Serii numerice rapid convergente

Fie doua serii convergente avand aceeasi suma S

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = S$$

Care dintre ele converge mai repede?

Notam $S_n = \sum_{k=0}^{n} a_k$ si $T_n = \sum_{k=0}^{n} b_k \implies \lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n = S \implies$

$$\lim_{n\to\infty} (S - S_n) = \lim_{n\to\infty} (S - T_n) = 0$$

DEF : Spunem ca seria $\sum_{n=0}^{\infty} a_n$ <u>converge</u> <u>mai</u> <u>repede</u> decat seria $\sum_{n=0}^{\infty} b_n$ daca

$$\lim_{n\to\infty} \frac{S - S_n}{S - T_n} = 0 \qquad (*)$$

OBS : Se poate arata ca daca sirul (a_n) <u>tinde</u> <u>la</u> <u>zero</u> <u>mai</u> <u>repede</u>

decat sirul (b_n), adica $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, atunci are loc relatia $(*)$

Ex : $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 = \sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

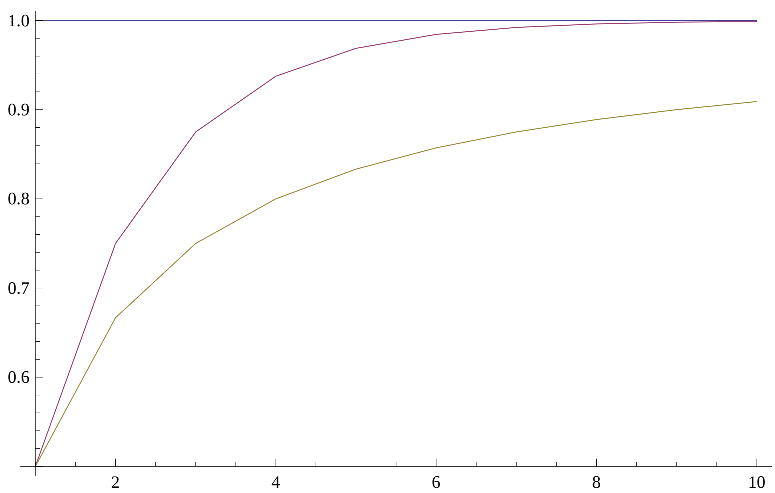
$S_n = \sum_{k=1}^{n} \frac{1}{2^k}$ si $T_n = \sum_{k=1}^{n} \frac{1}{k^2 + k}$

```
S[n_] := Sum[1 / 2^k, {k, 1, n}]
T[n_] := Sum[1 / (k^2 + k), {k, 1, n}]
1. - S[10]
1. - T[10]
```

0.000976563

0.0909091

```
DiscretePlot[{1, S[n], T[n]}, {n, 1, 10},
 Filling → None, Joined → True, PlotRange → Full]
```



Cum putem construi o serie rapid convergenta pornind de la o serie data ?

Ex : Se da seria $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Fie seria $\sum_{n=1}^{\infty} \frac{1}{n^2 + n} = 1$

Prin scadere termen cu termen obtinem

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^2 + n} \right) = \frac{\pi^2}{6} - 1 \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^3 + n^2} = \frac{\pi^2}{6} - 1 \Longrightarrow$$

$$1 + \sum_{n=1}^{\infty} \frac{1}{n^3 + n^2} = \frac{\pi^2}{6}$$

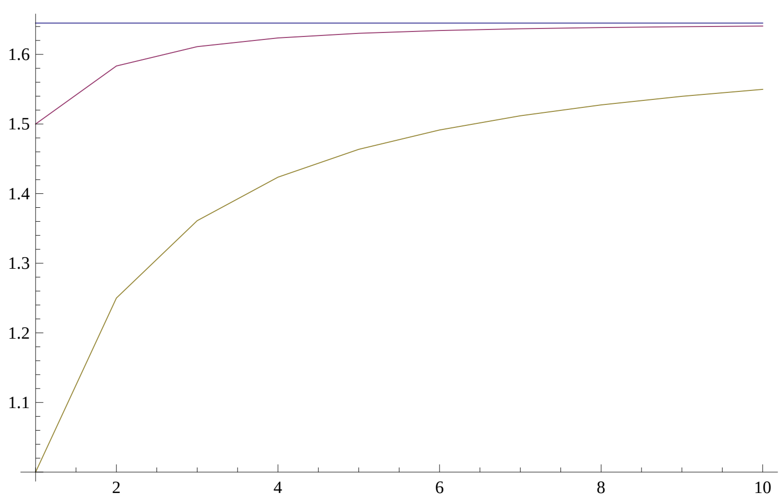
$S_n = 1 + \sum_{k=1}^{n} \frac{1}{k^3 + k^2}$ **si** $T_n = \sum_{k=1}^{n} \frac{1}{k^2}$

```
S[n_] := 1 + Sum[1 / (k^3 + k^2), {k, 1, n}]
T[n_] := Sum[1 / k^2, {k, 1, n}]
Pi^2 / 6. - S[10]
Pi^2 / 6. - T[10]
```

```
0.00425724
```

```
0.0951663
```

```
DiscretePlot[{Pi^2 / 6, S[n], T[n]}, {n, 1, 10},
 Filling → None, Joined → True, PlotRange → Full]
```



Transformarea lui Kummer

Fie $\sum_{n=0}^{\infty} b_n$ o serie convergenta ce se doreste a fi accelerata, iar

$\sum_{n=0}^{\infty} c_n$ o serie convergenta avand suma cunoscuta C si cu proprietatea

$$\lim_{n\to\infty} \frac{b_n}{c_n} = r \in (0, +\infty)$$

Atunci seria

$\sum_{n=0}^{\infty} a_n = rC + \sum_{n=0}^{\infty} (b_n - r c_n)$ converge mai repede decat seria initiala

$\sum_{n=0}^{\infty} b_n$, spre aceeasi suma S.

Ex : Sa acceleram convergenta seriei

$$\sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) =$$

$$4\left(1 - \frac{1}{3}\right) + 4\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots = \frac{8}{1\cdot 3} + \frac{8}{5\cdot 7} + \frac{8}{9\cdot 11} + \ldots =$$

$$\sum_{n=0}^{\infty} \frac{8}{(4n+1)\cdot(4n+3)} = \frac{8}{3} + \sum_{n=1}^{\infty} \frac{8}{(4n+1)\cdot(4n+3)}$$

Utilizam seria cunoscuta de la seminar $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

Cu notatiile din transformare avem $C = \frac{1}{2}$ **si** $r = 2$, **deci seria**

$$\sum_{n=0}^{\infty} a_n = \frac{11}{3} - \sum_{n=1}^{\infty} \frac{32n+14}{(2n-1)\cdot(2n+1)\cdot(4n+1)\cdot(4n+3)}$$

converge mai repede spre suma S.

Ce valoare are S?

Serii numerice remarcabile

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e \quad (\text{rapid})$$

$$\sum_{n=0}^{\infty} \frac{2}{n! \left(1 + n^2 + n^4\right)} = e \quad (\text{rapid})$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1} = \pi \quad (\text{lent})$$

$$\sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right) = \pi \quad (\text{rapid})$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln(2) \quad (\text{lent})$$

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \ln(2) \quad (\text{rapid})$$

$$\frac{3}{2} + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{(2n-3)!!}{(2n)!!} = \sqrt{2} \quad (\text{lent})$$

$$\sum_{n=0}^{\infty} \frac{(2n+1)!}{2^{3n+1} (n!)^2} = \sqrt{2} \quad (\text{rapid})$$

$$S_n = \sum_{k=0}^{n} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

```
S[n_] :=
 Sum[16^(-k) * (4 / (8 * k + 1) - 2 / (8 * k + 4) - 1 / (8 * k + 5) - 1 / (8 * k + 6)), {k, 0, n}]
N[S[1], 10]
N[S[3], 10]
N[S[5], 10]
N[Pi, 15]
```

3.141422466

3.141592458

3.141592653

3.141592653589793

$$S_n = \sum_{k=1}^{n} \frac{1}{k \cdot 2^k} \text{ si } T_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}$$

```
S[n_] := Sum[1 / (k * 2^k), {k, 1, n}]
T[n_] := Sum[- (-1)^k / k, {k, 1, n}]
Log[2.] - S[10]
Log[2.] - T[10]
```

0.0000823244

0.0475123

```
DiscretePlot[{Log[2], S[n], T[n]}, {n, 1, 20},
 Filling → None, Joined → True, PlotRange → Full]
```