-X_1)^2 + \cdots + (X_n-X_n)^2\end{bmatrix}$$

$$(x_1-x_1)^2 \begin{bmatrix} (x_1-x_1)^2+(x_2-x_2)^2+\cdots & +(x_1-x_n)^2 \\ +(x_2-x_1)^2+(x_2-x_1)^2+\cdots & (x_2-x_n)^2 \\ \cdots & \\ +(x_n-x_1)^2+\cdots & +(x_n-x_n)^2 \end{bmatrix}$$

$$= (x_1-x_1)^4 + (x_1-x_1)^2(x_1-x_2)^2 + \cdots + (x_1-x_1)^2(x_1-x_n)^2$$

$$+ (x_1-x_1)^2(x_2-x_1)^2 + (x_1-x_1)^2(x_2-x_2)^2 + \cdots + (x_1-x_1)^2(x_2-x_n)^2$$

$$\vdots \qquad\qquad \vdots$$

$$(x_1-x_1)(x_n-x_1)^2 \cdots \cdots (x_1-x_1)^2(x_n-x_n)^2$$

$$\theta_2 = E\,x_i^2$$

① 当 $i=j$ $(x_i-x_j)^2$. $A_r = 0$

$i \neq j$

$\underbrace{(x_1-x_1)^2}_{4}$ $\underbrace{\quad}_{16\text{ terms}}$ + $\underbrace{\quad}_{4\text{ tems}}$ $\underbrace{(x_i-x_j)^2}_{16-4}$ in each [ ]

$4\times 16 + 4\times(16-4) = 112$

② $(x_i-x_j)^4$ $i\neq j$ $i=1,2,3,4$ $j=1,2,3,4$

24个

$(x_1-x_2)^2$ $(x_1-x_2)^2$ $(x_2-x_1)^2$ 2

$(x_1-x_3)^2$ 2

$(x_1-x_4)^2$ 2

4 3 2 2

$(2\times3+2\times3)\times2$

$E(x_i-x_j)^4$

$= E(x_i^4 - 4x_i^3x_j + 6x_i^2x_j^2 - 4x_ix_j^3 + x_j^4)$

$= 2\cdot(E\,x_i^4 - \underbrace{4E(x_i^3x_j)}_{0} + 3E\,x_i^2x_j^2)$

$= 2(\theta_4 + 3\theta_2^2)$

③ $(x_i-x_j)^2(x_i-x_k)^2$ $i\neq j\neq k$

96个.

$(x_1-x_2)^2$ $(x_2-x_1)^2$ 4 8

$4-2=2$ x_1-x_3 x_2-x_4 x_1-x_4 x_2-x_3

3 (x_1-x_2) (x_1-x_3) (x_1-x_4) (x_2-x_3) (x_2-x_4) (x_3-x_4) 6 6

$(6+6)8 = 96$

$E[(x_i-x_j)^2(x_i-x_k)^2]$

$= E[(x_i^2-2x_ix_j+x_j^2)(x_i^2-2x_ix_k+x_k^2)]$

$= E[x_i^2(x_i^2-2x_ix_k+x_k^2)] - 2E[x_ix_j(x_i^2-2x_ix_k+x_k^2)] + E[x_j^2(x_i^2-2x_ix_k+x_k^2)]$

$= \underbrace{E\,x_i^4}_{\theta_4} - 2E(x_i^3x_k) + \underbrace{E\,x_i^2x_k^2}_{3\theta_2^2} - 2E(x_i^3x_j) + 4E(x_i^2x_jx_k) - 2E(x_ix_jx_k^2) + E(x_j^2x_i^2) - 2E(x_j^2x_ix_k) + E(x_j^2x_k^2)$

$= \theta_4 + 3\theta_2^2$

$$\frac{1}{24^2}\left[24\cdot 2(\theta_4+3\theta_2^2) + 96(\theta_4+3\theta_2^2) + 24\times 4\theta_2^2 + \frac{16}{9}\theta_2^2\right]$$

$$\frac{1}{4}\left[\frac{1}{3}(\theta_4+3\theta_2^2) + \frac{2}{3}(\theta_4+3\theta_2^2) + \frac{2}{3}\theta_2^2 - \frac{1}{9}\theta_2^2\right]$$

$$\frac{1}{4}\left[\theta_4 + \theta_2^2 + 2\theta_2^2 + \frac{2}{3}\theta_2^2 - \quad\right)$$

⑤ $(x_i-x_j)^2(x_k-x_l)^2$

$256-112-24-96 = 24$个

$E[(x_i-x_j)^2(x_k-x_l)^2]$

$= E[(x_i^2-2x_ix_j+x_j^2)(x_k^2-2x_kx_l+x_l^2)]$

$= E[x_i^2(x_k^2-2x_kx_l+x_l^2)] - 2E[x_ix_j(x_k^2-2x_kx_l+x_l^2)] + E[x_j^2(x_k^2-2x_kx_l+x_l^2)]$

$= E(x_i^2x_k^2) - 2E(x_i^2x_kx_l) + E(x_i^2x_l^2) - 2E(x_ix_jx_k^2) + 4E(x_ix_jx_kx_l) - 2E(x_ix_jx_l^2) = 4\theta_2^2$

$+ E(x_j^2x_k^2) - 2E(x_j^2x_kx_l) + E(x_j^2x_l^2)$

$$\sum x_i^2 x_j x_k$$

$$x_3^2 x_1 x_2 \qquad x_3^2 x_1 x_3 \qquad x_2^2 x_1 x_4$$
$$x_4^2 x_1 x_2 \qquad x_4^2 x_1 x_3 \qquad x_4^2 x_1 x_4$$

$$x_3^2 x_2 x_1 \qquad \underline{x_1^2 x_2 x_3} \qquad \underline{x_1^2 x_2 x_4}$$
$$x_4^2 x_2 x_1 \qquad x_4^2 x_2 x_3 \qquad x_3^2 x_2 x_4$$

$$x_2^2 x_3 x_1 \qquad \underline{x_1^2 x_3 x_2} \qquad \underline{x_1^2 x_3 x_4}$$
$$x_4^2 x_3 x_1 \qquad x_4^2 x_3 x_2 \qquad x_2^2 x_3 x_4$$

$$x_2^2 x_4 x_1 \qquad \underline{x_1^2 x_4 x_2} \qquad \underline{x_1^2 x_4 x_3}$$
$$x_3^2 x_4 x_1 \qquad x_3^2 x_4 x_2 \qquad x_2^2 x_4 x_3$$

$$2 x_1^2 (x_2 x_3 + x_2 x_4 + x_3 x_4)$$

$$2 x_2^2 (x_1 x_3 + x_1 x_4 + x_3 x_4)$$

$$2 x_3^2 (x_1 x_2 + x_1 x_4 + x_2 x_4)$$

$$2 x_4^2 (x_1 x_2 + x_1 x_3 + x_2 x_3)$$

$$2x_1 (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4)$$
$$2x_2 (x_1 x_2 x_3 + x_1 x_2 x_4 + x_2 x_3 x_4)$$
$$2x_3 (x_1 x_2 x_3 + x_1 x_3 x_4 + x_2 x_3 x_4)$$
$$2x_4 (x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)$$

$$= 2 x_1 x_2 x_3 (-x_4) \qquad = -8 x_1 x_2 x_3 x_4$$
$$+ 2 x_1 x_2 x_4 (-x_3) \qquad = -8 E(x_1 x_2 x_3 x_4)$$
$$+ 2 x_1 x_3 x_4 (-x_2) \qquad = 0$$
$$+ 2 x_2 x_3 x_4 (-x_1)$$

$$24 \cdot 2 \left[4\theta_2^2 + \tfrac{4}{3}\theta_4\right] + 96 \left[4\theta_2^2 + \tfrac{4}{3}\theta_4\right] + 24 \cdot \left[4\theta_2^2 + \tfrac{4}{3}\theta_4\right] - \left(32\theta_2^2\right)^2$$

$$= \frac{(24 \cdot 2 + 96 + 24)}{24^2} \left(4\theta_2^2 + \tfrac{4}{3}\theta_4\right) - \frac{32^2 \theta_2^2}{24^2}$$

5.9. Establish Lagrange Identity

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2$$

Induction: when $n=2$

$\sum_{1}^{2} a_i^2 = a_1^2 + a_2^2$

$\sum_{1}^{2} b_i^2 = b_1^2 + b_2^2$

$\left(\sum_{i=1}^{2} a_i b_i\right)^2 = (a_1 b_1 + a_2 b_2)^2$

$(a_1^2+a_2^2)(b_1^2+b_2^2) - (a_1b_1 + a_2b_2)^2$

$= \cancel{a_1^2b_1^2} + a_2^2b_1^2 + a_1^2b_2^2 + \cancel{a_2^2b_2^2} - \cancel{a_1^2b_1^2} - 2a_1a_2b_1b_2 - \cancel{a_2^2b_2^2}$

$= (a_2b_1 - a_1b_2)^2$ □

Assume that it is true for n then:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \underline{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2}$$

for $n+1$:

$\sum_{i=1}^{n+1} a_i^2 = a_{n+1}^2 + \sum_{i=1}^{n} a_i^2$ ①

$\sum_{i=1}^{n+1} b_i^2 = b_{n+1}^2 + \sum_{i=1}^{n} a_i^2$ ②

$\sum_{i=1}^{n+1} (a_i b_i)^2 = \left(a_{n+1}b_{n+1} + \sum_{i=1}^{n} a_i b_i\right)^2$ ③

① ② − ③

$= (a_{n+1}^2 + \Sigma a_i^2)(b_{n+1}^2 + \Sigma b_i^2) - (a_{n+1}b_{n+1} + \Sigma a_i b_i)^2$

$= a_{n+1}^2 b_{n+1}^2 + b_{n+1}^2 \Sigma a_i^2 + a_{n+1}^2 \Sigma b_i^2 + \Sigma a_i^2 \cdot \Sigma b_i^2 - \left[\cancel{a_{n+1}^2 b_{n+1}^2} + 2a_{n+1}b_{n+1}\Sigma a_i b_i + (\Sigma a_i b_i)^2\right]$

$= b_{n+1}^2 \Sigma a_i^2 + a_{n+1}^2 \Sigma b_i^2 - 2a_{n+1}b_{n+1}\Sigma a_i b_i + \sum_{i=1}^{n+1}\sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2$

$= \sum_{i=1}^{n} (b_{n+1}a_i - a_{n+1}b_i)^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (a_i b_j - a_j b_i)^2$

$= \sum_{i=1}^{n}\sum_{j=i+1}^{n+1} (a_i b_j - a_j b_i)^2$ □

$\therefore$ $n+1$ is true

$b_{n+1}^2 (a_1^2 + a_2^2 + \cdots + a_n^2)$

$+ a_{n+1}^2 (b_1^2 + b_2^2 + \cdots + b_n^2)$

$-2a_{n+1}b_{n+1}(a_1b_1 + a_2b_2 + \cdots + a_nb_n)$

$= (b_{n+1}a_1 - a_{n+1}b_1)^2 + \cdots + (b_{n+1}a_n + a_{n+1}b_n)^2$

$= \sum_{i=1}^{n}\sum_{j=i+1}^{n+1} (b_j a_i - a_j b_i)^2$

Show that the squared correlation coefficient $= 1$ if and only if all the sample points lies on a straight line

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2)-(EX)^2}\sqrt{E(Y^2)-(EY)^2}} = 1 \quad \square$$

$\sigma_X^2 = EX^2 - (EX)^2$

$\sigma_Y^2 = EY^2 - (EY)^2$

If all samples on a straight line,

then $Y - EY = c(X - EX)$

$b_i = (y_i - EY)$ $\quad a_i = (x_i - EX)$

$b_i = c\,a_i$

$\sum_{i=1}^{n}\sum_{j=i+1}^{n+1} (a_i b_j - a_j b_i)^2 = \sum\sum (a_i \cdot c\, a_j - a_j \cdot c\, a_i)^2$

$= 0$

$\therefore \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 = \left(\sum_{i=1}^{n} a_i b_i\right)^2$

$\underline{X}Y - EXEY = \cancel{EYEX + C}$

that is $\left[\sum_{1}^{n} (y_i - EY)^2\right]\left[\sum_{1}^{n} (x_i - EX)^2\right] = \left(\sum_{i=1}^{n} (y_i - EY)(x_i - EX)\right)^2$

$[E(Y^2) - (EY)^2][E(X^2) - (EX)^2] = [E(XY) - E(X)E(Y)]^2$

5.10. Let $Z_1, \dots, Z_n \sim n(\mu, \sigma^2)$

(a) Find expression for $\theta_1, \dots, \theta_4$ in terms of μ and σ^2 where $\theta_j = E(Z_i - \theta_1)^j$

$\theta_1 = \mu$

$\theta_2 = E(Z_i - \mu)^2 = E(Z_i^2 - 2\mu Z_i + \mu^2)$

$= E(Z_i^2) - (EZ_i)^2$

$E(Z_i^2) - \mu^2 = \sigma^2$ $\quad = \sigma^2$

$\theta_3 = E(Z_i - \mu)^3$

$= E(Z_i^3 - 3\mu Z_i^2 + 3Z_i\mu^2 - \mu^3)$

$= EZ_i^3 - 3\mu \cdot EZ_i^2 + 3\mu^3 - \mu^3$

$= EZ_i^3 - 3\mu(\sigma^2 + \mu^2) + 2\mu^3$

$= EZ_i^3 - 3\mu\sigma^2 - \mu^3$

$= E(Z_i - \mu)^2 (Z_i - \mu)$

Stein's lemma $\quad Eg(x) \cdot (Z - \theta) = \sigma^2 Eg'(Z)$

$\therefore E\underbrace{(Z_i - \mu)^2}_{g(Z)} (Z_i - \mu) = \sigma^2 E[2(Z_i - \mu)]$

$= 2\sigma^2 \underbrace{E(Z_i - \mu)}_{0}$

$= 0$

$\therefore \theta_3 = 0$ $\quad \square$

$\theta_4 = E(Z_i - \mu)^4$

$= E\underbrace{(Z_i - \mu)^3}_{g(x)} \cdot (Z_i - \mu)$

$= \sigma^2 \cdot E[3(Z_i - \mu)^2]$

$= 3\sigma^2 \cdot E(Z_i - \mu)^2$

$= 3\sigma^2 \cdot \sigma^2$

$= 3\sigma^4$ $\quad \square$

mgf $Z \sim (0, 1)$

$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$= e^{t^2/2}$

$= \sum_{n=0}^{\infty} \frac{(t^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n! \cdot 2^n}$

$e^{tZ} = \sum_{n=0}^{\infty} \frac{t^n Z^n}{n!}$

$\because (M_Z(0))^{(2n)} = (e^{t^2/2})^{(2n)}_{t=0}$

$= \frac{(2n)!}{n! \cdot 2^n}$

$= E(e^{tZ})^{(2n)}_{t=0}$

$= E(Z^{2n} e^0) = E(Z^{2n})$

that is $E(Z^{2n}) = \frac{(2n)!}{n! \cdot 2^n}$

$\therefore E(Z^4) = \frac{4!}{2! \cdot 2^2} = \frac{\cancel{4} \times 3 \times \cancel{2}}{\cancel{2} \times \cancel{4}}$

$= 3$ $\quad \square$

(b). From 5.8, we have:

$$\operatorname{Var} S^2 = \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\theta_2^2\right),$$

together with results of part (a), calculate $\operatorname{Var} S^2$

$\operatorname{Var} S^2 = \frac{1}{n}\left[3\sigma^4 - \frac{n-3}{n-1} \cdot (\sigma^2)^2\right]$

$= \frac{1}{n}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right)$

$= \frac{1}{n}\sigma^4\left(3 - \frac{n-3}{n-1}\right)$

$= \frac{1}{n}\sigma^4 \cdot \frac{2n}{n-1}$

$= \frac{2}{n-1}\sigma^4$ $\quad \square$

(c). Calculate $\operatorname{Var} S^2$: Use the fact that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$

$(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$

$\operatorname{Var}[(n-1)S^2/\sigma^2] = 2(n-1)$

$\operatorname{Var}[(n-1)S^2/\sigma^2] = \frac{(n-1)^2}{\sigma^4} \operatorname{Var} S^2$

$\therefore \operatorname{Var} S^2 = \frac{2(n-1)\sigma^4}{(n-1)^2}$

$= \frac{2}{n-1}\sigma^4$

Proof:

$Z_i \sim N(\mu, \sigma^2)$

$\frac{Z_i - \mu}{\sigma} \sim N(0, 1)$

$\bar{Z} \sim N(\mu, \frac{\sigma^2}{n})$

$\frac{\bar{Z} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

σ^2/n

$W = \sum_{i=1}^{n} \left(\frac{Z_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$

$= \sum \left[\frac{(Z_i - \bar{Z}) + (\bar{Z} - \mu)}{\sigma}\right]^2$

$= \sum \left[\frac{(Z_i - \bar{Z})^2}{\sigma^2} + \frac{(\bar{Z} - \mu)^2}{\sigma^2} + 2\frac{(Z_i - \bar{Z})}{\sigma} \cdot (\bar{Z} - \mu)\right]$

$= \frac{1}{\sigma^2}\sum(Z_i - \bar{Z})^2 + \frac{n}{\sigma^2}\sum(\bar{Z} - \mu)^2 + \boxed{\frac{2}{\sigma^2}(\bar{Z} - \mu)\underbrace{\sum(Z_i - \bar{Z})}_{\sum Z_i - n\bar{Z} = 0}}$

By the definition of chi-squared dis

$\operatorname{Var}(W) = \operatorname{Var}(Z_1^2 + Z_2^2 + \cdots + Z_n^2) = \sum \underbrace{\left(\frac{Z_i - \bar{Z}}{\sigma}\right)^2}_{\chi^2_{n-1}} + \sum \underbrace{\left(\frac{\bar{Z} - \mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2_1} = \chi^2_n$

$= \operatorname{Var}(Z_1^2) + \operatorname{Var}(Z_2^2) + \cdots + \operatorname{Var}(Z_n^2)$

$= E(Z_1^4) - (EZ_1^2)^2 + \cdots + E(Z_n^4) - (E(Z_n^2))^2$

$\Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ $\quad \square$

$= E(Z_1^4) + E(Z_2^4) + \cdots + E(Z_n^4) - \underbrace{\left[(EZ_1^2)^2 + \cdots + E(Z_n^2)^2\right]}_{n}$

$= 3n - n = 2n$ $\quad \square$

5.11. Prove that $ES \le \sigma$, and if $\sigma^2 > 0$ $ES < \sigma$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$S = \frac{1}{\sqrt{n-1}}\sqrt{\sum (X_i - \bar{X})^2}$$

$$ES = \frac{1}{\sqrt{n-1}} E\sqrt{\sum (X_i - \bar{X})^2}$$

$$\le \frac{1}{\sqrt{n-1}}\sqrt{\underbrace{E\left(\sum (X_i - \bar{X})^2\right)}_{=(n-1)\sigma^2}}$$

$$(X_i - \bar{X})^2$$

$$= X_i^2 - 2X_i\bar{X} + \bar{X}^2$$

$$\sum (X_i - \bar{X})^2$$

$$= \sum X_i^2 - 2n\bar{X}\cdot\sum X_i + n\bar{X}^2$$

$$E\left[\sum (X_i - \bar{X})^2\right]$$

$$= E\sum X_i^2 - 2nE\bar{X}\underbrace{E\sum X_i} + nE\bar{X}^2$$

$$= n\cdot E\bar{X}^2 - nE\bar{X}^2$$

$$= n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= n\sigma^2 - \sigma^2$$

$$\mathrm{Var}\,\bar{X} = E\bar{X}^2 - (E\bar{X})^2 \qquad X_i \sim \mu, \sigma^2$$

$$\sigma^2 = E\bar{X}^2 - \mu^2 \qquad \bar{X} \sim \mu, \frac{\sigma^2}{n}$$

$$E\bar{X}^2 = \sigma^2 + \mu^2$$

$$E\bar{X}^2 = \frac{\sigma^2}{n} + \mu^2$$

Aim: $E\sqrt{\theta} \le \sqrt{E\theta}$

$$\frac{1}{n}(\sqrt{\theta_1} + \sqrt{\theta_2} + \cdots + \sqrt{\theta_n}) \le \sqrt{\frac{\theta_1 + \theta_2 + \cdots + \theta_n}{n}}$$

Induction:

when $n = 2$: $\because \left(\frac{1}{2}\sqrt{\theta_1} - \frac{1}{2}\sqrt{\theta_2}\right)^2 \ge 0$

$$\therefore \frac{1}{4}\theta_1 - \frac{1}{2}\sqrt{\theta_1\theta_2} + \frac{1}{4}\theta_2 \ge 0$$

$$\frac{1}{4}\theta_1 + \frac{1}{4}\theta_2 \ge \frac{1}{2}\sqrt{\theta_1\theta_2}$$

$$\frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 \ge \frac{1}{4}\theta_1 + \frac{\sqrt{\theta_1\theta_2}}{2} + \frac{1}{4}\theta_2$$

that is $\sqrt{\frac{\theta_1 + \theta_2}{2}} \ge \frac{1}{2}(\sqrt{\theta_1} + \sqrt{\theta_2})$

$$\therefore ES = \frac{1}{\sqrt{n-1}} E\sqrt{\sum (X_i - \bar{X})^2} \le \frac{1}{\sqrt{n-1}}\sqrt{E\left[\sum (X_i - \bar{X})^2\right]}$$

$$= \frac{1}{\sqrt{n-1}}\sqrt{(n-1)\sigma^2}$$

$$= \sigma$$

$\therefore ES \le \sigma$ if $\sigma \ne 0$, all equal cannot be reached. $\square$

Assume it is correct:

$$\sqrt{\frac{\theta_1 + \theta_2 + \cdots + \theta_n}{n}} \ge \frac{1}{n}(\sqrt{\theta_1} + \sqrt{\theta_2} + \cdots + \sqrt{\theta_n})$$

$$\frac{\theta_1 + \cdots + \theta_n}{n} \ge \frac{1}{n^2}\left[(\theta_1 + \cdots + \theta_n) + 2\sum\sum \theta_i\theta_j\right]$$

$$\therefore (n-1)(\theta_1 + \cdots + \theta_n) \ge 2\sum_{i}^{n}\sum_{j}^{n} \theta_i\theta_j$$

$$\begin{array}{lll} \theta_1(\theta_2 + \theta_3 + \cdots + \theta_n) & \theta_1\cdot\theta_{n+1} \\ \theta_2(\theta_1 + \theta_3 + \cdots + \theta_n) & \theta_2\cdot\theta_{n+1} \\ \vdots & \vdots \\ \theta_n(\theta_1 \cdots + \theta_{n-1}) & \theta_{n+1}(\theta_1 + \theta_2 \cdots \theta_n) \end{array}$$

when $n+1$:

$$\sqrt{\frac{\theta_1 + \theta_2 + \cdots + \theta_n + \theta_{n+1}}{n+1}} \;?\; \frac{1}{n+1}(\sqrt{\theta_1} + \cdots + \sqrt{\theta_n} + \sqrt{\theta_{n+1}})$$

$$\frac{\theta_1 + \cdots + \theta_n + \theta_{n+1}}{n+1} \;?\; \frac{1}{(n+1)^2}\left[(\theta_1 + \cdots + \theta_{n+1}) + 2\sum\sum\sqrt{\theta_i}\sqrt{\theta_j}\right]$$

$$n(\theta_1 + \cdots + \theta_n) \;?\; 2\sum_{i}^{n}\sum_{j=i}^{n+1}\sqrt{\theta_i}\sqrt{\theta_j}$$

$$\theta_1 + \cdots + \theta_n + n\theta_{n+1} \qquad 2\sum_{1}^{n}\sqrt{\theta_i}\sqrt{\theta_{n+1}}$$

$$\because \theta_1 + \cdots + \theta_n + n\theta_{n+1} - 2\sum_{i=1}^{n}\sqrt{\theta_i}\sqrt{\theta_{n+1}} \ge 0$$

$$\therefore (n-1)(\theta_1 + \cdots + \theta_n) + \theta_1 + \cdots + \theta_n + n\theta_{n+1}$$

$$\ge 2\sum_{i=1}^{n}\sum_{j}^{n}\theta_i\theta_j + 2\sum_{i=1}^{n}\sqrt{\theta_i}\sqrt{\theta_{n+1}}$$

that is $n(\theta_1 + \cdots + \theta_{n+1}) \ge 2\sum_{i}^{n}\sum_{j}^{n+1}\sqrt{\theta_i}\sqrt{\theta_j}$

$$\therefore (n+1)(\theta_1 + \cdots + \theta_{n+1}) \ge \theta_1 + \cdots + \theta_{n+1} + 2\sum\sum\sqrt{\ }$$

$$\therefore \sqrt{\frac{\theta_1 + \cdots + \theta_{n+1}}{n+1}} \ge \sqrt{\frac{1}{(n+1)^2}\left[(\theta_1 + \cdots + \theta_{n+1}) + 2\sum\sum\sqrt{\ }\right]}$$

5.12. Let $Z_1, \cdots, Z_n \sim n(0,1)$

$$Y_1 = \left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \quad Y_2 = \frac{1}{n}\sum_{i=1}^{n}|Z_i|, \text{ calculate } EY_1, EY_2$$

$Y_1 = \frac{1}{n}|\Sigma X_i| \qquad Y_2 = \frac{1}{n}\Sigma|X_i|$

Jensen's equality $Eg(S) \geq g\,E(S)$

$EY_1 = \frac{1}{n}\cdot E|\Sigma X_i| = 0 \qquad E|\Sigma X_i| = E(|X_1+X_2+\cdots+X_n|) = 0 \qquad \because |X_1+X_2+\cdots+X_n| \leq |X_1|+|X_2|+\cdots+|X_n|$

$\therefore EY_1 \leq EY_2 \quad \square$

$EY_2 = \frac{1}{n}\cdot E\Sigma|X_i| = \frac{2}{n\sqrt{2\pi}} \qquad E\Sigma|X_i| = E(|X_1|+|X_2|+\cdots+|X_n|) = E|X| = \frac{2}{\sqrt{2\pi}}$

$$\begin{aligned}
E|Z_1| &= \int_{-\infty}^{\infty}|x|\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \\
&= 2\int_0^{\infty} x\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \\
&= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} \underbrace{x\cdot e^{-\frac{x^2}{2}}}\,dx \\
&= \frac{2}{\sqrt{2\pi}}\left(-e^{-\frac{x^2}{2}}\Big|_0^{\infty}\right) \\
&= \frac{2}{\sqrt{2\pi}}
\end{aligned}$$

5.13. Let $Z_1, Z_2 \cdots Z_n \sim n(\mu, \sigma^2)$, find a function of S^2 $g(S^2)$ satisfy $Eg(S^2) = \sigma$

Hint try $g(S^2) = c\cdot\sqrt{S^2}$ c is a constant.

Assume that $g(S^2) = c\cdot\sqrt{S^2} \qquad S^2 = \frac{\Sigma(X_i-\bar{X})^2}{n-1}$

$$\begin{aligned}
Eg(S^2) &= c\cdot E(\sqrt{S^2}) \\
&= c\cdot\sqrt{\frac{\sigma^2}{n-1}}\,\underbrace{E\left(\sqrt{\frac{S^2(n-1)}{\sigma^2}}\right)}_{\sqrt{\chi^2_{n-1}}} \\
&= c\cdot\sqrt{\frac{\sigma^2}{n-1}}\cdot\int_0^{\infty}\underbrace{\sqrt{x}}\cdot\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\,x^{\frac{n-1}{2}-1}e^{-\frac{x}{2}}\,dx \\
&= c\cdot\sqrt{\frac{\sigma^2}{n-1}}\,\underbrace{\int_0^{\infty}\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\cdot x^{\frac{n}{2}-1}\cdot e^{-\frac{x}{2}}\,dx}_{\text{看 } \chi^2_n} \\
&= c\cdot\sqrt{\frac{\sigma^2}{n-1}}\cdot\frac{2^{\frac{1}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\underbrace{\int_0^{\infty}\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\,x^{\frac{n}{2}-1}\cdot e^{-\frac{x}{2}}\,dx}_{\text{pdf of } \chi^2_n = 1} \\
&= c\cdot\sqrt{\frac{\sigma^2}{n-1}}\cdot 2^{\frac{1}{2}}\cdot\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}
\end{aligned}$$

$\therefore$ when $c = \sqrt{\frac{n-1}{2}}\cdot\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}$, $Eg(S^2) = \sigma \quad \square$

5.14 (a) show that if $Z_j = \sigma_j Z_j + \mu_j$, $Z_j \sim n(0,1)$,

$Cov(\sum_{j=1}^{n} a_{ij}Z_j, \sum_{j=1}^{n} b_{rj}Z_j) = 0 \Rightarrow \sum_{j=1}^{n} a_{ij}Z_j$ and $\Sigma b_{rj}Z_j$ are independent.

$Cov(\Sigma a_{ij}Z_j, \Sigma b_{rj}Z_j) = 0 \Rightarrow \Sigma a_{ij}Z_j$ and $\Sigma b_{rj}Z_j$ are independent.

$$\begin{aligned}
\because Cov(\sum_{j=1}^{n} a_{ij}Z_j, \Sigma b_{rj}Z_j) &= E\left[(\Sigma a_{ij}Z_j - \overline{\Sigma a_{ij}Z_j})(\Sigma b_{rj}Z_j - \overline{\Sigma b_{rj}Z_j})\right] \\
&= E(\Sigma a_{ij}Z_j\cdot\Sigma b_{rj}Z_j) - \overline{\Sigma a_{ij}Z_j}\cdot\overline{\Sigma b_{rj}Z_j} \\
&= 0
\end{aligned}$$

$\because E(\Sigma a_{ij}Z_j\cdot\Sigma b_{rj}Z_j) = E(\Sigma a_{ij}Z_j)E(\Sigma b_{rj}Z_j)$

that is $\Sigma a_{ij}Z_j$ and $\Sigma b_{rj}Z_j$ are independent.

Similarly we can prove $\Sigma a_{ij}Z_j$ and $\Sigma b_{rj}Z_j$ are independent. $\square$

5.15. Show that

(a) $\bar{X}_{n+1} = \dfrac{X_{n+1} + n\bar{X}_n}{n+1}$

(b) $nS_{n+1}^2 = (n-1)S_n^2 + \left(\frac{n}{n+1}\right)(X_{n+1} - \bar{X}_n)^2$

(a). $\because \bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$

$\therefore X_1 + X_2 + \cdots + X_n = n \cdot \bar{X}_n$

$\bar{X}_{n+1} = \frac{1}{n+1}(X_1 + X_2 + \cdots + X_n + X_{n+1})$

$= \frac{1}{n+1}(n \cdot \bar{X}_n + X_{n+1})$ □

(b) $S_n^2 = \frac{1}{n-1}\sum_1^n (X_i - \bar{X}_n)^2$

$\therefore (n-1)S_n^2 = \sum_1^n (X_i - \bar{X}_n)^2$ ①

$= (X_1 - \bar{X}_n)^2 + (X_2 - \bar{X}_n)^2 + \cdots + (X_n - \bar{X}_n)^2$

$= X_1^2 + X_2^2 + \cdots + X_n^2 - 2\bar{X}_n(X_1 + X_2 + \cdots + X_n) + n\bar{X}_n^2$

$S_{n+1}^2 = \frac{1}{n}\sum_1^{n+1}(X_i - \bar{X}_{n+1})^2$

$nS_{n+1}^2 = \sum_1^{n+1}(X_i - \bar{X}_{n+1})^2$ ②

$= (X_1 - \bar{X}_{n+1})^2 + (X_2 - \bar{X}_{n+1})^2 + \cdots$

$= X_1^2 + X_2^2 + \cdots + X_{n+1}^2 - 2\bar{X}_{n+1}(X_1 + X_2 + \cdots + X_{n+1}) + (n+1)\bar{X}_{n+1}^2$

Let $A = X_1 + X_2 + \cdots + X_n$

$\bar{X}_n = \frac{A}{n}$ $\quad \bar{X}_{n+1} = \frac{A + X_{n+1}}{n+1}$

① $= X_1^2 + X_2^2 + \cdots + X_n^2 - 2\frac{A}{n} \cdot A + n\left(\frac{A}{n}\right)^2$

② $= X_1^2 + X_2^2 + \cdots + X_{n+1}^2 - 2\frac{(A + X_{n+1})^2}{n+1} + (n+1)\left(\frac{A + X_{n+1}}{n+1}\right)^2$

② - ① $= X_{n+1}^2 - 2\left[\frac{(A+X_{n+1})^2}{n+1} - \frac{A^2}{n}\right] + \frac{(A+X_{n+1})^2}{n+1} - \frac{A^2}{n}$

$= X_{n+1}^2 + \frac{A^2}{n} - \frac{(A + X_{n+1})^2}{n+1}$

$= \dfrac{n(n+1)X_{n+1}^2 + (n+1)A^2 - n(A + X_{n+1})^2}{n(n+1)}$

Molecular: $n(n+1)X_{n+1}^2 + (n+1)A^2 - nA^2 - 2nAX_{n+1} - nX_{n+1}^2$

$= nX_{n+1}[(n+1)X_{n+1} - 2A - X_{n+1}] + A^2$

$= nX_{n+1}(nX_{n+1} - 2A) + A^2$

$= A^2 - 2AnX_{n+1} + n^2X_{n+1}^2$

$= (A - nX_{n+1})^2$

$= [A + X_{n+1} - (n+1)X_{n+1}]^2$

$\therefore$ ② - ① $= \frac{n+1}{n}\left[\frac{A + X_{n+1}}{n+1} - \frac{(n+1)X_{n+1}}{n+1}\right]^2$

$= \frac{n+1}{n}(\bar{X}_{n+1} - X_{n+1})^2$

That is $nS_{n+1}^2 = (n-1)S_n^2 + \frac{n+1}{n}(\bar{X}_{n+1} - X_{n+1})^2$ □

5.16. Let X_i, $i = 1, 2, 3$ $\sim n(i, i^2)$, use X_is to construct a statistic:

(a) chi-squared with 3 degree of freedom

$Y_1 = \frac{X_1 - 1}{1} = X_1 - 1$ $\quad Y_2 = \frac{X_2 - 2}{2}$ $\quad Y_3 = \frac{X_3 - 3}{3}$

$Y = Y_1^2 + Y_2^2 + Y_3^2 \sim \chi_3^2$

(b) t distribution with 2 degree of freedom

$Y_1 = X_1 - 1$ $\quad Y_2 = \frac{X_2 - 2}{2}$ $\quad Y_3 = \frac{X_3 - 3}{3}$ $\sim (0, 1)$

$\bar{Y} = \frac{1}{3}(Y_1 + Y_2 + Y_3)$ $\quad S^2 = \frac{1}{3-1}\Sigma(Y_i - \bar{Y})^2 = \frac{1}{2}\Sigma(Y_i - \bar{Y})^2$

$\frac{\bar{Y} - 0}{S/\sqrt{3}} \sim$ t distribution with freedom $= 2$

(c) F distribution with 1 and 2 degree of freedom.

$Y_1 = X_1 - 1$, $Y_1^2 \sim \chi_1^2$ $\quad d_1 = 1$

$Y_2 = \frac{X_2 - 2}{2}$ $\quad Y_3 = \frac{X_3 - 3}{3}$, $Y_2^2 + Y_3^2 \sim \chi_2^2$ $\quad d_2 = 2$

$Y = \dfrac{Y_1^2/d_1}{(Y_2^2 + Y_3^2)/d_2} = \dfrac{Y_1^2/1}{(Y_2^2 + Y_3^2)/2} \sim$ F distribution with 1 & 2 degree of freedom

5.17 Let Z be a random variable with an $F_{p,q}$ dis

(a) Derive the pdf of Z

$$Z = \frac{Y_1/p}{Y_2/q} \qquad Y_1 \sim \chi^2_p \qquad Y_2 \sim \chi^2_q$$

$$f(Y_1; p) = \frac{y_1^{\frac{p}{2}-1} \cdot e^{-\frac{y_1}{2}}}{2^{\frac{p}{2}}\,\Gamma(\frac{p}{2})} \quad x>0$$

$$f(Y_2; q) = \frac{y_2^{\frac{q}{2}-1} e^{-\frac{y_2}{2}}}{2^{\frac{q}{2}}\,\Gamma(\frac{q}{2})} \quad x>0$$

Let $U = \frac{Y_1/p}{Y_2/q}$ $\quad Y_1 = p \cdot U \cdot V$

$V = Y_2/q$ $\quad Y_2 = q \cdot V$

$$|J| = \begin{vmatrix} pV & pU \\ 0 & q \end{vmatrix} = pqV > 0$$

$$f(U,V) = f_{Y_1,Y_2}(pUV, qV) \cdot |J|$$

$$= \frac{(pUV)^{\frac{p}{2}-1} \cdot e^{-\frac{pUV}{2}}}{2^{\frac{p}{2}} \cdot \Gamma(\frac{p}{2})} \cdot \frac{(qV)^{\frac{q}{2}-1} \cdot e^{-\frac{qV}{2}}}{2^{\frac{q}{2}} \cdot \Gamma(\frac{q}{2})} \cdot pqV$$

$$f_Z(x) = f_U(U) = \int_0^\infty f(U,V)\,dV$$

$$= \frac{p^{\frac{p}{2}} q^{\frac{q}{2}} U^{\frac{p}{2}-1}}{2^{\frac{p}{2}}\,\Gamma(\frac{p}{2})} \cdot \underbrace{\int_0^\infty \frac{V^{\frac{p+q}{2}-1} e^{-\frac{(pU+q)V}{2}}}{2^{\frac{p+q}{2}} \cdot \Gamma(\frac{q}{2})}\,dV}$$

$$= \frac{p^{\frac{p}{2}} q^{\frac{q}{2}} U^{\frac{p}{2}-1}}{2^{\frac{p}{2}} \cdot 2^{\frac{q}{2}}\,\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \int V^{\frac{p+q}{2}-1} e^{-\frac{(pU+q)}{2}V}\,dV$$

$$= \frac{p^{\frac{p}{2}} \cdot q^{\frac{q}{2}} \cdot U^{\frac{p}{2}-1}}{\cancel{2}^{\frac{p+q}{2}} \cdot \Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \cdot \Gamma\left(\frac{p+q}{2}\right) \left(\frac{\cancel{2}}{pU+q}\right)^{\frac{p+q}{2}} \underbrace{\int_0^\infty \frac{V^{\frac{p+q}{2}-1} e^{\cdots}}{\left(\frac{2}{pU+q}\right)^{\frac{p+q}{2}} \cdot \Gamma(\frac{p+q}{2})}\,dV}_{1}$$

$$= \frac{U^{\frac{p}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \left(\frac{p}{q}\right)^{\frac{p}{2}} q^{\frac{p+q}{2}} \left(\frac{1}{pU+q}\right)^{\frac{p+q}{2}}$$

Gamma:

$$f(x, \alpha, \beta) = \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)}$$

$$\alpha = \frac{p+q}{2}$$

$$\beta = \frac{2}{pU+q}$$

$$\beta^\alpha = \left(\frac{2}{pU+q}\right)^{\frac{p+q}{2}}$$

$$\Gamma(\alpha) = \Gamma\left(\frac{p+q}{2}\right)$$

$$\because \left(\frac{1}{pU+q}\right)^{\frac{p+q}{2}} = (pU+q)^{-\frac{p+q}{2}} = q^{-\frac{p+q}{2}} \left(\frac{p}{q} \cdot U + 1\right)^{-\frac{p+q}{2}}$$

$$\therefore f_Z(x) = \frac{x^{\frac{p}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \left(\frac{p}{q}\right)^{\frac{p}{2}} \left(\frac{p}{q}x + 1\right)^{-\frac{p+q}{2}} \quad \square$$

(b) Derive the mean and variance of Z

$$E(Z) = \int_0^\infty f_Z(x)\, x\, dx$$

$$= \int_0^\infty \frac{x^{\frac{p}{2}}}{B(\frac{p}{2}, \frac{q}{2})} \cdot \left(\frac{p}{q}\right)^{\frac{p}{2}} \cdot \left(\frac{p}{q}x + 1\right)^{-\frac{p+q}{2}} dx$$

$$= \frac{\left(\frac{p}{q}\right)^{\frac{p}{2}}}{B(\frac{p}{2}, \frac{q}{2})} \underbrace{\int_0^\infty x^{\frac{p}{2}} \left(\frac{p}{q}x + 1\right)^{-\frac{p+q}{2}} dx}_{A}$$

let $\frac{p}{q}x = r \quad x = \frac{q}{p}r \quad dx = \frac{q}{p}dr$

$$A = \int_0^\infty \left(\frac{q}{p}\right)^{\frac{p}{2}} \frac{r^{\frac{p}{2}}}{(r+1)^{\frac{p+q}{2}}} \cdot \frac{q}{p}\,dr$$

$$= \left(\frac{q}{p}\right)^{\frac{p}{2}+1} \cdot \int_0^\infty \frac{r^{\frac{p}{2}}}{(r+1)^{\frac{p+q}{2}}}\,dr$$

$$= \left(\frac{q}{p}\right)^{\frac{p}{2}+1} \cdot B\left(\frac{p}{2}+1, \frac{q}{2}-1\right)$$

Beta:

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt$$

$$x = \frac{p}{2}+1 \quad y = \frac{q}{2}-1$$

$$\therefore E(Z) = \frac{q}{p} \cdot \frac{B(\frac{p}{2}+1, \frac{q}{2}-1)}{B(\frac{p}{2}, \frac{q}{2})} = \frac{q}{p} \cdot \frac{\Gamma(\frac{p}{2}+1) \cdot \Gamma(\frac{q}{2}-1)}{\cancel{\Gamma(\frac{p+q}{2})}} \cdot \frac{\cancel{\Gamma(\frac{p+q}{2})}}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} = \frac{q}{p} \cdot \frac{\frac{p}{2}}{\frac{q}{2}-1} = \frac{q}{q-2} \quad \square$$

$$Var(Z) = E(Z^2) - (EZ)^2$$

$$E(Z^2) = E\left[\left(\frac{Y_1/p}{Y_2/q}\right)^2\right] = \frac{q^2}{p^2} \cdot E\left(\frac{y_1^2}{y_2^2}\right) = \frac{q^2}{p^2} \cdot E(y_1^2) \cdot E\left(\frac{1}{y_2^2}\right)$$

$$E(y_1^2) = \int_0^\infty y_1^2 \cdot f(Y_1, p)\, dy$$

$$= \int_0^\infty \frac{y_1^{\frac{p}{2}+1} \cdot e^{-\frac{y_1}{2}}}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})} dy_1$$

$$= \underbrace{\int_0^\infty \frac{y_1^{\frac{p}{2}+1} \cdot e^{-\frac{y_1}{2}}}{2^{\frac{p+4}{2}} \Gamma(\frac{p+4}{2})} dy_1}_{1} \cdot 2^2 \cdot \frac{\Gamma(\frac{p+4}{2})}{\Gamma(\frac{p}{2})}$$

$$= \frac{4 \cdot \Gamma(\frac{p}{2}+2)}{\Gamma(\frac{p}{2})}$$

$$E(\frac{1}{y_2^2}) = \int_0^\infty (y_2)^{-2} \cdot f(Y_2, p)\, dy_2$$

$$= \int_0^\infty \frac{y_2^{\frac{q}{2}-3} \cdot e^{-\frac{y_2}{2}}}{2^{\frac{q}{2}} \Gamma(\frac{q}{2})} dy_2$$

$$= \underbrace{\int_0^\infty \frac{y_2^{\frac{q-4}{2}-1} \cdot e^{-\frac{y_2}{2}}}{2^{\frac{q-4}{2}} \Gamma(\frac{q-4}{2})} dy_2}_{1} \cdot \frac{1}{2^2} \cdot \frac{\Gamma(\frac{q-4}{2})}{\Gamma(\frac{q}{2})}$$

$$= \frac{1}{4} \cdot \frac{\Gamma(\frac{q}{2}-2)}{\Gamma(\frac{q}{2})}$$

$$\therefore E(Z^2) = \frac{q^2}{p^2} \cdot \frac{\cancel{4} \cdot \Gamma(\frac{p}{2}+2)}{\Gamma(\frac{p}{2})} \cdot \frac{\Gamma(\frac{q}{2}-2)}{\cancel{4}\,\Gamma(\frac{q}{2})}$$

$$\because E(Z) = \frac{q}{q-2}$$

$$= \frac{q^2}{p^2} \cdot \frac{(\frac{p}{2}+1)!}{(\frac{p}{2}-1)!} \cdot \frac{(\frac{q}{2}-3)!}{(\frac{q}{2}-1)!}$$

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$$= \frac{q^2}{p^2} \cdot (\frac{p}{2}+1) \cdot \frac{p}{2} \cdot \frac{1}{(\frac{q}{2}-1)(\frac{q}{2}-2)}$$

$$= \frac{q^2}{p^2} \cdot \frac{(p+2)p}{2 \cdot 2} \cdot \frac{1}{\frac{q-2}{2} \cdot \frac{q-4}{2}}$$

$$= \frac{q^2}{p^2} \cdot \frac{p(p+2)}{(q-2)(q-4)} \quad \square$$

$$\cancel{pq} + 2q - 2p - \cancel{pq} + 4p$$

$$2q + 2p$$

$$2(q+p)$$

$$\therefore Var(Z) = E(Z^2) - (EZ)^2$$

$$= \frac{q^2}{p^2} \cdot \frac{p(p+2)}{(q-2)(q-4)} - \frac{q^2}{(q-2)^2}$$

$$= \frac{q^2}{q-2} \left[\frac{p+2}{p(q-4)} - \frac{1}{q-2}\right]$$

$$= \frac{q^2}{q-2} \cdot \frac{(p+2)(q-2) - p(q-4)}{p(q-4)(q-2)}$$

$$= \frac{2q^2}{(q-2)^2} \cdot \frac{(p+q-2)}{p(q-4)} \quad \square$$

(c) Show that $\frac{1}{Z}$ has an $F_{q,p}$ distribution

$$(Z)^{-1} = \left[\frac{Y_1/p}{Y_2/q}\right]^{-1} \qquad Y_1 \sim \chi^2_p \qquad Y_2 \sim \chi^2_q$$

Let $U_1 = \frac{Y_2/q}{Y_1/p} = \frac{1}{Z}$ $\qquad Y_1 = pU_2$

$U_2 = Y_1/p$ $\qquad Y_2 = qU_1U_2$

$$|J| = \begin{vmatrix} 0 & p \\ qU_2 & qU_1 \end{vmatrix} = -pqU_2$$

$$f(Y_1; p) = \frac{y_1^{\frac{p}{2}-1} \cdot e^{-\frac{y_1}{2}}}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})} \quad x>0$$

$$f(Y_2; q) = \frac{y_2^{\frac{q}{2}-1} \cdot e^{-\frac{y_2}{2}}}{2^{\frac{q}{2}} \Gamma(\frac{q}{2})} \quad x>0$$

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}(u_1^{-1}, u_1^{-1}) \cdot pqu_2$$

$$= \frac{(pu_2)^{\frac{p}{2}-1} \cdot e^{-\frac{pu_2}{2}}}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})} \cdot \frac{(qu_1u_2)^{\frac{q}{2}-1} \cdot e^{-\frac{qu_1u_2}{2}}}{2^{\frac{q}{2}} \Gamma(\frac{q}{2})} \cdot pqu_2$$

$$f(x, \alpha, \beta) = \frac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)}$$

$$f_Z(\tfrac{1}{Z}) = f_{U_1}(u_1) = \int_0^\infty f_{U_1U_2}(u_1, u_2)\, du_2$$

$$= \frac{p^{\frac{p}{2}} q^{\frac{q}{2}} u_1^{\frac{q}{2}-1}}{2^{\frac{p+q}{2}} \Gamma(\frac{p}{2}) \Gamma(\frac{q}{2})} \int_0^\infty u_2^{\frac{p+q}{2}-1} e^{-\frac{(p+qu_1)u_2}{2}} du_2$$

let $\alpha = \frac{p+q}{2}$ $\quad \beta = \frac{2}{p+qu_1}$

$$\int_0^\infty u_2^{\frac{p+q}{2}-1} \cdot e^{-\frac{(p+qu_1)u_2}{2}} du_2$$

$$= \underbrace{\int_0^\infty \frac{u_2^{\alpha-1} \cdot e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} du_2}_{1} \cdot \underline{\underline{\beta^\alpha \cdot \Gamma(\alpha)}}$$

$$\beta^\alpha \Gamma(\alpha) = \frac{1}{2^{\frac{p+q}{2}}} (p+qu_1)^{-\frac{p+q}{2}} \Gamma(\frac{p+q}{2})$$

$$\therefore f_Z(\tfrac{1}{Z}) = f_{U_1}(u_1) = \frac{p^{\frac{p}{2}} q^{\frac{q}{2}} u_1^{\frac{q}{2}-1}}{\cancel{2^{\frac{p+q}{2}}} \Gamma(\frac{p}{2}) \Gamma(\frac{q}{2})} \cdot \cancel{2^{\frac{p+q}{2}}} (p+qu_1)^{-\frac{p+q}{2}} \Gamma(\frac{p+q}{2})$$

$$= B(\tfrac{p}{2}, \tfrac{q}{2})^{-1} p^{\frac{p}{2}} q^{\frac{q}{2}} \left[p(1+\tfrac{q}{p}u_1)\right]^{-\frac{p+q}{2}} \cdot u_1^{\frac{q}{2}-1}$$

$$= B(\tfrac{p}{2}, \tfrac{q}{2})^{-1} p^{-\frac{q}{2}} \cdot q^{\frac{q}{2}} (1+\tfrac{q}{p}u_1)^{-\frac{p+q}{2}} \cdot u_1^{\frac{q}{2}-1}$$

$$= B(\tfrac{p}{2}, \tfrac{q}{2})^{-1} \cdot (\tfrac{q}{p})^{\frac{q}{2}} (1+\tfrac{q}{p}u_1)^{-\frac{p+q}{2}} \cdot u_1^{\frac{q}{2}-1} \quad \square$$

$$\sim F_{q,p}$$

$$\therefore f_Z(x) = \frac{x^{\frac{p}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \left(\frac{p}{q}\right)^{\frac{p}{2}} \left(\frac{p}{q}x+1\right)^{-\frac{p+q}{2}} \quad \square$$

(d) Show that $(p/q)X / [1 + p/q\, X] \sim \text{Beta}\left(\frac{p}{2}, \frac{q}{2}\right)$

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}$$

$$\because f_X(x) = \frac{x^{\frac{p}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \left(\frac{p}{q}\right)^{\frac{p}{2}} \left(\frac{p}{q}x + 1\right)^{-\frac{p+q}{2}}$$

Let $Y = \dfrac{(p/q)X}{1+(p/q)X}$

$Y + Y(p/q)X = (p/q)X$

$(Y-1)(p/q)X = -Y$

$X = \dfrac{Y}{1-Y} \cdot \dfrac{q}{p} = \dfrac{q}{p}\left(\dfrac{1}{1-Y} - 1\right)$

$\therefore \dfrac{dX}{dY} = \dfrac{q}{p} \dfrac{1}{(1-Y)^2} (-1)$

$$f_Y(Y) = f_X(Y^{-1}) \cdot \left|\frac{dX}{dY}\right|$$

$$= \frac{\left(\frac{y}{1-y} \cdot \frac{q}{p}\right)^{\frac{p}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \left(\frac{p}{q}\right)^{\frac{p}{2}} \left(\cancel{\frac{p}{q}} \cdot \frac{y}{1-y} \cancel{\frac{q}{p}} + 1\right)^{-\frac{p+q}{2}} \cdot \frac{q}{p} \frac{1}{(1-y)^2}$$

$$= \frac{\left(\frac{y}{1-y}\right)^{\frac{p}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \cancel{\left(\frac{q}{p}\right)^{-1}} \underbrace{\left(\frac{y}{1-y} + 1\right)^{-\frac{p+q}{2}}}_{\Downarrow} \cdot \cancel{\frac{q}{p}} \cdot \frac{1}{(1-y)^2}$$

$$= \frac{(y)^{\frac{p}{2}-1}}{B(\frac{p}{2}, \frac{q}{2})} \cdot (1-y)^{1-\frac{p}{2}} \cdot (1-y)^{\frac{p+q}{2}} \cdot (1-y)^{-2}$$

$$= \frac{1}{B(\frac{p}{2}, \frac{q}{2})} (1-y)^{\frac{q}{2}-1} \cdot y^{\frac{p}{2}-1} \sim \text{Beta}\left(y, \frac{p}{2}, \frac{q}{2}\right)$$

5.18 Let $X \sim$ Student's t distribution with p degrees of freedom

(a) Derive the mean & variance of X

P223 [252]

$$f_T(t) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \cdot \frac{1}{(p\pi)^{1/2}} \cdot \frac{1}{(1+t^2/p)^{(p+1)/2}}$$

$$E(X) = \int_0^\infty x \cdot f_X(x)\, dx$$

$$= \int_0^\infty x \cdot \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \cdot \frac{1}{(p\pi)^{1/2}} \cdot \frac{1}{(1+x^2/p)^{\frac{(p+1)}{2}}}$$

$$= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \cdot \frac{1}{(p\pi)^{1/2}} \cdot \int_0^\infty \frac{x\, p^{\frac{1+p}{2}}}{(p+x^2)^{\frac{1+p}{2}}}\, dx$$

$$= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \cdot \frac{p^{\frac{1+p}{2}}}{(p\pi)^{1/2}} \int_0^\infty \frac{x}{(p+x^2)^{\frac{1+p}{2}}}\, dx$$ 做不下去 ...

换思路：定义解题

$$X = \frac{U}{\sqrt{V/p}}$$

$U \sim n(0,1)$ $V \sim \chi^2_p$

$E(X) = E\dfrac{U}{\sqrt{V/p}}$ $\because$ U and V are independent

$\therefore E(X) = EU / E\sqrt{V/p}$ $\because \text{Var}\, V = p$

$\therefore EU = 0$

$\therefore E(X) = 0$

$X = \dfrac{U}{\sqrt{V/p}}$ $\underbrace{X^2 = \dfrac{U^2}{V/p} \sim F_{1,p}}$ $\therefore E(X^2) = \dfrac{p}{p-2}$ (5.17 b)

$\text{Var}(X^2) = E(X^2) - (EX)^2 = \dfrac{p}{p-2} - 0 = \dfrac{p}{p-2}$ □

(b) Show that X^2 has $\sim F(1, p)$

(c) Let $f(x|p)$ denote the pdf of X

show that $\lim_{p\to\infty} f(x|p) \to \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Hint: Use Stirling's Formula

$$f_X(x) = \underbrace{\frac{\Gamma(\frac{p+1}{2})}{\Gamma(p/2)\sqrt{p\pi}}}_{①} \underbrace{\frac{1}{(1+x^2/p)^{(p+1)/2}}}_{②}$$

Stirling's formula:

$\ln N! \approx N \cdot \ln N - N$ $(N \to +\infty)$

$\therefore N! \approx e^{N\ln N - N} = e^{N\ln N} \cdot e^{-N} = N^N \cdot e^{-N}$

$\Gamma(\frac{p+1}{2}) = (\frac{p+1}{2} - 1)! = (\frac{p-1}{2})!$

let $N = \frac{p-1}{2}$, $\Gamma(\frac{p-1}{2}) = (\frac{p-1}{2})^{\frac{p-1}{2}} \cdot e^{-\frac{p-1}{2}}$

let $N = \frac{p-2}{2}$ $\Gamma(\frac{p}{2}) = (\frac{p-2}{2})^{\frac{p-2}{2}} \cdot e^{-\frac{p-2}{2}}$

$$\therefore \frac{\Gamma(\frac{p-1}{2})}{\Gamma(\frac{p-2}{2})\sqrt{\pi}} = \frac{(p-1)^{\frac{p-1}{2}}}{(p-2)^{\frac{p-2}{2}}} \frac{1}{\sqrt{2}} \cdot e^{-\frac{1}{2}} \cdot p^{-\frac{1}{2}} \frac{1}{\sqrt{\pi}}$$

$$= \frac{e^{-\frac{1}{2}}}{\sqrt{2}\sqrt{\pi}} \left[\frac{(p-1)^{p-1}}{(p-2)^{p-2} \cdot p}\right]^{\frac{1}{2}}$$

$$\lim_{p\to\infty} = \frac{e^{-\frac{1}{2}}}{\sqrt{2}\sqrt{\pi}} \left[\frac{p-1}{p} \left(\frac{p-1}{p-2}\right)^{p-2}\right]^{\frac{1}{2}}$$

$$= \frac{e^{-\frac{1}{2}}}{\sqrt{2}\sqrt{\pi}} \left[\left(1+\frac{1}{p-2}\right)^{p-2}\right]^{\frac{1}{2}}$$

$\because \lim_{n\to\infty} (1+\frac{a}{n})^n = e^a \quad \therefore (1+\frac{1}{p-2})^{p-2} = e$

$$\therefore ① = \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}$$

$$② = \frac{1}{(1+x^2/p)^{\frac{p+1}{2}}} \quad \lim_{p\to\infty} ② = \frac{1}{[(1+\frac{x^2}{p})^p]^{\frac{1}{2}}} = \frac{1}{(e^{x^2})^{\frac{1}{2}}}$$

$$\therefore f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \square$$

(d) Use (a), (b) argue that $p\to\infty$, Z^2 converges in distribution to a χ_1^2 random variable.

$Z^2 = \frac{U^2}{V/p} \sim F_{1,p}$ $\quad \chi_1^2 \sim U^2$

$p\to\infty$ $V/p \to 0$ $\quad Z^2 \approx U^2$

(e) what might you conjecture about the distribution limit as $p\to\infty$, of $q \cdot F_{q,p}$?

$q\, F_{q,p}$

$F_{q,p} = \frac{U/q}{V/p}$ $\quad U \sim \chi_q^2$ $\quad V \sim \chi_p^2$

$p\to\infty$ $F_{q,p} \to U/q$ $\quad$ the distribution will be chi distribution with the freedom of q $\square$

$q\, F_{q,p} \to U$

5.19 (a) Prove if $p>q$. $P(\chi_p^2 > a) \geq P(\chi_q^2 > a)$

$$f_p(x) = \frac{1}{2^{\frac{p}{2}-1}\Gamma(\frac{p}{2})} x^{p-1} e^{-x/2}$$

$\because p > q$ Let $p = q + d$

then $\chi_p^2 = \chi_q^2 + \chi_d^2$

$P(\chi_p^2 > a) = P(\chi_q^2 + \chi_d^2 > a) \geq P(\chi_q^2 > a)$

(b) Use the result (a) prove for any v, $kF_{k,v}$ is stochastically increasing in k.

Aim: if $k_1 > k_2$

$P(k_1 F_{k_1,v} > a) \geq P(k_1 F_{k_2,v} > a)$

$F_{k_1,v} = \frac{U_1/k_1}{S/v}$ $\quad U_1 \sim \chi_{k_1}^2$ $\quad S \sim \chi_v^2$

$k_1 F_{k_1,v} = \frac{U_1}{S/v}$

$k_2 F_{k_2,v} = \frac{U_2}{S/v}$ $\quad U_2 \sim \chi_{k_2}^2$

let $k_1 = k_2 + d$

$k_1 F_{k_1,v} = \frac{U_1}{S/v} = \frac{\chi_{k_1}^2}{S/v} = \frac{\chi_{k_2}^2 + \chi_d^2}{S/v}$

$= k_2 F_{k_2,v} + d\, F_{d,v}$

$\therefore P(k_1 F_{k_1,v} > a) = P(k_2 F_{k_2,v} + d\, F_{d,v} > a) \geq P(k_2 F_{k_2,v})$ o

5.20 (a) $f_{T_v}(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2 x/2v} \frac{\sqrt{x}}{\sqrt{v}} \frac{1}{\Gamma(v/2) 2^{v/2}} x^{\frac{v}{2}-1} \cdot e^{-x/2} dx$

Aim: Integral $f_{T_v}(t)$, simplify $\frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)} \frac{1}{(v\pi)^{\frac{1}{2}}} \frac{1}{(1+t^2/v)^{(v+1)/2}}$

$$f_{T_v}(t) = \int_0^\infty \frac{1}{(v\pi)^{\frac{1}{2}}} \frac{1}{\Gamma(v/2)} \frac{1}{2^{\frac{1}{2}}} \frac{1}{2^{v/2}} \cdot x^{\frac{1}{2}} e^{-t^2 x/2v} x^{\frac{v}{2}-1} e^{-x/2} dx$$

$$= \frac{1}{(v\pi)^{\frac{1}{2}}} \frac{1}{\Gamma(v/2)} \int_0^\infty \frac{1}{2^{\frac{v+1}{2}}} x^{\frac{v+1}{2}-1} e^{-\frac{x(t^2+v)}{2v}} dx$$

$x^{\frac{v-1}{2}}$

$(x^{\frac{1}{2}})^{v-1}$

$$= \frac{1}{(v\pi)^{\frac{1}{2}}} \frac{1}{\Gamma(v/2)} 2^{-\frac{v+1}{2}} \int_0^\infty x^{\frac{v+1}{2}-1} e^{-\frac{x(\frac{t^2}{v}+1)}{2}} dx$$

$$= \frac{1}{(v\pi)^{\frac{1}{2}}} \frac{1}{\Gamma(v/2)} \underbrace{2^{-\frac{v+1}{2}} \int_0^\infty (x^{\frac{1}{2}})^{v-1} \cdot e^{-\frac{x(\frac{t^2}{v}+1)}{2}} dx}_{A}$$

let $y = x \cdot (\frac{t^2}{v}+1)$ $\quad x=0 \quad y=0$, $x=\infty \quad y=\infty$

$x = (\frac{t^2}{v}+1)^{-1} y \qquad dx = (\frac{t^2}{v}+1)^{-1} dy$

$$A = 2^{-\frac{v+1}{2}} \int_0^\infty \left[y^{\frac{1}{2}} (\frac{t^2}{v}+1)^{-\frac{1}{2}}\right]^{v-1} e^{-\frac{y}{2}} \cdot (\frac{t^2}{v}+1)^{-1} dy$$

$$= \frac{1}{2^{\frac{v+1}{2}}} \int_0^\infty y^{\frac{v+1}{2}-1} (\frac{t^2}{v}+1)^{-\frac{v+1}{2}} \cdot e^{-\frac{y}{2}} dy$$

$$= \frac{1}{(\frac{t^2}{v}+1)^{\frac{v+1}{2}}} \Gamma(\tfrac{v+1}{2}) \underbrace{\int_0^\infty \frac{1}{2^{\frac{v+1}{2}} \Gamma(\frac{v+1}{2})} \cdot y^{\frac{v+1}{2}-1} e^{-\frac{y}{2}} dy}_{\text{chi-dis } (v) \quad 1}$$

$$A = \frac{1}{(\frac{t^2}{v}+1)^{\frac{v+1}{2}}} \Gamma(\tfrac{v+1}{2})$$

Chi:

$$f_p(x) = \frac{1}{2^{\frac{p}{2}} \Gamma(\frac{p}{2})} x^{\frac{p}{2}-1} e^{-\frac{x}{2}}$$

$$\therefore f_{T_v}(t) = \frac{1}{(v\pi)^{\frac{1}{2}}} \frac{1}{\Gamma(v/2)} A = \frac{1}{(v\pi)^{\frac{1}{2}}} \frac{\Gamma(\frac{v+1}{2})}{\Gamma(v/2)} \cdot (\frac{t^2}{v}+1)^{-\frac{v+1}{2}} \quad \square$$

(b) $F_{1,v} = \frac{U}{S/v}$ $\qquad U \sim \chi_1^2$, $n(0,1)$ $\qquad S \sim \chi_v^2$:

$$P(F_{1,v} \le vt) = P(\frac{U}{S/v} \le vt) = P(\frac{U}{S} \le t)$$

Let $S = f_v(y)$

$$P(F_{1,v} < vt) = P(\frac{U}{y} \le t) = \int_0^\infty P(\chi_1^2 \le ty) \cdot f_v(y) dy \quad , \; ? \; f_F(vt)$$

$$\frac{d\, P(F_{1,v} < vt)}{dt} = \frac{d \int_0^\infty P(\chi_1^2 \le ty) f_v(y) dy}{dt}$$

Chi:

$$f_1(x) = \frac{1}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})} x^{-\frac{1}{2}} e^{-\frac{x}{2}}$$

$$f_v(x) = \frac{1}{2^{\frac{v}{2}} \Gamma(\frac{v}{2})} x^{\frac{v}{2}-1} e^{-\frac{x}{2}}$$

$$v \cdot f_F(vt) = \int_0^\infty y \, f_1(ty) \cdot f_v(y) dy$$

$$= \int_0^\infty y \frac{1}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})} (ty)^{-\frac{1}{2}} e^{-\frac{ty}{2}} \frac{1}{2^{\frac{v}{2}} \Gamma(\frac{v}{2})} y^{\frac{v}{2}-1} e^{-\frac{y}{2}} dy$$

$$= \int_0^\infty \frac{t^{-\frac{1}{2}}}{2^{\frac{1+v}{2}} \Gamma(\frac{1}{2}) \Gamma(\frac{v}{2})} y^{\frac{v-1}{2}} e^{-\frac{y(t+1)}{2}} dy$$

$$= \frac{t^{-\frac{1}{2}} \Gamma(\frac{v+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{v}{2}) (t+1)^{\frac{v+1}{2}-1}} \underbrace{\frac{1}{t+1} \int_0^\infty \frac{1}{2^{\frac{v+1}{2}} \Gamma(\frac{v+1}{2})} [(t+1) y]^{\frac{v+1}{2}-1} e^{-\frac{y(t+1)}{2}} dy}_{1}$$

$$= \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{v}{2})} \frac{t^{-\frac{1}{2}}}{(t+1)^{\frac{v+1}{2}}}$$

$$\therefore f_F(vt) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{v}{2})} \frac{t^{-\frac{1}{2}}}{(t+1)^{\frac{v+1}{2}}} \cdot v^{-1} \quad ①$$

Let $vt = y$, $t = \frac{y}{v}$

Substitute t in ① with $\frac{y}{v}$

$$f_F(y) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{v}{2})} \frac{(\frac{y}{v})^{-\frac{1}{2}}}{(\frac{y}{v}+1)^{\frac{v+1}{2}}} v^{-1}$$

$$= \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{v}{2})} \cdot \frac{(vy)^{-\frac{1}{2}}}{(\frac{y}{v}+1)^{\frac{v+1}{2}}} \quad \square \sim F_{1,v}$$

(c) Verify that $P(F_{m,v} \le \frac{v}{m} t) = \int_0^\infty P(\chi_m^2 \le ty) \cdot f_v(y)\,dy$

Differentiate both sides: dt.

$$\frac{v}{m} f_{m,v}\left(\frac{v}{m} t\right) = \int_0^\infty y\, f_m(ty)\, f_v(y)\,dy$$

$$\frac{v}{m} f_{m,v}\left(\frac{v}{m} t\right) = \int_0^\infty y \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} (ty)^{\frac{m}{2}-1} e^{-\frac{ty}{2}} \frac{1}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} y^{\frac{v}{2}-1} e^{-\frac{y}{2}}\,dy$$

$f_m(x) \sim \chi_m^2$

$f_v(x) \sim \chi_v^2$

$f_{m,v}(x) \sim \frac{f_m(x)/m}{f_v(x)/v}$

cha

$f_p(x) = \frac{1}{2^{\frac{p}{2}}\Gamma(\frac{p}{2})} x^{\frac{p}{2}-1} e^{-\frac{x}{2}}$

$$f_{m,v}\left(\frac{v}{m} t\right) = \frac{m}{v} \int_0^\infty \frac{t^{\frac{m}{2}-1}}{2^{\frac{m+v}{2}}\Gamma(\frac{m}{2})\Gamma(\frac{v}{2})} y^{\frac{m+v}{2}-1} e^{-\frac{y(t+1)}{2}}\,dy$$

$$f_{m,v}\left(\frac{v}{m} t\right) = \frac{m}{v} \frac{t^{\frac{m}{2}-1}}{(t+1)^{\frac{m+v}{2}-1}} \frac{\Gamma(\frac{m+v}{2})}{t+1} \frac{1}{\Gamma(\frac{m}{2})\Gamma(\frac{v}{2})} \underbrace{\int_0^\infty \frac{1}{2^{\frac{m+v}{2}}\Gamma(\frac{m+v}{2})} [y(t+1)]^{\frac{m+v}{2}-1} e^{-\frac{y(t+1)}{2}}\,d\,y(t+1)}_{1}$$

$$= \frac{m}{v} \frac{t^{\frac{m}{2}-1}}{(t+1)^{\frac{m+v}{2}}} \frac{1}{B(\frac{m}{2}, \frac{v}{2})}$$

Let $\frac{v}{m} t = y$

$t = \frac{m}{v} y$

$$\therefore f_{m,v}(y) = \frac{m}{v t} \frac{t^{\frac{m}{2}}}{(t+1)^{\frac{m+v}{2}}} \cdot \frac{1}{B(\frac{m}{2}, \frac{v}{2})}$$

$$= y^{-1} \frac{(\frac{m}{v} y)^{\frac{m}{2}}}{(1+\frac{m}{v} y)^{\frac{m+v}{2}}} \frac{1}{B(\frac{m}{2}, \frac{v}{2})} \quad \square$$

5.21. Z_1, Z_2 are iid

$P(\max(Z_1, Z_2) > \text{median})$? Generalize to size n.

Assume that median is m.

$P(\max(Z_1, Z_2) > m) = 1 - P(Z_1 \le m) \cdot P(Z_2 \le m)$

$\because$ m is median

$\therefore P(Z_i \le m) = \frac{1}{2}$

$\therefore P(\max(Z_1, Z_2) > m) = 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

Generalization: $Z_1, Z_2, Z_3, \ldots, Z_n$

$P(\max(Z_1, Z_2, \ldots, Z_n) > m) = 1 - P(Z_i \le m) \quad (i = 1, 2, \cdots, n)$

$= 1 - (\frac{1}{2})^n \quad \square$

5.22. X, Y are iid, $\sim n(0,1)$, $Z = \min(X, Y)$

Prove $Z^2 \sim \chi_1^2$

$\because Z = \min(X, Y)$

Z, Y are iid

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$\therefore f_Z(z) = \frac{1}{2} f_X(z) + \frac{1}{2} f_Y(z)$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sim n(0,1)$

$\therefore$ According to the definition of Chi distribution,

$Z^2 \sim \chi_1^2 \quad \square$

5.23. Let $U_i, i = 1, 2, \cdots \sim U(0,1)$, iid.

Z: $P(Z = x) = \frac{c}{x!}$, $c = \frac{1}{e-1}$

Find the distribution of $Z = \min\{U_1, \cdots, U_Z\}$

Hint $Z | Z = x$

$$P(Z \mid X = x) = \underbrace{\min\{U_1, U_2, \cdots, U_x\}}_{\frac{1}{(e-1)x!}}$$

Assume that $\min\{U_1, U_2, \quad, U_x\} = z$

that is

$$P_2(z) = \sum_{x=1}^{\infty} \underbrace{P(z \mid X = x)}\ P(x)$$

$\because$ $U_1, U_2, \cdots, U_x$ are independent

$\therefore$ $P(z \mid X = x) = \prod_{i=1}^{x} P(U_i \geq z) = (1-z)^x$

$\because$ $U_i \sim$ Uniform $(0,1)$

$\therefore$ $P(U_i \geq z) = 1 - z$

$$\therefore P_2(z) = \sum_{x=1}^{\infty} (1-z)^x \frac{1}{(e-1)x!}\, dx$$

$$= \frac{1}{e-1} \sum_{x=1}^{\infty} \frac{(1-z)^x}{x!}\, dx$$

$$= \frac{e^{1-z} - 1}{e-1} \qquad \square$$

[Maclaurin Series]

$$f(x) = f^0(x) + f^1(x)\, x + f^2(x) \cdot \frac{x^2}{2!}$$

$$f(x) = f^0(x_0) + f^1(x_0)(x - x_0) + f^2(x_0) \frac{(x-x_0)^2}{2!} + \cdots + f^n(x_0) \frac{(x-x_0)^n}{n!}$$

P18 If $x_0 = 0$, $f(x) = e^x$, then

$$e^x = 1 + 1 \cdot x + 1 \frac{x^2}{2!} + \cdots + 1 \frac{x^n}{n!}$$

$$= \sum_{i=0}^{n} 1 \frac{x^i}{i!} \qquad \because e^0 = 1 \quad \because f^1(x_0) = f^2(x_0) = \quad = f^n(x_0) = 1$$

$$\therefore \sum_{i=1}^{n} \frac{(1-z)^x}{x!} = e^{1-z} - 1$$

5.24. Let $Z_1, \cdots, Z_n$ be random sample from

$$f_Z(x) = \begin{cases} 1/\theta & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

Let $Z_{(1)} < \cdots < Z_{(n)}$ be the order statistics.

Show that $Z_{(1)} / Z_{(n)}$ and $Z_{(n)}$ are independent

5.4.6 P230 [260] $1 \leq i < j \leq n$

$$f_{Z_{(i)}, Z_{(j)}}(u, v) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f_Z(u) f_Z(v)$$

$$F_Z(u)^{i-1} \cdot [F_Z(v) - F_Z(u)]^{j-i-1} [1 - F_Z(v)]^{n-j}$$

$$f_Z(x) = \frac{1}{\theta} \qquad 0 < x < \theta$$

$$F_Z(x) = \int_0^x f(x)\, dx = \frac{x}{\theta} \qquad \text{Assume } Z_{(1)} = U \qquad Z_{(n)} = V$$

From 5.4.6.

$$f_{Z_{(1)} Z_{(n)}}(u, v) = \frac{n!}{\underset{1}{0!}(n-2)!\,\underset{1}{0!}} \frac{1}{\theta} \frac{1}{\theta} \underset{1}{\left(\frac{u}{\theta}\right)^0} \left(\frac{v}{\theta} - \frac{u}{\theta}\right)^{n-2} \underset{1}{\left(\frac{v}{\theta}\right)^0} \qquad 0 < u < v < \theta$$

$$= \frac{n(n-1)}{\theta^2} \left(\frac{v-u}{\theta}\right)^{n-2}$$

Aim:

$$f_{Z_{(1)}/Z_{(n)}}\left(\frac{u}{v}\right)$$

Assume $M = \frac{U}{V}$; $U = Mk$. $|J| = \begin{vmatrix} k & m \\ 0 & 1 \end{vmatrix} = k > 0 \qquad k \in (0, \theta)$

$K = V \qquad V = k$

$$f_{Z_{(1)}/Z_{(n)}}\left(\frac{u}{v}\right) = f_M(m) = \int_0^\theta f_{M,K}(m, k)\, dk = \int_0^\theta f_{U,V}(m^{-1}, k^{-1}) \cdot k\, dk$$

$$= \int_0^\theta \frac{n(n-1)}{\theta^2} \left(\frac{k - mk}{\theta}\right)^{n-2} k\, dk = (n-1)(1-m)^{n-2}$$

which is independent with $f_{Z_{(n)}}(x) = \frac{1}{\theta}$ $\square$

$$\frac{n(n-1)}{\theta^n} \int_0^\theta [(1-m)k]^{n-2} k$$

$$\frac{n(n-1)}{\theta^n} (1-m)^{n-2} \int_0^\theta k^{n-1}\, dk$$

$$\frac{1}{n} k^n \Big|_0^\theta$$

$$\frac{n(n-1)}{\theta^n} (1-m)^{n-2} \cdot \frac{1}{n} \theta^n$$

5.25. $Z_1, \cdots, Z_n$ iid

$$f_Z(x) = \begin{cases} \frac{a}{\theta^a} x^{a-1} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

let $Z_{(1)} < \cdots < Z_{(n)}$

Show that $Z_{(1)}/Z_{(2)}$, $Z_{(2)}/Z_{(3)}, \cdots, Z_{(n-1)}/Z_{(n)}$ and Z_n are mutually independent.

5.46 P230 [260] $1 \le i < j \le n$

$$f_{Z_{(i)}, Z_{(j)}}(u, v) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f_Z(u) f_Z(v)$$
$$F_Z(u)^{i-1} \left[F_Z(v) - F_Z(u)\right]^{j-i-1} \left[1 - F_Z(v)\right]^{n-j}$$

Aim: $f_{Z_{(i-1)}/Z_{(i)}}(u/v)$

Assume $Z_{(i-1)} = U$ $\quad Z_{(i)} = V$ $\quad Z_{(i-1)} < Z_{(i)}$ $\quad i \geq 2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad U < V$

$$f_{Z_{(1)}}(u) = \frac{a}{\theta^a} u^{a-1} \qquad 0 < u < \theta$$

$$F_{Z_{(1)}}(u) = \int_0^u \frac{a}{\theta^a} u^{a-1} du = \left(\frac{u}{\theta}\right)^a$$

$$\therefore f_{Z_{(i-1)}, Z_{(i)}}(u, v) = \frac{n!}{(i-2)!\,0!\,(n-i)!} \frac{a}{\theta^a} u^{a-1} \frac{a}{\theta^a} v^{a-1} \left(\frac{u}{\theta}\right)^{a(i-2)} \underbrace{\left[\left(\frac{v}{\theta}\right)^a - \left(\frac{u}{\theta}\right)^a\right]^0}_{1} \left[1 - \left(\frac{v}{\theta}\right)^a\right]^{n-i}$$

$$= \frac{n!}{(i-2)!(n-i)!} \frac{a^2}{\theta^{2a}} (uv)^{a-1} \left(\frac{u}{\theta}\right)^{a(i-2)} \left[1 - \left(\frac{v}{\theta}\right)^a\right]^{n-i}$$

$$= \frac{n!}{(i-2)!(n-i)!} \frac{a^2}{\theta^{an}} u^{ai-a-1} v^{a-1} (\theta^a - v^a)^{n-i}$$

Let $m = u/v$ $\quad U = mk$ $\quad |J| = \begin{vmatrix} k & m \\ 0 & 1 \end{vmatrix} = k$

$\qquad k = v$ $\quad v = k$

$$f_{Z_{(i-1)}/Z_{(i)}}(u/v) = \int_0^\theta f_{Z_{(i-1)}, Z_{(i)}}(m^{-1}, k^{-1}) |J| \, dk$$

$$= \int_0^\theta \frac{n!}{(i-2)!(n-i)!} \frac{a^2}{\theta^{an}} (mk)^{ai-a-1} k^{a-1} (\theta^a - k^a)^{n-i} k \, dk$$

$$= \frac{n!}{(i-2)!(n-i)!} \frac{a^2}{\theta^{an}} m^{ai-a-1} \int_0^\theta k^{ai-a} k^{a-1} (\theta^a - k^a)^{n-i} dk$$

$$\int_0^\theta k^{ai-1} (\theta^a - k^a)^{n-i} dk \quad \text{与 } i \text{ 有关}$$

换思路 Joint pdf of $Z_{(1)}, Z_{(2)}, \cdots, Z_{(n)}$

$$f(u_1, u_2, \cdots, u_n) = n! \left(\frac{a}{\theta^a}\right)^n u_1^{a-1} u_2^{a-1} \cdots u_n^{a-1}$$

对角线元素相乘

Assume $Y_1 = Z_{(1)}/Z_{(2)}$ $\quad Z_{(1)} = Y_n Y_{n-1} \cdots Y_1$

$Y_2 = Z_{(2)}/Z_{(3)}$ $\quad \Rightarrow$

$\vdots$

$Y_{n-1} = Z_{(n-1)}/Z_{(n)}$ $\quad Z_{(n-1)} = Y_n Y_{n-1}$

$Y_n = Z_{(n)}$ $\quad Z_{(n)} = Y_n$

$$|J| = \begin{vmatrix} Y_2 \cdots Y_n & \cdots & Y_1 \cdots Y_{n-1} \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & \cdots & 1 \end{vmatrix} = y_2 y_3^2 \cdots y_n^{n-1}$$

$2a-2+2$

$$\therefore f(y_1, \cdots, y_n) = n! \left(\frac{a}{\theta^a}\right)^n (y_1 \cdots y_n)^{a-1} (y_2 \cdots y_n)^{a-1} \cdots (y_n)^{a-1} \cdot y_2 y_3^2 \cdots y_n^{n-1}$$

$$= (y_1 y_2^2 \cdots y_n^n)^{a-1} y_2 y_3^2 \cdots y_n^n = y_1^{a-1} y_2^{2a-1} y_3^{3a-1} \cdots y_n^{na-1}$$

that is $f(y_1, y_2, \cdots, y_n)$ factors

$f_{Y_1}(y_1) = C_1 y_1^{a-1}$ $\quad$ C_1 is a constant

$\because \int_0^1 f_{Y_1}(y_1)\, dy_1 = 1$

$Y_1 = Z_{(1)}/Z_{(2)}$ $\quad 0 < Z_{(1)} < Z_{(2)} < \theta$

$Y_1 \in (0,1)$ $\quad Y_1, \cdots Y_{n-1} \in (0,1)$

$\therefore \frac{C_1}{a} y_1^a \Big|_0^1 = 1$

$Y_n = Z_n$

$Y_n \in (0, \theta)$

$\therefore C_1 = a$

$f_{Y_k}(y_k) = C_k y_k^{ka-1}$ $\quad 1 \le k \le n-1$

$\int_0^1 f_{Y_k}(y_k)\, dy_k = 1$

$\therefore \frac{C_k}{ka} y_k^{ka} \Big|_0^1 = 1$

$C_k = ka$

$\therefore f_{Y_i}(y_i) = \begin{cases} (ia)\, y_i^{ia-1} & i = 1, \cdots, n-1 \\ n \cdot \frac{a}{\theta^{an}} \cdot y_n^{na-1} & i = n \end{cases}$ $\quad \square$

5.26. (a). U: counts the number of $Z_1, \cdots, Z_n \le u$

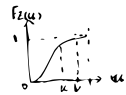
V: counts the number of $Z_1, \cdots, Z_n \ge u$, $\le v$

Show that $(U, V, n-U-V)$

multinomial random vector with n trials

with cell probabilities $(F_Z(u), F_Z(v) - F_Z(u), 1 - F_Z(v))$

$P_U(u) = C_n^u (F_Z(u))^u (1 - F_Z(u))^{n-u}$



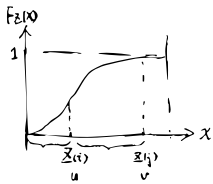
$P_V(v) = C_n^v (F_Z(v) - F_Z(u))^v (1 - F_Z(v) - F_Z(u))^{n-v}$

$P_{n-U-V}(k) = C_n^k (1 - F_Z(v))^k (F_Z(v))^{n-k}$ $\quad \square$

(b) Show that $\quad Z_{(1)} \quad Z_{(2)} \quad \underset{u}{Z_{(i)}} \cdots \underset{v}{Z_{(j)}} \cdots Z_{(n)}$

$F_{Z_{(i)}, Z_{(j)}}(u, v)$

$= P(U \ge i,\ U + V \ge j) = \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} P(U = k,\ V = m) + P(U \ge j)$



$= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k!\, m!\, (n-k-m)!} \cdot [F_Z(u)]^k \cdot [F_Z(v) - F_Z(u)]^m [1 - F_Z(v)]^{n-k-m}$

$+ P(U \ge j)$

至少有 i 个 $x \le u \Rightarrow U \ge i$ $\quad$ $P(U \ge i,\ U + V \ge j)$ $\quad$ $i \le j$

至少有 j 个 $x \le v \Rightarrow j \le U + i$ $\quad$ 1° $U < j$ 至少 2° $U \ge j$ $\quad$ $U = V$

$u \le j - i \le v$ $\quad$ $j \le V + U$

From (a)

$P(\overset{k}{U} \ge i,\ \overset{k}{U} + \overset{m}{V} \ge j)$

$= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k!\, m!\, (n-k-m)!} [F_Z(u)]^k [F_Z(v) - F_Z(u)]^m [1 - F_Z(v)]^{n-k-m} + P(U \ge j)$

$P(U \ge j) = \sum_{k=j}^{n} [F_Z(u)]^k$ $\quad \square$

(c) Find $f_{Z_{(i)}, Z_{(j)}}(u,v)$ by computing mixed partial

Hint: mixed partial of $P(U \geq j)$ is 0

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

P147 [175]

$$P(\overset{k}{U \geq i}, \overset{k}{U+V} \geq j)$$

$$= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k!\, m!\, (n-k-m)!} \underbrace{[F_Z(u)]^k [F_Z(v) - F_Z(u)]^m [1 - F_Z(v)]^{n-k-m}}_{A} + P(U \geq j)$$

$$\frac{\partial^2 P(U \geq j)}{\partial u \partial v} = 0$$

$$f_{Z_{(i)}, Z_{(j)}}(u,v) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f_Z(u) f_Z(v)$$

$$F_Z(u)^{i-1} \cdot [F_Z(v) - F_Z(u)]^{j-i-1} [1 - F_Z(v)]^{n-j}$$

$$\frac{\partial A}{\partial u} = k \cdot f_Z(u) [F_Z(v) - F_Z(u)]^m [1 - F_Z(v)]^{n-k-m}$$

$$+ [F_Z(u)]^k \cdot m [F_Z(v) - F_Z(u)]^{m-1} (-1) f_Z(u) [1 - F_Z(v)]^{n-k-m}$$

$$= [1 - F_Z(v)]^{n-k-m} f_Z(u) (F_Z(v) - F_Z(u))^{m-1} [k (F_Z(v) - F_Z(u)) - (F_Z(u))^k m]$$

$$\frac{\partial^2 A}{\partial u \partial v} = k \cdot f_Z(u) \Big[m (F_Z(v) - F_Z(u))^{m-1} (1 - F_Z(v))^{n-k-m} f_Z(v) + (F_Z(v) - F_Z(u))^m (n-k-m)(1 - F_Z(v))^{n-k-m-1} (-1) \cdot f_Z(v) \Big]$$

$$- (F_Z(u))^k m \cdot f_Z(u) \Big[(m-1)(F_Z(v) - F_Z(u))^{m-2} (1 - F_Z(v))^{n-k-m} f_Z(v) + (F_Z(v) - F_Z(u))^{m-1} \cdot (n-k-m)(1 - F_Z(v))^{n-k-m-1} (-1) \cdot f_Z(v) \Big]$$

$$= m (F_Z(v) - F_Z(u))^{m-1} (1 - F_Z(v))^{n-k-m} f_Z(v) \Big[k \cdot f_Z(u) - \cancel{m} f_Z(u) (F_Z(u))^k \frac{m-1}{\cancel{m}} \frac{1}{F_Z(v) - F_Z(u)} \Big]$$

$$+ (F_Z(v) - F_Z(u))^{m-1} \cdot (n-k-m)(1 - F_Z(v))^{n-k-m-1} (-1) \cdot f_Z(v) \Big[-(F_Z(u))^k m \cdot f_Z(u) + k f_Z(u) (F_Z(v) - F_Z(u)) \Big]$$

$$= m k (F_Z(v) - F_Z(u))^{m-1} (1 - F_Z(v))^{n-k-m} f_Z(v) f_Z(u) \Big[1 - \frac{(F_Z(u))^k (m-1)}{k (F_Z(v) - F_Z(u))} \Big]$$

$$mk$$

$$- (F_Z(v) - F_Z(u))^{m-1} \cdot (n-k-m)(1 - F_Z(v))^{n-k-m-1} \cdot f_Z(v) \; f_Z(u) \Big[\frac{k (F_Z(v) - F_Z(u))}{m} - \frac{m (F_Z(u))^k}{k} \Big]$$

5.27. $Z_1, \cdots, Z_n$ iid. $\sim f_Z(x)$ $F_Z(x)$

$Z_{(1)} < \cdots < Z_{(n)}$

(a) Find $f(Z_{(i)} \mid Z_{(j)})$

$$f_{Z_{(i)}|Z_{(j)}}(u|v) = \frac{f_{Z_{(i)},Z_{(j)}}(u,v)}{f_{Z_{(j)}}(v)}$$

1° $i<j$

$$f_{Z_{(i)},Z_{(j)}}(u,v) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} f_Z(u)\, f_Z(v)\, F_Z(u)^{i-1} \left(F_Z(v)-F_Z(u)\right)^{j-i-1} \left(1-F_Z(v)\right)^{n-j} \quad ①$$

△ $$f_{Z_{(j)}}(v) = \frac{n!}{(j-1)!\,(n-j)!} f_Z(v) \left(F_Z(v)\right)^{j-1} \left(1-F_Z(v)\right)^{n-j} \quad ②$$

$$\therefore f_{Z_{(i)}|Z_{(j)}}(u|v) = \frac{①}{②} = \frac{(j-1)!}{(i-1)!\,(j-i-1)!} \frac{f_Z(u)}{F_Z(v)} \left(\frac{F_Z(u)}{F_Z(v)}\right)^{i-1} F_Z(v)^{i-1-j+1+1} \left(F_Z(v)-F_Z(u)\right)^{j-i-1}$$

$$= \frac{(j-1)!}{(i-1)!\,(j-i-1)!} \frac{f_Z(u)}{F_Z(v)} \left(\frac{F_Z(u)}{F_Z(v)}\right)^{i-1} \left(1-\frac{F_Z(u)}{F_Z(v)}\right)^{j-i-1}$$

2° $i>j$

$$f_{Z_{(j)},Z_{(i)}}(v,u) = \frac{n!}{(j-1)!\,(i-j-1)!\,(n-i)!} f_Z(u)\cdot f_Z(v)\, F_Z(v)^{j-1} \left(F_Z(v)-F_Z(u)\right)^{i-j-1} \left(1-F_Z(u)\right)^{n-i} \quad ③$$

$$f_{Z_{(i)}|Z_{(j)}}(u|v) = \frac{③}{②} = \frac{(n-j)!}{(i-j-1)!\,(n-i)!} f_Z(u) \left(F_Z(u)-F_Z(v)\right)^{i-j-1} \left(1-F_Z(u)\right)^{n-i} \left(1-F_Z(v)\right)^{j-n}$$

$$= \frac{(n-j)!}{(i-j-1)!\,(n-i)!} \frac{f_Z(u)}{1-F_Z(v)} \left(\frac{F_Z(u)-F_Z(v)}{1-F_Z(v)}\right)^{i-j-1} \underbrace{\left(1-F_Z(v)\right)^{i-j-1+j-n+1} \left(1-F_Z(u)\right)^{n-i}}$$

$$= \frac{(n-j)!}{(i-j-1)!\,(n-i)!} \frac{f_Z(u)}{1-F_Z(v)} \left(\frac{F_Z(u)-F_Z(v)}{1-F_Z(v)}\right)^{i-j-1} \left(\frac{1-F_Z(u)}{1-F_Z(v)}\right)^{n-i}$$

$$\therefore f_{Z_{(i)}|Z_{(j)}}(u|v) = \begin{cases} \dfrac{(j-1)!}{(i-1)!\,(j-i-1)!} \dfrac{f_Z(u)}{F_Z(v)} \cdot \left(\dfrac{F_Z(u)}{F_Z(v)}\right)^{i-1} \left(1-\dfrac{F_Z(u)}{F_Z(v)}\right)^{j-i-1} & i<j \\ \dfrac{(n-j)!}{(i-j-1)!\,(n-i)!} \dfrac{f_Z(u)}{1-F_Z(v)} \left(\dfrac{F_Z(v)-F_Z(u)}{1-F_Z(v)}\right)^{i-j-1} \left(\dfrac{1-F_Z(u)}{1-F_Z(v)}\right)^{n-i} & i>j \end{cases}$$

□

(b). Find the pdf of $V|R=r$ $V = (Z_{(1)} + Z_{(n)})/2$

$V = (Z_{(1)} + Z_{(n)})/2$ $\quad \therefore Z_{(1)} = \frac{2V-R}{2} = V - \frac{R}{2}$ $\quad R = Z_{(n)} - Z_{(1)}$

$R = Z_{(n)} - Z_{(1)}$ $\quad Z_{(n)} = \frac{R+2V}{2} = V + \frac{R}{2}$

$|J| = \begin{vmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{vmatrix} = 1$ $\quad i=1 \quad j=n$

$$f_{V,R}(v,r) = f_{Z_{(1)},Z_{(n)}}(v^{-1}, r^{-1}) = \frac{n!}{0!\,(n-2)!\,0!} \cdot \overset{\frac{1}{a}}{f_Z\left(v-\frac{r}{2}\right)} \cdot \overset{\frac{1}{a}}{f_Z\left(v+\frac{r}{2}\right)} \cdot \left(F_Z\left(v-\frac{r}{2}\right) - F_Z\left(v+\frac{r}{2}\right)\right)^{n-2} = \frac{n(n-1)\cdot r^{n-2}}{a^n}$$

$$f_R(r) = \int_{\frac{r}{2}}^{a-\frac{r}{2}} f_{V,R}(v,r)\,dv = \frac{n(n-1)\cdot r^{n-2}}{a^n}(a-r)$$

$\left(\frac{r}{2} < v < a - \frac{r}{2}\right)$

$$f_{V|R}(v|r) = \frac{f_{V,R}(v,r)}{f_R(r)} = \frac{1}{a-r}$$ □

5.28. $Z_1, \cdots, Z_n$ iid $\sim f_Z(x)$

$Z_{(i_1)} < \cdots < Z_{(i_\ell)}$ $\quad Z_{(j_1)} < \cdots < Z_{(j_m)}$ $\quad$ P259 [229]

(a) The marginal cdf & pdf of $Z_{(i_1)}, \cdots, Z_{(i_\ell)}$.

Aim: $$f_{Z_{(i_1)},\cdots,Z_{(i_\ell)}}(x_{(i_1)}, x_{(i_2)} \cdots x_{(i_\ell)}) = \prod_{i_m=1}^{\ell} \frac{n!}{(i_m-1)!\,(n-i_m)!} \cdot f_Z(x_{i_m}) \cdot F_Z(x_{i_m})^{i_m-1} \left(1-F_Z(x_{i_\ell})\right)^{n-i_m}$$

$Y = \#$ of $Z_1, \cdots Z_{i_m} \le x$

? $Z_{i_m} \le x$ $\quad i_1, i_2 \cdots i_m$ 1

$i_{m+1} \cdots i_n$ 0

$$F_{Z_{(i_1)},\cdots,Z_{(i_\ell)}}(x_{(i_1)}, x_{(i_2)} \cdots x_{(i_\ell)}) = P(Z_{(i_1)} \le x_{(i_1)}) \cdots P(Z_{(i_\ell)} \le x_{(i_\ell)})$$

$$= \prod_{i_m=1}^{\ell} \cdot P(Y \ge i_m) = \prod_{i_m=1}^{\ell} \sum_{k=i_m}^{n} \binom{n}{k} F_Z(x_{i_m})^k \left[1-F_Z(x_{i_m})\right]^{n-k}$$ □

(b) The conditional cdf & pdf of $X_{(i_1)}, \cdots, X_{(i_\ell)}$ given $X_{(j_1)}, \cdots, X_{(j_m)}$

$\because$ $X_{(i_1)}, \cdots, X_{(i_\ell)}$ is independent with $X_{(j_1)}, \cdots, X_{(j_m)}$

$\therefore f_{X_{(i_1)} \cdots X_{(i_\ell)} \mid X_{(j_1)} \cdots X_{(j_m)}}(x_{i_1}, \cdots, x_{i_\ell} \mid x_{j_1}, \cdots x_{j_m}) = f(x_{i_1}, \cdots x_{i_\ell})$ (a)

$F_{X_{(i_1)} \cdots X_{(i_\ell)} \mid X_{(j_1)} \cdots X_{(j_m)}}(x_{i_1}, \cdots, x_{i_\ell} \mid x_{j_1}, \cdots, x_{j_m}) = P(X_{i_1} \le x_{i_1}) \cdot \cdots \cdot P(X_{i_\ell} \le x_{i_\ell})$ □

A, B are independent

$P(A \mid B) = \frac{P(AB)}{P(B)}$

$= \frac{P(A) \cdot P(B)}{P(B)}$

$= P(A)$

5.29. $\sigma = 0.05$ $\bar{x} = 1$

P(100 booklets weight more than 100.4 ounces)

From CLT: $\sqrt{100}\,(\bar{X} - 1)/0.05 \sim n(0,1)$ $\bar{X} \sim$

Central Limit Theorem P236 [66]

$\sqrt{n}\,(\bar{X}_n - \mu)/\sigma \sim n(0,1)$

P(100 booklets weight more than 100.4)

$= P(\bar{X} \geq 1.004)$ Let $A = \frac{10(\bar{X}-1)}{0.05} \sim n(0,1)$

$= P(\frac{0.05A}{10} + 1 \geq 1.004)$ $\bar{X} = \frac{0.05A}{10} + 1$

$= P(A \geq 0.8)$

$= \int_{0.8}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{a^2}{2}}\, da$

$= \frac{1}{\sqrt{2\pi}} \cdot \int_{0.8}^{\infty} e^{-\frac{a^2}{2}}\, da$

$= \frac{1}{\sqrt{2\pi}} \cdot (-1) \cdot e^{-\frac{a^2}{2}} \Big|_{0.8}^{\infty}$

$= \frac{1}{\sqrt{2\pi}} \cdot e^{-0.32}$ □

5.30. $\bar{X}_1, \bar{X}_2$ iid n. σ^2, Find the value for n so that $P(|\bar{X}_1 - \bar{X}_2| < \sigma/5) \approx 0.99$

From CLT: $\sqrt{n}\,(\bar{X} - \mu)/\sigma \sim n(0,1)$

Let $A = \sqrt{n}\,(\bar{X}_1 - \mu)/\sigma$ $\bar{X}_1 = \frac{\sigma A}{\sqrt{n}} + \mu$

$B = \sqrt{n}\,(\bar{X}_2 - \mu)/\sigma$ $\bar{X}_2 = \frac{\sigma B}{\sqrt{n}} + \mu$

$|\bar{X}_1 - \bar{X}_2| = |\frac{\sigma}{\sqrt{n}}(A - B)| = \frac{\sigma}{\sqrt{n}}|A - B|$

$\because A \sim n(0,1)$

$B \sim n(0,1)$

A, B are independent

Let $K = A - B$ $K \sim n(0,2)$

$\therefore A - B \sim n(0,2)$ $\frac{K}{\sqrt{2}} \sim n(0,1)$

$P(|\bar{X}_1 - \bar{X}_2| < \sigma/5) = P(\frac{\sigma}{\sqrt{n}}|A-B| < \sigma/5) = P(\frac{1}{\sqrt{n}}|A-B| < \frac{1}{5})$

$= P(-\frac{\sqrt{n}}{5} < K < \frac{\sqrt{n}}{5}) \approx 0.99$

$\therefore P(K > \frac{\sqrt{n}}{5}) \approx 0.005$

$P(\frac{K}{\sqrt{2}} > \sqrt{\frac{n}{2}} \cdot \frac{1}{5}) \approx 0.005$

$P(\frac{K}{\sqrt{2}} \le \sqrt{\frac{n}{2}} \cdot \frac{1}{5}) \approx 0.995$

$\sqrt{\frac{n}{2}} \cdot \frac{1}{5} = 2.575$

$n \approx 332$ □

5.31. $n=100$, $\bar{X}$, μ, $\sigma^2=9$

Find δ makes $P(|\bar{X}-\mu|<\delta)\geq 0.9$, use Chebychev's Iequality and Central Limit Theorem.

1° Chebyshev inequality

$\sigma^2=9$

$P(|\bar{X}-\mu|\geq k\sigma_{\bar{X}})\leq \frac{1}{k^2}$

$\sigma_{\bar{X}}=\sqrt{\frac{9}{100}}=\frac{3}{10}$

$P(|\bar{X}-\mu|<\delta)\geq 0.9 \iff P(|\bar{X}-\mu|\geq\delta)<0.1$

$P(|\bar{X}-\mu|\geq\sqrt{10}\cdot\frac{3}{10})\leq\frac{1}{(\sqrt{10})^2}$ $\quad \delta\geq\frac{3}{\sqrt{10}}\approx 0.948$

$Var(\bar{X})=E(\bar{X}^2)-E^2(\bar{X})=\frac{\sigma^2}{n}+\mu^2-\mu^2=\frac{\sigma^2}{n}=\frac{9}{100}$

$\bar{X}=\frac{1}{n}(X_1+X_2+\cdots+X_n)$

$E(\bar{X})=\frac{1}{n}\cdot E(X_1+X_2+\cdots+X_n)$

$=\frac{1}{n}\cdot n\mu=\mu$

$E(\bar{X})^2=E[(\frac{1}{n}\cdot(X_1+X_2+\cdots+X_n))^2]$

$=\frac{1}{n^2}E[(X_1+X_2+\cdots+X_n)^2]$

$=\frac{1}{n^2}\cdot E(X_1^2+X_2^2+\cdots+X_n^2$

$n-1$ $+$ $X_1X_2+X_1X_3+\cdots+X_1X_n$

$X_2X_1+X_2X_3+\cdots+X_2X_n$

$\vdots$

$X_nX_1+\cdots X_nX_{n-1}$)

$E(X_iX_j)$

$=E(X_i)\cdot E(X_j)$

$=\mu^2$

$Var(\bar{X})=\sigma^2=E(\bar{X}^2)-E^2(\bar{X})$

$E(\bar{X}^2)=\sigma^2+\mu^2$

$=\frac{1}{n^2}[nE(X^2)+n(n-1)\mu^2]$

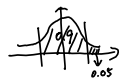
$=\frac{1}{n}\cdot E(X^2)+\frac{n-1}{n}\mu^2$

$=\frac{1}{n}(\sigma^2+\mu^2)+\frac{n-1}{n}\mu^2$

$=\frac{1}{n}(\sigma^2\cancel{+\mu^2}+n\mu^2\cancel{-\mu^2})$

$=\frac{1}{n}(\sigma^2+n\mu^2)$

$=\frac{1}{n}\sigma^2+\mu^2$



$P(|\bar{X}-\mu|<\delta)=P(|\frac{3A}{10}|<\delta)=P(|A|<\frac{10\delta}{3})\geq 0.9$

$\therefore P(A<\frac{10\delta}{3})\geq 0.95$ $\quad \frac{10\delta}{3}=1.65$

$\delta\geq 0.495$

$1^\circ \Rightarrow P(|\bar{X}-\mu|<0.948)\geq 0.9$ □

$2^\circ \Rightarrow P(|\bar{X}-\mu|<0.495)\geq 0.9$

Comments: Chebyshev inequality is conservative

5.32. $X_1, X_2, \cdots$ converge in probability to constant a.

Assume that $P(X_i>0)=1$ for all i $\lim_{n\to\infty}P(|X_n-a|<\varepsilon)=1$

(a). Varify that sequences $Y_i=\sqrt{X_i}$ and $Y_i'=a/X_i$ converge in probability

$\lim_{n\to\infty}P(|X_n-a|<\varepsilon)=1$

Aim: $\lim_{n\to\infty}P(|Y_n-\sqrt{a}|<\alpha)=1$

$Y_i=\sqrt{X_i}$

$\because \lim_{n\to\infty}P(|X_n-a|<\varepsilon)=1$ and $P(X_i>0)=1$, $a>0$

$\therefore \lim_{n\to\infty}P(|\sqrt{X_n}-\sqrt{a}|<\alpha)=1$ $\quad \alpha$ is a very small number $\alpha>0$

that is $\lim_{n\to\infty}P(|Y_n-\sqrt{a}|<\alpha)=1$

$\therefore Y_i=\sqrt{X_i}$ converges in probability

$\frac{a}{X_i}$

Aim: $\lim_{n\to\infty}P(|Y_n'-1|<\theta)=1$

$Y_n'=\frac{a}{X_n}$

$\because \lim_{n\to\infty}P(|X_n-a|<\varepsilon)=1$

$\therefore \lim_{n\to\infty}P(|1-\frac{a}{X_n}|<\frac{\varepsilon}{a})=1$ $\quad$ Let $\frac{\varepsilon}{a}=\theta$

$\therefore \lim_{n\to\infty}P(|\frac{a}{X_n}-1|<\theta)=1$

that is $\lim_{n\to\infty}P(|Y_n-1|<\theta)=1$

$\therefore Y_i'$ converges in probability.

(b) P240 [2]0] Suppose $\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}\sim n(0,1)$

Prove that σ/S_n converges in probability to 1.

Idea: $\Leftarrow P(|S_n-\sigma|<\varepsilon)=1$

$\Leftarrow S_n^2$ converge in probability to σ^2

$\because n\to\infty$ $Var S_n^2\to 0$

$\therefore P(|S_n^2-\sigma^2|<\varepsilon)=1$

that is S_n^2 converges to σ^2

From (a) S_n converges to σ in probability and σ/S_n converges to 1 in probability □

$S_n^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2$

By Chebychev's Iequality.

$P(|S_n^2-\sigma^2|\geq\varepsilon)=P((S_n^2-\sigma^2)^2\geq\varepsilon^2)$

$\leq\frac{E(S_n^2-\mu)^2}{\varepsilon^2}=\frac{Var S_n^2}{\varepsilon^2}=0$

5.33. X_n converges in distribution to X $\quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Y_n : $\lim_{n\to\infty} P(Y_n > c) = 1$

Show that $\lim_{n\to\infty} P(X_n + Y_n > c) = 1$

$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

$P(A+B>c)$

$\lim_{n\to\infty} F_{X_n}(-m) = P_X(x \geq -m)$, we can find a $m>0$ $\quad F_{X_n}(c) = F_X(c)$

make that $P_X(x \leq -m) \geq 1 - \varepsilon/2$, ε is a very small number.

Assume $P(Y_n > c+m) \geq 1 - \varepsilon/2$

$$P(X_n + Y_n > c) = P(X_n + m + Y_n - m > c)$$
$$\geq P(x \geq -m,\ Y > c+m)$$
$$= P(x \geq -m) \cdot P(Y > c+m)$$
$$= (1 - \varepsilon/2)^2$$
$$= \underbrace{1 - \varepsilon + \frac{\varepsilon^2}{2}}$$

Let $\frac{\varepsilon^2}{2} - \varepsilon = -\alpha$ (a small value)

$\lim_{n\to\infty} P(X_n + Y_n > c) \geq 1 - \alpha$

5.34. Let $X_1, \dots X_n \sim \mu, \sigma^2$

Show that $E\,\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = 0$ and $Var\,\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = 1$

$$E\,\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}}{\sigma} E(\bar{X}_n - \mu) = \frac{\sqrt{n}}{\sigma}\left(E(\bar{X}_n) - \mu\right)$$

$\because E(\bar{X}_n) = E\left(\frac{X_1 + \dots + X_n}{n}\right) = \frac{n}{n} E(X_1) = \mu$

$\therefore E\,\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = 0$

$$Var\,\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$$
$$= \frac{n}{\sigma^2} \cdot Var(\bar{X}_n - \mu)$$
$$= \frac{n}{\sigma^2} \cdot \frac{\sigma^2 - \mu^2}{n}$$
$$= \frac{n}{\sigma^2} \cdot \frac{\sigma^2}{n}$$
$$= 1 \quad \square$$

$Var\,X = E(X-\mu)^2$

$\therefore Var(\bar{X}_n - \mu)$

$$= E\Big[(\bar{X}_n - \mu) - \underbrace{E(\bar{X}_n - \mu)}_{0}\Big]^2$$
$$= E(\bar{X}_n - \mu)^2$$
$$= E(\bar{X}_n)^2 - \mu^2$$
$$= E\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right)^2 - \mu^2$$
$$= \frac{1}{n^2} E(X_1^2 + X_2^2 + \dots + X_n^2 + X_1X_2 \dots + X_1X_n + X_2X_1 \quad X_2X_3 \dots X_2X_n \ \vdots \ X_nX_1 \dots X_nX_{n-1}) - \mu^2$$

$EX_1^2 = \sigma^2 + \mu^2$

$$= \frac{1}{n^2}\left[n(\sigma^2 + \mu^2) + n(n-1)\mu^2\right] - \mu^2$$
$$= \frac{1}{n^2} \cdot (n\sigma^2 + \cancel{n\mu^2} + n^2\mu^2 - \cancel{n\mu^2}) - \mu^2$$
$$= \frac{\sigma^2}{n} + \mu^2 - \mu^2$$
$$= \frac{\sigma^2}{n}$$

5.35 (a) If $X_i \sim$ exponential(1) , $i = 1, 2, \dots$, all independent,

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) \to P(Z \leq x) \quad \text{where } Z \sim n(0,1)$$

$\because X_i \sim$ exponential (1) $\qquad \lambda \cdot e^{-\lambda x}$

$\therefore f_{X_i}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \lambda = 1 \quad e^{-x}$

$E(X_i) = \int_0^\infty x e^{-x}\,dx = \frac{1}{\lambda} = 1$

$Var(X_i) = E X_i^2 - (E X_i)^2 = 1 = \sigma^2$

$E(X_i^2) = \frac{2}{\lambda}$

$\int_0^\infty x \cdot e^{-x}\,dx$

$u = x \quad v' = -e^{-x}$

$u' = 1 \quad v = e^{-x}$

$(x \cdot e^{-x})' = x \cdot (-e^{-x}) + e^{-x}$

$x \cdot e^{-x}\Big|_0^\infty = -\int_0^\infty x \cdot e^{-x}\,dx + \int_0^\infty e^{-x}\,dx$

$\int_0^\infty x \cdot e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = 1$

According to Central Limit Theory

$$\frac{\sqrt{n}(\bar{X}_n - E(X))}{\sigma} \sim n(0,1)$$

that is

$$\frac{\bar{X}_n - 1}{1/\sqrt{n}} \sim n(0,1)$$

$\therefore P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) \to P(z \leq x)$

$z \sim n(0,1)$ $\square$

$\int_0^\infty x^2 \cdot \lambda \cdot e^{-\lambda x}\,dx$

$= \lambda \int_0^\infty \underbrace{x^2 \cdot e^{-\lambda x}}\,dx = \lambda \cdot \frac{2}{\lambda^2} = \frac{2}{\lambda}$

Let $u = x^2 \quad v' = -\lambda e^{-\lambda x}$

$u' = 2x \quad v = e^{-\lambda x}$

$\therefore x^2 e^{-\lambda x}\Big|_0^\infty = -\lambda \int x^2 \cdot e^{-\lambda x}\,dx + 2\int x\, e^{-\lambda x}\,dx$

$\therefore \int x^2 e^{-\lambda x}\,dx = 0 + 2 \cdot \frac{1}{\lambda}$

$\int x \cdot e^{-\lambda x}\,dx$

$u = x \quad v' = -\lambda \cdot e^{-\lambda x}$

$u' = 1 \quad v = e^{-\lambda x}$

$x \cdot e^{-\lambda x}\Big|_0^\infty = -\int x \lambda \cdot e^{-\lambda x}\,dx + \int e^{-\lambda x}\,dx$

$\therefore \int x e^{-\lambda x} = \int e^{-\lambda x}\,dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda}$

(b) Differentiating both sides in (a) suggests :

$$\frac{\sqrt{n}}{\Gamma(n)}(x\sqrt{n}+n)^{n-1}\cdot e^{-(x\sqrt{n}+n)} \approx \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}$$

$$P\left(\frac{\bar{X}_n-1}{1/\sqrt{n}} \le x\right) \to P(Z \le x)$$

$f(x) = \lambda \cdot e^{-\lambda x}$

$= e^{-x}$

Left Part '

$P\left(\frac{\bar{X}_n-1}{1/\sqrt{n}} \le x\right)$

$= P\left(\bar{X}_n \le \frac{x}{\sqrt{n}}+1\right) = P\left(X_1+X_2+\cdots+X_n \le n\left(\frac{x}{\sqrt{n}}+1\right)\right)$

$X_i \sim$ exponential (1)

$X_1+X_2+\cdots+X_n \sim$ gamma $(n, \frac{1}{\lambda})$

mean of expone.

Let $S = X_1+X_2+\cdots+X_n$

$f_S(s) = \frac{s^{\alpha-1}\cdot e^{-s/\beta}}{\beta^\alpha \cdot \Gamma(\alpha)}$ $\quad \alpha = n \quad \beta = \frac{1}{\lambda} = 1$

$= \frac{s^{n-1}\cdot e^{-s}}{\Gamma(n)}$

$\left[F\left(S \le n\left(\frac{x}{\sqrt{n}}+1\right)\right)\right]'$

$= F_S\left(n\left(\frac{x}{\sqrt{n}}+1\right)\right)'$

$= f_S\left(n\left(\frac{x}{\sqrt{n}}+1\right)\right)\cdot n\cdot\frac{1}{\sqrt{n}}$

$= \frac{\left[n\left(\frac{x}{\sqrt{n}}+1\right)\right]^{n-1}\cdot e^{-n\left(\frac{x}{\sqrt{n}}+1\right)}}{\Gamma(n)}\sqrt{n}$ ①

Right Part '

$P(Z \le x)$ $\quad Z \sim n(0,1)$

$f_Z(z) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{z^2}{2}}$

$P(Z \le x)$

$= F(x)$

$F(x)' = f_Z(x) = \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x}{2}}$ ②

$\therefore$ ① $\approx$ ② □

when $x=0$ ① $= \frac{n^{n-1}\cdot e^{-n}\cdot\sqrt{n}}{\Gamma(n)}$ ② $= \frac{1}{\sqrt{2\pi}}$

that is $\frac{n^{n-1}\cdot e^{-n}\cdot\sqrt{n}}{\Gamma(n)} = \frac{1}{\sqrt{2\pi}}$ □

$n! \approx \sqrt{2\pi}\cdot n^{n+1/2}\cdot e^{-n}$

5.36 $N=n$, $f_{Y|N}(y|n) = \frac{x^{k-1}\cdot e^{-x/2}}{2^{k/2}\,\Gamma(\frac{k}{2})} \sim \chi^2_{2n}$

Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\cdot e^{-t}\,dt$.

$N \sim$ Poisson (θ) $\quad f_N(n) = \frac{\lambda^n\cdot e^{-\lambda}}{n!}$

(a) Calculate EY and VarY

$f_{Y|N}(y|n) = \frac{y^{2n-1}\cdot e^{-y^2/2}}{2^n\cdot\Gamma(n)}$ $\quad k=2n$

$f_N(n) = \frac{\theta^n\cdot e^{-\theta}}{n!}$ $\quad \lambda=\theta$

$f_Y(y) = \sum_{n=1}^{\infty} f_{Y|N}(y|n)\cdot f_N(n)$

$EY = E_N(\underbrace{E(Y|N)}_{2n}) = E_N(2n) = 2E_N(n) = 2\theta$

$VarY = E[Var(Y|N)] + Var[E(Y|N)]$

$= E[\underbrace{(2n+2)(2n)}_{E(Y^2|N)} - \underbrace{(2n)^2}_{E^2(Y|N)}] + Var(2n)$

$= 4\theta + 4E(n) = 8\theta$ □

Chi-square distribution Mean & Var Proof

χ^2_P $\quad f(x|p) = \frac{x^{p/2-1}\cdot e^{-x/2}}{2^{\frac{p}{2}}\,\Gamma(\frac{p}{2})}$

$EX = \int_0^\infty x\cdot f(x|p)\,dx = \int_0^\infty \frac{x^{p/2}\cdot e^{-x/2}}{2^{\frac{p}{2}}\,\Gamma(\frac{p}{2})}\,dx$

$= \frac{1}{2^{\frac{p}{2}}\Gamma(\frac{p}{2})}\underbrace{\int_0^\infty x^{p/2}\cdot e^{-\frac{x}{2}}\,dx}_{A}$

Let $t=\frac{x}{2}$ then $x=2t$ $\quad dx = 2\cdot dt$

$A = \int_0^\infty (2t)^{p/2}\cdot e^{-\frac{t}{2}}\cdot 2\cdot dt$

$= 2^{p/2+1}\cdot\underbrace{\int_0^\infty t^{p/2}\cdot e^{-\frac{t}{2}}\,dt}$

$= 2^{p/2+1}\,\Gamma(p/2+1)$

$\therefore EX = \frac{2^{p/2+1}\cdot\Gamma(p/2+1)}{2^{\frac{p}{2}}\cdot\Gamma(\frac{p}{2})}$

$= 2\cdot\frac{(\frac{p}{2})!}{(\frac{p}{2}-1)!} = 2\cdot\frac{p}{2} = p$

$Var X = E(X^2) - E^2X = (p+2)\cdot p - p^2 = 2p$

$E(X^2) = \int_0^\infty x^2\cdot f(x|p)\,dx = \int_0^\infty \frac{x^{p/2+1}\cdot e^{-x/2}}{2^{\frac{p}{2}}\,\Gamma(\frac{p}{2})}\,dx$

$= \frac{1}{2^{\frac{p}{2}}\cdot\Gamma(\frac{p}{2})}\cdot\underbrace{\int_0^\infty x^{p/2+1}\cdot e^{-x/2}\,dx}$

$= \frac{1}{2^{\frac{p}{2}}\,\Gamma(\frac{p}{2})}\cdot 2^{\frac{p}{2}+2}\,\Gamma(p/2+2)$

$= \frac{4\cdot(p/2+2)!}{(\frac{p}{2}-1)!} = 4\cdot(p/2+1)\,\frac{p}{2} = (p+2)\cdot p$

- Chi-square distribution Mean & Var Proof 2

  - Moment Generation Function (MGF)

$$M_Z(t) = E(e^{tZ}) = \int_0^\infty e^{tx} \cdot f(x)\, dx$$

$$= \int_0^\infty e^{tx} \cdot \frac{x^{\frac{p}{2}-1} \cdot e^{-\frac{x}{2}}}{2^{\frac{p}{2}} \cdot \Gamma(\frac{p}{2})}\, dx$$

Gamma fun:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx$$

$$= \frac{1}{2^{\frac{p}{2}} \cdot \Gamma(\frac{p}{2})} \int_0^\infty x^{\frac{p}{2}-1} \cdot e^{tx - \frac{x}{2}}\, dx$$

$$= \frac{1}{2^{\frac{p}{2}} \cdot \Gamma(\frac{p}{2})} \underbrace{\int_0^\infty x^{\frac{p}{2}-1} \cdot e^{(1-2t)(-\frac{x}{2})}\, dx}_{A}$$

Assume $y = (1-2t)x/2$ $\quad x = \frac{2}{1-2t} y = 2(1-2t)^{-1} y$

$dx = 2(1-2t)^{-1} \cdot dy$

$$A = \int_0^\infty \left[(1-2t)^{-1} \cdot y 2\right]^{\frac{p}{2}-1} \cdot e^{-y} \cdot 2(1-2t)^{-1} \cdot dy$$

$$= 2^{\frac{p}{2}} \cdot (1-2t)^{-\frac{p}{2}} \underbrace{\int_0^\infty y^{\frac{p}{2}-1} \cdot e^{-y}\, dy}$$

$$= 2^{\frac{p}{2}} \cdot (1-2t)^{-\frac{p}{2}} \cdot \Gamma(\tfrac{p}{2})$$

$$\therefore M_Z(t) = \frac{1}{\cancel{2^{\frac{p}{2}}}\, \Gamma(\frac{p}{2})} \cancel{2^{\frac{p}{2}}} \cdot (1-2t)^{-\frac{p}{2}} \cdot \Gamma(\tfrac{p}{2}) = (1-2t)^{-\frac{p}{2}}$$

$$= \frac{1}{(1-2t)^{\frac{p}{2}}}$$

$$M_Z^r(0) = \left[\frac{d^r M_Z(t)}{dt}\right]_{t=0} = E(Z^r)$$

$$E(Z) = \left((1-2t)^{-\frac{p}{2}}\right)'_{t=0} = -\frac{p}{2}(1-2t)^{-\frac{p}{2}-1}(-2)_{t=0} = p$$

$$E(Z^2) = \left(p(1-2t)^{-\frac{p}{2}-1}\right)'_{t=0} = p\left(-\frac{p}{2}-1\right)(-2) \cdot \underbrace{(1-2t)^{-\frac{p}{2}-2}}_{1}{}_{t=0} \quad \square$$

$$= p(p+2)$$

$$M_Z'(0) = \lim_{t\to 0} E(x e^{tx}) = E(x)$$

$$M_Z^2(0) = \lim_{t\to 0} E(x^2 e^{tx}) = E(x^2)$$

- Poisson distribution mean & variance. 1

$$f(n) = \frac{\lambda^n \cdot e^{-\lambda}}{n!}$$

$$E(N) = \sum_{n=0}^{\infty} n \cdot \frac{\lambda^n \cdot e^{-\lambda}}{n!}$$

$$= \sum \frac{\lambda^n \cdot e^{-\lambda}}{(n-1)!}$$

$$= \sum \frac{\lambda \cdot \lambda^{n-1} \cdot e^{-\lambda}}{(n-1)!}$$

$$= \lambda \underbrace{\sum \cdots}_{1}$$

$$= \lambda$$

$$E(N^2) = \sum n^2 \cdot \frac{\lambda^n \cdot e^{-\lambda}}{n!}$$

$$= \sum \frac{(n-1+1)\lambda^n e^{-\lambda}}{(n-1)!}$$

$$= \sum\left[(n-1) \cdot \frac{\lambda^n \cdot e^{-\lambda}}{(n-1)!} + \frac{\lambda^n \cdot e^{-\lambda}}{(n-1)!}\right]$$

$$= \sum \frac{\lambda^2 \lambda^{n-2} \cdot e^{-\lambda}}{(n-2)!} + \sum \frac{\lambda \cdot \lambda^{n-1} \cdot e^{-\lambda}}{(n-1)!}$$

$$= \lambda^2 + \lambda$$

$$Var(N) = E(N^2) - E^2(N) = \lambda^2 + \lambda - \lambda^2 = \lambda \quad \square$$

- Derivation of the **Law of total variance**

$$Var\, Z = \underbrace{E[Var(Z|Y)]}_{①} + \underbrace{Var[E(Z|Y)]}_{②} \qquad Var(Z) = E(Z^2) - E^2(Z)$$

$$Var(Z|Y=y) = E(Z^2|Y=y) - E^2(Z|Y=y) \quad \text{for all } y$$

$$Var(Z|Y) = E(Z^2|Y) - E^2(Z|Y)$$

$$①: E[Var(Z|Y)] = E[E(Z^2|Y)] - E[E^2(Z|Y)]$$

$$= EZ^2 - E[E^2(Z|Y)]$$

$$②: Var[E(Z|Y)] = E[E^2(Z|Y)] - E^2[E(Z|Y)] = E[E^2(Z|Y)] - E^2(Z)$$

$$Var(A) = E(A^2) - E^2(A)$$

$$\therefore ① + ② = E(Z^2) - \cancel{E[E^2(Z|Y)]} + \cancel{E[E^2(Z|Y)]} - E^2(Z)$$

$$= E(Z^2) - E^2 Z$$

$$= Var(Z) \quad \square$$

(b) Show that $\lim_{\theta \to \infty} \frac{Y - EY}{\sqrt{Var Y}} \to n(0,1)$ in distribution

According to Central Limit Theory

$$\frac{\sqrt{2n}\left(\frac{Y}{2n} - E(Z_i^2)\right)}{\sigma_{Z_i^2}} \sim n(0,1)$$

Y is the sum of $2n$ Z^2 $\quad Z \sim (0,1)$

that is $Y = Z_1^2 + Z_2^2 + \cdots + Z_{2n}^2$

Assume that $\bar{Y}_n = \frac{Y}{2n}$

$E(Z_i^2) = 1$

$\sigma^2(Z_i^2) = 2 \qquad \sigma(Z_i^2) = \sqrt{2}$

$$\frac{\sqrt{2n}\left(\frac{Y}{2n} - 1\right)}{\sqrt{2}} \sim n(0,1)$$

$$\therefore \frac{Y - 2n}{2\sqrt{2n}} \sim n(0,1)$$

that is $\frac{Y - EY}{\sqrt{Var Y}} \sim n(0,1)$ $\square$

$Var(Y) = 2p = Var(2n\,\bar{Z}^2)$

$2(2n) = 4n^2\, Var(\bar{Z}^2)$

$Var(\bar{Z}^2) = \frac{4n}{4n^2}$

$= \frac{1}{n}$

5.37. $X_1, \cdots, X_n \sim$ negative binomial (r, p)

$EX = \frac{r(1-p)}{p} \qquad Var X = \frac{r(1-p)}{p^2}$

$U_n = \sum_{i=1}^{n} X_i$, For $n = 10$, $p = 0.7$, $r = 2$

Calculate $P(U_n = u)$ $\quad u = 0, 1, \cdots, 10$ by

(1) Exact calculation

$\because X_i \sim$ negative binomial (r, p) $\qquad x_i$: number of failure

$\therefore f_{X_i}(x_i) = \binom{x_i + r - 1}{r - 1} (1-p)^{x_i} p^r \qquad n = 10,\ p = 0.7\ \ r = 2$

$= \binom{x_i + 1}{1} (0.3)^{x_i} \cdot 0.7^2$

$= (x_i + 1)(0.3)^{x_i} \cdot 0.49 \qquad x_i = 0, 1, \cdots$

$U_n = X_1 + X_2 + \cdots + X_{10}$

$P(U_n = 0) = P(X_1 = 0, X_2 = 0, \cdots, X_n = 0) \qquad P(x_i = 0) = 1 \cdot 0.3^0 \cdot 0.49 = 0.49$

$= (0.49)^{10} = 0.0008$

$P(U_n = 1) = \binom{10}{1} (0.49)^9 \cdot 0.6 \cdot 0.49 \qquad P(x_i = 1) = 2 \cdot 0.3 \cdot 0.49 = 0.6 \cdot 0.49$

$= 6(0.49)^{10} = 0.0048$

$P(U_n = 2) = \overset{2,0,0,0\cdots}{\binom{10}{1}} (0.49)^9 \cdot 0.27 \cdot 0.49 \qquad P(x_i = 2) = 3 \cdot 0.3^2 \cdot 0.49 = 0.27 \cdot 0.49$

$+ \overset{1,1,0,0\cdots}{\binom{10}{2}} (0.49)^8 (0.6 \cdot 0.49)^2 \qquad \binom{10}{2} = \frac{10!}{2!(8)!} = \frac{10 \cdot 9}{2 \cdot 1}$

$= 2.7\,(0.49)^{10} + 16.2\,(0.49)^{10} \qquad 45 \cdot 0.36$

$= 18.9\,(0.49)^{10} = 0.015 \qquad \binom{10}{7} = \binom{10}{3} = \frac{10!}{3! \cdot 7!} = \frac{\cancel{10} \cdot \cancel{9}^{5} \cdot \cancel{8}^{3}}{\cancel{3} \times \cancel{8}}$

$= 15 \times 8$

$= 120$

$P(U_n = 3) = \overset{3,0,0,\cdots}{\binom{10}{1}} (0.49)^9 \cdot 4(0.3)^3 (0.49)$

$+ \overset{2,1,0,0,\cdots}{\underbrace{\binom{10}{2}}_{45 \times 8} (0.49)^8 \cdot \underbrace{\binom{2}{1} 0.6 \cdot 0.49}_{1} \quad \binom{1}{1} 0.27 \cdot 0.49}$

$+ \binom{10}{7} (0.49)^7 \cdot [0.6 \cdot 0.49]^3 = 0.0352$

(b) normal approximation

From Central Limit Theory

$V_n = Z_1 + Z_2 + \cdots + Z_{10}$

$\bar{Z} = V_n / 10$

$$\frac{\sqrt{n}\,(\bar{Z} - E(Z))}{\sigma_Z} \sim N(0,1)$$

that is

$$\frac{\sqrt{10}\cdot\left(\frac{V_n}{10} - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}} \sim Z(0,1)$$

$$E(Z) = \frac{r(1-p)}{p} = \frac{2(1-0.7)}{0.7} = \frac{6}{7}$$

$$\sigma_Z = \sqrt{\frac{r(1-p)}{p^2}} = \frac{1}{p}\sqrt{r(1-p)} = \frac{1}{0.7}\cdot\sqrt{2\cdot 0.3} = \frac{\sqrt{0.6}}{0.7}$$

$P(V_n \le 0) = P(V_n = 0)$

$$= P\left(\frac{\sqrt{10}\left(\frac{V_n}{10} - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}} \le \frac{\sqrt{10}\cdot\left(0 - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}}\right)$$

$$= P\left(Z \le \frac{-6\sqrt{10}}{\frac{\sqrt{0.6}}{0.7}}\right) = 0.0071$$

$P(V_n = 1) = P(V_n \le 1) - P(V_n \le 0)$

$$= P\left(Z \le \frac{\left(\frac{1}{10} - \frac{6}{7}\right)\sqrt{10}}{\frac{\sqrt{0.6}}{0.7}}\right) - 0.0071 = 0.0153 - 0.0071$$

$= 0.0082$

$P(V_n = 2) = P(V_n \le 2) - P(V_n \le 1)$

$$= P\left(Z \le \frac{\left(\frac{2}{10} - \frac{6}{7}\right)\sqrt{10}}{\frac{\sqrt{0.6}}{0.7}}\right) - 0.0153$$

$= 0.03 - 0.0153$

$= 0.0147$

$P(V_n = 3) = P(V_n \le 3) - P(V_n \le 2)$

$$= P\left(Z \le \frac{\left(\frac{3}{10} - \frac{6}{7}\right)\sqrt{10}}{\frac{\sqrt{0.6}}{0.7}}\right) - 0.0147$$

$= 0.0559 - 0.0147$

$= 0.0412$

(c) normal approximation with continuity correction

$P(V_n = 0) \approx P(V_n \le 0.5) - P(V_n \le -0.5)$

$$= P\left(Z \le \frac{\sqrt{10}\left(\frac{0.5}{10} - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}}\right) - P\left(Z \le \frac{\sqrt{10}\left(\frac{-0.5}{10} - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}}\right)$$

$= P(Z \le -2.31) - P(Z \le -2.59)$

$= 0.0104 - 0.0048$

$= 0.0056$

$P(V_n = 1) \approx P(V_n \le 1.5) - P(V_n \le 0.5)$

$$= P\left(Z \le \frac{\sqrt{10}\left(\frac{1.5}{10} - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}}\right) - P(Z \le -2.31)$$

$= P(Z \le -2.02) - P(Z \le -2.31)$

$= 0.0217 - 0.0104$

$= 0.0113$

$P(V_n = 2) \approx P(V_n \le 2.5) - P(V_n \le 1.5)$

$$= P\left(Z \le \frac{\sqrt{10}\left(\frac{2.5}{10} - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}}\right) - P(Z \le -2.02)$$

$= P(Z \le -1.74) - P(Z \le -2.02)$

$= 0.0409 - 0.0217$

$= 0.0192$

$P(V_n = 3) \approx P(V_n \le 3.5) - P(V_n \le 2.5)$

$$= P\left(Z \le \frac{\sqrt{10}\left(\frac{3.5}{10} - \frac{6}{7}\right)}{\frac{\sqrt{0.6}}{0.7}}\right) - P(Z \le -1.74)$$

$= P(Z \le -1.45) - P(Z \le -1.74)$

$= 0.0735 - 0.0409$

$= 0.0326$

Summary

	(a)	(b)	(c)
$V_n = 0$	0.0008	0.0071	0.0056
$V_n = 1$	0.0048	0.0082	0.0113
$V_n = 2$	0.015	0.0147	0.0192
$V_n = 3$	0.0332	0.0412	0.0326

5.38. Let $Z_1, Z_2, \cdots, Z_n$ be iid with mgf $M_Z(t)$, $-h<t<h$

Let $S_n = \sum_{i=1}^{n} Z_i$ and $\bar{Z}_n = S_n/n$

(a) Show that $P(S_n > a) \le e^{-at}[M_Z(t)]^n \quad 0<t<h$

$P(S_n \le a) \le e^{-at}[M_Z(t)]^n \quad -h<t<0$

$M_Z(t) = \int_{-\infty}^{\infty} e^{tx} f_Z(x)\,dx$

$M_{S_n}(t) = \int_{-\infty}^{\infty} e^{t\cdot s_n} f_{S_n}(s_n)\,ds_n$

$= \int_{-\infty}^{\infty} e^{t(x_1+x_2+\cdots+x_n)} f_{Z_1 Z_2 \cdots Z_n}(x_1, x_2, \cdots x_n)\,d(x_1, x_2 \cdots)$

$\because x_1, x_2, \cdots x_n$ are i.i.d

$\therefore f_{Z_1 Z_2 \cdots Z_n}(x_1, x_2, \cdots x_n) = f_Z(x_1)\cdot f_Z(x_2)\cdots f_Z(x_n)$

$\therefore M_{S_n}(t) = \int_{-\infty}^{\infty} e^{tx_1} f_Z(x_1)\cdot e^{tx_2} f_Z(x_2)\cdots e^{tx_n} f_Z(x_n)\,d(x_1, x_2 \cdots x_n)$

$= \int_{-\infty}^{\infty} e^{tx_1} f_Z(x_1)\,dx_1 \int_{-\infty}^{\infty} e^{tx_2} f_Z(x_2)\,dx_2 \cdots \int_{-\infty}^{\infty} e^{tx_n} f_Z(x_n)\,dx_n$

$= [M_Z(t)]^n \ge \int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm \qquad 1-[M_Z(t)]^n \le 1-\int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm$

$u = e^{tm} \qquad v = F_{S_n}(m)$

$u' = t e^{tm} \qquad v' = f_{S_n}(m)$

$(uv)' = [e^{tm} F_{S_n}(m)]'$

$uv' + u'v = e^{tm} f_{S_n}(m) + t e^{tm} F_{S_n}(m)$

$\int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm = e^{tm} F_{S_n}(m)\Big|_{-\infty}^{a} - \int t e^{tm} F_{S_n}(m)\,dm$

$t<0 \quad \int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm = e^{at} F_{S_n}(a) - t\int e^{tm} F_{S_n}(m)\,dm$

$\int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm + t\int_{-\infty}^{a} e^{tm} F_{S_n}(m)\,dm = e^{at} P(S_n \le a)$

if $t<0$

$[M_Z(t)]^n \ge \int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm \ge e^{at} P(S_n \le a)$

$\therefore P(S_n \le a) \le e^{-at}[M_Z(t)]^n$ □

$t>0 \quad \int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm + t\int_{-\infty}^{a} e^{tm} \underbrace{F_{S_n}(m)}_{\le 1}\,dm = e^{mt} F_{S_n}(m)\Big|_{-\infty}^{a}$

非常巧妙的偷放

$\int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm + t\int_{-\infty}^{a} e^{tm}\,dm \ge e^{at} \underbrace{F_{S_n}(a)}_{P(S_n \le a)}$

$\int_{-\infty}^{a} e^{tm} f_{S_n}(m)\,dm + \underbrace{e^{at}}_{e^{-at}[M_Z(t)]^n} \ge e^{at} F_{S_n}(a)$

$\ge 1 - F_{S_n}(a) = P(S_n \ge a)$ □

(b) Use the fact that $M_Z(0)=1$ and $M'_Z(0)=E(Z)$ to show if $E(Z)<0$, then there is a $0<c<1$ with $P(S_n>a)\le c^n$

If $M_Z(0)=1$. $M'_Z(0)=E(Z)<0$ then $0<M_Z(t)<1$ $(0<t<h)$

that is $0<[M_Z(t)]^n<1$

Assume that $M_Z(t)=c$, $0<c<1$.

then from (a): $P(S_n>a)\le [M_Z(t)]^n$ $0<t<h$

$P(S_n>a)\le c^n$ □

Siminarly if $M_Z(0)=1$ $M'_Z(0)=E(Z)>0$ then $0<M_Z(t)<1$ $(-h<t<0)$

Assume that $M_Z(t)=c$, $0<c<1$

then from (a): $P(S_n\le a)\le [M_Z(t)]^n$ $-h<t<0$

$P(S_n\le a)\le c^n$ □

(c). Define $Y_i=X_i-\mu-\varepsilon$ and use the above argument, $a=0$, establish that $P(\bar{X}_n-\mu>\varepsilon)\le c^n$

$S_n=\sum_{i=1}^{n} X_i$ and $\bar{X}_n=S_n/n$

$P(\frac{S_n}{n}-\mu-\varepsilon>a)\le e^{-at}[M_Z(t)]^n$

let $a=0$

$P(\frac{S_n}{n}-\mu>\varepsilon)\le [M_Z(t)]^n$

$\Leftarrow$ $P(S_n>n(\varepsilon+\mu))\le [M_Z(t)]^n$

If $\varepsilon+\mu>0$

$P(S_n>\underbrace{n(\varepsilon+\mu)}_{+})\le P(S_n>0)\le [M_Z(t)]^n$

$\therefore P(S_n>n(\varepsilon+\mu))\le c^n$

that is $P(\frac{S_n}{n}-\mu>\varepsilon)\le c^n$ □

If $\varepsilon+\mu<0$

$P(S_n>\underbrace{n(\varepsilon+\mu)}_{-})\le\left(\int_{-\infty}^{\infty} e^{tx_1}f_Z(x_1)\,dx_1\right)^n=[M_{S_n}(t)]^n=c^n$ □

(d) $Y_i=-X_i+\mu-\varepsilon$, establish an equality similar to (c) and get $P(|\bar{X}_n-\mu|>\varepsilon)\le 2c^n$ $0<c<1$

$\Leftarrow P(-\bar{X}+\mu>\varepsilon)\le c^n$

$P(-\frac{S_n}{n}+\mu-\varepsilon>0)\le [M_Z(t)]^n$

$P(S_n<(\mu-\varepsilon)n)\le [M_Z(t)]^n$

If $\mu-\varepsilon<0$

$P(S_n<(\mu-\varepsilon)n)\le P(S_n<0)\le [M_Z(t)]^n=c^n$

$\therefore P(-\frac{S_n}{n}+\mu-\varepsilon>0)\le c^n$

that is $P(-\bar{X}+\mu>\varepsilon)\le c^n$

If $\mu-\varepsilon>0$

$P(S_n<(\mu-\varepsilon)n)\le\left(\int_{-\infty}^{\infty} e^{tx_1}f_Z(x_1)\,dx_1\right)^n=c^n$ that is $P(-\bar{X}+\mu>\varepsilon)\le c^n$.

$\therefore$ Combine (d) and (c) we have $P(|\bar{X}-\mu|>\varepsilon)\le 2c^n$ □

5.39 Convergence

(a). Prove $X_1, X_2, \cdots$ converges in probability to X, h is a continuous function, then $h(X_1), h(X_2), \cdots$ converges in probability to $h(X)$

Converge in probability: $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$

$\because h(x)$ is continus $\therefore$ can find $|X_n - X| \leq \delta$ that makes $|h(X_n) - h(X)| \leq \varepsilon$

$\because X_1, X_2, \cdots$ converges in probability to X

$\therefore \lim_{n\to\infty} P(|X_n - X| \leq \delta) = 1$

$\therefore \lim_{n\to\infty} P(|h(X_n) - h(X)| \leq \varepsilon) = 1$ $\therefore h(X_1), h(X_2) \cdots$ converges in probability to $h(X)$ □

(b). Exampe 5.5.8

$X_1(S) = S + I_{[0,1]}(S)$ $\quad X_2(S) = S + I_{[0,\frac{1}{2}]}(S)$ $\quad X_3(S) = S + I_{[\frac{1}{2},1]}(S)$

$X_4(S) = S + I_{[0,\frac{1}{3}]}(S)$ $\quad X_5(S) = S + I_{[\frac{1}{3},\frac{2}{3}]}(S)$ $\quad X_6(S) = S + I_{[\frac{2}{3},1]}(S)$

Find a subsequence of X_is that converges almost surely.

$X_i(S) = S + I_{[0,\frac{1}{i}]}(S)$

$\lim_{i\to\infty} X_i(S) = S + \underbrace{I_{[0,\frac{1}{i}]}(S)}_{0} \approx S$ □

5.40. X_n, X are continuous random variables

(a) Given t and ε, show that

$P(X \leq t - \varepsilon) \leq P(X_n \leq t) + P(|X_n - X| \geq \varepsilon)$

$P(X \leq t-\varepsilon) = P(-X \geq \varepsilon - t) = P(X_n - X \geq X_n + \varepsilon - t)$ → 更严格

1° $X_n \leq t$ $\quad P(X_n \leq t) = 1$

$0 \leq P(X_n - X \geq \varepsilon + \underbrace{X_n - t}_{\geq 0}) \leq 1 \leq \underbrace{P(X_n \leq t)}_{1} + P(|X_n - X| \geq \varepsilon)$

that is $P(X \leq t-\varepsilon) \leq P(X_n \leq t) + P(|X_n - X| \geq \varepsilon)$

2° $X_n > t$ $\quad P(X_n \leq t) = 0$

$P(X_n - X \geq \varepsilon + \underbrace{X_n - t}_{+}) \leq \underbrace{P(X_n - X \geq \varepsilon)}_{更宽松} + 0 \leq P(|X_n - X| \geq \varepsilon) + P(X_n \leq t)$

that is $P(X \leq t-\varepsilon) \leq \underbrace{P(X_n \leq t)} + P(|X_n - X| \geq \varepsilon)$

With 1° & 2°. □

Converge in probability: $P(\lim_{n\to\infty} |\bar{X}_n - \mu| < \varepsilon) = 1$

Converge in distribution: $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

(b) Use a similar strategy to get upper bound on $P(X_n \leq t)$

Aim: $P(X_n \leq t) \leq P(X \leq t+\varepsilon) + P(|X_n - X| \geq \varepsilon)$

$\Leftarrow P(X_n \leq t) - P(|X_n - X| \geq \varepsilon) \boxed{\leq} P(X \leq t+\varepsilon)$

Proof: $P(X \leq t+\varepsilon) = P(-X \geq -t-\varepsilon) = P(X_n - X \geq X_n - t - \varepsilon)$

1° $\varepsilon \leq 0$ $\quad P(|X_n - X| \geq \varepsilon) = 1$ $\quad P(X_n \leq t) \in (0,1)$ $\quad P(X_n \leq t) - P(|X_n - X| \geq \varepsilon) \leq 0 \leq P(X \leq t+\varepsilon)$

2° $\varepsilon > 0$ $\quad P(|X_n - X| \geq \varepsilon) = \underbrace{P(X_n - X \geq \varepsilon)}_{X_n > X} + \underbrace{P(X - X_n \leq -\varepsilon)}_{X_n < X}$

2° $\varepsilon > 0$ $\quad P(|Z_n - Z| \geq \varepsilon) = \underbrace{P(Z_n - Z \geq \varepsilon)}_{Z_n > Z} + \underbrace{P(Z - Z_n \leq -\varepsilon)}_{Z_n < Z}$

Assume ① $Z_n \geq Z$

$$P(Z_n \leq t) - P(|Z_n - Z| \geq \varepsilon) \leq P(Z \leq t) - \underbrace{P(|Z_n - Z| \geq \varepsilon)}_{\geq 0} \leq P(Z \leq t) \leq P(Z \leq t+\varepsilon)$$

that is $P(Z \leq t+\varepsilon) \geq P(Z_n \leq t) - P(|Z_n - Z| \geq \varepsilon)$

② $Z_n < Z$ $\quad -Z_n > -Z$

$$P(Z \leq t+\varepsilon) = P(-Z \geq -t-\varepsilon) \geq P(-Z_n \geq -t) = P(Z_n \leq t)$$

that is $P(Z \leq t+\varepsilon) \geq P(Z_n \leq t) \geq P(Z_n \leq t) - P(|Z_n - Z| \geq \varepsilon)$

$\therefore$ Combine 1° and 2°, we have $P(Z \leq t+\varepsilon) \geq P(Z_n \leq t) - P(|Z_n - Z| \geq \varepsilon)$ $\square$

(c) By pinching, deduce that $P(Z_n \leq t) \to P(Z \leq t)$

from (a): $\quad P(Z_n \leq t) \geq P(Z \leq t-\varepsilon) - P(|Z_n - Z| \geq \varepsilon)$

(b): $\quad P(Z_n \leq t) \leq P(Z \leq t+\varepsilon) + P(|Z_n - Z| \geq \varepsilon)$

$\because$ Z and Z_n converge in probability

$\therefore \lim_{n\to\infty} \lim_{\varepsilon \to 0} P(|Z_n - Z| \geq \varepsilon) = 0$

$\lim_{\varepsilon \to 0} P(Z \leq t-\varepsilon) = P(Z \leq t+\varepsilon) = P(Z \leq t)$

$\therefore$ $P(Z_n \leq t) \geq P(Z \leq t) + 0 \quad \Rightarrow P(Z_n \leq t) \to P(Z \leq t)$

$P(Z_n \leq t) \leq P(Z \leq t) + 0$

5.41. Proov $P(|Z_n - \mu| > \varepsilon) \to 0 \quad \Leftrightarrow \quad P(Z_n \leq x) \to \begin{cases} 0 & \text{if } x < \mu \\ 1 & \text{if } x > \mu \end{cases}$

(a) Set $\varepsilon = |x - \mu|$, show that

if $x > \mu$, $P(Z_n < x) \geq P(|Z_n - \mu| \leq \varepsilon)$

if $x \leq \mu$, $P(Z_n \leq x) \leq P(|Z_n - \mu| \geq \varepsilon)$

1° $x > \mu$: $\quad \varepsilon = x - \mu > 0$

if $Z_n < x$ $\quad P(Z_n < x) = 1 \geq P(|Z_n - \mu| \leq \varepsilon)$

if $Z_n > x$ $\quad P(Z_n < x) = 0$

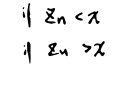
$Z_n - x > Z_n - \mu > 0$. $\quad P(|Z_n - \mu| \leq \varepsilon) = 0$

$\therefore P(Z_n < x) = P(|Z_n - \mu| \leq \varepsilon)$

$\therefore P(Z_n < x) \geq P(|Z_n - \mu| \leq \varepsilon) \quad x > \mu$

2° $x \leq \mu$ $\quad \varepsilon = \mu - x \geq 0$

if $Z_n < x$ $\quad P(Z_n \leq x) = 1 = P(|Z_n - \mu| \leq \varepsilon)$



$Z_n - x < 0$

$|Z_n - x| \leq |Z_n - \mu|$

$P(|Z_n - x| \leq \varepsilon) \geq \underbrace{P(|Z_n - \mu| \leq \varepsilon)}_{} = 1$

$\therefore P(Z_n \leq x) \leq P(|Z_n - \mu| \geq \varepsilon) \quad x \leq \mu$

if $Z_n > x$ $\quad P(Z_n \leq x) = 0 \leq \underbrace{P(|Z_n - \mu| \geq \varepsilon)}_{\in [0,1]}$

$\because P(Z_n \leq x) \leq P(|Z_n - \mu| \geq \varepsilon) \quad x \leq \mu$

$P(Z_n \leq x) \geq P(|Z_n - \mu| \leq \varepsilon) \quad x > \mu$

$\therefore P(|Z_n - \mu| > \varepsilon) \to 0$ means $P(|Z_n - \mu| \geq \varepsilon) = 0 \quad \Rightarrow \quad P(Z_n \leq x) = 0 \quad x \leq \mu$

$P(|Z_n - \mu| \leq \varepsilon) = 1 \quad P(Z_n \leq x) = 1 \quad x > \mu$ $\square$

(b) $\{x: |x-\mu|>\varepsilon\} = \{x: x-\mu<-\varepsilon\} \cup \{x: x-\mu>\varepsilon\}$ to deduce

$P(X_n \le x) \to \begin{cases} 0 & x<\mu \\ 1 & x>\mu \end{cases} \Rightarrow P(|X_n-\mu|>\varepsilon) \to 0.$

$P(|X_n-\mu|>\varepsilon) = P(X_n-\mu<-\varepsilon) + P(X_n-\mu>\varepsilon)$

1° $x<\mu$ $\because P(X_n \le x)=0$ $\therefore X_n > x$ — X_n — x

$\varepsilon = |x-\mu| = \mu - x$ $\quad x = \mu-\varepsilon$

$$\begin{aligned} P(|X_n-\mu|>\varepsilon) &= P(X_n-\mu<-\varepsilon) + P(X_n-\mu>\varepsilon) \\ &= P(X_n<\mu-\varepsilon) + P(X_n>\mu+\varepsilon) \\ &= \underbrace{P(X_n<x)}_{0} + P(X_n>\mu+\varepsilon) \\ &= \underset{\uparrow}{P(X_n>\mu+\varepsilon)} < P(X_n>\mu) \end{aligned}$$

2° $x>\mu$ $\because P(X_n \le x)=1$ $\therefore X_n \le x$

$\varepsilon = |x-\mu| = x-\mu$ $\quad x = \mu+\varepsilon$

$$\begin{aligned} P(|X_n-\mu|>\varepsilon) &= P(X_n-\mu<-\varepsilon) + P(X_n-\mu>\varepsilon) \\ &= P(X_n<\mu-\varepsilon) + P(X_n>\underbrace{\mu+\varepsilon}_{x}) \\ &= \underbrace{P(X_n<\mu-\varepsilon)}_{0} \\ &< P(X_n<\mu) \end{aligned}$$

$\therefore P(|X_n-\mu|>\varepsilon) < P(X_n>\mu)$ $\quad x<\mu$

$P(|X_n-\mu|>\varepsilon) < P(X_n<\mu)$ $\quad x>\mu$

$\because \lim_{n\to\infty} P(|X_n-x|>\delta) \to 0$

$\therefore X_n \to x$

$P(X_n>\mu) \to P(x>\mu)$

if $x<\mu$ $\quad P(X_n>\mu) \to P(x>\mu) \to 0$

if $x>\mu$ $\quad P(X_n<\mu) \to P(x<\mu) \to 0$

$\therefore \lim_{n\to\infty} P(|X_n-\mu|>\varepsilon) \to 0$ $\quad \square$

$A_n = n^v(1-Z_{(n)})$

$\lim_{n\to\infty} F_{A_n}(a) = F_Y(a)$

$\lim_{n\to\infty} A_n = n^v \cdot \underbrace{(1-Z_{(n)})}_{\to 0} = a$

$v>0$ $\quad \lim_{n\to\infty} a \to +\infty$

5.42. $Z_1, Z_2, \cdots$, iid, $Z_{(n)} = \max_{1\le i\le n} Z_i$ 前 i 个中最大.

(a). If $Z_i \sim$ beta $(1, \beta)$, find a value v, $n^v(1-Z_{(n)})$ converges in distribution.

$\because Z_i \sim$ beta $(1, \beta)$ $\qquad f(x, \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1}$

$$\begin{aligned} \therefore f(Z_i, 1, \beta) &= \frac{\Gamma(1+\beta)}{\Gamma(1)\Gamma(\beta)} \cdot x^0 \cdot (1-x)^{\beta-1} \\ &= \frac{\Gamma(1+\beta)}{\Gamma(\beta)} 1 \cdot (1-x)^{\beta-1} \\ &= \frac{\beta!}{(\beta-1)!}(1-x)^{\beta-1} \\ &= \beta(1-x)^{\beta-1} \end{aligned}$$

$\Gamma(0) = 0! = 1$

$Z_{(n)} = \max_{1\le i\le n} Z_i$

$n^v(1-Z_{(n)})$ converge in probability

$\lim_{n\to\infty} Z_{(n)} \to 1$ $\quad n^v(1-\underset{0}{Z_{(n)}})$

$\lim_{n\to\infty} n^v(1-Z_{(n)}) \to 0$

$\lim_{n\to\infty} F_{Z_n}(x) = F_Z(x)$

$\lim_{n\to\infty} Z_{(n)} \to 1$

$\therefore Z_{(n)}$ converges to 1 in probability.

$$F_Z(x) = \frac{1}{B(\alpha,\beta)} \int_0^x z^{\alpha-1}(1-z)^{\beta-1}\,dz$$

$\alpha = 1$

$$F_Z(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta)\Gamma(1)} \int_0^x (1-z)^{\beta-1}\,dz \qquad z=0 \quad 1-z=1$$

$$z = x \quad 1-z = 1-x$$

$$= \beta \int_1^{1-x} (1-(1-z))^{\beta-1}\,d(1-z)$$

$$= \beta \int_1^{1-x} z^{\beta-1}\cdot(-1)\cdot dz$$

$$= \beta\cdot(-1)\, z^{\beta}\left(\frac{1}{\beta}\right)\Big|_1^{1-x} \qquad F_Z(1-s) = 1-(1-(1-s))^{\beta}$$

$$= (-1)\cdot\left[(1-x)^{\beta}-1\right] \qquad = 1-s^{\beta}$$

$$= 1-(1-x)^{\beta} \qquad x\in(0,1]$$

$$\therefore P(n^{v}(1-Z_n)\le t) = P(1-Z_n \le t\cdot n^{-v}) = P(Z_n \ge t\cdot n^{-v})$$

$$= (1-t\cdot n^{-v})^{\beta n}$$

if $v=-1$

$$P(n^{v}(1-Z_n)\le t) = \left[\left(1-\frac{t}{n}\right)^n\right]^{\beta} \to e^{-t\beta} \qquad \square$$

(b) If X_i exponential (1) find a sequence a_n $Z_{(n)}-a_n$ converge in dis

$\because X_i \sim$ exponential dis

$$\therefore f_X(x) = \lambda\cdot e^{-\lambda x} = e^{-x}$$

$$F_Z(x) = \int_0^x \lambda\cdot e^{-\lambda z}\,dz$$

$$= \lambda\cdot(-1)\frac{1}{\lambda}e^{-\lambda z}\Big|_0^x$$

$$= (-1)\cdot(e^{-\lambda x}-1)$$

$$= 1-e^{-\lambda x}$$

$$= 1-e^{-x} \qquad x>0$$

$$P(Z_{(n)}-a_n \le s) = P(Z_{(n)} \le s+a_n)$$

$$= (1-e^{-(s+a_n)})^n \longrightarrow e^{-t}$$

$$= (1-e^{-s}\cdot e^{-a_n})^n = \left(1-\frac{e^{-s}}{e^{a_n}}\right)^n$$

if $e^{a_n} = \frac{1}{n}$ $\quad a_n = \log\frac{1}{n} = -\log n$ $\square$

5.43.

(a). Show that if $\sqrt{n}(Y_n-\mu) \to n(0,\sigma^2)$ in distribution, then $Y_n \to \mu$ in probability.

$\because \sqrt{n}(Y_n-\mu) \to n(0,\sigma^2)$ in distribution

$$\therefore P(|Y_n-\mu|\le t) = P(\sqrt{n}|Y_n-\mu| \le \sqrt{n}\,t)$$

$$\lim_{n\to\infty} P(\sqrt{n}\,|Y_n-\mu| \le \sqrt{n}\,t) = P(|z| \le \underbrace{\sqrt{n}\,t}_{\infty}) = 1$$

$\lim_{n\to\infty} P(|Y_n-\mu|\le t) = 1 \quad \therefore Y_n \to \mu$ in probability. $\square$

$$\because \quad P(x\le t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\cdot \underbrace{\int_{-\infty}^{t} e^{-\frac{x^2}{2\sigma^2}}\,dx}_{A}$$

(b) Give the details for Slutsky's Theorem.

If $Z_n \to Z$ in distribution $Y_n \to a$ in probability, then

a. $Y_n Z_n \to aZ$ in distribution

b. $Z_n + Y_n \to Z + a$ in distribution.

From a: $g'(\theta) \cdot \sqrt{n}(Y_n - \theta) \to$

Assume $Z_n = \sqrt{n}(Y_n - \theta)$ $\quad Z \sim n(0, \sigma^2)$

$Y_n = g'(\theta)$ $\quad a = g'(\theta)$

$g'(\theta) \sqrt{n}(Y_n - \theta) \to g'(\theta) Z$

$\because \lim_{Y_n \to \theta} \sqrt{n}[g(Y_n) - g(\theta)] = g'(\theta) \sqrt{n}(Y_n - \theta)$

$\therefore \lim_{Y_n \to \theta} \sqrt{n}[g(Y_n) - g(\theta)] \to n(0, \sigma^2 [g'(\theta)]^2)$ □

$\sqrt{n}(Y_n - \theta) \to n(0, \sigma^2)$

5.47. Verify the distributions of random variables in

$U_j \sim$ uniform $(0, 1)$

(a) $Y_j = -\frac{1}{\lambda} \log(U_j) \sim$ exponential (λ)

(b) $Y = -2 \sum_{j=1}^{\nu} \log(U_j) \sim \chi^2_{2\nu}$

(c) $Y = -\beta \sum_{j=1}^{\alpha} \log(U_j) \sim$ gamma(α, β)

(d) $Y = \dfrac{\sum_{j=1}^{a} \log(U_j)}{\sum_{j=1}^{a+b} \log(U_j)} \sim$ beta(a, b)

(a) If $Y \sim$ exponential (λ) $\quad f_Y(y) = \lambda \cdot e^{-\lambda y}$

$F_Y(y) = \int_0^y f_Y(x)\,dx = \int_0^y \lambda \cdot e^{-\lambda x}\,dx = (-1)e^{-\lambda x}\Big|_0^y = (-1)(e^{-\lambda y} - 1) = 1 - e^{-\lambda y}$

$F_Y^{-1}(y) = -\frac{1}{\lambda} \cdot \log(U_j)$ □

$1 - e^{-\lambda x} = y$

$1 - y = +e^{-\lambda x}$

$\log(1-y) = -\lambda x$

(b) Let $\lambda = \frac{1}{2}$ $\quad Y_j = -2\log(U_j) \sim$ exponential $(\frac{1}{2}) \sim \chi^2_2$

$\therefore \sum_{j=1}^{\nu} (-2\log(U_j)) \sim \chi^2_{2\nu}$ □

(c) Let $\lambda = \frac{1}{\beta}$ $\quad Y_j = -\beta \log(U_j) \sim$ exponential $(\frac{1}{\beta}) \sim$ gamma$(1, \beta)$

$\therefore \sum_{j=1}^{\alpha} -\beta \log(U_j) \sim$ gamma(α, β)

(d) Let $\lambda = 1$ $\quad Y_j = -\log(U_j) \sim$ exponential $(1) \sim$ gamma$(1, 1)$

$V = \sum_{j=1}^{a} \log(U_j) \sim$ gamma$(a, 1)$

$U = \sum_{j=1}^{b} \log(U_j) \sim$ gamma$(b, 1)$

$\frac{V}{V+U} \sim$ beta(a, b) □

5.48. How to generate an $F_{m,n}$ random variable, m, n are even integers.

If $X \sim F(m,n)$

$$f_X(x) = \frac{S_1/m}{S_2/n} \qquad S_1 \sim \chi^2_m \qquad S_2 \sim \chi^2_n$$

$U_j \sim \text{uniform}(0,1)$

$-\frac{1}{\lambda}\log(U_j) \sim \text{exponential}(\lambda)$

let $\lambda = \frac{1}{2}$

$-2\log(U_j) \sim \text{exponential}(\frac{1}{2}) \sim \chi^2_2$

$\sum_{j=1}^{\frac{m}{2}} -2\log(U_j) \sim \chi^2_m$

$\sum_{j=1}^{\frac{n}{2}} -2\log(U_j) \sim \chi^2_n$

$$Y = \frac{\left(\sum_{j=1}^{m}[-2\log(U_j)]\right)/m}{\left(\sum_{j=1}^{n}[-2\log(U_j)]\right)/n} = \frac{-2\cdot\frac{m}{2}\left(\sum_{j=1}^{m/2}\log(U_j)\right)/m}{-2\cdot\frac{n}{2}\left(\sum_{j=1}^{n/2}\log(U_j)\right)/n} = \frac{\sum_{j=1}^{m/2}\log(U_j)}{\sum_{j=1}^{n/2}\log(U_j)} \sim F_{m,n} \quad \square$$

5.49. $U \sim \text{uniform}(0,1)$

(a) Show that both $-\log U$ and $-\log(1-U)$ are exponential random variables

$X \sim \text{exponential}(\lambda)$

$f_X(x) = \lambda \cdot e^{-\lambda y} \qquad \lambda = 1 : f_X(x) = e^{-y}$

$F_X(x) = \int_0^x \lambda \cdot e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^x = -e^{-\lambda x} + 1 = 1 - e^{-\lambda x}$

$F_X^{-1}(x) = -\frac{1}{\lambda}\log(1-x)$

when $\lambda = 1$ $T(x) = F_X^{-1}(x) = -\log(1-x)$

$T(u) = -\log(1-u)$, $U \sim \text{uniform}(0,1)$

$\sim \text{exponential}(1)$

$P(u \geq a) = 1 - P(u \leq a)$

$T(1-U) = -\log(1-(1-U)) = -\log U \sim \text{exponential}(1) \ \square$

e^{-y}

$F_X(x) = P(T(U) \leq x)$

$= P(U \leq T^{-1}(x))$

$\because U \sim \text{uniform}(0,1)$

$\therefore P(U \leq T^{-1}(x)) = T^{-1}(x)$

that is $F_X(x) = T^{-1}(x)$

$T(x) = F_X^{-1}(x)$ Just Replace

$F_X(x) = P(T(1-U) \leq x)$ $\quad U \to 1-U$

$= P(1-U \leq T^{-1}(x))$

$\because U \sim \text{uniform}(0,1)$

$1-U \sim \text{uniform}(0,1)$

$\therefore P(1-U \leq T^{-1}(x))$

$= P(1-U \leq T^{-1}(x)) = T^{-1}(x)$

(b) Show that $X = \log\frac{U}{1-U}$ is a logistic $(0,1)$ random variable.

$$f_X(x) = \frac{e^{-x}}{(1+e^{-x})^2} = \left(e^{x/2} + e^{-x/2}\right)^{-2}$$

$$F_X(x) = \int_0^x \left(e^{\frac{y}{2}} + e^{-\frac{y}{2}}\right)^{-2} dy$$

$$= \int_0^x \left(e^{y} + 2 + e^{-y}\right)^{-1} dy = (1+e^{-x})^{-1}$$

$F_X(x) = P(T(U) \leq x)$

$= P(U \leq T^{-1}(x)) = T^{-1}(x) = -\log\frac{1-x}{x} = \log\frac{x}{1-x}$

$\therefore T(x) = F_X^{-1}(x) = \log\frac{x}{1-x}$

Replace x with U, we have $\log\frac{U}{1-U} \sim \text{logistic}(0,1)$, where $U \sim \text{uniform}(0,1)$

(c) Show how to generate a logistic (μ, β) random variable.

If $X \sim \text{logistic}(\mu, \beta)$

$$f_X(x) = \frac{e^{-(x-\mu)/\beta}}{\beta(1+e^{-(x-\mu)/\beta})^2} \qquad \ln a + \ln$$

$$= \frac{1}{\beta\left(e^{(x-\mu)/2\beta} + e^{-(x-\mu)/2\beta}\right)^2}$$

$$F_X(x) = \frac{1}{\beta}\int_{-\infty}^{x} \frac{1}{\left(e^{(y-\mu)/2\beta} + e^{-(y-\mu)/2\beta}\right)^2}\, dy$$

$$= \frac{1}{\beta}\int_{-\infty}^{x}\left[e^{(y-\mu)/\beta} + 2 + e^{-(y-\mu)/\beta}\right]^{-1} dy$$

Assume $e^{(y-\mu)/\beta} = w \qquad y = \beta \ln w + \mu$

$y = -\infty \quad w = 0$

$y = x \quad w = e^{(x-\mu)/\beta} \qquad dy = \beta \cdot \frac{1}{w} \cdot dw$

$$F_X(x) = \frac{1}{\beta}\int_0^{e^{(x-\mu)/\beta}} (w + 2 + w^{-1})^{-1}\, \beta \frac{1}{w}\, dw$$

$$= \int_0^{e^{(x-\mu)/\beta}} \frac{1}{\left(w + 2 + \frac{1}{w}\right)} \cdot \frac{1}{w}\, dw$$

$$= \int_0^{e^{(x-\mu)/\beta}} (w+1)^{-2}\, dw$$

$$= (-1)(w+1)^{-1}\Big|_1^{e^{(x-\mu)/\beta}}$$

$$= (-1)\left[\left(e^{(x-\mu)/\beta}+1\right)^{-1} - 1\right]$$

$$= 1 - \frac{1}{e^{(x-\mu)/\beta}+1}$$

$$= \frac{e^{(x-\mu)/\beta}}{e^{(x-\mu)/\beta}+1} \quad \checkmark$$

$f(\theta) \qquad f(r)$

$$F_X(x) = P(T(U) \le x) = P(U \le T^{-1}(x))$$

$\because U \sim \text{uniform}(0,1)$

$\therefore P(U \le T^{-1}(x)) = T^{-1}(x)$

$\therefore T(x) = F_X^{-1}(x)$

$= -\beta \cdot \ln \frac{1-x}{x} + \mu$

Replace x with U

$-\beta \cdot \ln \frac{1-U}{U} + \mu \sim \text{Logistic}(\mu, \beta)$

$$\frac{e^{(x-\mu)/\beta}}{e^{(x-\mu)/\beta}+1} = y = \frac{1}{1+e^{-(x-\mu)/\beta}} = y$$

$1 + e^{-(x-\mu)/\beta} = \frac{1}{y}$

$e^{-(x-\mu)/\beta} = \frac{1}{y} - 1$

$-(x-\mu)/\beta = \ln \frac{1-y}{y}$

$x = -\beta \ln \frac{1-y}{y} + \mu$

$\cos(2\pi U_1)$

5.50. $X_1 = \cos(2\pi U_1)\sqrt{-2\log U_2} \qquad X_2 = \sin(2\pi U_1)\cdot\sqrt{-2\log U_2}$

U_1, U_2 are iid uniform $(0,1)$. Prove that X_1, X_2 are iid. $n(0,1)$ random variables.

If X_1, X_2 are iid $\sim n(0,1)$

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}}$$

$$= \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}$$

Change into polar coordination: $f_{\theta,r}(r\cos\theta, r\sin\theta) = \frac{1}{2\pi} e^{-\frac{r^2}{2}} \cdot r = f(\theta)\cdot f(r)$

Assume $\Theta = 2\pi U_1 \quad \because U_1 \sim (0,1)$

$\therefore f(\theta) = \frac{1}{2\pi}$

$R = \sqrt{-2\log U_2}$

$P(R \le r) = P(\sqrt{-2\log U_2} < r) = P(-\log U_2 < \frac{r^2}{2})$

$= 1 - e^{-\frac{r^2}{2}}$

$f_R(r) = r \cdot e^{-\frac{r^2}{2}}$

$\therefore X_1 = \cos(2\pi U_1)\sqrt{-2\log U_1}$

$= \cos(\theta)\cdot r$

$X_2 = \sin(2\pi U_1)\cdot\sqrt{\ }$

$= \sin\theta\, r$

5.51. $Z = \sum_{i=1}^{12} U_i - 6$, U_is are iid uniform $(0,1)$

(a). Justify the fact that Z is approximately $n(0,1)$

$U_i \sim (0,1)$ $\quad \mu_{U_i} = \frac{1}{2}$ $\quad Var(U) = \frac{1}{12} = \sigma^2$ $\quad Z = \sum_{i=1}^{12} U_i - 6 = 12\bar{U}_i - 6 = 12(\bar{U}_i - \frac{1}{2}) = \sqrt{12} \cdot \frac{\bar{U} - 1/2}{\frac{1}{\sqrt{12}}} = \sqrt{n} \cdot \frac{\bar{U}_{12} - \mu_U}{\sigma}$

$f_U(u) = \frac{1}{b-a}$

from central limit theory.

no matter the distribution of U_i

$Z \sim n(0,1)$ $\square$

$$Var(U) = \int_a^b u^2 f_U(u)\,du - E^2(U)$$
$$= \int_a^b \frac{x^2}{b-a}\,du - \left(\frac{a+b}{2}\right)^2$$
$$= \frac{1}{b-a} \cdot \frac{1}{3} x^3 \Big|_a^b - \left(\frac{a+b}{2}\right)^2$$
$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$
$$= \frac{4(b-a)(a^2+ab+b^2)}{12(b-a)} - \frac{(a+b)^2}{4}$$
$$= \frac{4(a^2+ab+b^2) - 3(a+b)^2}{12}$$
$$= \frac{a^2 - 2ab + b^2}{12}$$
$$= \frac{(a-b)^2}{12} = \frac{1}{12}$$

(b) Find an obvious way in which the approximation fails.

$\sum_{i=1}^{12} U_i - 6 \in [-6, 6]$ but $n(0,1) \in (-\infty, +\infty)$

(c) Show how good (bad) the approximation is by comparing the first four moments.

$$EZ = E(\sum U_i - 6) = E(\sum U_i) - 6 = 12\,EU_i - 6 = 12 \cdot \tfrac{1}{2} - 6 = 0$$

$$EZ^2 = E(\sum U_i - 6)^2 = E(\sum U_i^2 - 12\sum U_i + 36) = \underbrace{E(\sum U_i^2)} - 12\sum U_i + 36 = 37 - 12 \times 12 \times \tfrac{1}{2} + 36 = 37 - 6 \times 12 + 36 = 1$$

$E\{(U_1 + U_2 + \cdots + U_{12})^2\}$

$$= E(\underbrace{U_1^2 + U_1U_2 + U_1U_3 + \cdots + U_1U_n}_{11} + U_1U_2 + U_2^2 + \cdots + U_2U_{12} + U_3U_1 + \cdots U_3^2 + \cdots + U_3U_{12} + \cdots + U_{12}U_1 + \cdots + U_{12}^2)$$

$$= E(12U_i^2 + U_iU_j \times 11 \times 12)$$
$$= 12 \cdot E(U_i^2) + 11 \cdot 12\, E(U_i) \cdot E(U_j)$$
$$= 12 \times \tfrac{1}{3} + 11 \times 12 \times \tfrac{1}{2} \times \tfrac{1}{2}$$
$$= 4 + 33 = 37$$

$Var U_i = EU_i^2 - (EU_i)^2$

$EU_i^2 = Var U_i + (EU_i)^2$

$= \frac{1}{12} + (\frac{1}{2})^2$

$= \frac{1}{12} + \frac{1}{4}$

$= \frac{1+3}{12}$

$= \frac{1}{3}$

$EZ^3 = 0$ (Z is symmetric about 0, $EZ = 0$ $\quad -6 < Z < 6$)

$EZ^4 = E(\sum U_i - 6)^4$ $\qquad \sum U_i = a$

$(\sum U_i^2 - 12\sum U_i + 36)(\sum U_i^2 - 12\sum U_i + 36)$

$(a^2 - 12a + 36)(a^2 - 12a + 36)$

$= a^4 - 12a^3 + 36a^2 - 12a^3 + 144a^2 - 36 \times 12a + 36a^2 - 12 \times 36a + 36^2$

$= a^4 - 24a^3 + 216a^2 - 864a + 36^2$

$E[(\sum U_i)^4] - 24\,E[(\sum U_i)^3] + 216\,E[(\sum U_i)^2] - 864\,E(\sum U_i) + 36^2$

$= E[(\sum U_i)^4] - 24 \cdot 234 + 216 \cdot 37 - 864 \cdot 12 \cdot \frac{1}{2} + 36^2$

$EZ^3 = E(\sum U_i - 6)^3$

$= 234$

$(a-6)^3 = (a-6)(a^2 - 12a + 36)$

$= a^3 - 6a^2 - 12a^2 + 72a + 36a - 36 \times 6$

$= a^3 - 18a^2 + 108a - 36 \times 6$

$= E(a^3) - 18\,E(a^2) + 108\,E(a) - 216$

$= E(a^3) - 18 \times 37 + 108 \cdot 12 \cdot \frac{1}{2} - 216 = 0$

$E(a^3) = 666 - \frac{108 \times 6}{648} + 216 = 234$

$E(\sum U_i)^4$

$= E(U_1 + U_2 + \cdots + U_n)^4$

$= E \cdots$ 很不好求了 换方法.

用mgf $\quad Ee^{tU} = \int_0^1 e^{tu} f_U(u)\,du$

$= \int_0^1 e^{tu}\,du = \frac{1}{t} e^{tu} \Big|_0^1 = \frac{1}{t}(e^t - 1)$

$e^{t(U_1 + U_2 + \cdots + U_{12} - 6)}$

$= e^{tU_1} \cdot e^{tU_2} \cdot e^{tU_3} \cdots e^{tU_{12}} \cdot e^{-6t}$

$E[e^{t(\sum U_i - 6)}] = E e^{tU_1} \cdot E e^{tU_2} \cdots E e^{tU_{12}} \cdot E e^{-6t}$

$= (Ee^{tU_i})^{12} \cdot e^{-6t}$

$f(t) = [\frac{1}{t}(e^t - 1)]^{12} \cdot e^{-6t}$

$((t)^{-1})' = (-1) \cdot t^{-2}$

Calculate the 4th derivation:

$f'(t) = [\frac{1}{t}(e^t - 1)]^{12} \cdot e^{-6t}(-6) + e^{-6t}\, 12\, [\frac{1}{t}(e^t - 1)]^{11} (\frac{1}{t} e^t - e^t \cdot t^{-2})$

$EZ^4 = 29/10 < EZ^4 = 3$ $\quad [Z \sim n(0,1)]$

mgf $Z \sim (0,1)$ $f(t) = e^{t^2/2}$

$E(Z)$ $f'(t) = e^{t^2/2} \cdot \frac{1}{2} 2t = t e^{t^2/2}$ $t=0$ $f'(t) = 0$

$E(Z^2)$ $f''(t) = t \cdot t e^{t^2/2} + e^{t^2/2}$ $t=0$ $f''(t) = 1$

$E(Z^3)$ $f^3(t) = t^2 \cdot t e^{t^2/2} + 2t \cdot e^{t^2/2} + t \cdot e^{t^2/2}$ $t=0$ $f^3(t) = 0$

$E(Z^4)$ $f^4(t) = t^2 \cdot t e^{t^2/2} + 3t^2 e^{t^2/2} + 3t \cdot t e^{t^2/2} + 3 \cdot e^{t^2/2}$ $t=0$ $f^4(t) = 3$

not too far from 29/10, ∴ is not a bad approximation □

5.52. Write an algorithm for generating the indicated random variable

(a) $Y \sim$ binomial $(8, \frac{2}{3})$ 有放回

Java:

```
public static int getBinomial (int n, double p) {
    int x=0;
    for (int i=0 ; i<n ; i++) {
        if (math.random < p) { x++; }
    }
    return x;
}
```

$n=8$
$p=\frac{2}{3}$

□

(b) $Y \sim$ hypergeometric $N=10$, $M=8$, $K=4$ 无放回

Java:

```
public static int getHypergeometric (int n, int m, int k){
    int x=0;
    for (int i=0 ; i<k ; i++) {    // trails
        if (math.random < m/n) { x++ ; m--; }   // no put back
        n--;                                    // Update m and n
    }
    return x;
}
```

N: population size
M: num of success states in population
K: num of trials
$n=N$
$m=M$
$k=K$
x: num of observed success

(c) $Y \sim$ negative binomial $(5, \frac{1}{3})$

Java:

```
public static int getNegativeBinomial (int n, double p){
    int x=0;   // num of failures
    int s=0;   // num of success
    while (s<n){
        if (math.random < p) { s++; }
        else { x++; }
    }
    return x;
}
```

s: number of success
$\frac{1}{3}$: probability of being success
x: number of failures before success s times

$n=5$
$p=\frac{1}{3}$

□

5.54. Data 158, 143, 106, 57, 97, 80, 109, 109, 350, 224, 109, 214, 84

(a). Construct a simulated distribution of s^2 to see if the assumption of Poisson counts is tenable.

$n=13$ $\bar{x} = 141.5385$ $s^2 = 6370.603$

Assume its Possion distribution with $\lambda = 141.5385$

when $n=13$ calculate the var, repeating 50000 times

And find the probability of var $\geq s^2 = 0$

$\therefore$ this assumption is not tenable

(b) Negative binomial (r, p) is a reasonable alternative model. $\mu = r \cdot \frac{1-p}{p}$ $var = \mu + \frac{\mu^2}{r}$

For $\mu = \bar{x}$, what values of r lead to simulated distributions that are consistent with the data?

$n=13$, $\bar{x} = 141.5385$ $s^2 = 6370.603$

$\mu = \bar{x} = r \cdot \frac{1-p}{p} = r \cdot (\frac{1}{p} - 1)$ $p = (\frac{\bar{x}}{r} + 1)^{-1}$

$var = \mu + \frac{\mu^2}{r} = 6370.603 = s^2$

$r = (s^2 - \mu)^{-1} \mu^2 = 3.2 \approx 3$ □

5.55 Y is a discrete random variable

$y_1 < y_2 < \cdots < y_k$

$P[F_Y(y_i) < U \leq F_Y(y_{i+1})] = F_Y(y_{i+1}) - F_Y(y_i)$

$= P(Y = y_{i+1})$

Show that the expected num of comparisons is $E(Y+1)$

Let X denote the number of comparisons, then

$EX = \sum_{k=0}^{\infty} P(X > k) = 1 + \sum_{k=1}^{\infty} P(U > F_Y(y_{k-1}))$

$= 1 + \sum_{k=1}^{\infty} (1 - F_Y(y_{k-1}))$

$= 1 + \sum_{k=0}^{\infty} (1 - F_Y(y_k))$

$= 1 + EY$

$P(X>k)$: 在 k 次的时候是否都是坏

如果是 : $P(X>k) = 1$

不是 : $P(X>k) = 0$

$\therefore \sum_k P(X>k)$ is EX.

$k=0$ $k=1$ $k=2$

y_0 y_0 y_1 y_0 y_1 y_2

$\int_0^\infty (1 - F_Z(x))\,dx$

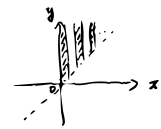
$= \int_0^\infty P(Z > x)\,dx$

$= \int_{0=x}^{\infty=x} \int_{x=y}^{\infty=y} f_Z(y)\,dy\,dx$

$= \int_0^\infty \int_0^y 1\,dx\, f_Z(y)\,dy$

$= \int_0^\infty y f_Z(y)\,dy$

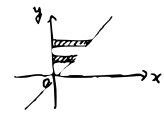
$= EX$ □



Integral y first

Integral x first

$\int_0^\infty \int_0^y f_Z(y)\,dx\,dy$



5.56 Let Y have the Cauchy distribution

$f_Y(y) = \frac{1/\pi}{1+y^2}$ $y \in (-\infty, \infty)$

(a) Show that $F_Y(y) = \tan^{-1}(y)$

$\because f_Y(y) = \frac{1}{\pi(1+y^2)}$

$\therefore F_Y(y) = \frac{1}{\pi}\int_{-\infty}^{y} f_Y(x)\,dx$

$= \frac{1}{\pi}\int_{-\infty}^{y} \frac{1}{1+x^2}\,dx$

Substitute $x = \tan\theta = \frac{\sin\theta}{\cos\theta} = \sin\theta\cdot(\cos\theta)^{-1}$

$dx = d\frac{\sin\theta}{\cos\theta} = (-\sin\theta\cdot(\cos\theta)^{-2}\cdot(-\sin\theta) + \cos\theta\cdot(\cos\theta)^{-1}$

$= \frac{+\sin^2\theta + \cos^2\theta}{\cos^2\theta}$

$= \frac{1}{\cos^2\theta}\,d\theta$

$x = -\infty \quad \tan\theta = -\infty \quad \theta = -\frac{\pi}{2}$

$x = y \quad \tan\theta = y \quad \theta = \arctan y$

$\frac{1}{1+x^2} = \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta}{1}$

$F_Y(y) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\arctan y} \cos^2\theta \cdot \frac{1}{\cos^2\theta}\,d\theta$

$= \frac{1}{\pi}\left(\theta\Big|_{-\frac{\pi}{2}}^{\arctan y}\right)$

$= \frac{1}{\pi}\left(\arctan y + \frac{\pi}{2}\right)$

$= \frac{1}{\pi}\arctan y + \frac{1}{2}$ □

(b) Show how to simulate a Cauchy(a,b) random variable starting from a uniform$(0,1)$ random variable.

$Z \sim$ Cauchy(a,b)

$f_Z(x) = \frac{1/(\pi b)}{1+\left(\frac{x-a}{b}\right)^2}$

$F_Z(x) = \frac{1}{\pi b}\cdot\arctan\left(\frac{x-a}{b}\right) + \frac{1}{2}$

$\therefore T(x) = b\tan[(x-\frac{1}{2})\pi b] + a$

$x \sim U(0,1)$

$T(x) \sim$ Cauchy(a,b)

$T(U)$

$P(T(U) \le x) = F_Z(x)$

$= P(U \le T^{-1}(x))$

$\because U \sim$ uniform$(0,1)$

$\therefore P(U \le T^{-1}(x))$

$= T^{-1}(x) = F_Z(x) = \frac{1}{\pi b}\arctan\left(\frac{x-a}{b}\right) + \frac{1}{2} = y$

$\frac{x-a}{b} = \tan(y-\frac{1}{2})\pi b$

$x = b\tan[y-\frac{1}{2})\pi b] + a$

5.57. Z_1, Z_2, Z_3 independent Poisson with mean $\lambda_1, \lambda_2, \lambda_3$

$Y_1 = Z_1 + Z_3 \qquad Y_2 = Z_2 + Z_3$

(a) Show that $Cov(Y_1, Y_2) = \lambda_3$

$Cov(Y_1, Y_2) = E[(Y_1 - \mu_{Y_1})(Y_2 - \mu_{Y_2})]$

$= E(Y_1Y_2) - \mu_{Y_1}\cdot E(Y_2) - \mu_{Y_2}\cdot E(Y_1) + \mu_{Y_1}\mu_{Y_2}$

$E(Y_1) = E(Z_1) + E(Z_3) = \lambda_1 + \lambda_3 = \mu_{Y_1}$

$E(Y_2) = E(Z_2) + E(Z_3) = \lambda_2 + \lambda_3 = \mu_{Y_2}$

$Cov(Y_1, Y_2) = E(Y_1Y_2) - (\lambda_1+\lambda_3)\cdot(\lambda_2+\lambda_3) - (\lambda_2+\lambda_3)\cdot(\lambda_1+\lambda_3)$

$+ (\lambda_1+\lambda_3)(\lambda_2+\lambda_3)$

$= E(Y_1Y_2) - (\lambda_1+\lambda_3)(\lambda_2+\lambda_3)$

$Y_1Y_2 = (Z_1+Z_3)(Z_2+Z_3)$

$= Z_1Z_2 + Z_1Z_3 + Z_2Z_3 + (Z_3)^2$

$E(Y_1Y_2) = E(Z_1Z_2) + E(Z_1Z_3) + E(Z_2Z_3) + E(Z_3^2)$

$\because Z_1, Z_2, Z_3$ are independent

$E(Z_1Z_2) = \lambda_1\lambda_2$

$E(Z_1Z_3) = \lambda_1\lambda_3$

$E(Z_2Z_3) = \lambda_2\lambda_3$

$Var(Z_3) = E(Z_3^2) - E^2(Z_3) = \lambda_3$

$E(Z_3^2) = Var(Z_3) + E^2(Z_3)$

$= \lambda_3 + \lambda_3^2$

$E(Y_1Y_2) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3 + \lambda_3^2$

$Cov(Y_1, Y_2) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3 + \lambda_3^2$

$- \lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 - \lambda_3^2 = \lambda_3$ □

(b) Define $Z_i = I(Y_i = 0)$ and $P_i = e^{-(\lambda_i + \lambda_3)}$, show that Z_i are Bernoulli (P_i) with

$$\mathrm{Corr}(Z_1, Z_2) = \frac{P_1 P_2 (e^{\lambda_3} - 1)}{\sqrt{P_1(1-P_1)}\sqrt{P_2(1-P_2)}}$$

$Z_i = \begin{cases} 1 & Y_i = 0 \ (Z_i = Z_3 = 0) \\ 0 & Y_i \neq 0 \end{cases}$ $\quad Y_1 = Z_1 + Z_3 \quad Y_2 = Z_2 + Z_3 \quad Z_1, Z_2, Z_3 \sim \mathrm{Poisson}(\lambda)$

$P_{Z_i}(x_i) = \dfrac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!}$ if $x_i = 0$

$P = e^{-(\lambda_i + \lambda_3)}$

when $Z_i = 1 \quad Z_1 = Z_3 = 0$

$P(Z_i = 1) = P(Z_1 = Z_3 = 0) = e^{-\lambda_1} \cdot e^{-\lambda_3} = e^{-(\lambda_1 + \lambda_3)} \quad P_{Z_1}(0) = e^{-\lambda_1}$

$P(Z_i = 0) = 1 - P(Z_1 = Z_3 = 0) = 1 - e^{-(\lambda_1 + \lambda_3)}$

$\therefore f_{Z_i}(z) = \begin{cases} e^{-(\lambda_1 + \lambda_3)} & z = 1 \\ 1 - e^{-(\lambda_1 + \lambda_3)} & z = 0 \end{cases}$ $\square$

$\mathrm{Corr}(Z_1, Z_2) = \dfrac{\mathrm{Cov}(Z_1, Z_2)}{\sigma_{Z_1} \sigma_{Z_2}}$

$\mathrm{Cov}(Z_1, Z_2) = E[(Z_1 - \mu_{Z_1})(Z_2 - \mu_{Z_2})]$

$= E(Z_1 Z_2) - \mu_{Z_1}\mu_{Z_2} - \mu_{Z_1}\mu_{Z_2} + \mu_{Z_1}\mu_{Z_2}$

$= E(Z_1 Z_2) - \mu_{Z_1}\mu_{Z_2}$

$\because Z_i \sim \mathrm{Bernoulli}(P_i) \quad \therefore E(Z_i) = P_i = e^{-(\lambda_i + \lambda_3)} = \mu_{Z_i}$

$\therefore \mu_{Z_1} = e^{-(\lambda_1 + \lambda_3)} \quad \mu_{Z_2} = e^{-(\lambda_2 + \lambda_3)}$

$f_{Z_1}(z_1) = \begin{cases} e^{-(\lambda_1 + \lambda_3)} & z_1 = 1 \\ 1 - e^{-(\lambda_1 + \lambda_3)} & z_1 = 0 \end{cases} \quad f_{Z_2}(z_2) = \begin{cases} e^{-(\lambda_2 + \lambda_3)} & z_2 = 1 \\ 1 - e^{-(\lambda_2 + \lambda_3)} & z_2 = 0 \end{cases}$

$f(z_1, z_2) = \begin{cases} e^{-(\lambda_1 + \lambda_2 + \lambda_3)} & z_1 = z_2 = 1 \iff Z_1 = Z_2 = Z_3 = 0 \\ \cdots & \text{otherwise} \end{cases}$

$E(Z_1 Z_2) = e^{-(\lambda_1 + \lambda_2 + \lambda_3)}$

★ $\therefore \mathrm{Cov}(Z_1 Z_2) = e^{-(\lambda_1 + \lambda_2 + \lambda_3)} - e^{-(\lambda_1 + \lambda_2 + 2\lambda_3)}$

$= e^{-(\lambda_1 + \lambda_2 + 2\lambda_3)}(e^{\lambda_3} - 1) = P_1 P_2 (e^{\lambda_3} - 1)$

$\mathrm{Var}(Z_i) = E(Z_i)^2 - E(Z_i)^2$

$= e^{-(\lambda_i + \lambda_3)} - e^{-2(\lambda_i + \lambda_3)}$

$= e^{-(\lambda_i + \lambda_3)}(1 - e^{-(\lambda_i + \lambda_3)}) = P_i(1 - P_i)$

$\mathrm{Var}(Z_1) = P_1(1 - P_1)$ ★ $\sigma_{Z_1} = \sqrt{\mathrm{Var}\, Z_1} = \sqrt{P_1(1-P_1)}$

$\mathrm{Var}(Z_2) = P_2(1 - P_2) \quad \sigma_{Z_2} = \sqrt{\mathrm{Var}\, Z_2} = \sqrt{P_2(1-P_2)}$

$\therefore \mathrm{Corr}(Z_1 Z_2) = \dfrac{\mathrm{Cov}(Z_1, Z_2)}{\sigma_{Z_1}\sigma_{Z_2}} = \dfrac{P_1 P_2 (e^{\lambda_3} - 1)}{\sqrt{P_1(1-P_1)}\sqrt{P_2(1-P_2)}}$ $\square$

(c) Show that the correlation of Z_1 & Z_2 is not unrestricted in the range $[-1, 1]$, but

$$\mathrm{Corr}(Z_1, Z_2) \le \min\left\{ \sqrt{\frac{P_2(1-P_1)}{P_1(1-P_2)}}, \sqrt{\frac{P_1(1-P_2)}{P_2(1-P_1)}} \right\}$$

$$\mathrm{Cov}(Z_1, Z_2) = E(Z_1 Z_2) - \mu_{Z_1}\mu_{Z_2}$$
$$= e^{-(\lambda_1+\lambda_2+\lambda_3)} - P_1 \cdot P_2$$

$$P_1 = e^{-(\lambda_1+\lambda_3)} \qquad e^{-(\lambda_1+\lambda_2+\lambda_3)} \le P_1$$
$$P_2 = e^{-(\lambda_2+\lambda_3)} \qquad e^{-(\lambda_1+\lambda_2+\lambda_3)} \le P_2$$

$$\therefore \mathrm{Cov}(Z_1, Z_2) \le P_1 - P_1 \cdot P_2 = P_1(1-P_2) \qquad \mathrm{Cov}(Z_1, Z_2) \le P_2 - P_1 \cdot P_2 = P_2(1-P_1)$$

$$\mathrm{Corr}(Z_1, Z_2) = \frac{\mathrm{Cov}(Z_1, Z_2)}{\sigma_{Z_1}\sigma_{Z_2}} \le \frac{P_1(1-P_2)}{\sqrt{P_1(1-P_1)}\sqrt{P_2(1-P_2)}} = \frac{\sqrt{P_1(1-P_2)}}{\sqrt{P_2(1-P_1)}}$$

$$\frac{\mathrm{Cov}(Z_1, Z_2)}{\sigma_{Z_1}\sigma_{Z_2}} \le \frac{P_2(1-P_1)}{\sqrt{P_1(1-P_1)}\sqrt{P_2(1-P_2)}} = \frac{\sqrt{P_2(1-P_1)}}{\sqrt{P_1(1-P_2)}} \qquad \square$$

5.58. Suppose $U_1, U_2, \cdots, U_n$ are iid uniform $(0, 1)$. Let $S_n = \sum_{i=1}^{n} U_i$. Define random variable N by.

$$N = \min\{k : S_k > 1\}$$

(a). Show that $P(S_k \le t) = t^k / k!$ if $0 \le t \le 1$

$$P(S_k \le t) = P\left(\sum_{i=1}^{k} U_i \le t\right) = P(S_1 \le t, S_2 \le t, S_3 \le t \cdots, S_k \le t)$$
$$= P(U_1 + U_2 + \cdots + U_k \le t) \qquad U_i \in (0,1) \quad t \in (0,1)$$

$U = U_1 + U_2 + \cdots + U_k \sim$ Irwin-Hall dis

$$P(U \le t) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (t-i)^k$$
$$= \frac{1}{k!} \cdot (t)^k$$

$f(x) = e^x$

Taylor-expand:

$f'(x) = f^2(x) = f''(x) = e^x$

$$f_{n=0}(x) = \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \cdots + \frac{f^n(0)}{n!}(x-0)^n$$
$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$
$$f_{n=0}(1) = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

(b) Show that $P(N=n) = P(S_{n-1} < 1) - P(S_n < 1)$ and $E(N) = e$

$$S_n = \sum_{i=1}^{n} U_i = S_{n-1} + U_i$$

$$N = \min\{k : S_k > 1\}$$

$$S_{k-1} \le 1 \quad S_k > 1 \Rightarrow \sum_{i=1}^{k-1} U_i \le 1 \quad \sum_{i=1}^{k} U_i > 1$$

$$P(N=n) = P(S_{n-1} \le 1, S_n > 1) = P\left(\sum_{i=1}^{n-1} U_i \le 1, \sum_{i=1}^{n} U_i > 1\right) = P(S_{n-1} < 1) \cdot \left[1 - \frac{P(S_n < 1)}{P(S_{n-1} < 1)}\right] = P(S_{n-1} < 1) - P(S_n < 1) \quad \square$$
$$= P(S_n > 1) - P(S$$

$$\text{Ⓐ} \quad P(S_{n-1} < 1) \begin{cases} P(S_n > 1) = P(S_{n-1} < 1) \cdot \underbrace{P(U_n > 1 - S_{n-1})}_{①} \\ P(S_n < 1) = P(S_{n-1} < 1) \cdot \underbrace{\frac{P(S_n < 1)}{P(S_{n-1} < 1)}}_{②} \end{cases}$$

① + ② = 1

$$\therefore ② = P(U_n > 1 - S_{n-1}) = 1 - \frac{P(S_n < 1)}{P(S_{n-1} < 1)}$$

$$P(S_n < 1) = \frac{1}{k!} \cdot 1^k = \frac{1}{k!}$$

$$E(N) = \sum_{n=0}^{\infty} n \cdot P(N)$$
$$= \sum n \cdot [P(S_{n-1} < 1) - P(S_n < 1)]$$
$$= \sum n \cdot \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)$$
$$= \sum n \cdot \frac{n-1}{n!} = \underset{n=1}{1} + \underset{n=2}{1} + \sum_{n=3}^{\infty} \frac{1}{(n-2)!} = e \quad \square$$

$$\frac{1}{(k-1)!} - \frac{1}{k!} = \frac{k-1}{k!}$$

(c) Use the result of part (b) to calculate the value of e by simulation

From (b), we know : $E(N) = e$

$\therefore$ To estimate e. we generate N : $n_1, n_2, \cdots n_k$

```
public static double estimateE () {
    Double u=0;
    Double[] N = new Double [5000] ;
    Int n = 0;
    for (int i=0; i<5000 ; i++){
        u=0;
        n=0;
        while (u ≤ 1){
            u = u+ math.random() ;
            n++;
        }
        N[i] = n ;
    }
    Return mean (N);
}
```

5.59 Prove that the algorithm of Example 5.6.7 generates a beta(a, b) random variable.

a. Generate (U, V) independent uniform $(0,1)$

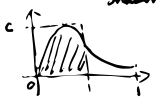
b. If $U < \frac{1}{c} f_Y(V)$, set $Y = V$

$C = \max_y f_Y(y)$ —— beta density

$Y \sim$ beta (a, b)

$P(y \le Y) = f_Y(y)$ $\quad$ $T(U, V)$

Shadow area



Q:

$\because$ U, V are independent

$\therefore f_Y(y) = P(V \le y,\ U < \frac{1}{c} f_Y(V)) = P(V \le y) \cdot P(U < \frac{1}{c} f_Y(y))$

$\therefore$ when $P(U < \frac{1}{c} f_Y(y)) = 1$

$f_Y(y) = P(V \le y)$ $\quad$ $V \sim$ beta (a, b)

that is when $U < \frac{1}{c} f_Y(y)$ $\quad$ assign V to Y,

we will get $\underset{Y \sim}{\text{beta}}(a, b)$ $\quad$ □

5.60. For an arbitrary bounded pdf $f(x)$ on $[a,b]$, define $c = \max_{a \le x \le b} f(x)$. Let X and Y be independent. $X \sim \text{uniform}(a,b)$ and $Y \sim \text{uniform}(0,c)$. , $d > b$

$$W = \begin{cases} X & \text{if } Y < f(X) \\ d & \text{if } Y \geq f(X) \end{cases}$$

(a) Show that $P(W \le w) = \int_a^w f(t)\,dt \,/\, c(b-a)$ $\quad a \le w \le b$

$X \sim \text{uniform}(a,b)$ $\quad P_X(x) = \frac{1}{b-a}$

$Y \sim \text{uniform}(0,c)$

$P(Y < f(X)) = \frac{f(X)}{c} = \frac{1}{c} f(x) \cdot P(x) = \frac{1}{c} f(x) \cdot \frac{1}{b-a}$

when $Y < f(X)$ $\quad W = X$

means we only pick $w(X)$ when $Y < f(X)$ $\quad P(W) = P(Y < f(X)) = \frac{f(x)}{c(b-a)}$

otherwise $W = d$ $(d > b)$

if $a \le w \le b$

$$P(W \le w) = \int_a^w \frac{f(t)}{c(b-a)}\,dt \qquad \square$$

(b) Explain how a random variable with pdf $f(x)$ can be generated.

$P(Z \le w) = \int_a^w f(z)\,dz$

from (a) we know $P(W \le w) = P(Z \le w)$ $\quad Z \sim f(z)$

they are the same shadow area

$\therefore W \sim f(z)$ $\qquad \square$

5.61. (a) To generate $Y \sim \text{beta}(a,b)$, a, b are not int

$V \sim \text{beta}([a],[b])$, show that

$M = \sup_y f_Y(y)/f_V(y)$ is a finite value.

$f_Y(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot y^{a-1}(1-y)^{b-1}$ ①

Assume that $[a] > a$ $\quad [b] > b$ $\qquad 0 < \theta_a < 1$

$[a] - a = \theta_a$ $\quad [b] - b = \theta_b$ $\qquad 0 < \theta_b < 1$

$$f_V(y) = \frac{\Gamma([a]+[b])}{\Gamma([a])\Gamma([b])} \cdot y^{[a]-1}(1-y)^{[b]-1}$$

$$= \frac{\Gamma(a+b+\theta_a+\theta_b)}{\Gamma(a+\theta_a)\Gamma(b+\theta_b)} \cdot y^{a+\theta_a-1}(1-y)^{b+\theta_b-1}$$

$$= \frac{\Gamma(a+b+\theta_a+\theta_b)}{\Gamma(a+\theta_a)\Gamma(b+\theta_b)} \cdot y^{a-1}(1-y)^{b-1} \cdot y^{\theta_a}(1-y)^{\theta_b} \quad ②$$

let $f(y) = y^{\theta_a}(1-y)^{\theta_b}$

$f'(y) = y^{\theta_a}\theta_b(1-y)^{\theta_b-1}(-1) + \theta_a y^{\theta_a-1}(1-y)^{\theta_b}$

$f'(y) = 0$ $\quad y = \frac{\theta_a}{\theta_a+\theta_b}$

$\lim_{y\to 0} f(y) > 0$

$\lim_{y\to 1} f'(y) < 0$

$\therefore M = k \frac{1}{\left(\frac{\theta_a}{\theta_a+\theta_b}\right)^{\theta_a}\left(1-\frac{\theta_a}{\theta_a+\theta_b}\right)^{\theta_b}}$

$= k\left(\frac{\theta_a+\theta_b}{\theta_a}\right)^{\theta_a}\left(\frac{\theta_a+\theta_b}{\theta_b}\right)^{\theta_b}$

$= k\left(1+\frac{\theta_b}{\theta_a}\right)^{\theta_a} \cdot \left(1+\frac{\theta_a}{\theta_b}\right)^{\theta_b}$

Assume $\theta_a > \theta_b$

$M = k \frac{(\theta_a+\theta_b)^{\theta_b}(\theta_a+\theta_a)^{\theta_a}}{(\theta_a\theta_b)^{\theta_b}\,\theta_a^{\theta_a-\theta_b}}$

$= k \underbrace{\left(\frac{\theta_a+\theta_b}{\theta_a\theta_b}\right)^{\theta_b}}_{S} \left(\frac{\theta_a+\theta_b}{\theta_a}\right)^{\theta_a}$ $\quad \geq 2 \quad \geq 1$

$S = \left(\frac{1}{\theta_a} + \frac{1}{\theta_b}\right)^{\theta_b}$

$\le \left(\frac{2}{\theta_a}\right)^{\theta_b} = 2^{\theta_b}\boxed{\left(\frac{1}{\theta_a}\right)^{\theta_b}}$

let $r = \theta_b$

$f = \left(\frac{1}{r}\right)^r$ $\quad r \in (0,1)$

$\ln f = r \cdot \ln\frac{1}{r} = -r\ln r$

$\frac{f'}{f} = (-r)\frac{1}{r} + (-1)\ln r$

$f' = \left(\frac{1}{r}\right)^r(-1-\ln r) = 0$

$r = \frac{1}{e}$ $\quad r > 0$ $\quad f' > 0$

$r > 1$ $\quad f' < 0$

$\therefore f$ $\quad f_{max} = (e)^{\frac{1}{e}}$

$\therefore S \le 2^{\theta_b} \cdot e^{\frac{1}{e}}$

$\therefore M \le 2S$ is finite $\square$

$$\frac{f_Y(y)}{f_V(y)} = \frac{①}{②} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+\theta_a)\Gamma(b+\theta_b)}{\Gamma(a+b+\theta_a+\theta_b)} \frac{y^{a-1}(1-y)^{b-1}}{y^{a-1}(1-y)^{b-1}\,y^{\theta_a}(1-y)^{\theta_b}}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+\theta_a)\Gamma(b+\theta_b)}{\Gamma(a+b+\theta_a+\theta_b)} \cdot \frac{1}{y^{\theta_a}(1-y)^{\theta_b}} = k \cdot \frac{1}{y^{\theta_a}(1-y)^{\theta_b}}$$

(b) Suppose it is desired to generate $Y \sim \text{gamma}(a, b)$, a, b are not int. Show $U \sim \text{gamma}([a], b)$ $M = \sup_y f_Y(y) / f_U(y)$ is finite.

$\because Y \sim \text{gamma}(a, b)$

$\therefore f_Y(y) = \dfrac{y^{a-1} e^{-y/b}}{b^a \cdot \Gamma(a)}$

Assume that $[a] = a + \theta$

$1 > \theta > 0$

$f_U(y) = \dfrac{y^{[a]-1} \cdot e^{-y/b}}{b^{[a]} \, \Gamma([a])}$

$f_U(y) = \dfrac{y^{a-1} \cdot y^{\theta} \cdot e^{-y/b}}{b^{\theta} \, b^{a} \, \Gamma(a+\theta)}$

$\therefore \dfrac{f_Y(y)}{f_U(y)} = \dfrac{y^{a-1} e^{-y/b}}{b^a \cdot \Gamma(a)} \cdot \dfrac{b^{\theta} \, b^{a} \, \Gamma(a+\theta)}{y^{a-1} e^{-y/b} \, y^{\theta}}$

$= \dfrac{b^{\theta}}{y^{\theta}} \dfrac{\Gamma(a+\theta)}{\Gamma(a)}$

$= \dfrac{\Gamma(a+\theta) \, b^{\theta}}{\Gamma(a)} \cdot \dfrac{1}{y^{\theta}}$

$\because \theta \in (0, 1)$

1° $y > 1$ $\quad \frac{1}{y} < 1 \quad \left(\frac{1}{y}\right)^{\theta} < 1 < \infty$

2° $0 < y < 1$ $\quad \frac{1}{y} > 1$

$\lim_{y \to 0} \left(\frac{1}{y}\right) \to +\infty \quad \left(\frac{1}{y}\right)^{\theta} < \infty$

$\therefore M$ is finite

(c) Show that, in each of parts (a) & (b). if U had parameter $[a]+1$, then M would be infinite.

(b) If $U \sim \text{gamma}([a]+1, b)$

it means $[a]+1 = a + \theta + 1 = a + \theta'$ $\quad \theta' \in (1, 2)$

when $0 < y < 1$ $\left(\frac{1}{y}\right)^{\theta'} \to +\infty$ $\quad \therefore M$ can be infinite.

(a) Assume $\theta_a > \theta_b$ $\quad$ if $\theta_a \in (1, 2)$, M is still finite.

$M = k \dfrac{(\theta_a + \theta_b)^{\theta_a} \, (\theta_a + a)^{\theta_a}}{(\theta_a \theta_b)^{\theta_b} \, a^{\theta_a - \theta_b}}$

$= k \underbrace{\left(\dfrac{\theta_a + \theta_b}{\theta_a \theta_b}\right)^{\theta_a}}_{\leq 2^{\theta_a} \, 2^{\frac{1}{2}}} \left(\dfrac{\theta_a + \theta_b}{\theta_a}\right)^{\theta_a} \dfrac{a^{\theta_a}}{a^{\theta_a - \theta_b}}$

5.62. Find the values of M, so that an Accept/Reject Algo can generate $Y \sim n(0,1)$ using $U \sim \text{uniform}(0,1)$ and

(a) $V \sim$ Cauchy

$U < \frac{1}{M} f_Y(V)/f_V(V)$, set $Y = V$

$V \sim$ Cauchy $(0,1)$

$Y \sim n(0,1)$

$f_V(y) = \frac{1}{\pi(1+y^2)}$

$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$

$M = \sup_y \frac{f_Y(y)}{f_V(y)}$

$= \sup_y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot \pi(1+y^2)$

$= \sup_y \frac{\sqrt{\pi}}{\sqrt{2}} \underbrace{e^{-\frac{y^2}{2}} (1+y^2)}$

let $y^2 = x \in [0, +\infty)$

$g(x) = e^{-\frac{x}{2}} (1+x)$

$g'(x) = e^{-\frac{x}{2}} + e^{-\frac{x}{2}} (-\frac{1}{2})(1+x)$

$= e^{-\frac{x}{2}} - \frac{1}{2}(1+x) e^{-\frac{x}{2}}$

$= e^{-\frac{x}{2}} (\frac{1}{2} - \frac{x}{2})$

$Y = \tan\left[\pi\left(U - \frac{1}{2}\right)\right]$

let $g'(x) = 0$ $x = 1$

when $0 < x < 1$ $g'(x) > 0$

when $x > 1$ $g'(x) < 0$

$\therefore g(x)$ $x=1$

$\therefore g(x)_{max} = g(1) = e^{-\frac{1}{2}}(1+1) = 2 \cdot e^{-\frac{1}{2}}$

$x = 1$ $y = \pm 1$

$\therefore M = \frac{\sqrt{\pi}}{2} \cdot 2 \cdot e^{-\frac{1}{2}} = \sqrt{\pi} \cdot e^{-\frac{1}{2}}$ □ ≈ 1.075

(b) $V \sim$ double exponential

$V \sim$ double exponential $(0, 1)$ (μ, λ)

$Y = -\ln(1-U)$

$f_V(y) = \frac{1}{2} \cdot e^{-|y|}$

$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$

$M = \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{2} = \frac{1}{\sqrt{2\pi}} \approx 0.399$

$M = \sup_y \frac{f_Y(y)}{f_V(y)}$

$= \sup_y \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \cdot 2 \cdot e^{|y|}$

1° $y \geq 0$

$M = \sup_y \frac{2}{\sqrt{2\pi}} \underbrace{e^{-\frac{y^2}{2} + y}}$

let $g(y) = -\frac{y^2}{2} + y$

$g'(y) = -\frac{1}{2} \cdot (2y) + 1$

$= -y + 1$

let $g'(y) = 0$ $y = 1$

$\therefore 1 > y \geq 0$ $g'(y) > 0$

$y > 1$ $g'(y) < 0$

$\therefore g(y)_{max} = g(1) = -\frac{1}{2} + 1 = \frac{1}{2}$

2° $y < 0$

$M = \sup_y \frac{2}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2} - y}$

let $g(y) = -\frac{y^2}{2} - y$

$g'(y) = -\frac{1}{2} \cdot (2y) - 1$

$= -y - 1$

let $g'(y) = 0$ $y = -1$

$\therefore -1 < y < 0$ $g'(y) < 0$

$y < -1$ $g'(y) > 0$

-1

$\therefore g(y)_{max} = g(-1) = -\frac{1}{2} - (-1) = \frac{1}{2}$

(c) Compare the algorithms, which one do you recommend?

$\because M_{exponential} \approx 0.399 < M_{cauchy} \approx 1.075$

$\therefore$ Uv exponential is faster.

5.63. $Y \sim n(0,1)$, $U \sim$ uniform $\pm$ $V \sim$ exponential (λ)

$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$

$U \sim$ uniform

$f(u) = f(-u) = \frac{1}{2} \cdot \lambda \cdot e^{-\lambda u}$

$= \frac{1}{2} \cdot \lambda \cdot e^{-\lambda |u|}$

$M = \sup_y \frac{f_Y(y)}{f_U(y)}$ $\quad M \downarrow \ \frac{1}{M} \uparrow \ P(\text{Accept}) \uparrow$

$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \cdot 2 \frac{1}{\lambda} \cdot e^{\lambda |y|}$

1° $y \geq 0$

$M = \sup_y \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\lambda} \underbrace{e^{-\frac{y^2}{2} + \lambda y}}$

$g(y) = -\frac{y^2}{2} + \lambda y$

$g'(y) = -\frac{1}{2} \cdot 2y + \lambda$

$g'(y) = 0 \quad y = \lambda$

$g'(y) \begin{cases} >0 & \lambda > y \geq 0 \\ <0 & y > \lambda \end{cases}$

$\nearrow y=\lambda \searrow \quad \therefore g(y)_{max} = -\frac{\lambda^2}{2} + \lambda^2 = \frac{\lambda^2}{2}$

2° $y < 0$

$M = \sup_y \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\lambda} \cdot e^{-\frac{y^2}{2} + \lambda y}$

Let $g(y) = -\frac{y^2}{2} - \lambda y$

$g'(y) = -\frac{1}{2} \cdot 2y - \lambda$

$= -y - \lambda$

$g'(y) = 0 \quad y = -\lambda$

$g'(y) \begin{cases} <0 & -\lambda < y < 0 \\ >0 & y < -\lambda \end{cases}$

$\nearrow -\lambda \searrow \quad \therefore g(y)_{max} = -\frac{\lambda^2}{2} + \lambda^2 = \frac{\lambda^2}{2}$

$\therefore M = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\lambda} \cdot e^{\frac{\lambda^2}{2}}$

$\min_\lambda M$? $\qquad \frac{\lambda^2 - 1}{\lambda^2}$

Let $f(\lambda) = \frac{1}{\lambda} \cdot e^{\frac{\lambda^2}{2}} \qquad \lambda > 0$

$f'(\lambda) = \underbrace{\frac{1}{\lambda}} \cdot e^{\frac{\lambda^2}{2}} \cdot \underbrace{\frac{1}{2}(2\lambda)} + (-1)(\lambda)^{-2} \cdot e^{\frac{\lambda^2}{2}}$

$= e^{\frac{\lambda^2}{2}} - \frac{1}{\lambda^2} \cdot e^{\frac{\lambda^2}{2}}$

$= \frac{\lambda^2 - 1}{\lambda^2} \cdot e^{\frac{\lambda^2}{2}}$

Let $f'(\lambda) = 0 \quad \lambda = 1$

$f'(\lambda) \begin{cases} <0 & 1 > \lambda > 0 \\ >0 & \lambda > 1 \end{cases}$

$\searrow \lambda = 1 \nearrow \quad \therefore \min f(\lambda) = 1 \cdot e^{\frac{1}{2}} = e^{\frac{1}{2}}$

$\therefore \min_\lambda M = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot e^{-\frac{1}{2}}$ when $\lambda = 1$ o

5.64. Importance Sampling.

$Z \sim f$, $Y_1, Y_2, \cdots, Y_m$ iid from g, for any function h,

$$\frac{1}{m} \cdot \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)$$

(a) Show that $E\left(\frac{1}{m} \cdot \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)\right) = Eh(Z)$

$$E\left(\frac{1}{m} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)\right)$$

$$E\left[\sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)\right] = \sum \frac{f(y)}{g(y)} h(y)\, g(y) = \sum \underbrace{f(y)}_{P} \underbrace{h(y)}_{Var} = Eh(Z) \quad \square$$

(b) Show that $\frac{1}{m} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i) \to Eh(Z)$ in probability.

$\Leftarrow \underbrace{\frac{1}{m} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)}_{A} \to Eh(Z)$ in probability.

$\Leftarrow \lim_{m \to \infty} P(|A_m - Eh(Z)| < \varepsilon) = 1$

$$Eh(Z) = \frac{h(Z_1) + h(Z_2) + \cdots h(Z_m)}{n}$$

$\because \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)$

$= \frac{1}{m} \lim_{m \to \infty} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i) = Eh(Z)$

$\therefore \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)$ converge to $Eh(Z)$ in probability

(c). $\sum_{i=1}^{m} \frac{f(Y_i)/g(Y_i)}{\sum_{j=1}^{m} f(Y_j)/g(Y_j)} h(Y_i) \to Eh(Z)$ in probability.

$\lim_{m \to \infty} \sum_{j=1}^{m} f(Y_j)/g(Y_j) = 1$ ②

$\lim_{m \to \infty} \sum_{i=1}^{m} f(Y_i)/g(Y_i) \cdot h(Y_i) \to Eh(Z)$ ① ... from (b)

$\therefore$ ①/② $\to Eh(Z)$ in probability $\square$

Show that when h is continuous, (c) is superior than (a)

$\because$ ② ≤ 1 $\therefore \frac{①}{②} \geq 1$ that is (c) $\geq$ (a) $\square$

5.65. Let $X \sim f$, $Y_1, Y_2, \cdots, Y_m$ iid from g.

$q_i = \dfrac{f(Y_i)/g(Y_i)}{\sum_{j=1}^{m} f(Y_j)/g(Y_j)}$, generate X^* $P(X^* = Y_k) = q_k$

Show that $X_1^*, X_2^* \cdots X_r^*$ is approximately a random sample from f.

($P(X^* \le x) = \sum_{i=1}^{m} q_i \cdot I(Y_i \le x)$ $\qquad I(Y_i \le x) = \begin{cases} 1 & Y_i \le x \\ 0 & Y_i > x \end{cases}$

$P(X^* = Y_k) = q_k$ $\qquad X^* = Y_k \quad k = 1, 2, \cdots m$

$$P(X^* \le x) = \sum_{i=1}^{m} P(X^* \le x \mid q_i) \cdot q_i = \sum_{i=1}^{m} \underbrace{I(Y_i \le x)} \cdot q_i = \frac{\sum_{i=1}^{m} \frac{1}{m} \cdot \frac{f(Y_i)}{g(Y_i)} I(Y_i \le x)}{\frac{1}{m} \sum_{j=1}^{m} \frac{f(Y_j)}{g(Y_j)}}$$

$P(AB) = P(A \mid B) \cdot P(B)$

$$\lim_{m \to \infty} = \frac{E\left[\frac{f(Y)}{g(Y)} \cdot I(Y \le x)\right]}{E \underbrace{\frac{f(Y)}{g(Y)}}} = \frac{\int_{-\infty}^{x} \frac{f(y)}{g(y)} \cdot g(y)\, dy}{\underbrace{\int_{-\infty}^{\infty} \frac{f(y)}{g(y)} \cdot g(y)\, dy}_{1}} = \int_{-\infty}^{x} f(y)\, dy$$

Variable value $\frac{f(Y)}{g(Y)}$

Frequency $g(Y)$

□

5.66. (a) Show that M is the median of X_is $\sim n(\mu, \sigma^2)$ if and only if $(M-\mu)/\sigma$ is the median of $(X_i - \mu)/\sigma$. Thus, we only need consider the distribution of the median from a $n(0,1)$ sample.

Assume that $X_1 \le X_2 \le \cdots \le X_{2n+1}$ in order (odds)

1° Proof if

Since $\frac{(M-\mu)}{\sigma}$ is the median of $\frac{X_i - \mu}{\sigma}$

that is $\frac{X_1 - \mu}{\sigma}, \frac{X_2 - \mu}{\sigma}, \cdots \frac{X_{2n+1} - \mu}{\sigma}$ in order (odd)

$\frac{(M-\mu)}{\sigma} = \frac{X_{2n} - \mu}{\sigma}$ $\therefore M = X_{2n}$

$\because X_{2n}$ is the median of X_i $\therefore M$ is the median

2° Proof only if

Assume that $\frac{M-\mu}{\sigma}$ is not the median of $\frac{X_i - \mu}{\sigma}$

that is $\frac{(M-\mu)}{\sigma} \ne \frac{X_{2n} - \mu}{\sigma}$, $M \ne X_{2n}$.

$\therefore$ if $\frac{M-\mu}{\sigma}$ is not the median of $\frac{X_i - \mu}{\sigma}$, then

M is not the median of X_i □

(b) $X_1, X_2, \cdots X_n$ are iid $n(\mu, \sigma^2)$ $n=15$, $\mu=0$, $\sigma=1$, simulate the distribution of the median M.

$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$

```
public static double[] estimateMofNorm () {
    Double u1=0;   // θ
    Double u2=0;   // R
    Double[] M = new Double[5000];
    Double[] X = new Double[15];
    Int n=0
    while (n<5000) {
        for (i=0; i<15 ; i++) {
            u1 = math.random ();
            u2 = math.random();
            X[i] = ((-2) * log(u2))^(1/2) *
                   cos (2*Pi*u1) ;
        }
        X = X.order ();          Xi ~ N(0,1)
        M[n] = X[8] ;
    }
  Return M ;
}
```

Box-Muller algorithm

$R = \sqrt{-2\log U_2}$ $\theta = 2\pi U_1$

$X = R \cdot \cos\theta$ $Y = R \cdot \sin\theta$

5.67. Show how to implement the Metropolis Algorithm to generate a sample of size 100 from the specified distri...

(a) $X \sim \frac{1}{\sigma} f[(x-\mu)/\sigma]$, f = Student's t with v degrees of freedom. v, μ, σ known.

$f_T(t) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\,\Gamma(\frac{v}{2})} (1+\frac{t^2}{v})^{-\frac{v+1}{2}}$

0. Generate $t \sim n(\mu, \sigma)$. $z_0 = t$
1. Generate $t_i \sim n(\mu, \sigma)$ $u_i \sim U(0,1)$ $i = 1, 2, \cdots 100$

$$P_i = \min\left\{ \frac{f_T(\frac{t_i-\mu}{\sigma})}{f_n(t_i)} \cdot \frac{f_n(z_{i-1})}{f_T(\frac{z_{i-1}-\mu}{\sigma})}, 1 \right\}$$

2. Set $z_i = \begin{cases} t_i & \text{if } u_i \le P_i \\ z_{i-1} & \text{if } u_i > P_i \end{cases}$

$Y \sim f_Y(y)$ $V \sim f_V(v)$

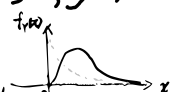
0. Generate $V \sim f_V$, $z_0 = V$
1. Generate $V_i \sim$ uniform $(0,1)$ $U_i \sim f_V$

$$P_i = \min\left\{ \frac{f_Y(V_i)}{f_V(V_i)} \cdot \frac{f_V(z_{i-1})}{f_Y(z_{i-1})}, 1 \right\}$$

2. Set $z_i = \begin{cases} V_i & \text{if } U_i \le P_i \\ z_{i-1} & \text{if } U_i > i \end{cases}$

(b) $X \sim$ lognormal (μ, σ^2), μ, σ^2 known

$f_T(t) = \frac{1}{t\sigma\sqrt{2\pi}} e\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right)$



0. Generate $f_n(x) \sim$ half normal dis $\Rightarrow hn(\mu, \sigma^2)$

$f_n(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$

$t \sim f_n(t)$ $z_0 = t$

1. Generate $t_i \sim hn(\mu, \sigma^2)$, $u_i \sim U(0,1)$ $i = 1, 2, \cdots, 100$

$$P_i = \min\left\{ \frac{f_T(t_i)}{f_n(t_i)} \cdot \frac{f_n(z_{i-1})}{f_T(z_{i-1})}, 1 \right\}$$

2. Set $z_i = \begin{cases} t_i & \text{if } u_i \le P_i \\ z_{i-1} & \text{if } u_i > i \end{cases}$

(c) $X \sim \text{Weibull}(\alpha, \beta)$, α, β known

$$f_Y(t) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \cdot e^{-(x/\alpha)^\beta} & x<0 \\ 0 & \end{cases}$$

$f_V(t) = \text{exponential}[(\alpha^\beta)^{-1}]$ $\quad$ exponential $((\alpha^\beta)^{-1})$

0. Generate $t \sim f_V(t)$ $\quad z_0 = t$

1. Generate $t_i \sim f_V(t)$ $\quad u_i \sim U(0,1)$ $\quad i=1,2,\cdots 100$ $\quad -\frac{x^\beta}{\alpha^\beta}$

$$p_i = \min\left\{\frac{f_Y(t_i^{-\beta})}{f_V(t_i)} \cdot \frac{f_V(z_{i-1})}{f_Y(z_{i-1}^{-\beta})}, 1\right\} \qquad (\alpha^\beta)^{-1}$$

2. Set $z_i = \begin{cases} t_i & \text{if } u_i \le p_i \\ z_{i-1} & \text{if } u_i > i \end{cases}$ $\quad \square$

5.68. (a). Show how to generate student's t distribution with v degree freedom, starting from $n(0,1)$

$f_V(t)$ be $n(0, \frac{v}{v-2})$

$f_Y(t)$ be student's t with v degree of freedom

0. Generate $t \sim n(0, \frac{v}{v-2})$ set $z_0 = t$

1. Generate $t_i \sim f_V(t)$ $\quad u_i \sim U(0,1)$ $\quad i=1,2,\cdots 100$

$$p_i = \min\left\{\frac{f_Y(t_i)}{f_V(t_i)} \cdot \frac{f_V(z_{i-1})}{f_Y(z_{i-1})}, 1\right\}$$

2. Set $z_i = \begin{cases} t_i & \text{if } u_i \le p_i \\ z_{i-1} & \text{if } u_i > p_i \end{cases}$

(b) Show how to use Accept/Reject Algorithm to generate Student's t distribution random var from Cauchy random var. $\quad$ Cauchy dis. $\frac{x_1}{x_2}$ $\quad x_1, x_2$ iid $\sim n(0, \sigma^2)$

$x \in (0, \infty)$

$f_Y(t)$ be student's t with v degree freedom.

$f_V(t)$ be Cauchy distribution (x_0, γ) $\quad x \in (0, \infty)$

$Y \sim f_Y(y)$ $\quad U \sim f_V(U)$ have common support $M = \sup_y f_Y(y)/f_V(y) < \infty$

a. Generate $U \sim \text{uniform}(0,1)$ $\quad U \sim f_V$

b. If $U < \frac{1}{M} f_Y(U)/f_V(U)$ set $Y = U$

(c) Show how to use transformations to generate random var $\sim$ student's t distribution with v degree freedom.

Let U_1, U_2 iid $\sim n(0,1)$

$R = \sqrt{-2\log U_2}$ $\quad \theta = 2\pi U_1$

Generate $v+1$ samples of U_1 and U_2.

we have $R_1, R_2, \cdots R_{v+1}$

$\theta_1, \theta_2, \cdots \theta_{v+1}$

$Z = R \cdot \cos\theta$

$Z_1 = R_1 \cdot \cos\theta_1, Z_2, \cdots Z_{v+1}$ $\qquad Y = \sqrt{v+1}\,\frac{\bar{Z}}{SD}$

$\bar{Z} = \frac{Z_1 + Z_2 + \cdots + Z_{v+1}}{v+1}$

$SD = \sqrt{\frac{(Z_1 - \bar{Z})^2 + (Z_2 - \bar{Z})^2 + \cdots + (Z_{v+1} - \bar{Z})^2}{v}}$

Repeat this process

we can generate

$y_1, y_2, \cdots, y_n$

these samples are of student's t distribution. $\square$

5.69. Show that the pdf $f_Y(y)$ is a stable point of Metropolis Algorithm $\Leftrightarrow$ If $z_i \sim f_Y(y)$, then $z_{i+1} \sim f_Y(y)$

let $w(v, z) = \dfrac{f_Y(v)\, f_V(z)}{f_V(v)\, f_Y(z)}$ $\quad \rho(v, z) = \min\{w(v, z), 1\}$ $\quad z_{i+1}$

Aim: if $z_i \sim f_Y$, the $z_{i+1} \sim f_Y$, that is $P(z_{i+1} \le a) = P(Y \le a)$

$$P(z_{i+1} \le a) = \underset{\substack{z_{i+1} = V_{i+1} \\ U_{i+1} \le \rho_{i+1}}}{P(V_{i+1} \le a \text{ and } U_{i+1} \le \rho_{i+1})} + \underset{\substack{z_{i+1} = z_i \\ U_{i+1} > \rho_{i+1}}}{P(z_i \le a \text{ and } U_{i+1} > \rho_{i+1})}$$

$\because$ $z_i \sim f_Y$ (known) and $V \sim f_V$, $U \sim n(0,1)$

$\therefore$

$$P(z_{i+1} \le a) = P(V \le a \text{ and } U \le \rho(V, z_i)) + P(Y \le a \text{ and } U > \rho(V, z_i))$$

$$= P(V \le a \text{ and } U \le \rho(V, Y)) + P(Y \le a \text{ and } U > \rho(V, Y))$$

$$= P(V \le a \text{ and } U \le \rho(V, Y)) + P(Y \le a)$$

$$- P(Y \le a \text{ and } U > \rho(V, Y))$$

$\Rightarrow$ Aim: $\underbrace{P(V \le a \text{ and } U \le \rho(V, Y))}_{①} = P(Y \le a \text{ and } U > \rho(V, Y))$

$$① = \int_{-\infty}^{a} \int_{-\infty}^{\infty} \rho(v, y) \cdot f_V(v)\, f_Y(y)\, dy\, dv$$

$$\rho(v, y) = \min\{w(v, z), 1\}$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} I(w(v, y) \le 1) \cdot \frac{f_Y(v) \cdot f_V(y)}{f_V(v) \cdot f_Y(y)}\, f_V(v) \cdot f_Y(y)\, dy\, dv$$

$$+ \int_{-\infty}^{a} \int_{-\infty}^{\infty} I(w(v, y) > 1) \cdot 1 \cdot f_V(v) + f_Y(y)\, dy\, dv$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} I(w(v, y) \le 1) \cdot f_Y(v) \cdot f_V(y)\, dy\, dv \quad ②$$

$$+ \int_{-\infty}^{a} \int_{-\infty}^{\infty} I(w(v, y) > 1) \cdot f_V(v) \cdot f_Y(y)\, dy\, dv \quad ③$$

$w(v, y) = \dfrac{1}{w(y, v)}$ $\quad \therefore$ $② = \displaystyle\int_{-\infty}^{a} \int_{-\infty}^{\infty} I(w(y, v) > 1) \cdot f_Y(v) \cdot f_V(y)\, dy\, dv$

$$= P(Y \le a \text{ and } \rho(Y, V) = 1) \quad ④$$

$$③ = \int_{-\infty}^{a} \int_{-\infty}^{\infty} I(w(y, v) < 1) \frac{f_Y(y)\, f_V(v)}{f_V(y)\, f_Y(v)}\, f_Y(y)\, f_V(v)\, dy\, dv$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} I(w(y, v) < 1) \cdot w(y, v) \cdot f_V(y)\, f_Y(v)\, dy\, dv.$$

$$= P(Y \le a,\ \rho(y, v) < 1 \text{ and } U < \rho(y, v)) \quad ⑤$$

$\therefore$ ④ + ⑤ $= P(Y \le a)$ $\quad$ □