# Aproximarea numerica a radacinilor

**DEF :** Un numar $r \in \mathbb{R}$ se numeste <u>radacina</u> a functiei $f : \mathbb{R} \longrightarrow \mathbb{R}$ daca $f(r) = 0$

**Radacinile polinomului de gradul 2**

$f(x) = ax^2 + bx + c, \quad a \neq 0$

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

unde $\Delta = b^2 - 4ac > 0$

**Radacinile polinomului de gradul 3**

$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$

$$x_1 = S + T - \frac{b}{3a}$$

$$x_2 = -\frac{S+T}{2} - \frac{b}{3a} + i\frac{\sqrt{3}}{2}(S - T) \qquad \text{(formulele lui Cardano)}$$

$$x_3 = -\frac{S+T}{2} - \frac{b}{3a} - i\frac{\sqrt{3}}{2}(S - T)$$

unde $S = \sqrt[3]{R + \sqrt{D}}$, $T = \sqrt[3]{R - \sqrt{D}}$, $D = Q^3 + R^2 < 0$

si $R = \dfrac{9abc - 27a^2 d - 2b^3}{54a^3}$, $Q = \dfrac{3ac - b^2}{9a^2}$

**Cum putem calcula radacinile polinoamelor de grad superior ?**

**OBS :**

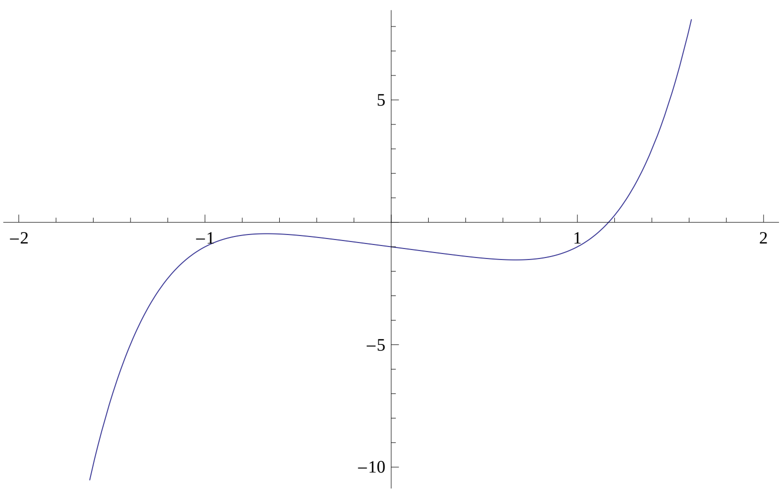
**Daca $f : [a, b] \longrightarrow \mathbb{R}$ este o functie continua si $f(a) \cdot f(b) < 0$ atunci exista cel putin o radacina a lui f in intervalul $(a, b)$**

$$f(x) = x^5 - x - 1$$

$$f(1) = -1 < 0, \quad f(2) = 2^5 - 3 > 0$$

$\Longrightarrow$ **in intervalul $(1, 2)$ avem radacina**

```
Plot[x^5 - x - 1, {x, -2, 2}]
```




**Cum putem calcula radacinile unei functii oarecare?**

$$x^x = 2, \quad x > 0$$

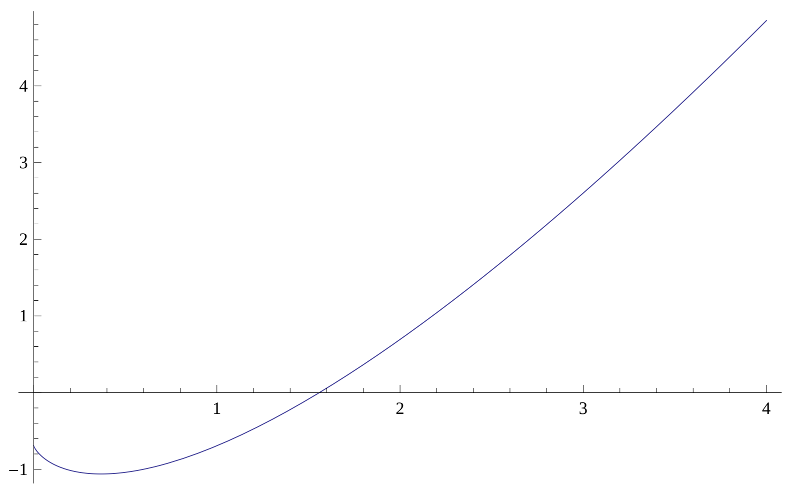
$$x \ln(x) = \ln(2)$$

$$f(x) = x \ln(x) - \ln(2), \quad x > 0$$

$$f(1) = -\ln(2) < 0, \; f(2) = \ln(2) > 0$$

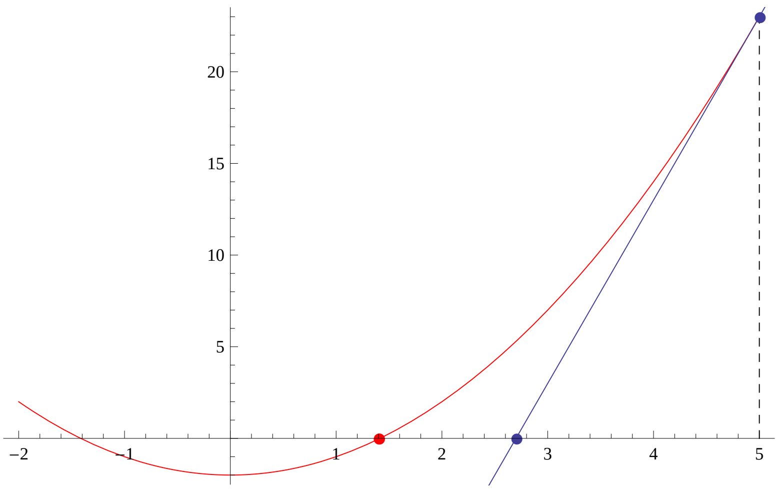
$\Longrightarrow$ **in intervalul (1, 2) avem radacina**

```
Plot[x * Log[x] - Log[2], {x, 0, 4}]
```



**Consideram graficul functiei $f(x) = x^2 - 2$ si dorim sa obtinem aproximatii succesive pentru radacina situata in intervalul (1, 2)**

```
g = Plot[x^2 - 2, {x, -2, 5}, PlotStyle → {Red}];
t1 = Plot[10 * x - 27, {x, -2, 6}];
l1 = Graphics[{Dashed, Line[{{5, 0}, {5, 23}}]}];
p1 = ListPlot[{{5.01, 23}, {2.71, 0}}, PlotMarkers → Automatic];
p2 = ListPlot[{{1.41, 0}}, PlotMarkers → Automatic, PlotStyle → {Red}];
Show[g, t1, l1, p1, p2]
```




**Fie $x_0 = 5$ o valoare de start aleasa arbitrar**

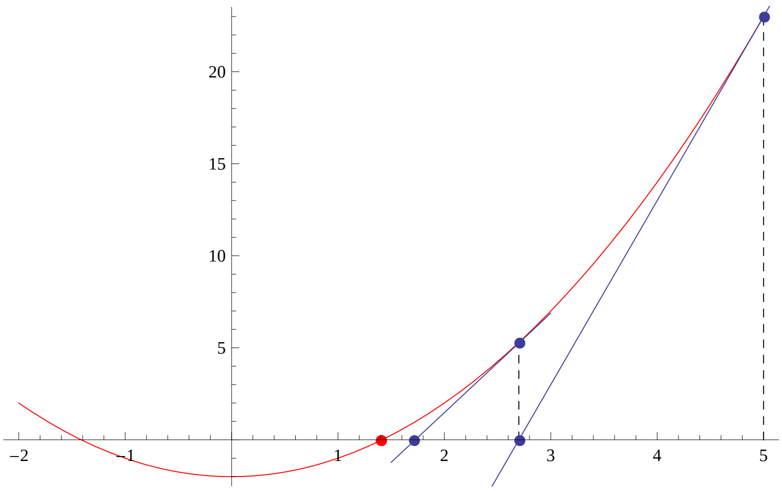
**Construim tangenta la grafic in punctul $(x_0, f(x_0))$ :**

$$y - f(x_0) = f'(x_0) \cdot (x - x_0)$$

**Aceasta tangenta intersecteaza axa Ox in punctul $x_1$ dat de ecuatia**

$$0 - f(x_0) = f'(x_0) \cdot (x_1 - x_0) \implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

```
g = Plot[x^2 - 2, {x, -2, 5}, PlotStyle → {Red}];
t1 = Plot[10 * x - 27, {x, -2, 6}];
t2 = Plot[5.4 * x - 9.32, {x, 1.5, 3}];
l1 = Graphics[{Dashed, Line[{{5, 0}, {5, 23}}]}];
l2 = Graphics[{Dashed, Line[{{2.70, 0}, {2.70, 5.3}}]}];
p1 =
  ListPlot[{{5.01, 23}, {2.71, 0}, {2.71, 5.3}, {1.72, 0}}, PlotMarkers → Automatic];
p2 = ListPlot[{{1.41, 0}}, PlotMarkers → Automatic, PlotStyle → {Red}];
Show[g, t1, t2, l1, l2, p1, p2]
```




**Fie $x_1$ valoarea obtinuta la pasul anterior**

**Construim tangenta la grafic in punctul $(x_1, f(x_1))$ :**

$$y - f(x_1) = f'(x_1) \cdot (x - x_1)$$

**Aceasta tangenta intersecteaza axa Ox in punctul $x_2$ dat de ecuatia**

$$0 - f(x_1) = f'(x_1) \cdot (x_2 - x_1) \implies x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

**si asa mai departe ...**

**ALGORITM (metoda lui Newton)**

**Fie $f : (a, b) \longrightarrow \mathbb{R}$ o functie derivabila ce are radacina $r \in (a, b)$**
**Urmatorul sir recurent converge (in anumite conditii) la r**

$$x_0 \in (a, b), \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \ n \geq 0 \qquad (1)$$

**OBS : Daca $f'(x) \neq 0$, $\forall x \in (a, b)$ si $f' : (a, b) \longrightarrow \mathbb{R}$ este monotona atunci f are radacina unica in intervalul (a, b),**
**iar sirul (1) este convergent la r.**

**Radacina polinomului $f(x) = x^5 - x - 1$**

$$x_0 = 1, \quad x_{n+1} = x_n - \frac{x_n^5 - x_n - 1}{5 x_n^4 - 1}, \ n \geq 0$$

```
f[x_] := x^5 - x - 1
x0 = 1.0;
For[n = 1; x = x0, n ≤ 5, n++, x = x - f[x] / f'[x];
 Print["n=", n, "    x=", SetAccuracy[x, 10]]]
```

```
n=1   x=1.2500000000

n=2   x=1.178459394

n=3   x=1.1675373894

n=4   x=1.1673040828

n=5   x=1.167303978
```

```
N[Solve[u^5 - u - 1 == 0, u, Reals], 15]
```

```
{{u → 1.167303978261418}}
```

```
f[x_] := x^5 - x - 1
x0 = 5.0;
For[n = 1; x = x0, n ≤ 11, n++, x = x - f[x] / f'[x];
 Print["n=", n, "    x=", SetAccuracy[x, 10]]]
```

```
n=1   x=4.001600512
n=2   x=3.204559970
n=3   x=2.5704193314
n=4   x=2.0704027222
n=5   x=1.6855532294
n=6   x=1.408109887
n=7   x=1.2404662170
n=8   x=1.1761906034
n=9   x=1.167452804
n=10   x=1.167304021
n=11   x=1.1673039783
```

```
f[x_] := x^5 - x - 1
x0 = 0.0;
For[n = 1; x = x0, n ≤ 10, n++, x = x - f[x] / f'[x];
 Print["n=", n, "    x=", SetAccuracy[x, 10]]]
```

```
n=1   x=-1.000000000
n=2   x=-0.7500000000
n=3   x=0.0872483221
n=4   x=-1.0003100461
n=5   x=-0.7503873775
n=6   x=0.082568178
n=7   x=-1.0002477999
n=8   x=-0.750309635
n=9   x=0.0835039877
n=10   x=-1.0002594112
```

**Sirul obtinut nu este convergent !**

**OBS : Viteza de convergenta a sirului $(x_n)$ din metoda lui Newton depinde de valoarea de start $x_0$**

**Sa aplicam metoda de mai sus functiei $f(x) = x^2 - 2, \quad x > 0$**

$$x_0 = 1, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2 x_n}, \; n \geq 0$$

$$x_0 = 1, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \; n \geq 0$$

**Acest sir converge la $\sqrt{2}$ (vezi seminar)**

# O metoda de aproximare a radicalilor de ordin superior

**Consideram functia** $f(x) = x^p - q, \ x > 0, \ q > 0$ **si** $p \in \mathbb{N}, \ p \geq 2$

**Aceasta functie are radacina unica in intervalul** $(0, q)$
**si anume** $\sqrt[p]{q}$

**Urmatorul sir recurent din metoda lui Newton converge la aceasta radacina**

$$x_0 = 1, \quad x_{n+1} = x_n - \frac{x_n{}^p - q}{p x_n{}^{p-1}}, \quad n \geq 0$$

$$x_0 = 1, \quad x_{n+1} = \frac{p-1}{p} x_n + \frac{q}{p x_n{}^{p-1}}, \quad n \geq 0$$

**Sa calculam o valoare aproximativa pentru** $\sqrt[3]{3}$

$p = q = 3$

$$x_0 = 1, \quad x_{n+1} = \frac{2}{3} x_n + \frac{1}{x_n{}^2}, \quad n \geq 0$$

```
x0 = 1.0;
For[n = 1; x = x0, n ≤ 5, n++, x = 2 * x / 3 + 1 / x^2;
 Print["n=", n, "    x=", SetAccuracy[x, 10]]]
```

```
n=1    x=1.666666667

n=2    x=1.4711111111

n=3    x=1.4428120982

n=4    x=1.442249790

n=5    x=1.4422495703
```

```
N[3^(1 / 3), 15]
```

```
1.442249570307408
```