Serii Taylor remarcabile si aplicatii in calculul de limite

Serii Taylor centrate in origine ($x_0 = 0$)

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \quad x \in (-1, 1)$$

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad x \in (-1, 1)$$

$$\sqrt{1 + x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n - 3)!!}{(2n)!!} x^n, \quad x \in (-1, 1]$$

$$(1 + x)^{\alpha} = \sum_{n=0}^{\infty} C_{\alpha}^{n} x^n, \quad x \in (-1, 1) \qquad \text{(\underline{seria} \underline{binomiala})}$$

unde

$$C_{\alpha}^{0} = 1, \quad C_{\alpha}^{n} = \frac{\alpha (\alpha - 1) \cdot \ldots \cdot (\alpha - n + 1)}{n!}, \quad \alpha \in \mathbb{R}, \quad n \in \mathbb{N}^*$$

(coeficient binomial generalizat)

$$\alpha \in \mathbb{N}, \ \alpha \ge n \implies C_{\alpha}^{n} = \frac{\alpha!}{n! \, (\alpha - n)!}, \quad n \ge 0$$

$$\alpha = -1 \implies C_{-1}^{n} = \frac{-1 \, (-2) \, (-3) \cdot \ldots \cdot (-n)}{n!} = (-1)^n \frac{n!}{n!} = (-1)^n, \quad n \ge 0$$

$$\alpha = \frac{1}{2} \implies C_{\frac{1}{2}}^{0} = 1, \ C_{\frac{1}{2}}^{1} = \frac{1}{2},$$

$$C_{\frac{1}{2}}^{n} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\ldots\cdot\left(-\frac{2n-3}{2}\right)}{n!} = (-1)^{n-1}\frac{(2n-3)!!}{2^n n!} =$$

$$(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!},\quad n \geq 2$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!},\qquad x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n},\qquad x \in (-1,1]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},\qquad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},\qquad x \in \mathbb{R}$$

$$\arcsin x = x + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!\,(2n+1)}x^{2n+1},\qquad x \in [-1,1]$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1},\qquad x \in [-1,1]$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1},\qquad x \in \mathbb{R}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = x + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n},\qquad x \in \mathbb{R}$$

Calcul de limite

OBS : Daca $f : (a, b) \longrightarrow \mathbb{R}$ este functie indefinit derivabila si dezvoltabila in serie Taylor intr - un punct $x_0 \in (a, b)$ pe multimea $I \subseteq (a, b)$, atunci

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \quad \forall x \in I$$

Fie $n \in \mathbb{N}$ fixat si $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$ polinomul Taylor

$$\Longrightarrow f(x) = T_n(x) + \sum_{k=n+1}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

$$\Longrightarrow \frac{f(x) - T_n(x)}{(x - x_0)^{n+1}} = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^{k-n-1}, \quad x \neq x_0$$

$$\Longrightarrow \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^{n+1}} = \frac{f^{(n+1)}(x_0)}{(n+1)!}$$

Notam $g(x) = \frac{f(x) - T_n(x)}{(x - x_0)^{n+1}}$, deci

$$f(x) = T_n(x) + (x - x_0)^{n+1} g(x)$$

$$\lim_{x \to x_0} g(x) = \frac{f^{(n+1)}(x_0)}{(n+1)!}$$

Exemple :

$$L = \lim_{x \to 0} \frac{\sin(x)\ln(x+1) - x^2}{x^3}$$

alegem $x_0 = 0$ si $n = 1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \Longrightarrow \sin x = x + x^2 g(x) \text{ cu } \lim_{x \to 0} g(x) = 0$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \Longrightarrow \ln(x+1) = x + x^2 h(x) \text{ cu } \lim_{x \to 0} h(x) = -\frac{1}{2}$$

$$\Longrightarrow L = \lim_{x \to 0} \frac{\left(x + x^2 g(x)\right)\left(x + x^2 h(x)\right) - x^2}{x^3} =$$

$$\lim_{x \to 0} \frac{x^3 (g(x) + h(x)) + x^4 g(x) h(x)}{x^3} = \lim_{x \to 0} (g(x) + h(x)) = -\frac{1}{2}$$

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Limit[(Sin[x] * Log[x + 1] - x^2) / x^3, x → 0]
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$-\frac{1}{2}$

$$L = \lim_{x \to 0} \frac{\text{tg}(x) - \text{arctg}(x)}{x^3}$$

alegem $x_0 = 0$ si $n = 2$

$$\text{tg}\,x = x + \frac{x^3}{3} + \ldots \Longrightarrow \text{tg}\,x = x + x^3 g(x) \text{ cu } \lim_{x \to 0} g(x) = \frac{1}{3}$$

$$\text{arctg}\,x = x - \frac{x^3}{3} + \ldots \Longrightarrow \text{arctg}\,x = x + x^3 h(x) \text{ cu } \lim_{x \to 0} h(x) = -\frac{1}{3}$$

$$\Longrightarrow L = \lim_{x \to 0} \frac{\left(x + x^3 g(x)\right) - \left(x + x^3 h(x)\right)}{x^3} = \lim_{x \to 0} (g(x) - h(x)) = \frac{2}{3}$$

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Limit[(Tan[x] - ArcTan[x]) / x^3, x → 0]
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$\frac{2}{3}$

$$\text{tg}\,x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} \left(2^{2n} - 1\right) B_{2n}}{(2n)!} x^{2n-1}, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

unde $B_n = -\sum_{k=0}^{n-1} C_n^k \frac{B_k}{n-k+1}$, $B_0 = 1$ (numerele lui Bernoulli)

$$1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, \ldots$$

$$L = \lim_{x \to 0} \frac{\sinh^4(x) - x^4}{(x - \sin x)^2}$$

alegem $x_0 = 0$ si $n = 2$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \Longrightarrow \sin x = x + x^3 g(x) \text{ cu } \lim_{x \to 0} g(x) = -\frac{1}{6}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \Longrightarrow \sinh x = x + x^3 h(x) \text{ cu } \lim_{x \to 0} h(x) = \frac{1}{6}$$

$$\Longrightarrow L = \lim_{x \to 0} \frac{\left(x + x^3 h(x)\right)^4 - x^4}{\left[x - \left(x + x^3 g(x)\right)\right]^2} =$$

$$= \lim_{x \to 0} \frac{1}{x^6 g^2(x)} \left(C_4^1 x^6 h(x) + C_4^2 x^8 h^2(x) + C_4^3 x^{10} h^3(x) + C_4^4 x^{12} h^4(x)\right) =$$

$$= \lim_{x \to 0} \frac{C_4^1 h(x)}{g^2(x)} = \lim_{x \to 0} \frac{\frac{4}{6}}{\left(-\frac{1}{6}\right)^2} = 24$$

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Limit[(Sinh[x]^4 - x^4) / (x - Sin[x])^2, x → 0]
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