Curbe in plan si calculul lungimii lor

Curbele in plan pot fi definite implicit, explicit sau parametric

Fie $f : [a, b] \longrightarrow \mathbb{R}$ o functie derivabila, cu derivata f' continua pe (a, b). Graficul lui f defineste o curba in intervalul $[a, b]$, avand <u>ecuatia</u> <u>explicita</u>

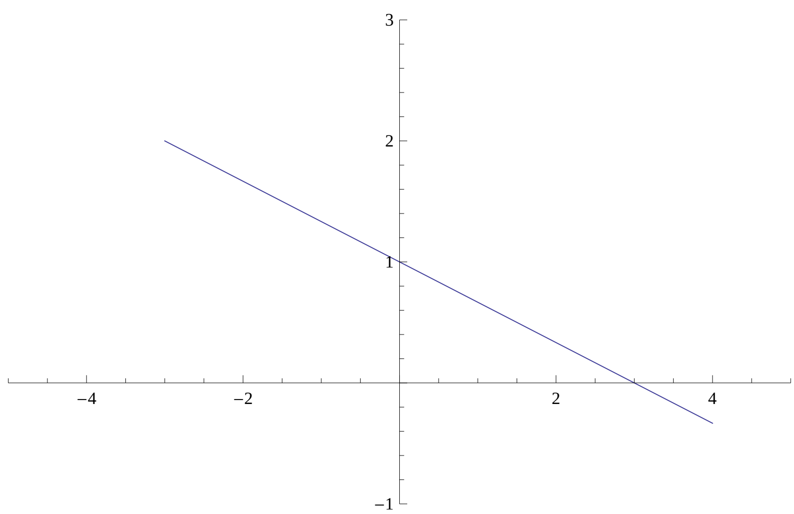
$$y = f(x), \quad x \in [a, b]$$

Exemple :

1. Ecuatia unui segment de dreapta

$$y = \alpha x + \beta, \quad x \in [a, b]$$

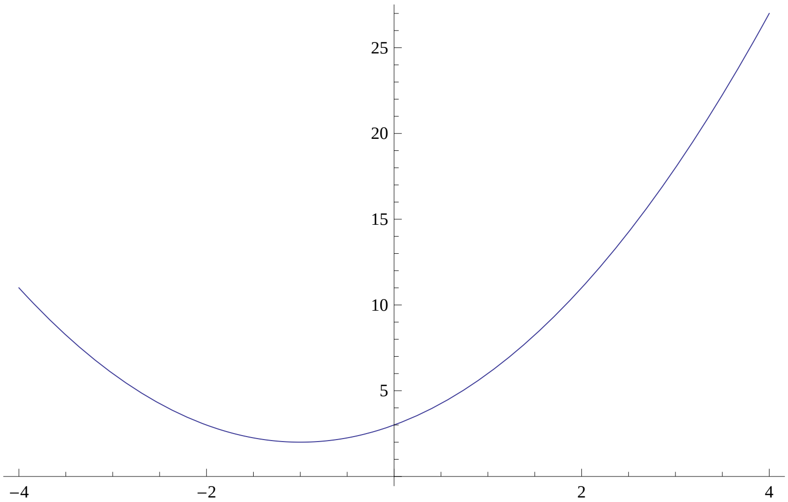
```
Plot[-x / 3 + 1, {x, -3, 4}, PlotRange → {{-5, 5}, {-1, 3}}]
```



2. Ecuatia parabolei

$$y = \alpha x^2 + \beta x + \gamma, \quad x \in [a, b]$$

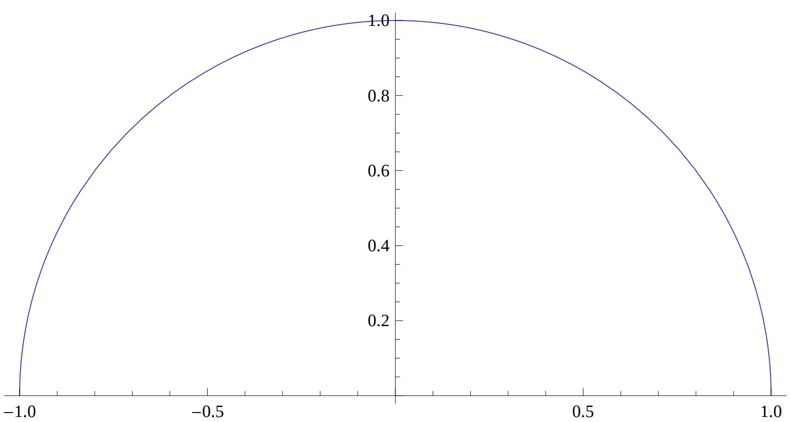
```
Plot[x^2 + 2 * x + 3, {x, -4, 4}]
```



3. Ecuatia semicercului de raza r si centrul in origine

$y = \sqrt{r^2 - x^2}$, $x \in [-r, r]$

```
Plot[Sqrt[1 - x^2], {x, -1, 1}, AspectRatio → Automatic]
```



O curba poate fi definita si prin <u>ecuatii</u> <u>parametrice</u>

$x = u(t)$
$y = v(t), \quad t \in [\alpha, \beta]$

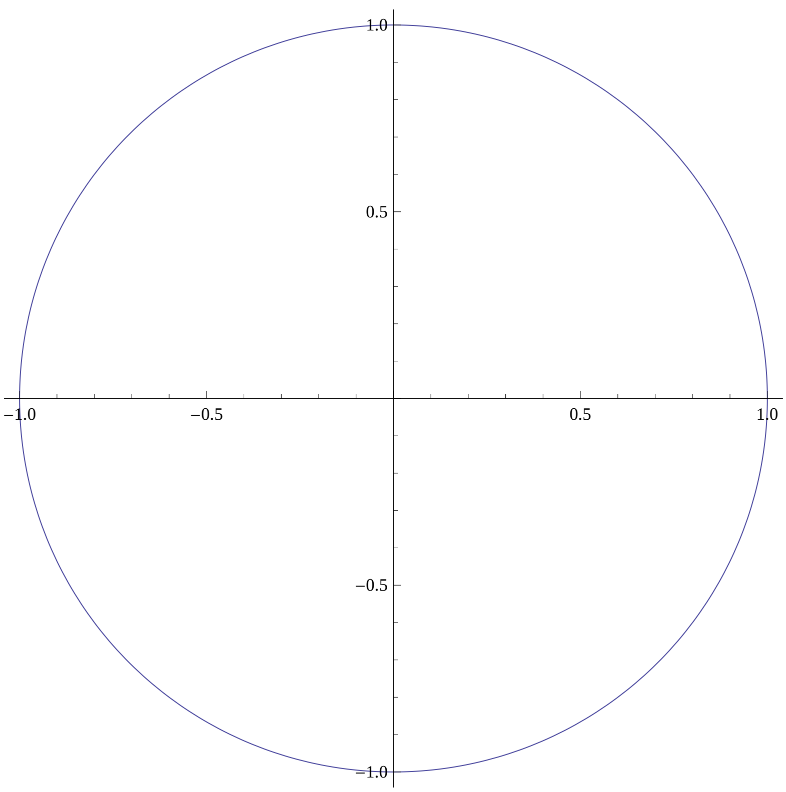
unde $u, v : [\alpha, \beta] \longrightarrow \mathbb{R}$ sunt functii derivabile,
cu derivatele u' si v' continue pe (α, β)

Exemple :

1. Ecuatia cercului de raza r si centrul in origine

$x = r \cdot \cos(t)$
$y = r \cdot \sin(t), \quad t \in [0, 2\pi]$

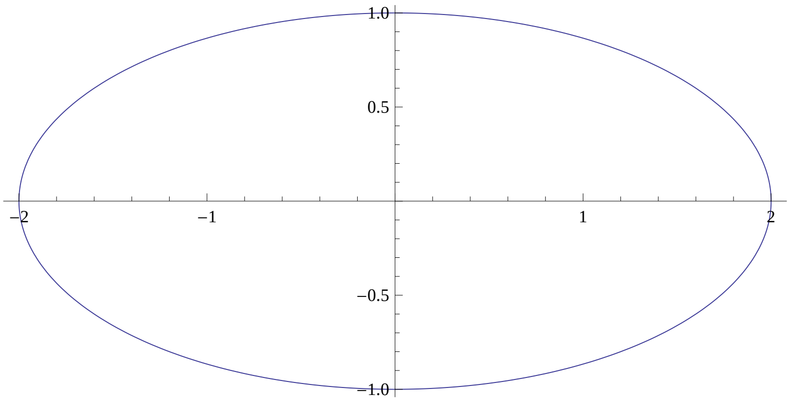
```
ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 * Pi}, AspectRatio → Automatic]
```



2. Ecuatia elipsei de semiaxe α, β si centrul in origine

$x = \alpha \cdot \cos(t)$
$y = \beta \cdot \sin(t)$, $t \in [0, 2\pi]$

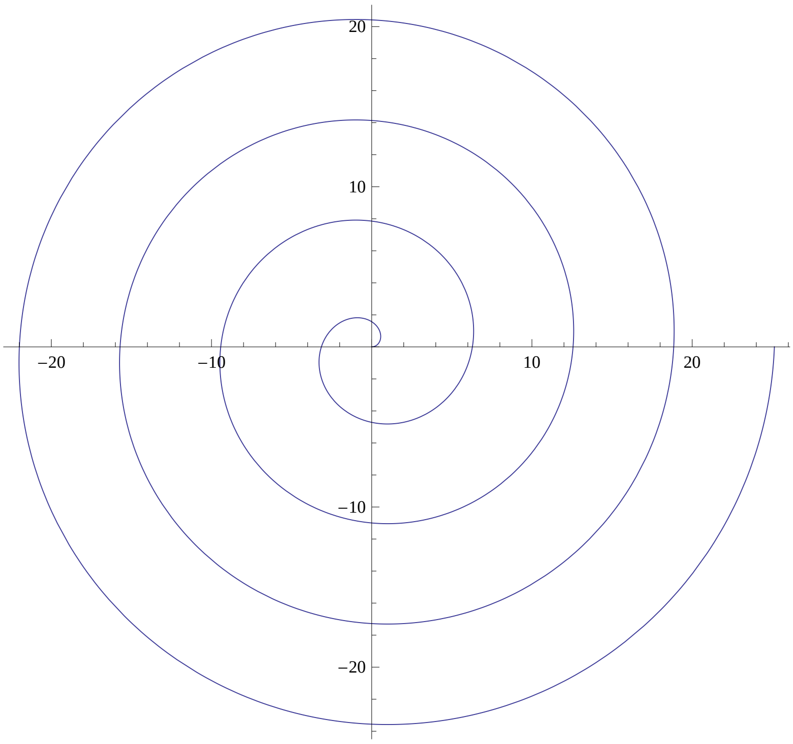
```
ParametricPlot[{2 * Cos[t], 1 * Sin[t]}, {t, 0, 2 * Pi}, AspectRatio → Automatic]
```

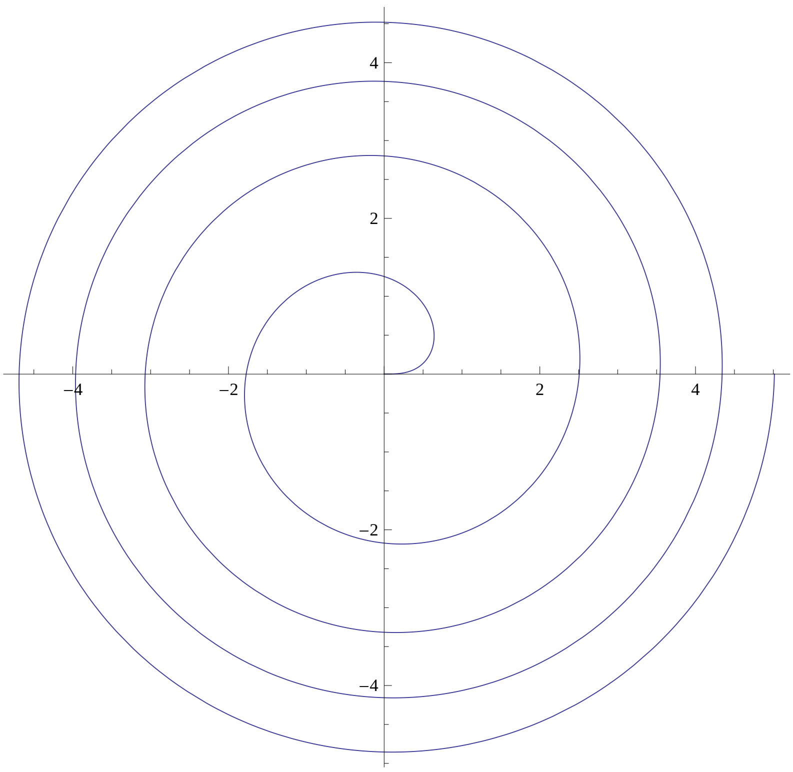


3. Ecuatia spiralei lui Arhimede cu centrul in origine

$x = t^{\alpha} \cdot \cos(t)$
$y = t^{\alpha} \cdot \sin(t)$, $t \in [0, 2k\pi]$, $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$

```
ParametricPlot[{t * Cos[t], t * Sin[t]}, {t, 0, 8 * Pi}, AspectRatio → Automatic]
ParametricPlot[{Sqrt[t] * Cos[t], Sqrt[t] * Sin[t]},
 {t, 0, 8 * Pi}, AspectRatio → Automatic]
```

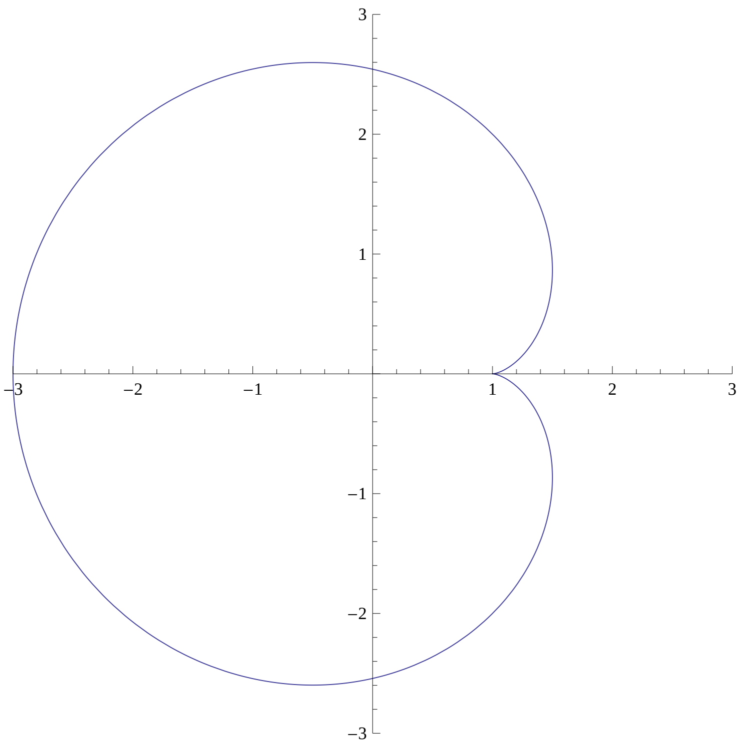


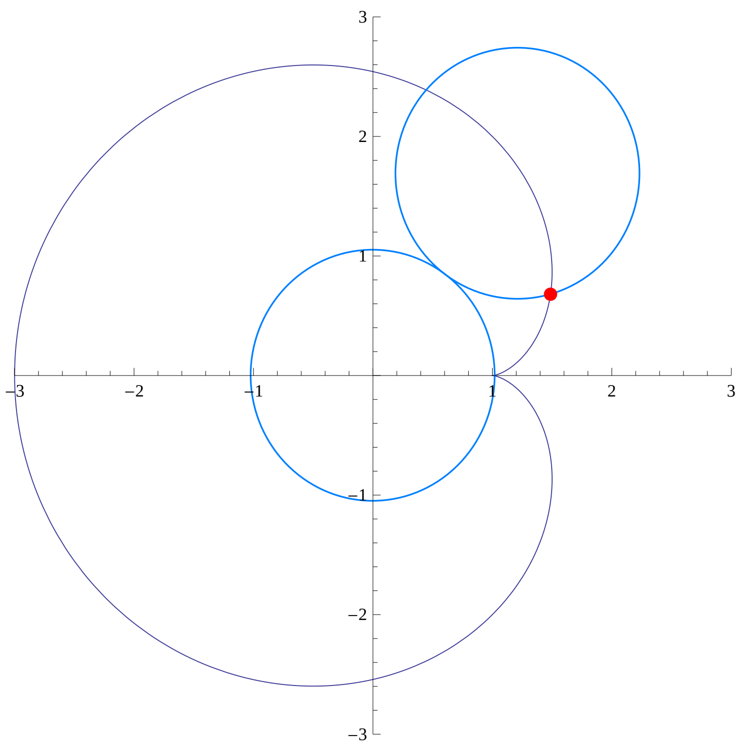


4. Ecuatia cardioidei

$$x = 2\cos(t) - \cos(2t)$$
$$y = 2\sin(t) - \sin(2t), \quad t \in [0, 2\pi]$$

```
ParametricPlot[{2 Cos[t] - Cos[2 t], 2 Sin[t] - Sin[2 t]},
 {t, 0, 2 Pi}, PlotRange → {{-3, 3}, {-3, 3}}, AspectRatio → Automatic]
```

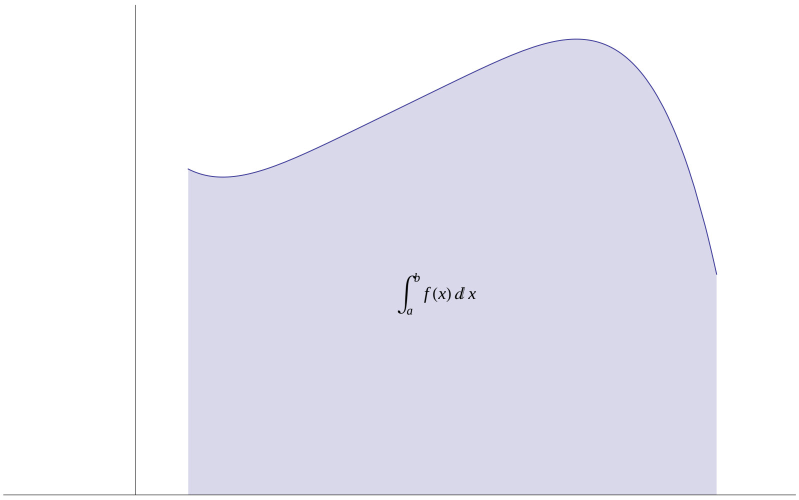




Cum putem calcula lungimea curbelor plane?

Interpretarea geometrica a integralei Riemann

```
Plot[{-(x-1)^5+x+2}, {x, 0.2, 2.2}, Filling → Bottom,
 Ticks → None, PlotRange → {{-0.5, 2.5}, {0, 3.8}}]
```




```
Plot[{-(x-1)^5+x+2}, {x, 0.2, 2.2}, Filling → None,
 Ticks → None, PlotRange → {{-0.5, 2.5}, {-0.5, 3.8}}]
```

Consideram o curba definita prin ecuatia explicita

$y = f(x),\ x \in [a, b]$

Lungimea L a acesteia este

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Daca curba este definita prin ecuatiile parametrice

$x = u(t)$
$y = v(t),\ t \in [\alpha, \beta]$

atunci

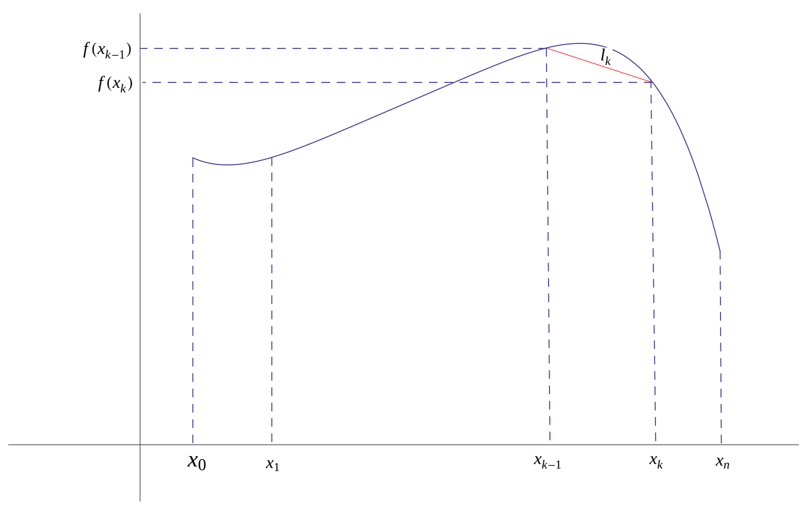
$$L = \int_\alpha^\beta \sqrt{(u'(t))^2 + (v'(t))^2}\, dt$$

Demonstratie

Fie $\Delta = (x_0, x_1, \ldots, x_n)$ **o diviziune a intervalului** $[a, b]$

Notam $l_k = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$, $k = \overline{1, n}$

$\Longrightarrow L \approx l_1 + l_2 + \ldots + l_n$



Din teorema de medie a lui Lagrange

$$\exists\, c_k \in (x_{k-1}, x_k) \text{ a.i. } f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$\xi = (c_1, c_2, \ldots, c_n)$ **este s.p.i. pentru diviziunea** Δ

$$L = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} l_k = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} \sqrt{1 + \left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)^2} \cdot (x_k - x_{k-1})$$

$$= \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2} \cdot (x_k - x_{k-1}) = \lim_{\|\Delta\|\to 0} \sigma_g(\Delta, \xi)$$

$$= \int_a^b g(x)\, dx$$

unde $g(x) = \sqrt{1 + (f'(x))^2}$

si astfel $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

Din legatura ecuatiilor parametrice cu ecuatia explicita

$y = f(x), \; x \in [a, b]$
$x = u(t), \; y = v(t), \; t \in [\alpha, \beta]$

$v(t) = f(u(t)), \; t \in [\alpha, \beta]$

$v'(t) = f'(u(t)) \cdot u'(t)$

facand substitutia $x = u(t), \; a = u(\alpha), \; b = u(\beta)$ **obtinem**

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_\alpha^\beta \sqrt{1 + (f'(u(t)))^2}\, u'(t)\, dt$$

$$= \int_\alpha^\beta \sqrt{(u'(t))^2 + (f'(u(t)) \cdot u'(t))^2}\, dt$$

$$= \int_\alpha^\beta \sqrt{(u'(t))^2 + (v'(t))^2}\, dt$$

Exemple :

1. Lungimea unui segment de dreapta

$f(x) = 1 - x, \quad x \in [0, 1]$

$f'(x) = -1$

$\sqrt{1 + (f'(x))^2} = \sqrt{2}$

$$L = \int_0^1 \sqrt{2} \, dx = \sqrt{2}$$

2. Lungimea semicercului de raza r

$f(x) = \sqrt{r^2 - x^2}, \quad x \in [-r, r]$

$f'(x) = -\frac{x}{\sqrt{r^2 - x^2}}$

$\sqrt{1 + (f'(x))^2} = \frac{r}{\sqrt{r^2 - x^2}}$

$$L = \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}} \, dx = \pi r$$

3. Lungimea unui arc de parabola

$$f(x) = x^2, \quad x \in [0, 1]$$

$$f'(x) = 2x$$

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + 4x^2}$$

$$L = \int_0^1 \sqrt{1 + 4x^2}\, dx = \frac{\sqrt{5}}{2} + \frac{\ln\left(2 + \sqrt{5}\right)}{4}$$

4. Lungimea unui arc din spirala lui Arhimede

$$u(t) = t \cdot \cos(t)$$
$$v(t) = t \cdot \sin(t), \quad t \in [0, 2\pi]$$

$$u'(t) = \cos(t) - t \cdot \sin(t)$$
$$v'(t) = \sin(t) + t \cdot \cos(t)$$

$$\sqrt{(u'(t))^2 + (v'(t))^2} = \sqrt{1 + t^2}$$

$$L = \int_0^{2\pi} \sqrt{1 + t^2}\, dt = \pi\sqrt{1 + 4\pi^2} + \frac{\mathrm{Ln}\left(2\pi + \sqrt{1 + 4\pi^2}\right)}{2}$$

5. Lungimea cardioidei

$u(t) = 2\cos(t) - \cos(2t)$
$v(t) = 2\sin(t) - \sin(2t), \quad t \in [0, 2\pi]$

$u'(t) = -2\sin(t) + 2\sin(2t)$
$v'(t) = 2\cos(t) - 2\cos(2t)$

$$\sqrt{(u'(t))^2 + (v'(t))^2} = \sqrt{8 - 8\cos(t)} = 4\sin\left(\frac{t}{2}\right)$$

$$L = \int_0^{2\pi} 4\sin\left(\frac{t}{2}\right) dt = 16$$