Functia factorial si integrala probabilitatilor

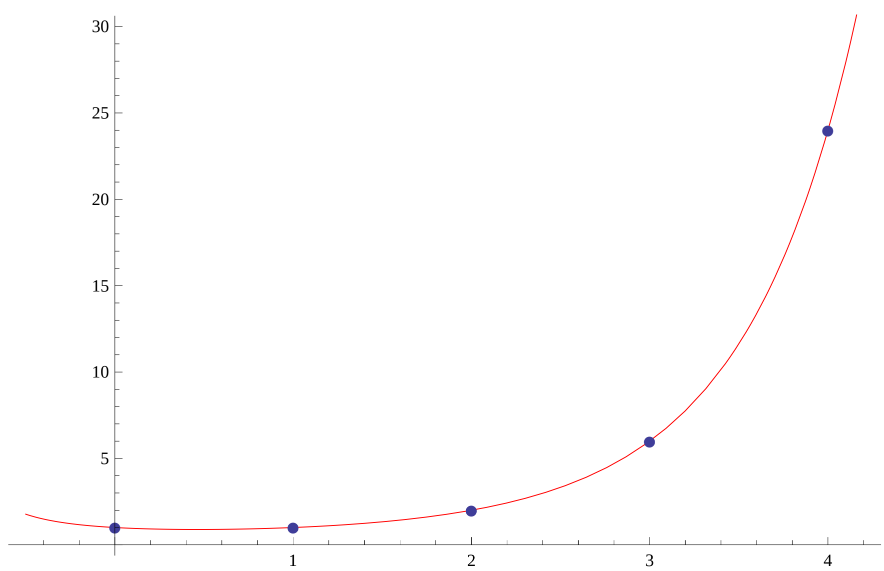
$0! = 1$
$1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$
…

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n, \quad n \in \mathbb{N}$

Se poate extinde notiunea de factorial si pentru $n \notin \mathbb{N}$?

Exista o functie continua $F : [0, \infty) \to \mathbb{R}$ a.i. $F(t) = t!, \ \forall t \in \mathbb{N}$?

```
g = Plot[Gamma[x + 1], {x, -1 / 2, 5}, PlotStyle → {Red}];
p = ListPlot[{{0, 0 !}, {1, 1 !}, {2, 2 !}, {3, 3 !}, {4, 4 !}}, PlotMarkers → Automatic];
Show[g, p, PlotRange → {{-0.5, 4.2}, {0, 30}}]
```



$F(t) = t! = t \cdot (t-1)! = t \cdot F(t-1), \ \forall t \in \mathbb{N}, \ t \geq 2$

Se poate extinde aceasta egalitate la

$F(t) = t \cdot F(t-1), \quad \forall t > 1?$

Consideram functia Gama a lui Euler

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} \, dx, \ \forall t > 0$$

La seminar se demonstreaza ca

$\Gamma(t)$ **convergenta**, $\forall t > 0$
$\Gamma(n+1) = n!, \ \forall n \in \mathbb{N}$
$\Gamma(t+1) = t \cdot \Gamma(t), \ \forall t > 0$

Mai mult, functia Γ este continua, chiar indefinit derivabila pe $(0, \infty)$

$$\Gamma^{(n)}(t) = \int_0^{\infty} x^{t-1} e^{-x} \ln^n(x) \, dx, \ \forall t > 0, \forall n \in \mathbb{N}$$

Deci

$F(t) = \Gamma(t+1), \ \forall t \geqslant 0$

OBS : Functia Γ extinde valorile factorialului la intervalul $(-1, \infty)$, adica

$$t! = \Gamma(t+1), \quad \forall t \in (-1, \infty)$$

Are sens sa vorbim despre

$$\left(-\frac{1}{2}\right)! = \Gamma\left(-\frac{1}{2}+1\right) = \Gamma\left(\frac{1}{2}\right)$$

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\left(\frac{4}{3}\right)! = \Gamma\left(\frac{4}{3}+1\right) = \frac{4}{3}\Gamma\left(\frac{4}{3}\right) = \frac{4}{3}\Gamma\left(\frac{1}{3}+1\right) = \frac{4}{3^2}\Gamma\left(\frac{1}{3}\right)$$

etc

Care sunt valorile numerice ale acestor factoriale?

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\, dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx$$

$$\Gamma\left(\frac{1}{3}\right) = \int_0^\infty x^{\frac{1}{3}-1} e^{-x}\, dx = \int_0^\infty \frac{e^{-x}}{\sqrt[3]{x^2}}\, dx$$

Are loc "formula complementului"

$$\Gamma(t) \cdot \Gamma(1-t) = \frac{\pi}{\sin(\pi t)}, \quad \forall\, t \in (0, 1)$$

$$t = \frac{1}{2} \Longrightarrow \Gamma^2\left(\frac{1}{2}\right) = \pi \Longrightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad \forall\, n \in \mathbb{N}^*$$

Pentru $\Gamma\left(\frac{1}{3}\right)$, $\Gamma\left(\frac{1}{4}\right)$, … nu exista formule de calcul,
dar se pot construi siruri care au ca limita aceste valori

Media aritmetico – geometrica a doua numere pozitive a si b

Definim sirurile de numere

$x_0 = a,\ \ y_0 = b$

$$x_{n+1} = \frac{x_n + y_n}{2},\ \ y_{n+1} = \sqrt{x_n\, y_n},\ \ \forall\, n \in \mathbb{N}$$

OBS : Cele doua siruri sunt convergente si au aceeasi limita

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n \overset{not}{=} MAG\,(a, b)$$

numita <u>media</u> <u>aritmetico</u> – <u>geometrica</u> a numerelor a si b

Ex : $a = 1,\ \ b = 2$

$x_0 = 1,\ \ y_0 = 2$

$$x_1 = \frac{3}{2},\ \ y_1 = \sqrt{2}$$

$$x_2 = \frac{3 + 2\sqrt{2}}{4},\ \ y_2 = \sqrt{\frac{3}{\sqrt{2}}}$$

...

```
a = 1; b = 2;
t := RecurrenceTable[{x[n + 1] == (x[n] + y[n]) / 2,
    y[n + 1] == Sqrt[x[n] * y[n]], x[0] == a, y[0] == b}, {x, y}, {n, 1, 5}];
N[{a, b}, 10]
N[t[[1]], 10]
N[t[[2]], 10]
N[t[[3]], 10]
N[t[[4]], 10]
```

```
{1.000000000, 2.000000000}

{1.500000000, 1.414213562}

{1.457106781, 1.456475315}

{1.456791048, 1.456791014}

{1.456791031, 1.456791031}
```

Au loc egalitatile

$$\Gamma\left(\frac{1}{3}\right) = \frac{(2\pi)^{3/4}}{\mathrm{MAG}\left(\sqrt{2}, 1\right)^{1/2}}$$

$$\Gamma\left(\frac{1}{4}\right) = \frac{2^{4/9}\,\pi^{2/3}}{3^{1/12}\,\mathrm{MAG}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}, 1\right)^{1/3}}$$

Generalizarea formulei lui Stirling

Reamintim formula lui Stirling pentru siruri

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

adica

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1$$

Are loc extinderea

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1$$

Integrala probabilitatilor

$$\int_0^{\infty} e^{-x^2} \, dx \quad \text{(integrala Euler – Poisson)}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx \quad \text{(integrala probabilitatilor)}$$

La seminar se arata

$$\int_0^{\infty} e^{-x^2} \, dx = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} \, dx = 1$$

Erori de masurare

DEF : Eroarea de masurare intr – un experiment este egala cu diferenta dintre valoarea masurata x si valoarea reala

OBS : Daca erorile de masurare au un caracter aleatoriu (cauzat de fluctuatiile imprevizibile ale aparatelor de masurare, sau de citire a acestora), atunci aceste erori urmeaza o <u>distributie</u> <u>normala</u> in jurul valorii medii, descrisa prin functia

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

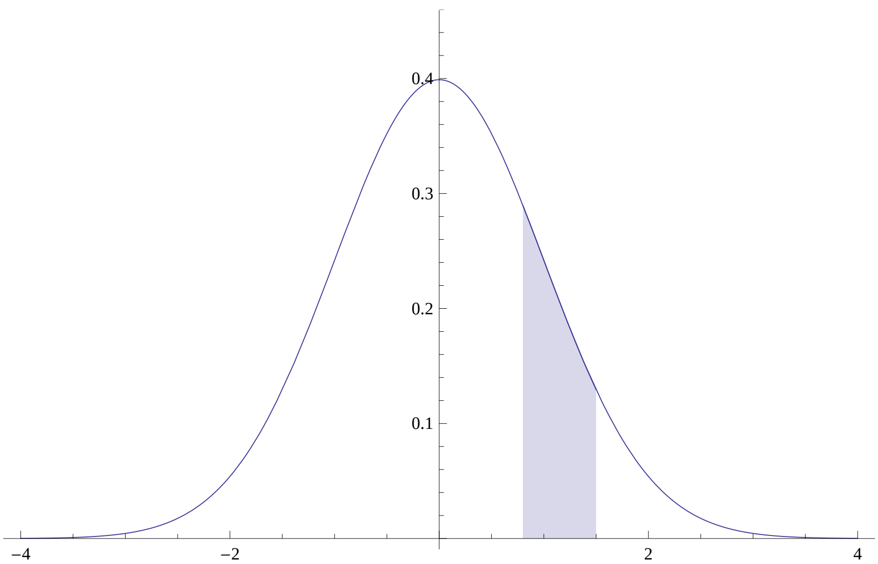
unde

σ – **<u>deviatia</u> <u>standard</u> (precizia de masurare)**
μ – **<u>valoarea</u> <u>medie</u> (media masuratorilor)**

Pentru simplitate consideram $\mu = 0$ si $\sigma = 1$

$$N(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

```
p1 := Plot[Exp[-x^2 / 2] / Sqrt[2 Pi], {x, -4, 4}];
p2 := Plot[Exp[-x^2 / 2] / Sqrt[2 Pi], {x, 0.8, 1.5},
   Filling → Bottom, Ticks → None, PlotRange → {{0.8, 1.5}, {0, 0.45}}];
Show[p1, p2, PlotRange → {{-4, 4}, {0, 0.45}}]
```



OBS : Probabilitatea ca o masuratoare efectuata sa returneze o valoare cuprinsa in intervalul [a, b] este

$$P\,(a < X < b) = \int_a^b N\,(X)\,dX$$

Daca punem $a = -\infty$ **si** $b = \infty$**, obtinem**

$$P\,(-\infty < X < \infty) = \int_{-\infty}^{\infty} N\,(X)\,dX = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dX = 1$$