# Functia zeta a lui Riemann

$\zeta$ = "zeta"

$\zeta : (1, \infty) \longrightarrow \mathbb{R},$

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1$$

$\zeta(1) = +\infty$

$\zeta(p) = ?, \quad p \in \mathbb{N}, \ p \geqslant 2$

"Demonstratia lui Euler" pentru valoarea $\zeta(2)$

OBS : Daca se cunosc cele n radacini reale $r_1, r_2, \ldots, r_n$ ale unui polinom $P(x)$ de gradul n, atunci

$$P(x) = A\,(x - r_1)\,(x - r_2) \cdot \ldots \cdot (x - r_n), \quad A \in \mathbb{R} \text{ const.}$$

sau

$$P(x) = B\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right) \cdot \ldots \cdot \left(1 - \frac{x}{r_n}\right), \quad B \in \mathbb{R} \text{ const.}$$

**Consideram polinomul de grad "infinit"**

$$P(x) = \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

**avand toate radacinile de forma** $\pm k\pi$, $k \in \mathbb{N}^*$,

**deoarece** $P(\pm k\pi) = \dfrac{\sin(\pm k\pi)}{\pm k\pi} = 0$, $k \in \mathbb{N}^*$

$$\begin{aligned} P(x) &= B\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdot\ldots \\ &= B\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdot\ldots \end{aligned}$$

**Egalam cele doua polinoame**

$$\begin{aligned} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots &= B\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdot\ldots \\ &= B\left(1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots\right)x^2 + \ldots\right) \end{aligned}$$

**si identificam termenul liber si coeficientul lui** $x^2$

$$B = 1$$

$$-\frac{1}{3!} = -\frac{1}{\pi^2}\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots\right)$$

$$\Longrightarrow$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

**Alte valori remarcabile ale functiei $\zeta$**

$\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, …

**Exista formula pentru $\zeta(2n)$, $n \in \mathbb{N}^*$**

**Nu exista formula pentru $\zeta(2n+1)$, $n \in \mathbb{N}^*$**

$\zeta(3) = ?$, $\zeta(5) = ?$, …

$\zeta(3) \approx 1.202\ldots \notin \mathbb{Q}$ (**constanta lui Apery**)

**Legatura cu functia eta a lui Dirichlet**

$$\eta(p) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}, \quad p \geqslant 1$$

$$\eta(1) = \ln(2)$$

$$\eta(p) = \left(1 - 2^{1-p}\right) \zeta(p), \quad p > 1$$

$$\eta(2) = 1 - \frac{1}{n^2} + \frac{1}{n^4} - \frac{1}{n^6} + \ldots = \left(1 - 2^{-1}\right) \zeta(2) = \frac{\pi^2}{12}$$

**Legatura functiei zeta cu numerele prime**

$$\prod_{n \in \text{prim}} \left(1 - \frac{1}{n^p}\right) = \frac{1}{\zeta\,(p)}, \quad p > 1$$

**OBS : Probabilitatea ca doua numere naturale alese la intamplare sa fie prime intre ele este** $\frac{6}{\pi^2}$

**ALGORITM**

**p = 0**

**Pentru n = 1 pana la $10^7$ executa**
**a = AlegeAleator[1, $10^{10}$]**
**b = AlegeAleator[1, $10^{10}$]**
**Daca CoPrime[a, b] atunci p = p + 1**

**Afiseaza (p / $10^7$)**

```
t = 10.^7;
For[n = 1; p = 0, n ≤ t, n++,
 p = If[ CoprimeQ[RandomInteger[{1, 10^10}], RandomInteger[{1, 10^10}]], p + 1, p]]
p / t
```

```
0.607992
```

```
6. / Pi^2
```

```
0.607927
```

**TEMA**

**Realizati un algoritm pentru a calcula probabilitatea ca alegand la intamplare o pereche de numere reale $(x, y)$ cu $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, acestea sa verifice inegalitatea $x^2 + y^2 \leq 1$**

**Raspuns :** $\approx \frac{\pi}{4}$