Multimi remarcabile in $\mathbb{R}^m$ si esantionarea Monte Carlo

$\mathbb{R}^m = \left\{(x_1, x_2, \ldots, x_m) \mid x_i \in \mathbb{R},\ i = \overline{1, m}\right\}$

Hiperplanul

$P = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid a_1 x_1 + a_2 x_2 + \ldots + a_m x_m + a_0 = 0\},$

unde $a_i \in \mathbb{R}$ sunt constante

Hipercubul inchis de centru O_m si latura $2r$

$\overline{H}(O_m, r) = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid -r \le x_1 \le r,\ -r \le x_2 \le r,\ \ldots,\ -r \le x_m \le r\}$
$\overset{not}{=} [-r, r]^m$

Cazuri particulare

$\overline{H}(O_1, r) = [-r, r]$, lungime $= 2r$

$\overline{H}(O_2, r) = [-r, r] \times [-r, r]$, arie $= (2r)^2$

$\overline{H}(O_3, r) = [-r, r] \times [-r, r] \times [-r, r]$, volum $= (2r)^3$

Care este "volumul" unui hipercub?

$\mathrm{vol}\,\overline{H}(O_m, r) = (2r)^m,\ m \in \mathbb{N},\ m \ge 1$

Bila inchisa de centru O_m si raza r

$$\overline{B}(O_m, r) = \left\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \,\middle|\, x_1^2 + x_2^2 + \ldots + x_m^2 \le r^2\right\}$$

Cazuri particulare

$$\overline{B}(O_1, r) = \left\{x_1 \in \mathbb{R} \,\middle|\, x_1^2 \le r^2\right\} = [-r, r], \text{ lungime} = 2\,r$$

$$\overline{B}(O_2, r) = \left\{(x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^2 + x_2^2 \le r^2\right\}, \text{ arie} = \pi r^2$$

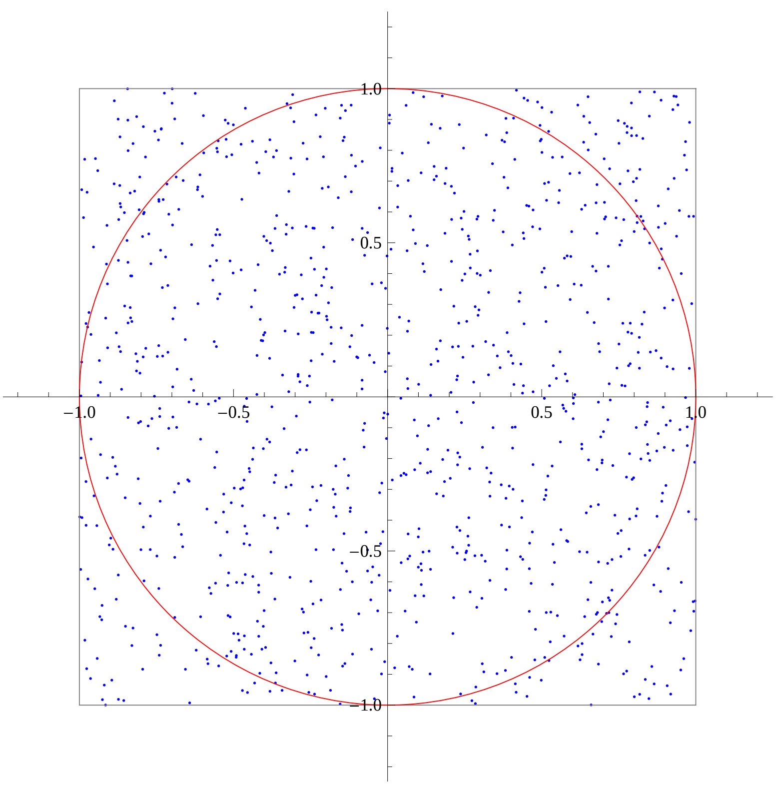
$$\overline{B}(O_3, r) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, x_1^2 + x_2^2 + x_3^2 \le r^2\right\}, \text{ volum} = \frac{4\,\pi}{3}\, r^3$$

Care este "volumul" unei (hiper) bile?

Vom da o definitie probabilistica a "volumului" bilei

Cazul plan

```
g = Graphics[{Gray, Line[{{1, 1}, {-1, 1}, {-1, -1}, {1, -1}, {1, 1}}], Red, Circle[{0, 0}, 1]}];
l = ListPlot[Table[{2 * RandomReal[] - 1, 2 * RandomReal[] - 1}, {n, 1000}],
   PlotStyle -> {Blue, PointSize[0.003]}, AspectRatio → Automatic];
Show[l, g, PlotRange → {{-1.2, 1.2}, {-1.2, 1.2}}]
```



Din cele "n" puncte generate aleatoriu in patratul $\overline{H}(O_2, 1)$, un numar "B(n)" vor fi in discul $\overline{B}(O_2, 1)$

$$\lim_{n\to\infty} \frac{B(n)}{n} = \frac{\text{aria}\,\overline{B}(O_2, 1)}{\text{aria}\,\overline{H}(O_2, 1)} = \frac{\pi}{4} \simeq 0.7853$$

```
n = 1 000 000;
For[k = 1; b = 0, k ≤ n, k++, b = If[RandomReal[]^2 + RandomReal[]^2 ≤ 1, b + 1, b]]
N[b / n]
```

```
0.785322
```

Cazul general

Cum generam puncte in hipercubul $\overline{H}(O_m, 1)$?

$$x_1, x_2, \ldots, x_m \in [-1, 1]$$

Cum testam apartenenta punctelor la bila $\overline{B}(O_m, 1)$?

$$x_1^2 + x_2^2 + \ldots + x_m^2 \le 1$$

Din cele "n" puncte generate aleatoriu in hipercubul $\overline{H}(O_m, 1)$, un numar "B(n)" vor fi in bila $\overline{B}(O_m, 1)$

$$\lim_{n\to\infty} \frac{B(n)}{n} = \frac{\text{vol}\,\overline{B}(O_m, 1)}{\text{vol}\,\overline{H}(O_m, 1)} = \frac{\text{vol}\,\overline{B}(O_m, 1)}{2^m}, \quad \text{deci}$$

$$\text{vol}\,\overline{B}(O_m, 1) \overset{\text{def}}{=} 2^m \lim_{n\to\infty} \frac{B(n)}{n}$$

Sa calculam vol $\overline{B}(O_4, 1)$

```
n = 1 000 000;
f[x1_, x2_, x3_, x4_] = x1^2 + x2^2 + x3^2 + x4^2;
For[k = 1; b = 0, k ≤ n, k++,
 b = If[f[RandomReal[], RandomReal[], RandomReal[], RandomReal[]] ≤ 1, b + 1, b]]
2^4 * N[b / n]
```

```
4.93968
```

Are loc formula

$$\text{vol}\,\overline{B}\,(O_m,\ r) = \frac{\pi^{m/2}}{\Gamma\left(1+\frac{m}{2}\right)}\,r^m,\ m\in\mathbb{N},\ m\geq 1$$

unde Γ este functia Gama a lui Euler

$$\Gamma\,(t) = \int_0^\infty x^{t-1}\,e^{-x}\,dx,\ \ \forall\, t>0$$

$$\text{vol}\,\overline{B}\,(O_4,\ r) = \frac{\pi^{4/2}}{\Gamma\left(1+\frac{4}{2}\right)}\,r^4 = \frac{\pi^2}{\Gamma\,(3)}\,r^4 = \frac{\pi^2}{2\,!}\,r^4 \simeq 4.934\,r^4$$

$$\text{vol}\,\overline{B}\,(O_3,\ r) = \frac{\pi^{3/2}}{\Gamma\left(1+\frac{3}{2}\right)}\,r^3 = \frac{\pi^{3/2}}{\frac{3}{2^2}\,\Gamma\left(\frac{1}{2}\right)}\,r^3 = \frac{4\,\pi}{3}\,r^3$$

$$\text{vol}\,\overline{B}\,(O_2,\ r) = \frac{\pi^{2/2}}{\Gamma\left(1+\frac{2}{2}\right)}\,r^2 = \frac{\pi}{\Gamma\,(2)}\,r^2 = \pi r^2$$

EXERCITIU

Calculati valoarea lui $\Gamma\left(1+\dfrac{m}{2}\right),\ \ m\in\mathbb{N}$