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# Math for Machine Learning

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## **Linear algebra - Week 1**



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# System of Linear Equations

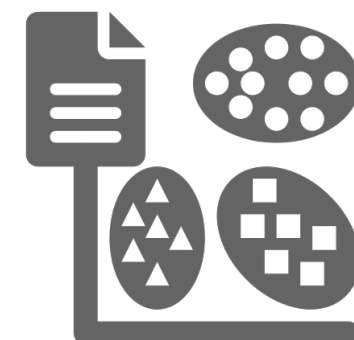
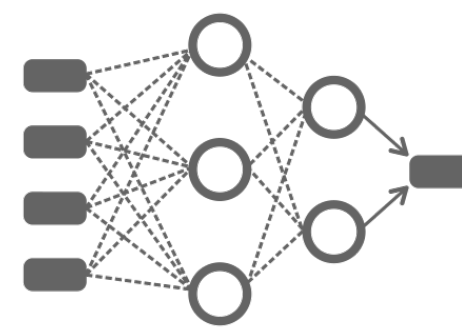
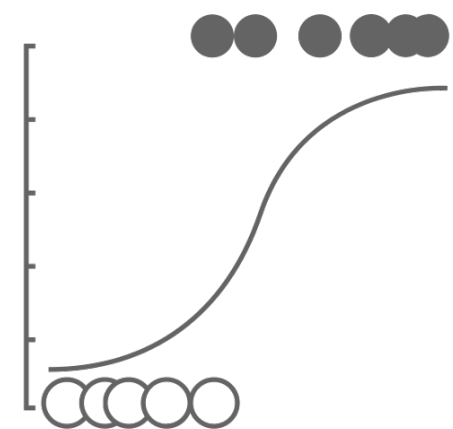
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## **Linear Algebra Applied I**

# Machine Learning



# Machine Learning



Don't worry about  
the math!

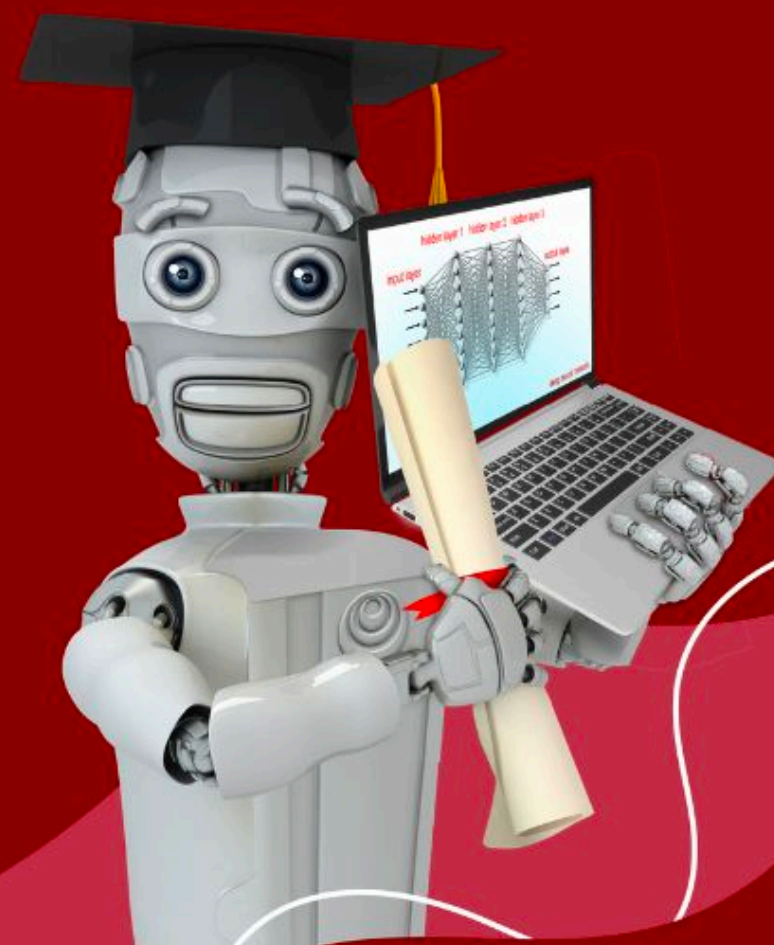


Don't worry about  
the machine learning!

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**Machine Learning  
Specialization**

**Enroll now**





# Linear Algebra and Machine Learning

## Linear Regression

*Supervised Machine Learning*



# Linear Algebra and Machine Learning

Input



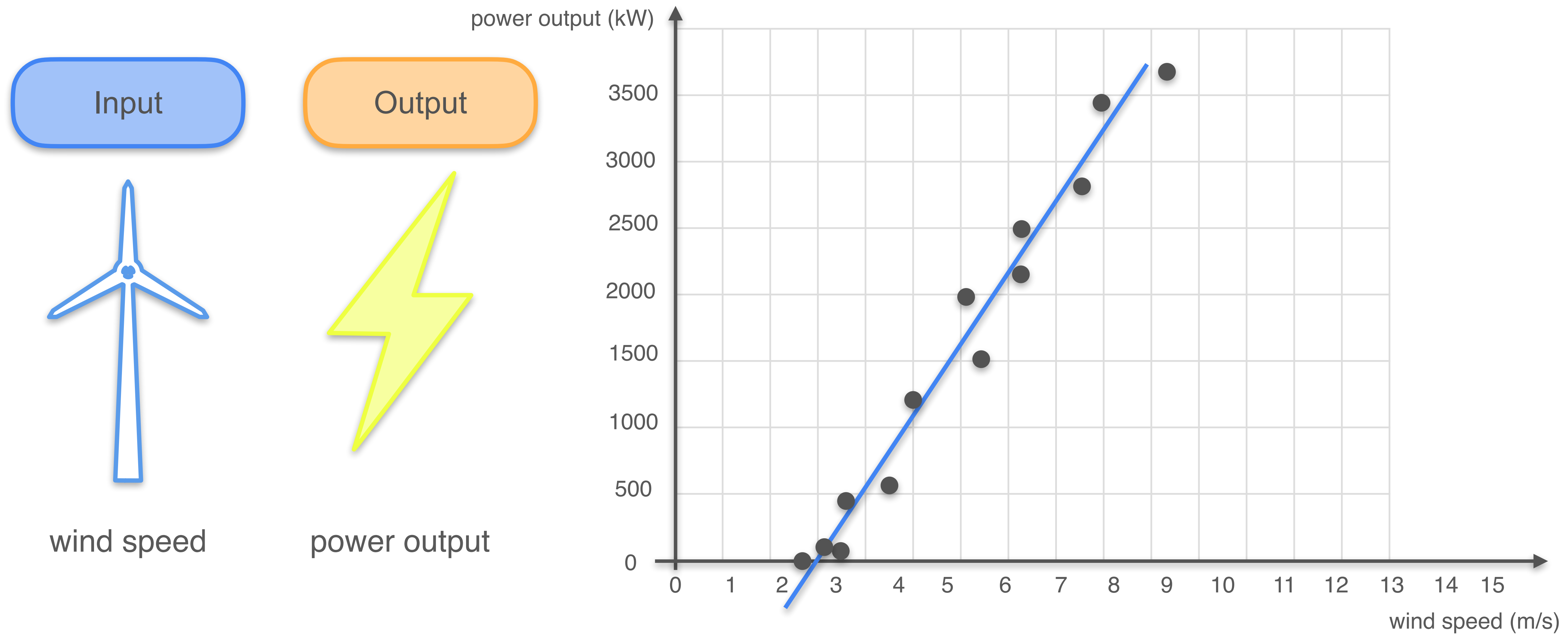
wind speed

Output



power output

# Linear Algebra and Machine Learning





# Linear Algebra and Machine Learning



$$m \times \text{wind speed} + b = \text{power output}$$

5m/s                      1500kW



# Linear Algebra and Machine Learning

Input



wind speed

Input



temperature

Output



power output

# Linear Algebra and Machine Learning



# Linear Algebra and Machine Learning

Input



wind speed

Input

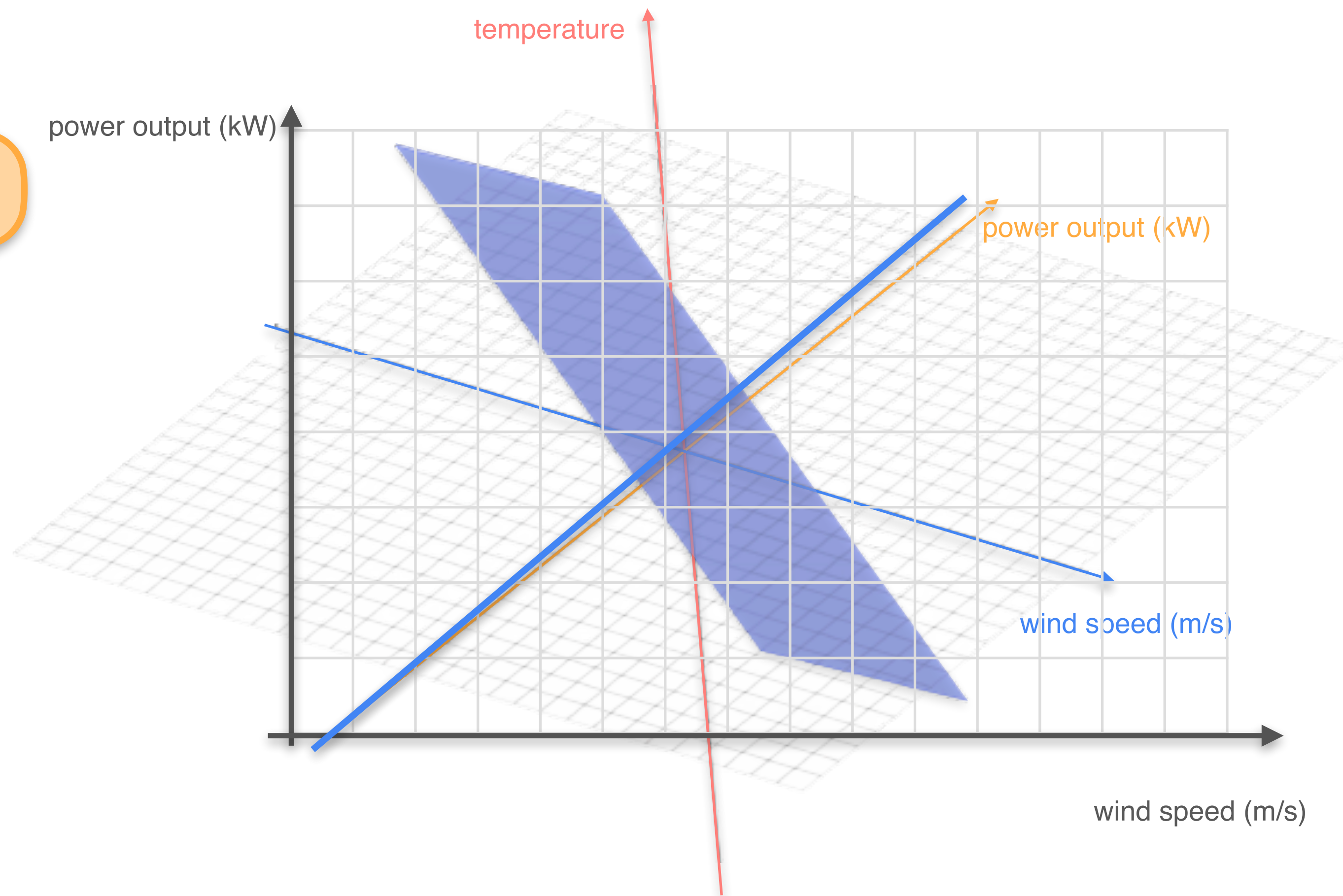


temperature

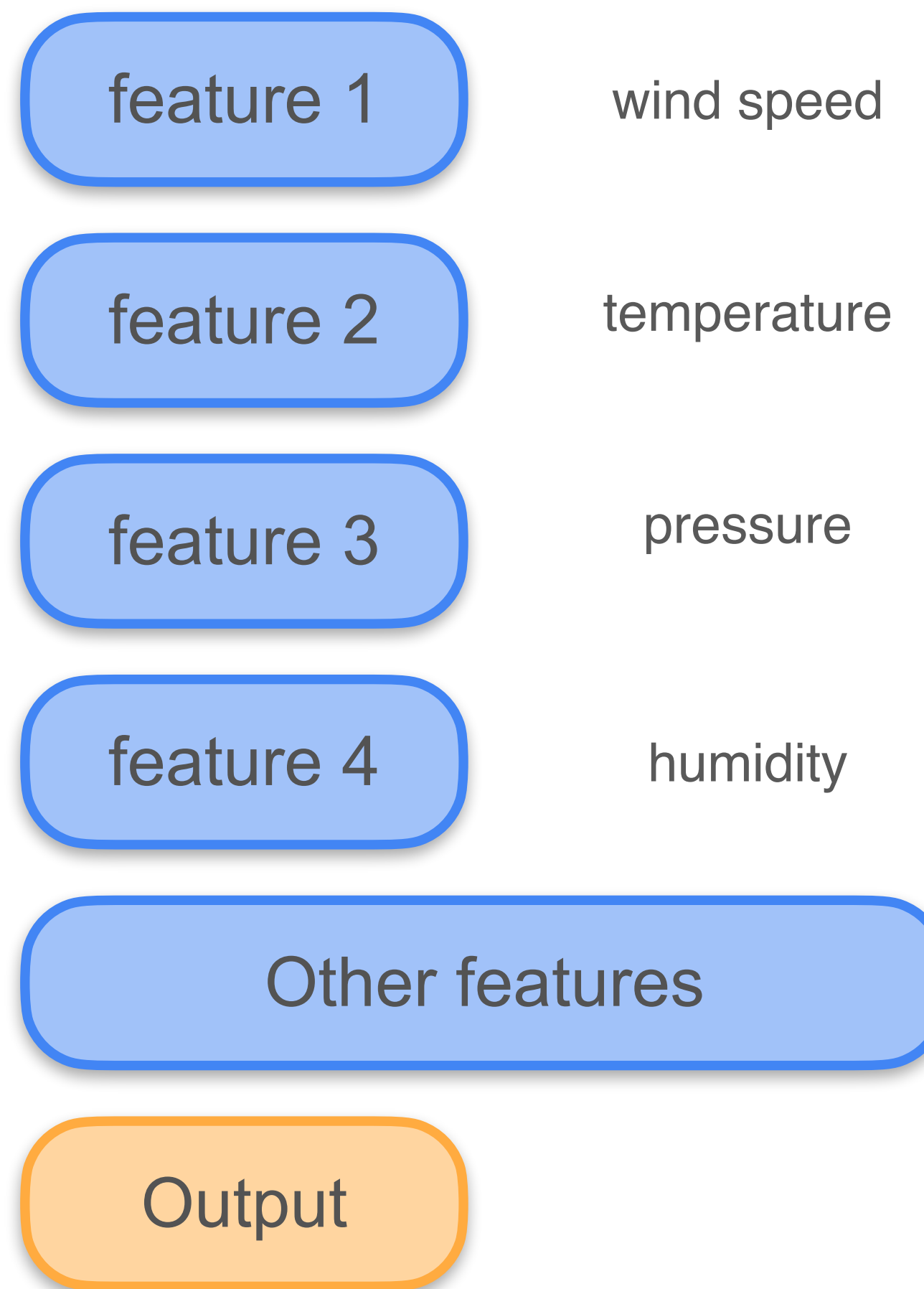
Output



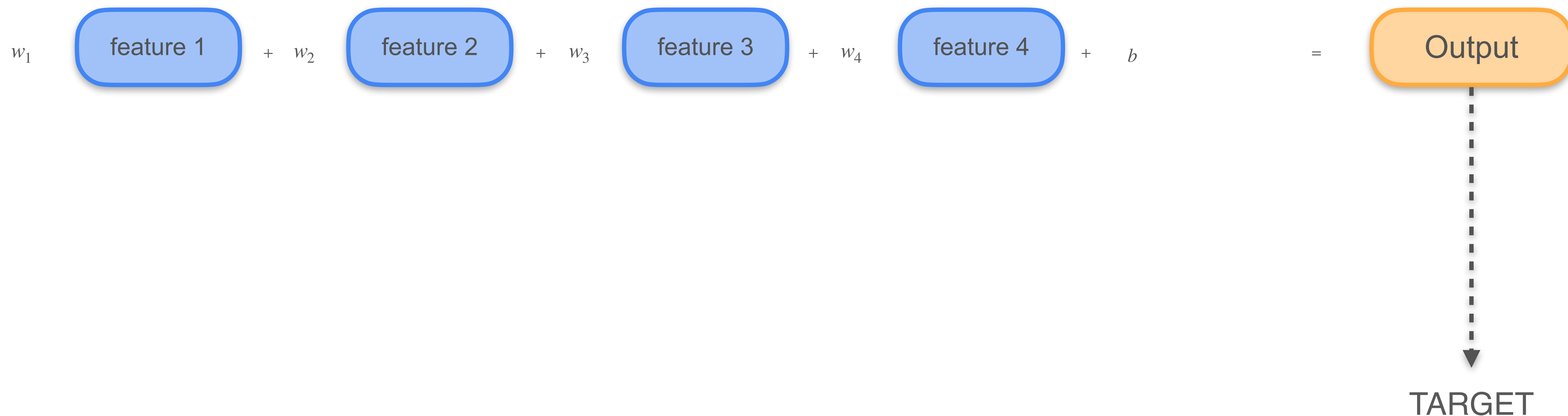
power output



# Linear Algebra and Machine Learning



# Linear Algebra and Machine Learning





# Linear Algebra and Machine Learning

$$w_1 x_1 + w_2 x_2 + \dots + w_n x_n + b = y$$


**TARGET**

# Linear Algebra and Machine Learning

$$w_1 x_1^{(1)} + w_2 x_2^{(1)} + \dots + w_n x_n^{(1)} + b = y^{(1)}$$

$$w_1 x_1^{(2)} + w_2 x_2^{(2)} + \dots + w_n x_n^{(2)} + b = y^{(2)}$$

$$w_1 x_1^{(3)} + w_2 x_2^{(3)} + \dots + w_n x_n^{(3)} + b = y^{(3)}$$

System of Linear Equations

$$\vdots$$
$$w_1 x_1^{(m)} + w_2 x_2^{(m)} + \dots + w_n x_n^{(m)} + b = y^{(m)}$$



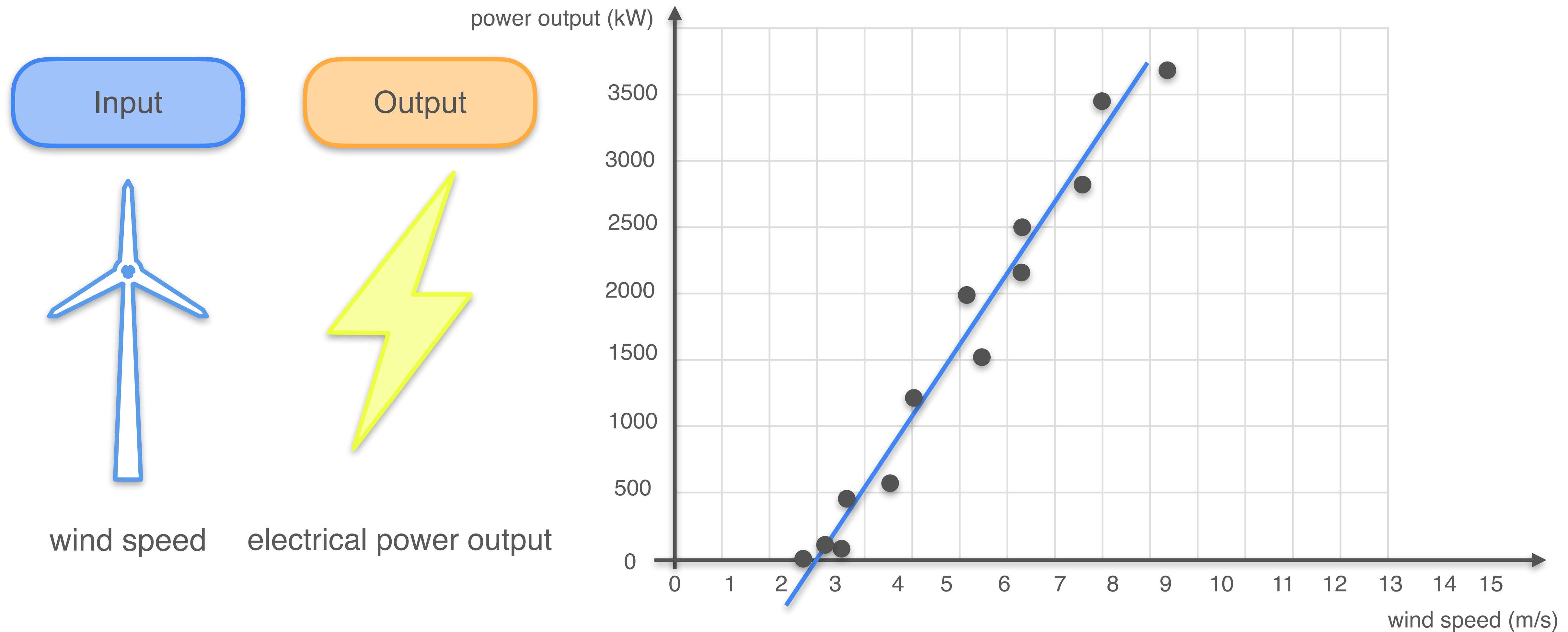
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# System of Linear Equations

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## **Linear Algebra Applied II**

# Linear Algebra and Machine Learning



# Linear Algebra and Machine Learning



# Linear Algebra and Machine Learning

|                 |            |                 |          |                 |           |                 |        |     |   |                 |   |     |   |           |
|-----------------|------------|-----------------|----------|-----------------|-----------|-----------------|--------|-----|---|-----------------|---|-----|---|-----------|
| wind speed      | temperatur | pressure        | humidity | ...             | feature n |                 | target |     |   |                 |   |     |   |           |
| $w_1 x_1^{(1)}$ | +          | $w_2 x_2^{(1)}$ | +        | $w_3 x_3^{(1)}$ | +         | $w_4 x_4^{(1)}$ | +      | ... | + | $w_n x_n^{(1)}$ | + | $b$ | = | $y^{(1)}$ |
| $w_1 x_1^{(2)}$ | +          | $w_2 x_2^{(2)}$ | +        | $w_3 x_3^{(2)}$ | +         | $w_4 x_4^{(2)}$ | +      | ... | + | $w_n x_n^{(2)}$ | + | $b$ | = | $y^{(2)}$ |
|                 |            |                 |          | ■               |           |                 |        |     |   |                 |   |     |   |           |
|                 |            |                 |          | ■               |           |                 |        |     |   |                 |   |     |   |           |
|                 |            |                 |          | ■               |           |                 |        |     |   |                 |   |     |   |           |
| $w_1 x_1^{(m)}$ | +          | $w_2 x_2^{(m)}$ | +        | $w_3 x_3^{(m)}$ | +         | $w_4 x_4^{(m)}$ | +      | ... | + | $w_n x_n^{(m)}$ | + | $b$ | = | $y^{(m)}$ |



# Linear Algebra and Machine Learning



# Linear Algebra and Machine Learning

$$\begin{array}{c} \mathbf{w} \\ \left[ \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 & \dots & w_n \end{array} \right] \\ \text{vector} \end{array} \bullet \begin{array}{c} X \\ \left[ \begin{array}{cccc} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} & \dots & x_n^{(1)} \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & x_4^{(2)} & \dots & x_n^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{(m)} & x_2^{(m)} & x_3^{(m)} & x_4^{(m)} & \dots & x_n^{(m)} \end{array} \right] \\ \text{matrix} \end{array} + \begin{array}{c} b \\ \left[ \begin{array}{cccc} y^{(1)} & y^{(2)} & \dots & y^{(m)} \end{array} \right] \\ \text{vector} \end{array} = \begin{array}{c} Y \end{array}$$

# Linear Algebra and Machine Learning



# Plan for the Week

Common vector and matrix operations



# Plan for the Week

Systems of Linear Equations

Representing systems as vectors and matrices

Computing the determinant of matrices

# Check your Knowledge

Linear  
Algebra

Your algebra score added to your calculus score minus your probability score was 6

---

Calculus

Your algebra score minus your calculus score plus double your probability score was 4.

---

Probability &  
Statistics

Four times your algebra score minus double your calculus score added to your probability score was 10

**Represent these statements as a system of linear equations.**



# Check your Knowledge

a

Linear  
Algebra

Your algebra score added to your calculus score minus your probability score was 6

$$a + c - p = 6$$

c

Calculus

Your algebra score minus your calculus score plus double your probability score was 4.

$$a - c + 2p = 4$$

p

Probability &  
Statistics

Four times your algebra score minus double your calculus score added to your probability score was 10

$$4a - 2c + p = 10$$

**Represent these statements as a system of linear equations.**

# Check your Knowledge

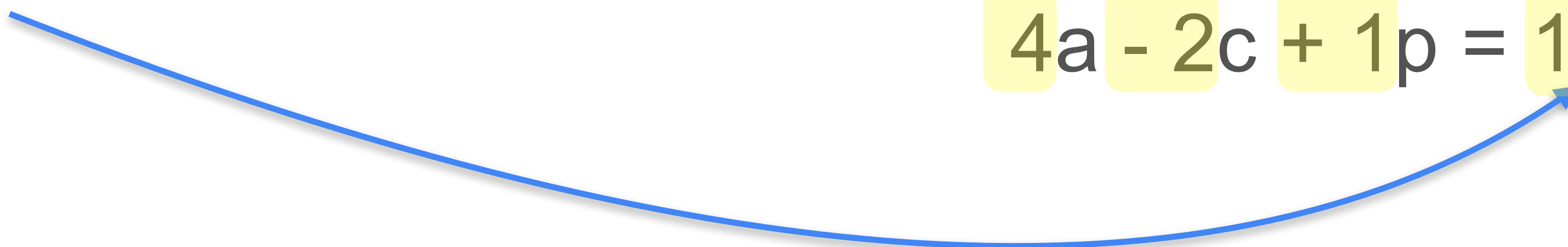
What are the weights,  $w$ ?  $a, c, p$

$$1a + 1c - 1p = 6$$

What are the features,  $x$ ?

$$1a - 1c + 2p = 4$$

The targets,  $y$ ?  $6, 4, 10$

$$4a - 2c + 1p = 10$$


# Check your Knowledge

Is this system singular or non-singular?

$$a + c - p = 6$$

Can you solve this system of equations?

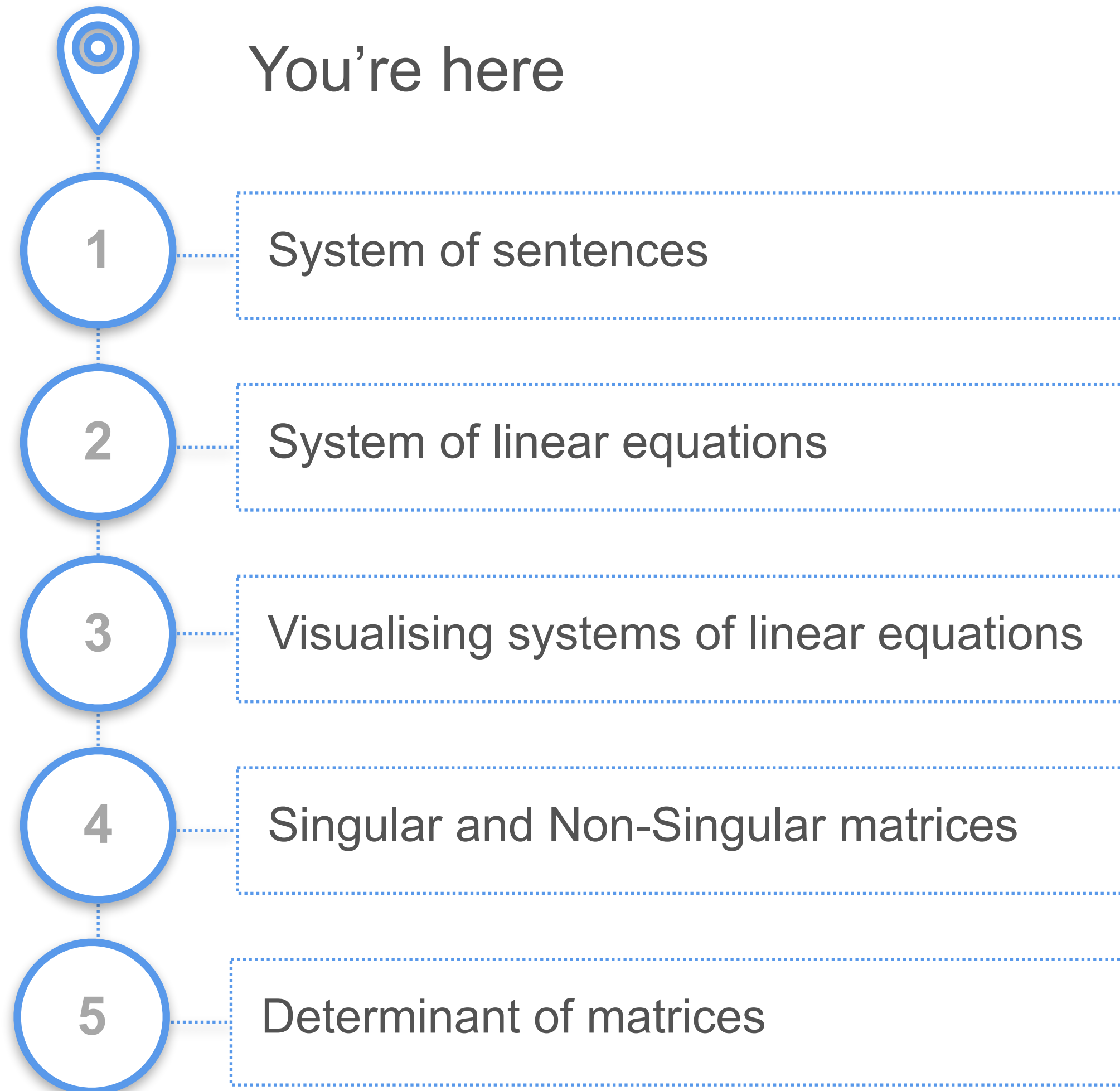
$$a - c + 2p = 4$$

Can you represent this system as a matrix and a vector?

$$4a - 2c + p = 10$$

Can you calculate the determinant of that matrix?

# What to expect





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# System of Linear Equations

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**System of sentences**

# Systems of sentences

## System 1

 The dog is **black**  
 The cat is **orange**

Complete

Non-singular


## System 2

 The dog is **black**  
 The dog is **black**

Redundant

Singular

## System 3

 The dog is **black**  
 The dog is **white**

Contradictory

Singular



# Systems of sentences

## System 1



Complete

Non-singular

## System 2



Redundant

Singular

## System 3



Redundant

Singular

## System 4



Contradictory

Singular

# Quiz: Systems of sentences

Given this system:

- Between the dog, the cat, and the bird, one is red.
- Between the dog and the cat, one is orange.
- The dog is black.

**Problem 1:**

What color is the bird?

**Problem 2:**

Is this system singular or non-singular?

# Solution: Systems of information

Given this system:

- Between the dog, the cat, and the bird, one is red.
- Between the dog and the cat, one is orange.
- The dog is black.



**Solution 1:**

The bird is red. 

**Solution 2:**

It is non-singular.   



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# System of Linear Equations

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**System of equations**

# Sentences → Equations

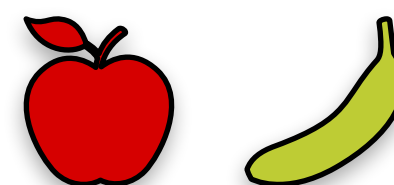
## Sentences

Between the dog and the cat, one is black.



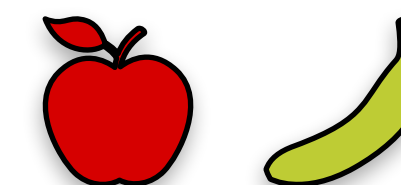
## Sentences with numbers

The price of an apple and a banana is \$10.



## Equations

$$a + b = 10$$



# Quiz: Systems of equations 1

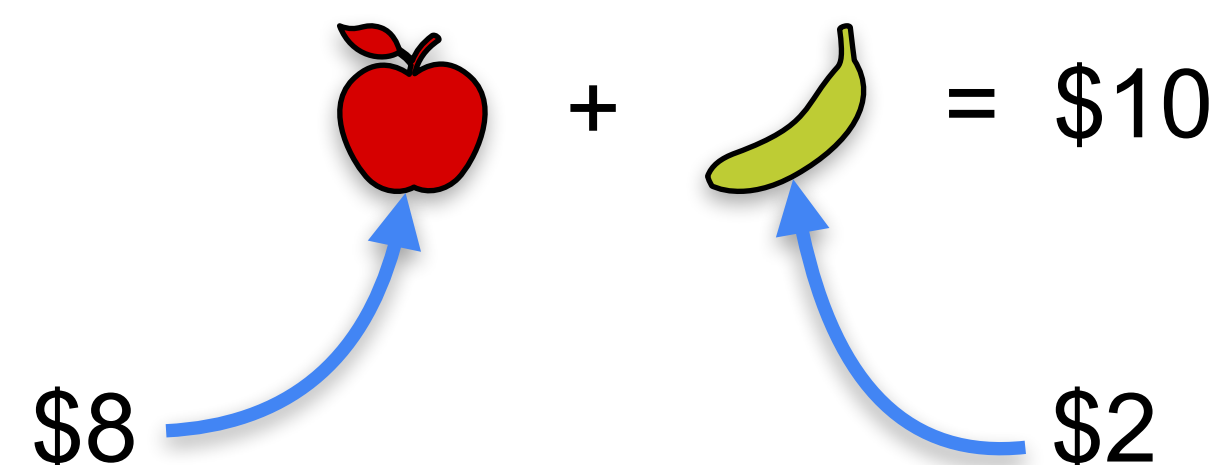
You go two days in a row and collect this information:

- **Day 1:** You bought an apple and a banana and they cost \$10.
- **Day 2:** You bought an apple and two bananas and they cost \$12.

**Question:** How much does each fruit cost?

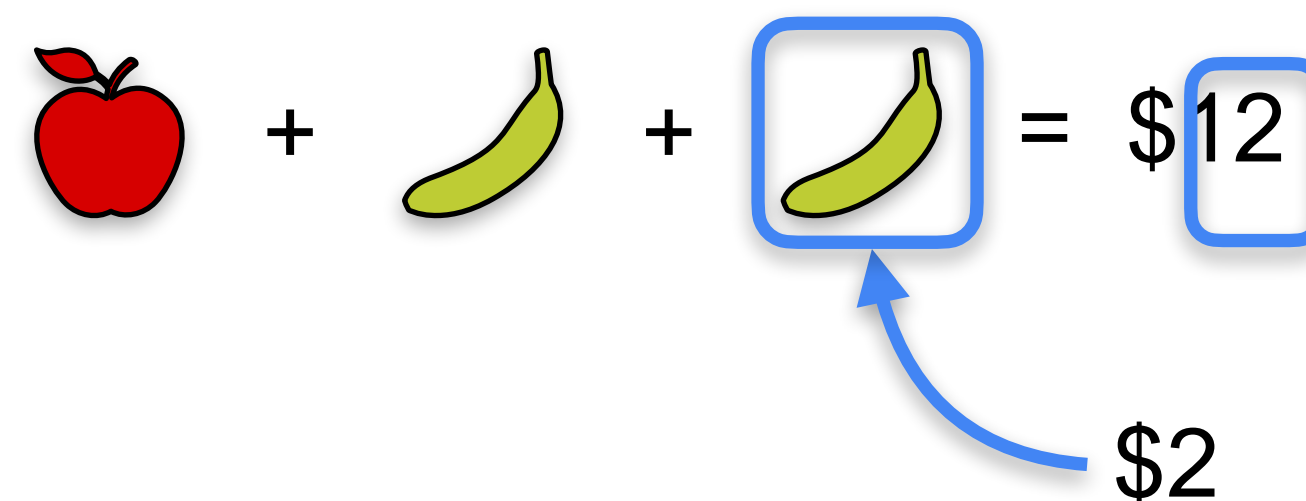
# Solution: Systems of equations 1

- **Day 1:** You bought an apple and a banana and they cost \$10.


$$\text{apple} + \text{banana} = \$10$$

\$8      \$2

- **Day 2:** You bought an apple and two bananas and they cost \$12.


$$\text{apple} + \text{banana} + \boxed{\text{banana}} = \$12$$

\$2

- **Solution:** An apple costs \$8, a banana costs \$2.

# Quiz: Systems of equations 2

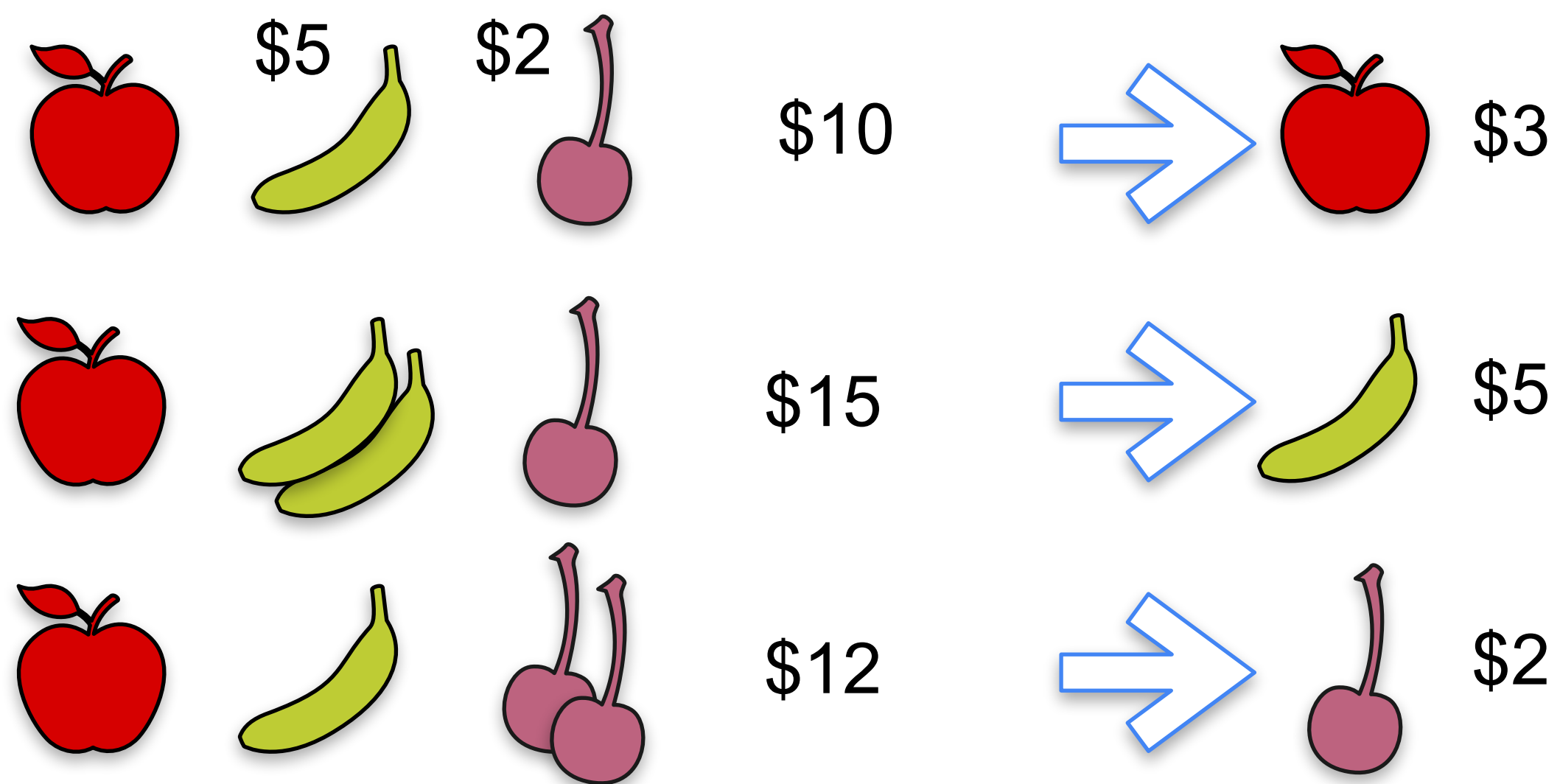
**Problem 1:** You're trying to figure out the price of apples, bananas, and cherries at the store. You go three days in a row, and bring this information.

- **Day 1:** You bought an apple, a banana, and a cherry, and paid \$10.
- **Day 2:** You bought an apple, two bananas, and a cherry, and paid \$15.
- **Day 3:** You bought an apple, a banana, and two cherries, and paid \$12.

How much does each fruit cost?



# Solution: Systems of equations 2



## System of equations 1

$$\begin{aligned}a + b + c &= 10 \\a + 2b + c &= 15 \\a + b + 2c &= 12\end{aligned}$$

## Solution

$$\begin{aligned}a &= 3 \\b &= 5 \\c &= 2\end{aligned}$$

# Quiz: Systems of equations 3

You go two days in a row and collect this information:

- **Day 1:** You bought an apple and a banana and they cost \$10.
- **Day 2:** You bought two apples and two bananas and they cost \$20.

**Question:** How much does each fruit cost?

# Solution: Systems of equations 3

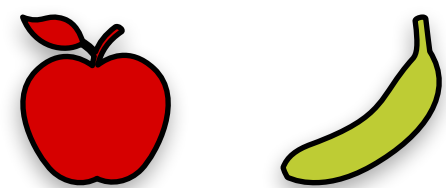
- **Day 1:** You bought an apple and a banana and they cost \$10.

$$\text{🍏} + \text{🍌} = \$10$$

- **Day 2:** You bought two apples and two bananas and they cost \$20.

$$\text{🍏🍏} + \text{🍌🍌} = \$20$$

Same thing!!!



8      2

5      5

8.3    1.7

0      10

Infinitely many solutions!

# Quiz: Systems of equations 4

You go two days in a row and collect this information:

- **Day 1:** You bought an apple and a banana and they cost \$10.
- **Day 2:** You bought two apples and two bananas and they cost \$24.

**Question:** How much does each fruit cost?

# Solution: Systems of equations 4

- **Day 1:** You bought an apple and a banana and they cost \$10.

$$\text{🍏} + \text{🍌} = \$10 \quad \Rightarrow \quad \text{🍏🍏} + \text{🍌🍌} = \$20$$

- **Day 2:** You bought two apples and two bananas and they cost \$24.

$$\text{🍏🍏} + \text{🍌🍌} = \$24$$

Contradiction!

No solutions!

# Systems of equations

## System 1

$$a + b = 10$$

$$a + 2b = 12$$

**Unique solution:**

$$a = 8$$

$$b = 2$$

**Complete**

**Non-singular**

## System 2

$$a + b = 10$$

$$2a + 2b = 20$$

**Infinite solutions**

$$\begin{array}{l} a = 8, 7, 6, \dots \\ b = 2, 3, 4 \end{array}$$

**Redundant**

**Singular**

## System 3

$$a + b = 10$$

$$2a + 2b = 24$$

**No solution**

**Contradictory**

**Singular**

# Quiz: More systems of equations

## System 1

$$\begin{aligned}a + b + c &= 10 \\a + b + 2c &= 15 \\a + b + 3c &= 20\end{aligned}$$

## System 2

$$\begin{aligned}a + b + c &= 10 \\a + b + 2c &= 15 \\a + b + 3c &= 18\end{aligned}$$

## System 3

$$\begin{aligned}a + b + c &= 10 \\2a + 2b + 2c &= 20 \\3a + 3b + 3c &= 30\end{aligned}$$

# Solutions: More systems of equations

## System 2

$$\begin{aligned}a + b + c &= 10 \\a + b + 2c &= 15 \\a + b + 3c &= 20\end{aligned}$$

## Infinitely many sols.

$$\begin{aligned}c &= 5 \\a + b &= 5 \\(0, 5, 5), (1, 4, 5), (2, 3, 5), \dots\end{aligned}$$

## System 3

$$\begin{aligned}a + b + c &= 10 \\a + b + 2c &= 15 \\a + b + 3c &= 18\end{aligned}$$

## No solutions

$$\begin{aligned}\text{From 1st and 2nd:} \\c &= 5 \\\text{From 2nd and 3rd:} \\c &= 3\end{aligned}$$

## System 4

$$\begin{aligned}a + b + c &= 10 \\2a + 2b + 2c &= 20 \\3a + 3b + 3c &= 30\end{aligned}$$

## Infinitely many solutions

$$\begin{aligned}\text{Any 3 numbers that add} \\ \text{to 10 work.} \\(0, 0, 10), (2, 7, 1), \dots\end{aligned}$$

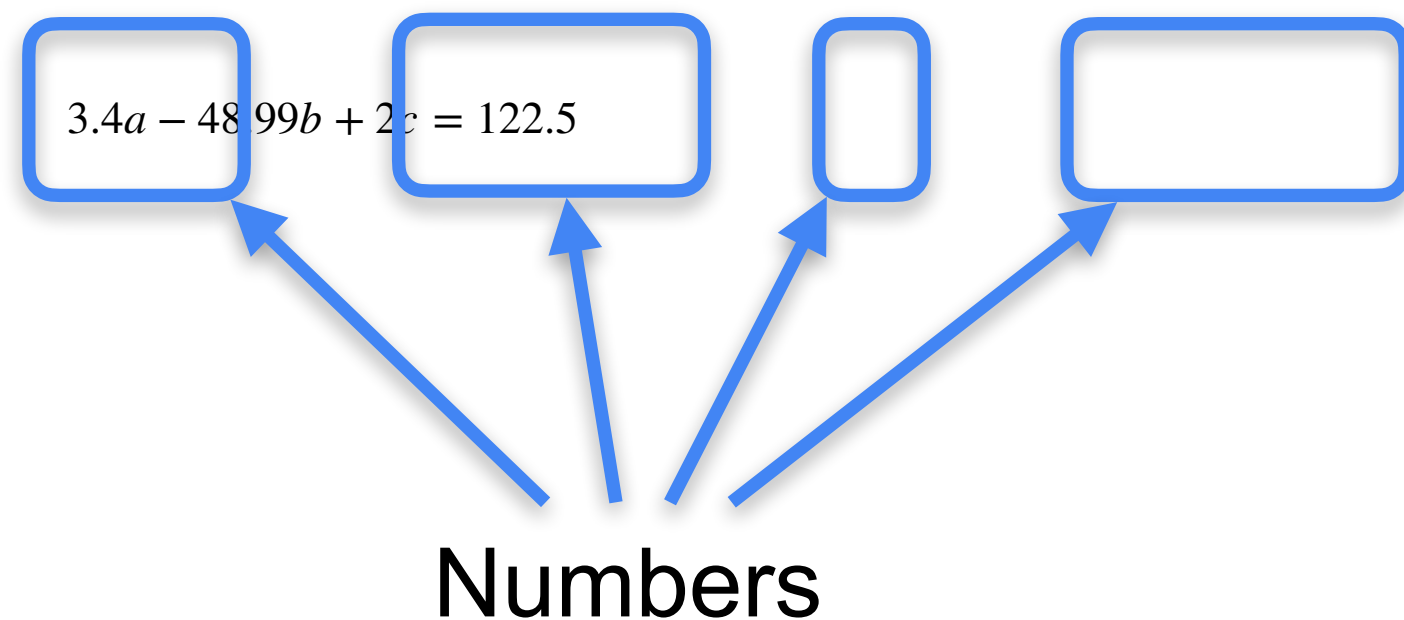


# What is a linear equation?

## Linear

$$a + b = 10$$

$$2a + 3b = 15$$



## Non-linear

$$a^2 + b^2 = 10$$

$$\sin(a) + b^5 = 15$$

$$2^a - 3^b = 0$$

$$ab^2 + \frac{b}{a} - \frac{3}{b} - \log(c) = 4^a$$



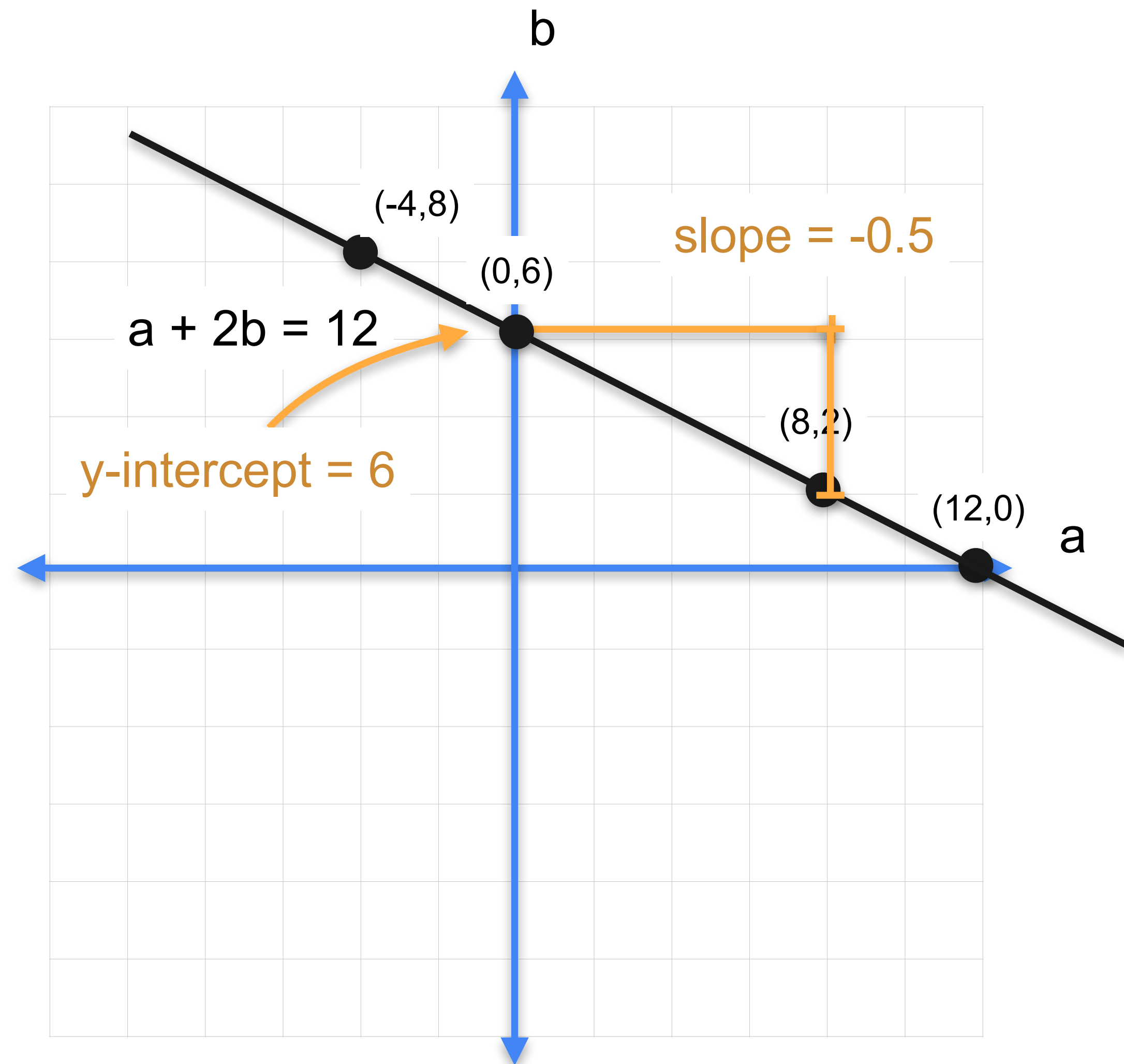
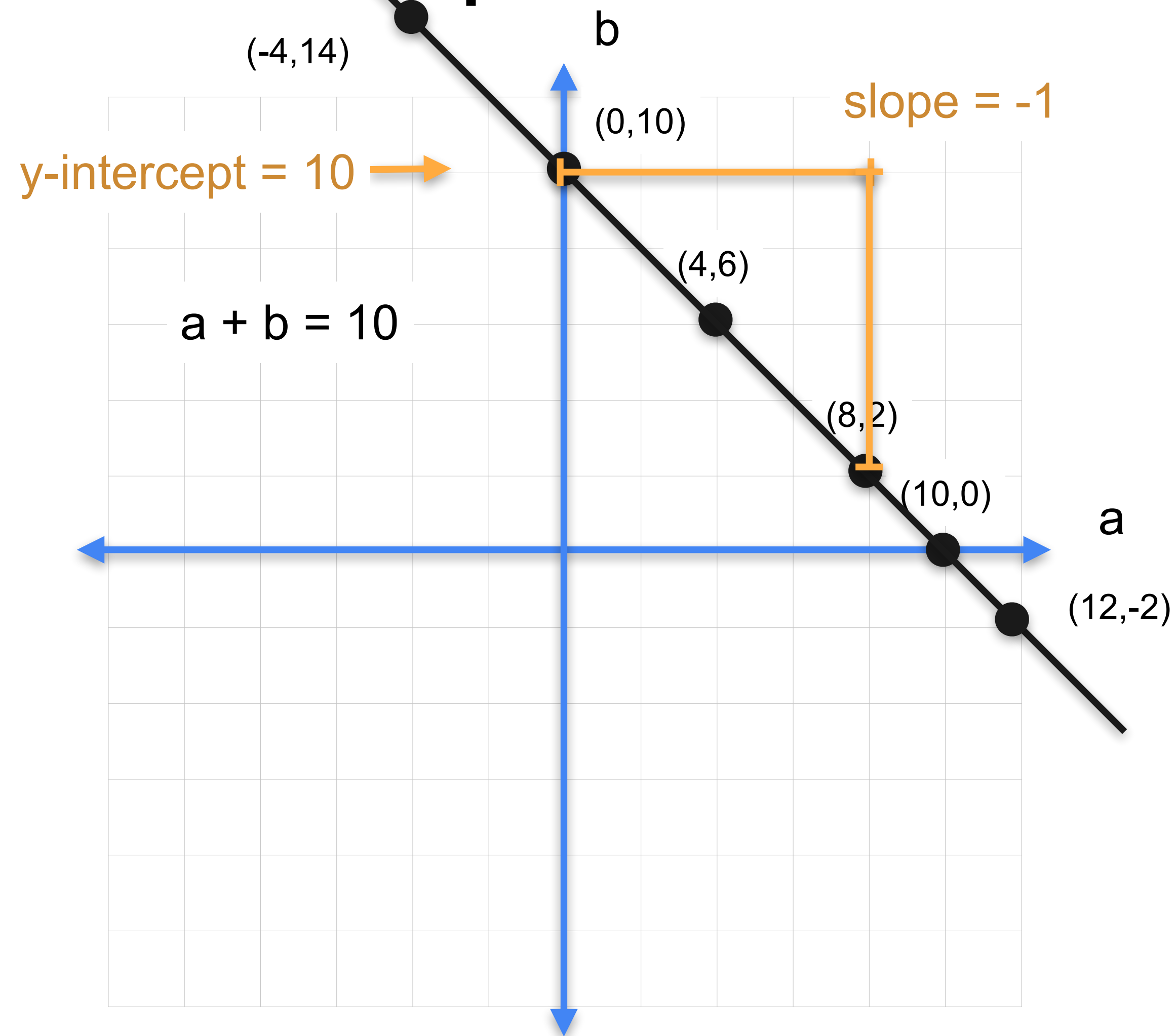
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# System of Linear Equations

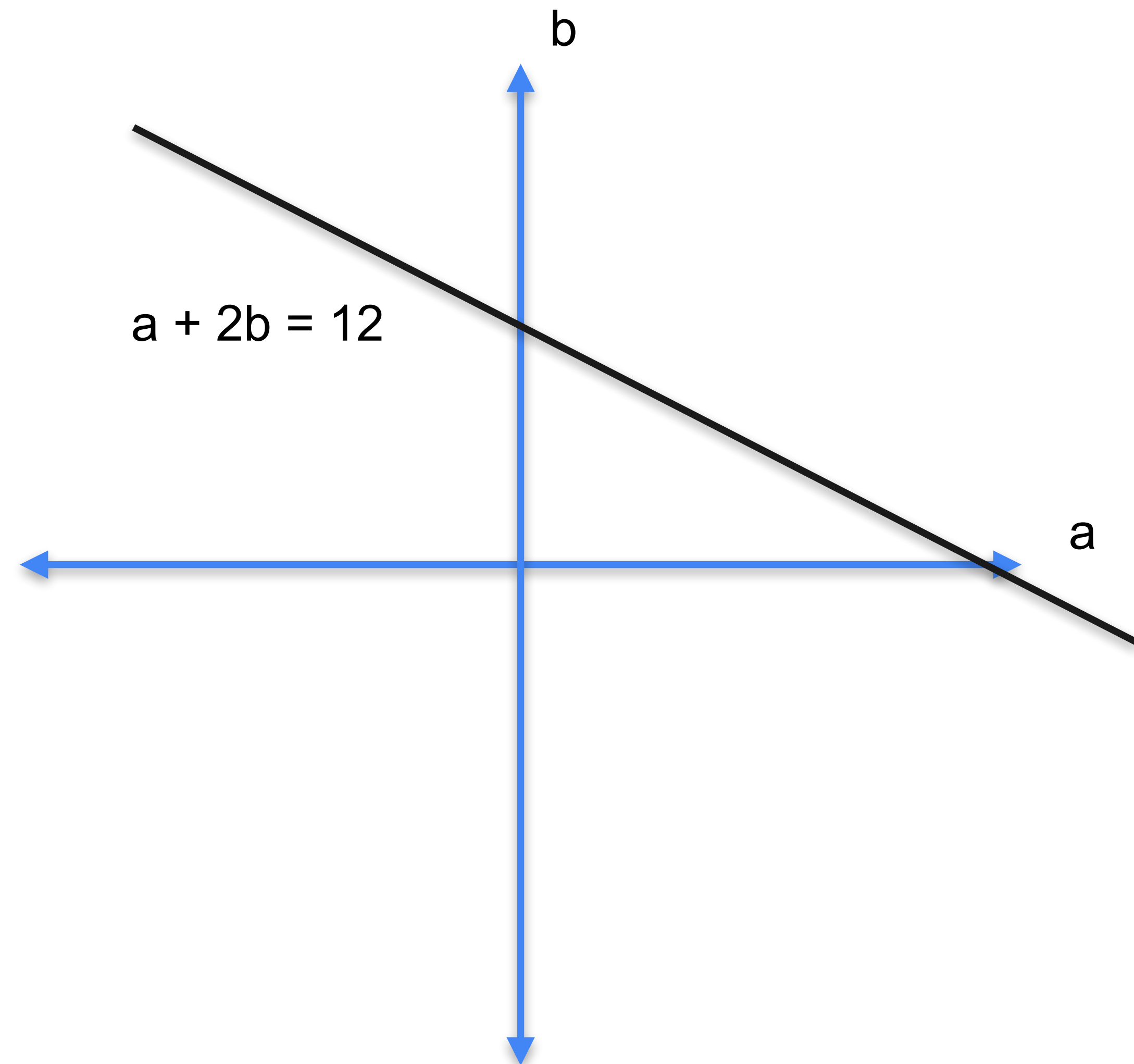
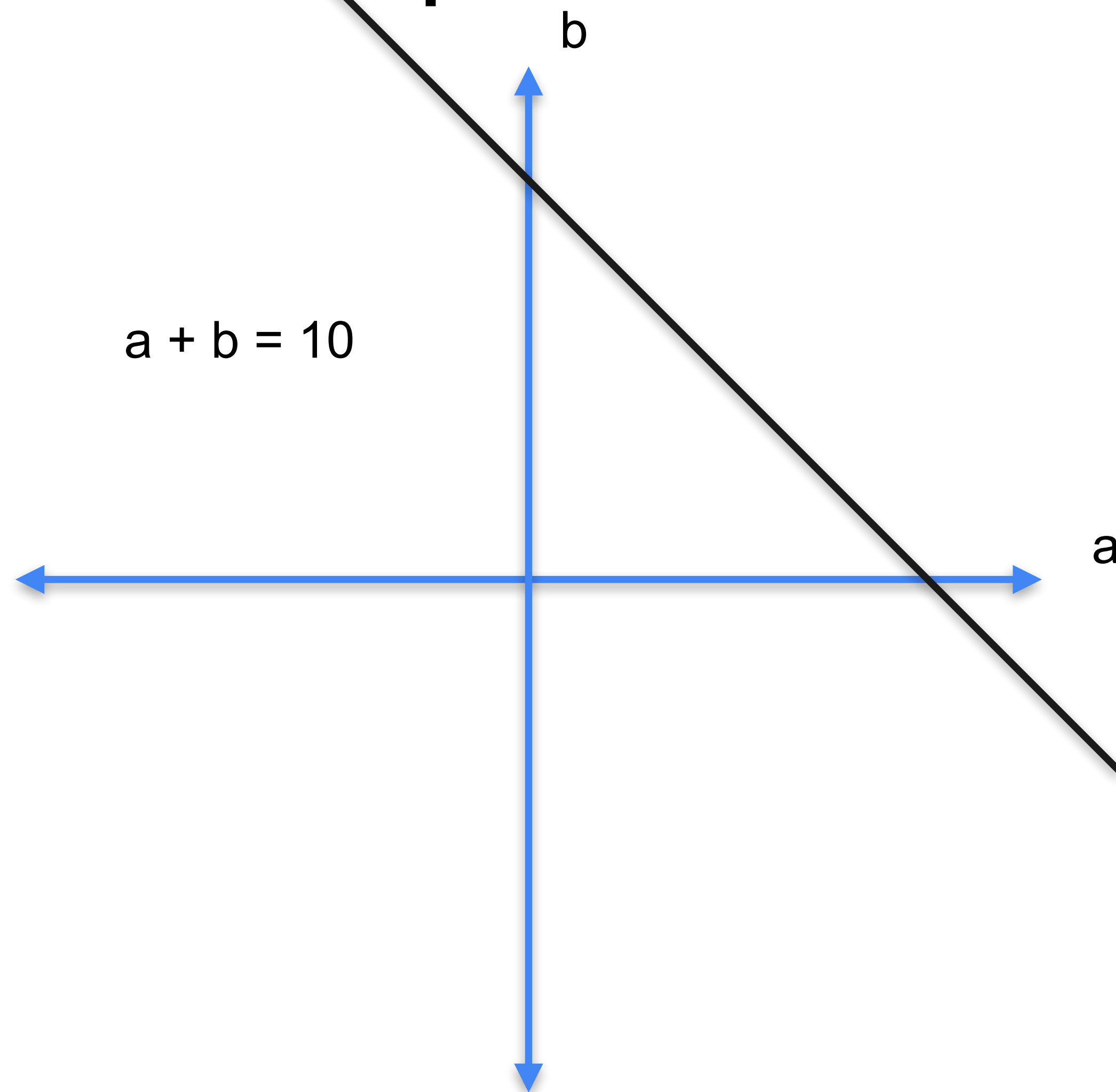
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**System of equations as lines  
and planes**

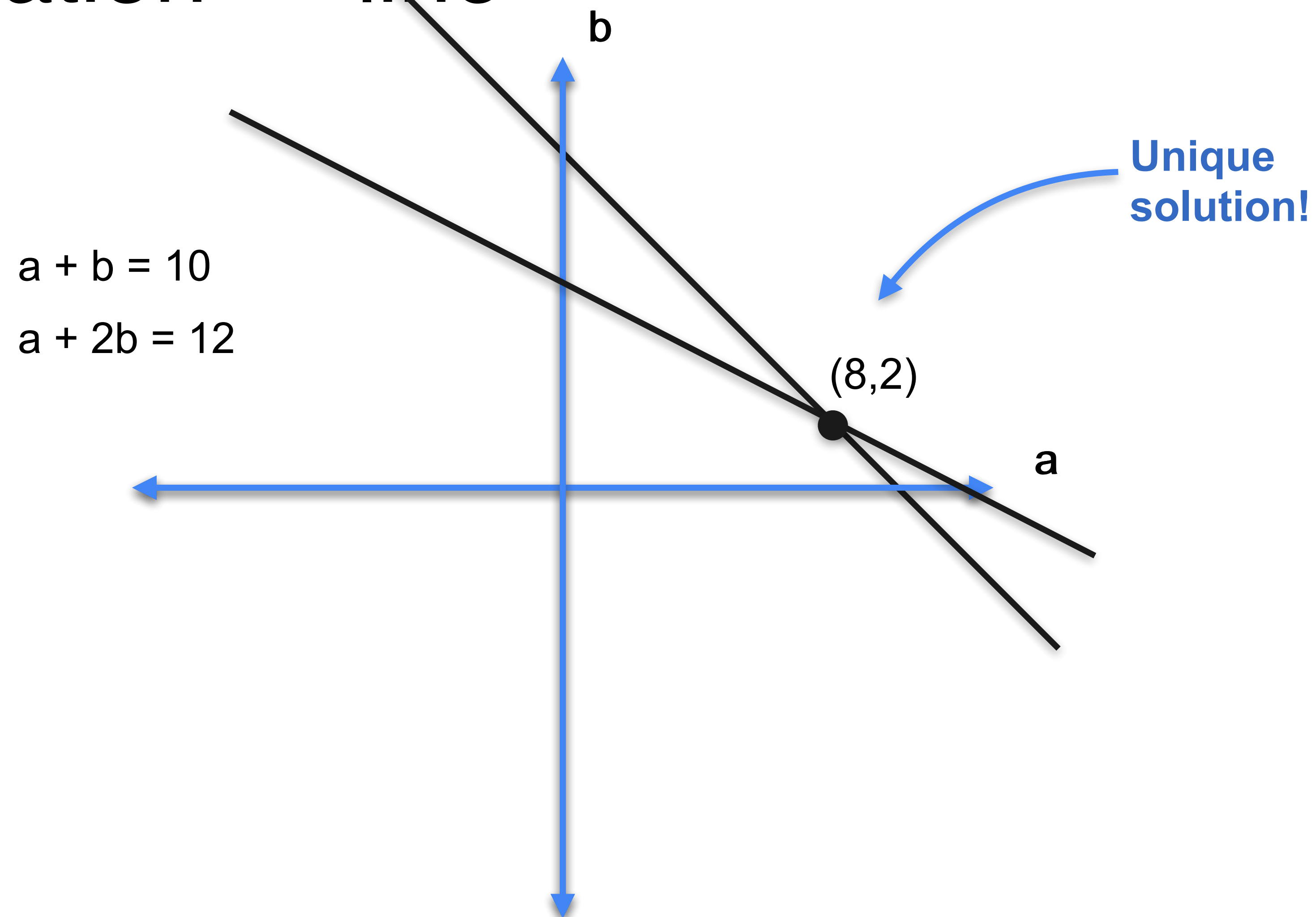
# Linear equation $\rightarrow$ line



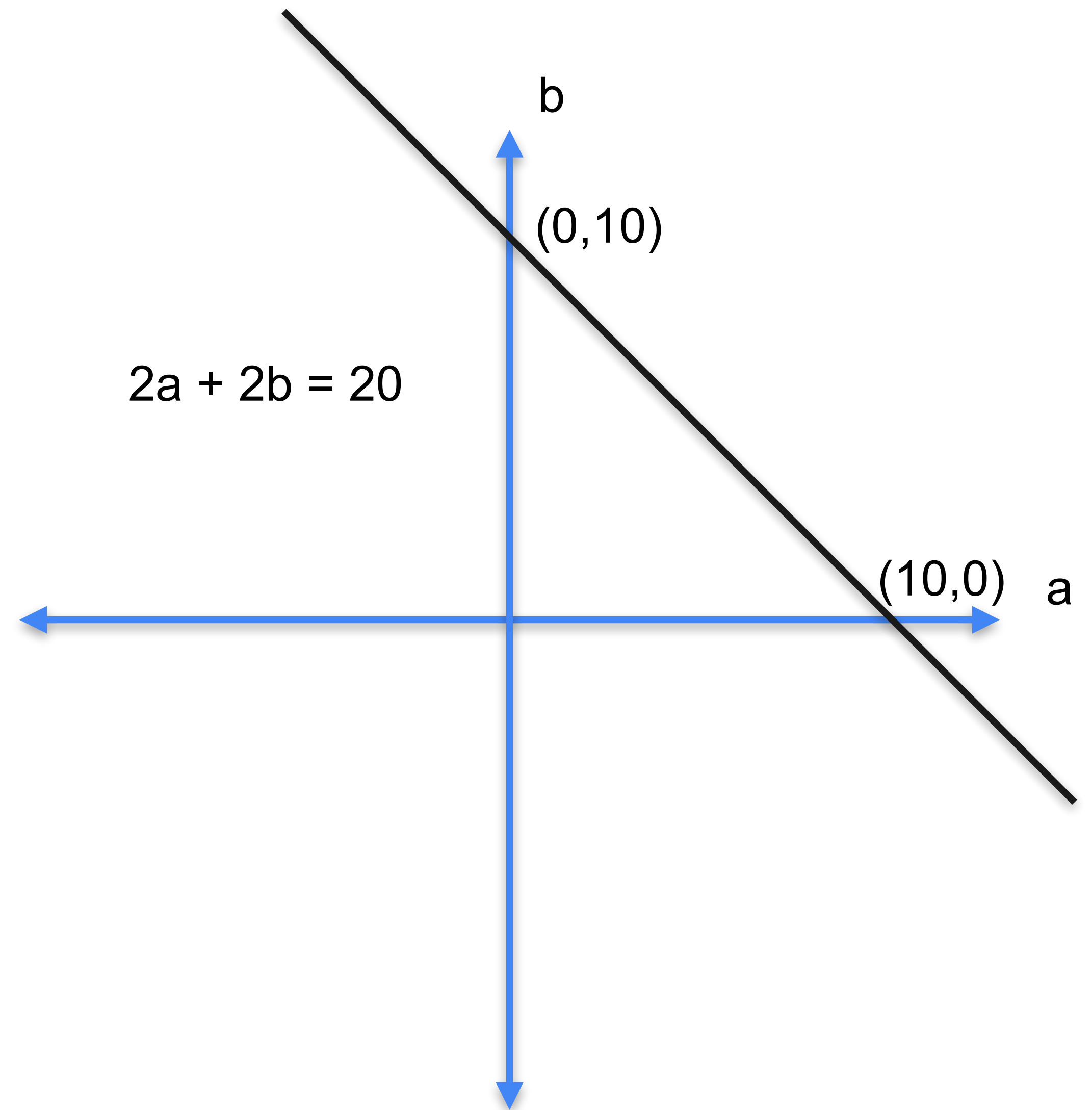
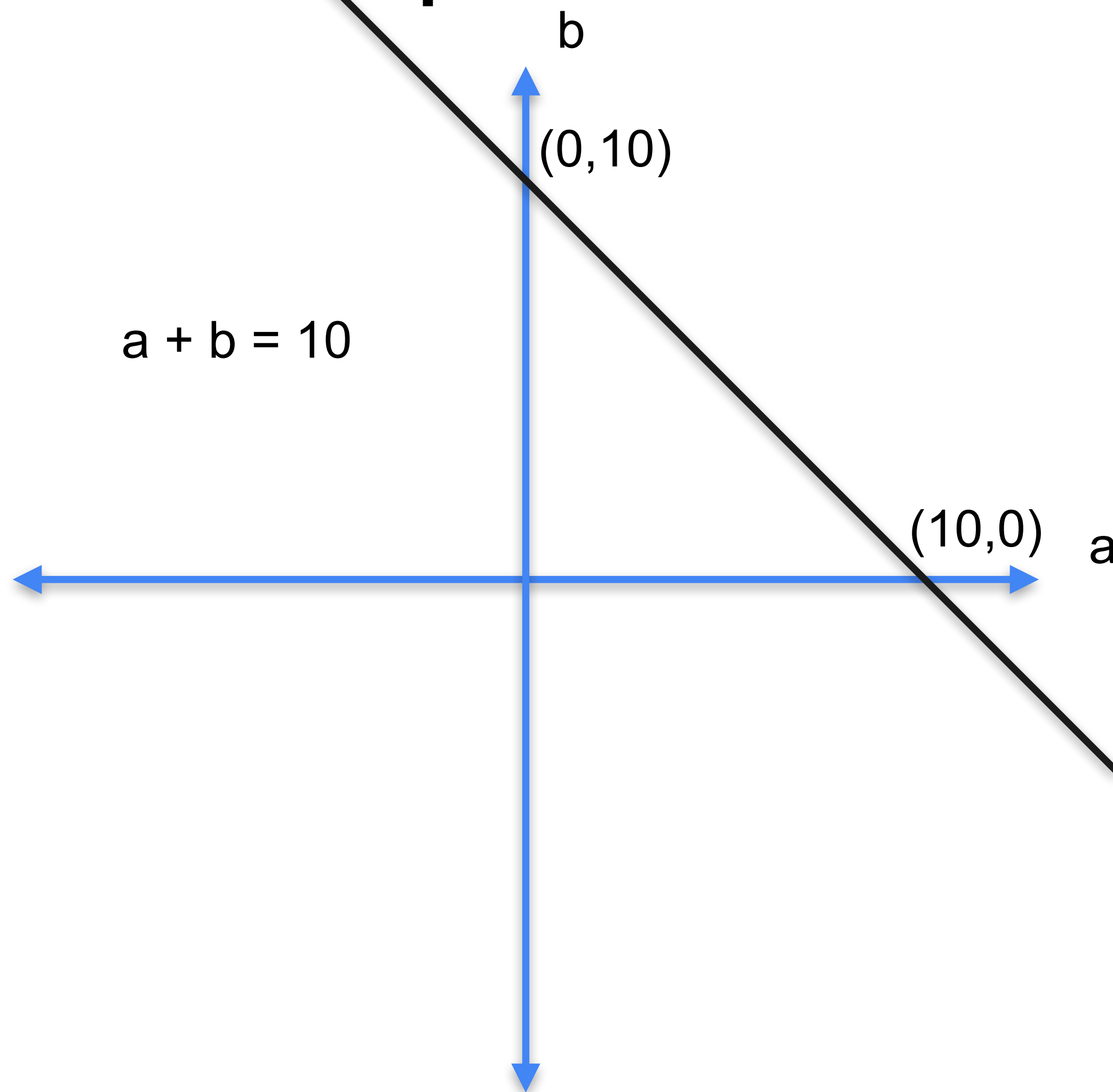
# Linear equation $\rightarrow$ line



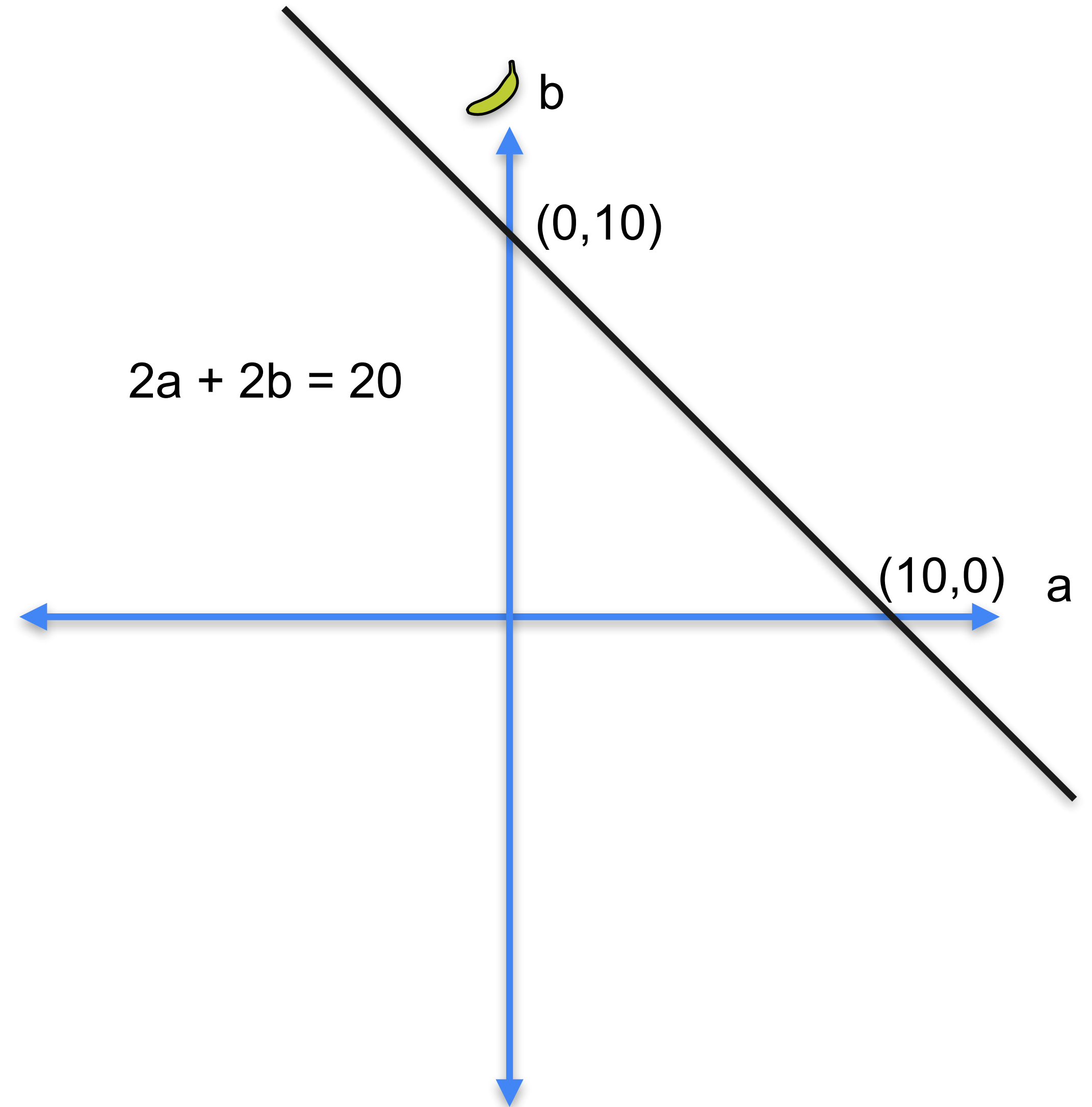
# Linear equation $\rightarrow$ line



# Linear equation $\rightarrow$ line

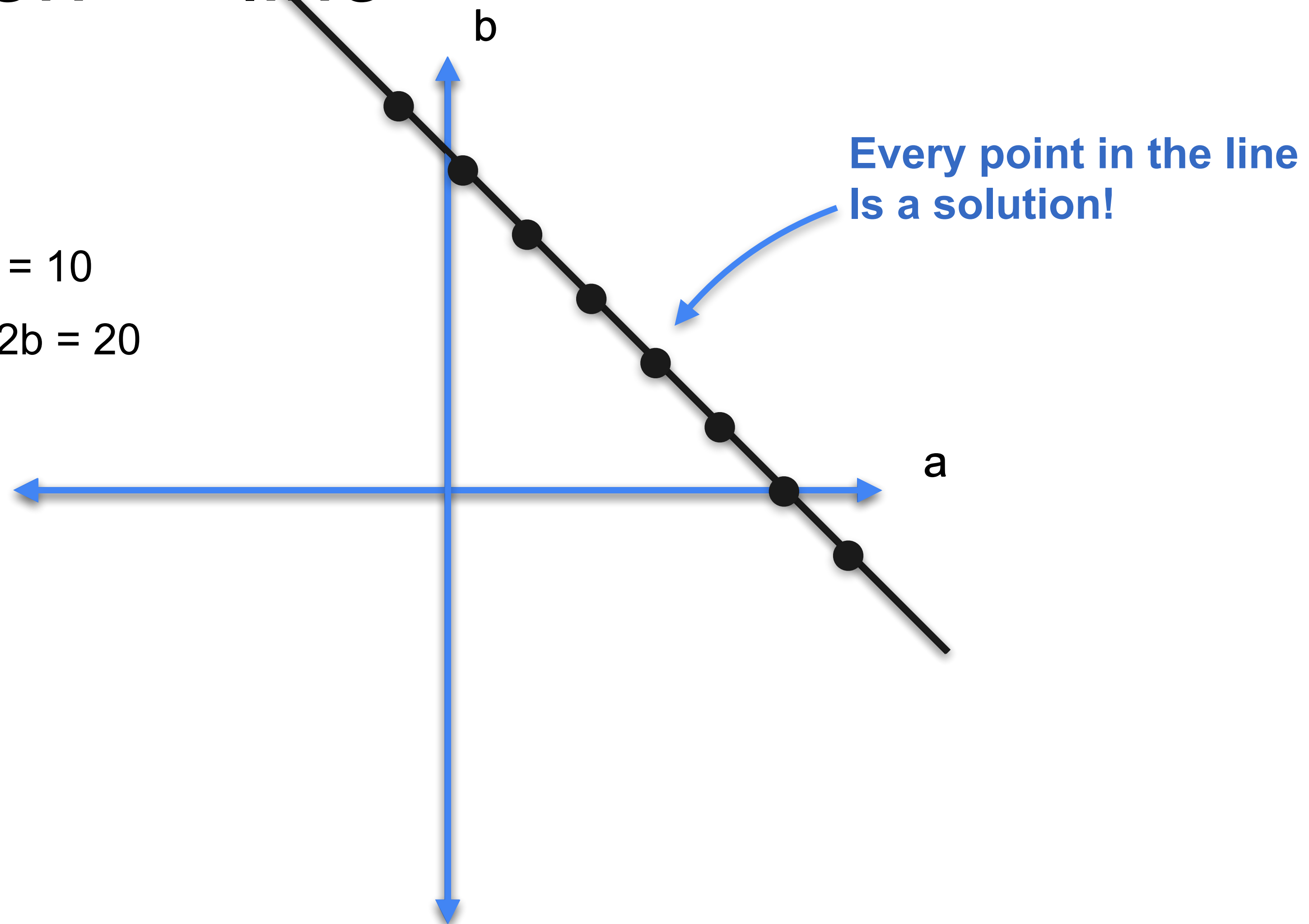


# Linear equation $\rightarrow$ line



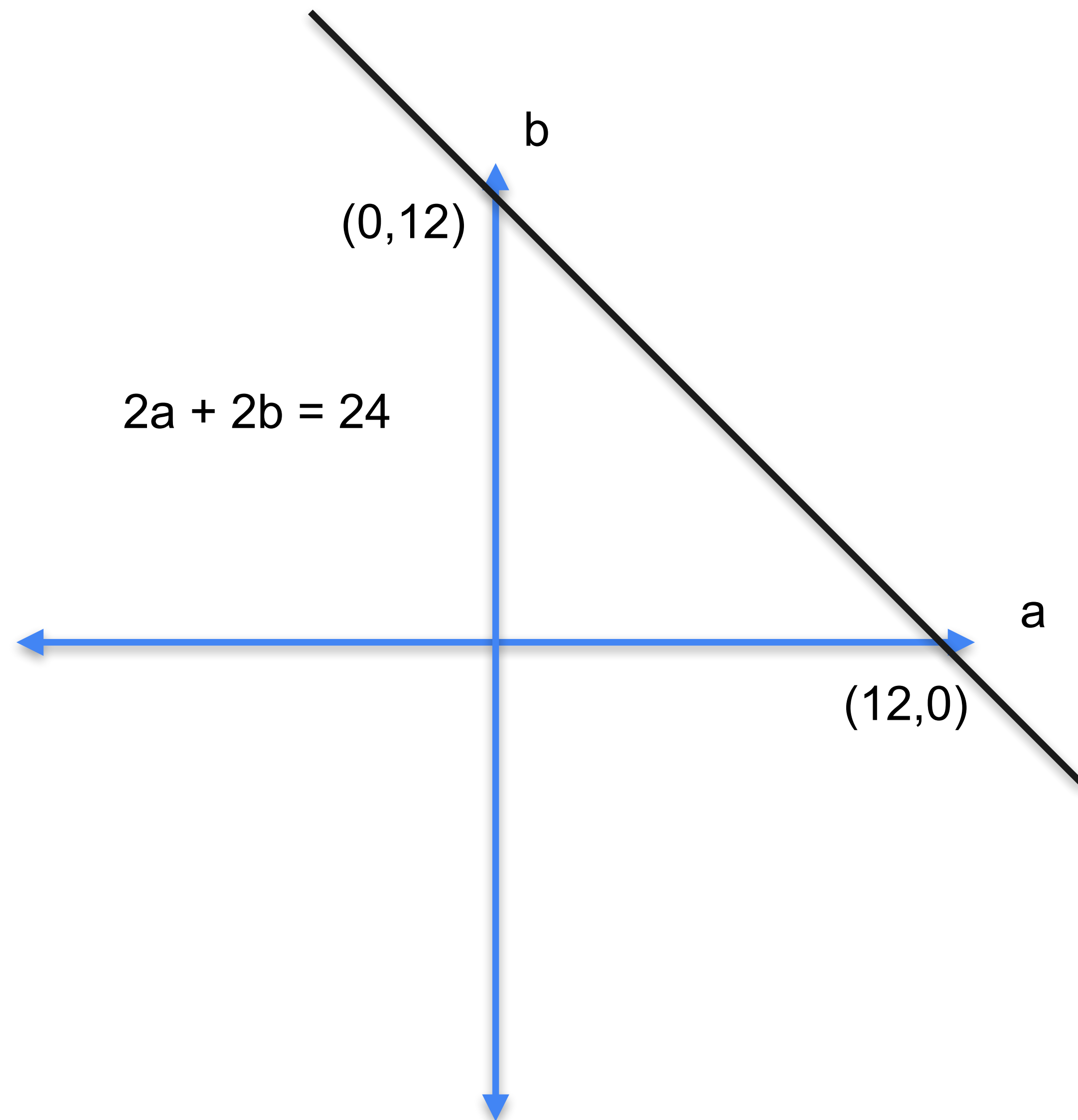
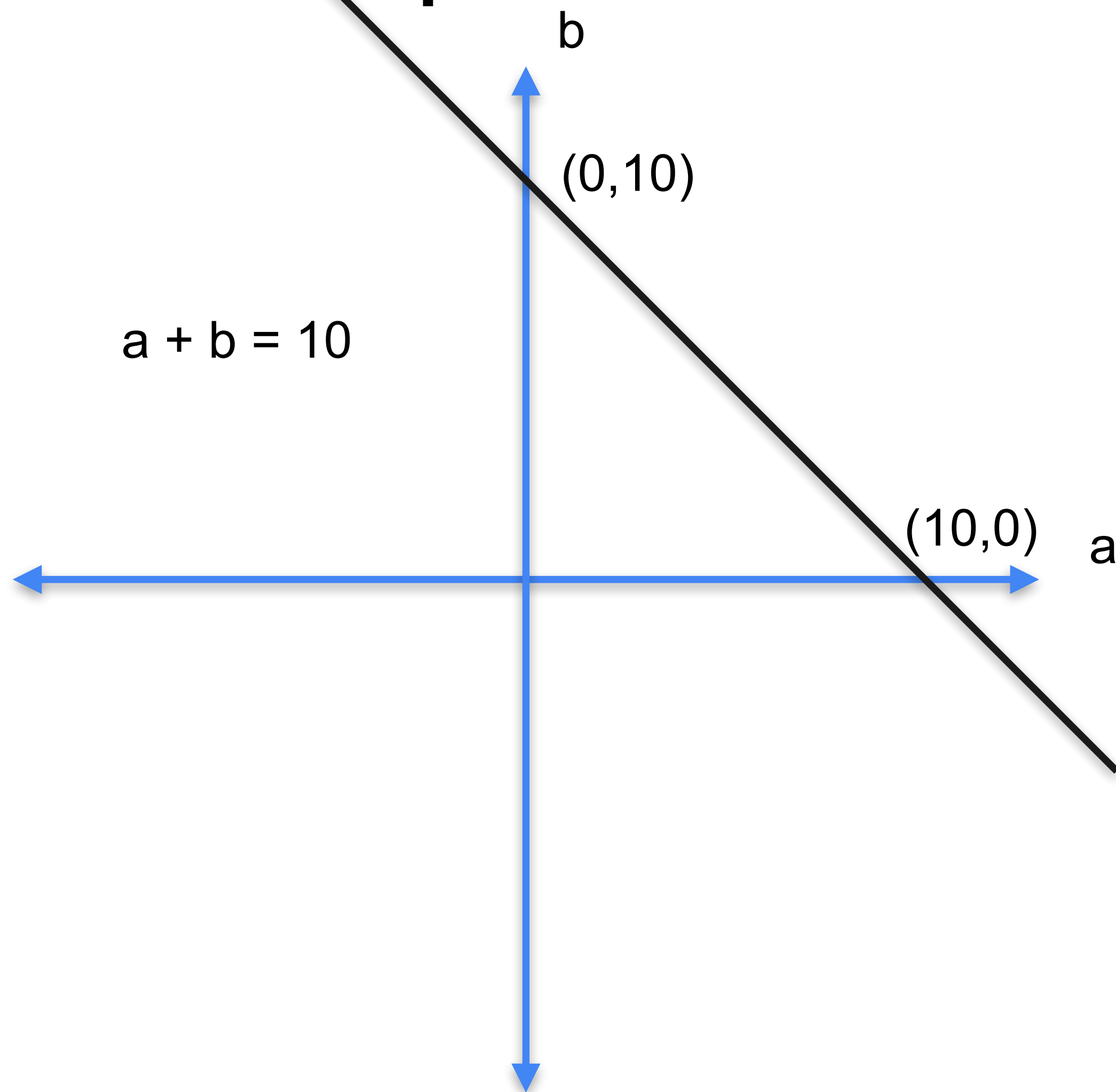
# Linear equation → line

$$a + b = 10$$
$$2a + 2b = 20$$

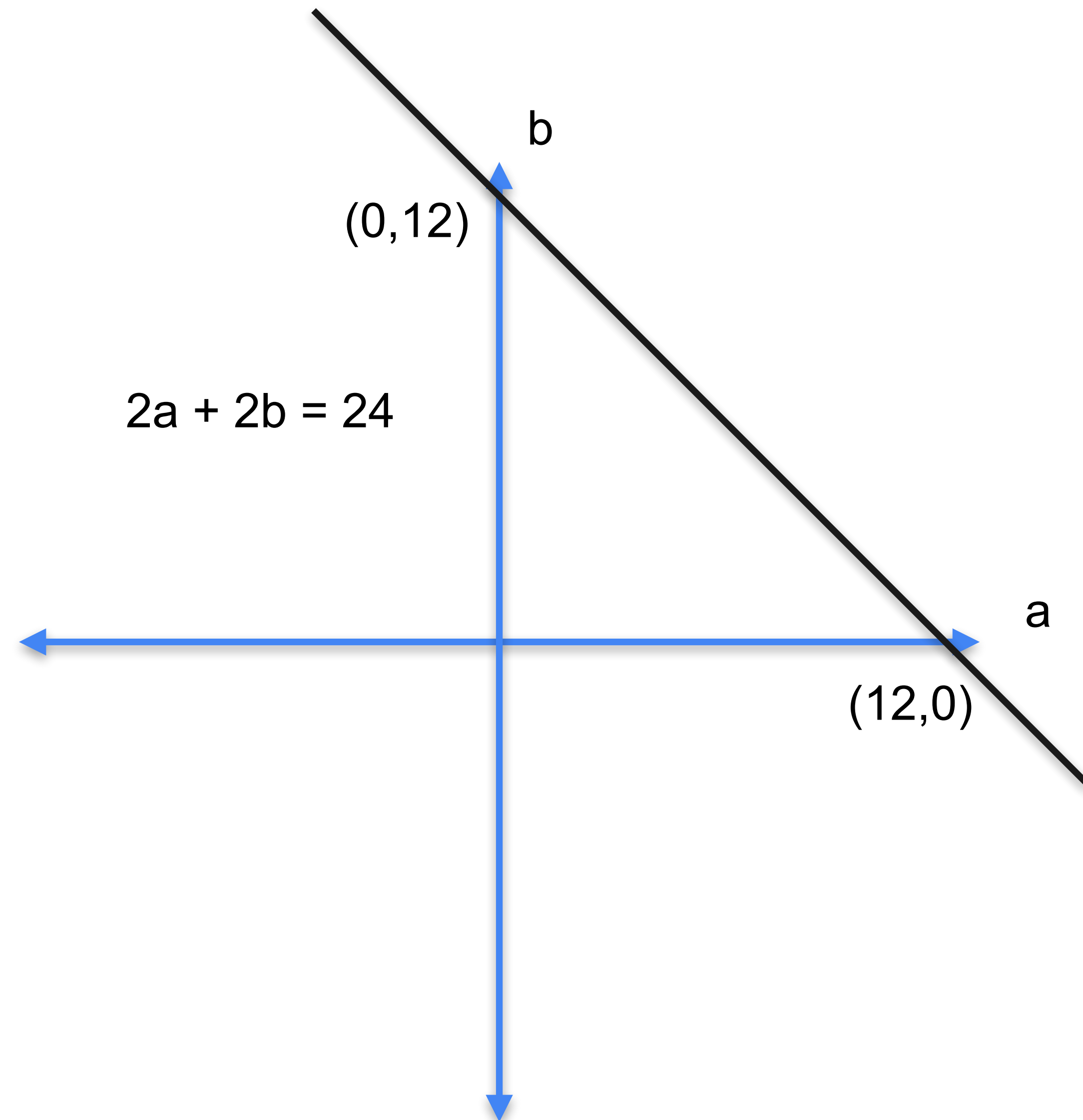




# Linear equation $\rightarrow$ line



# Linear equation $\rightarrow$ line



# Linear equation → line

$$a + b = 10$$

$$2a + 2b = 24$$



# Systems of equations as lines

System 1

$$a + b = 10$$

$$a + 2b = 12$$



System 2

$$a + b = 10$$

$$2a + 2b = 20$$



System 3

$$a + b = 10$$

$$2a + 2b = 24$$



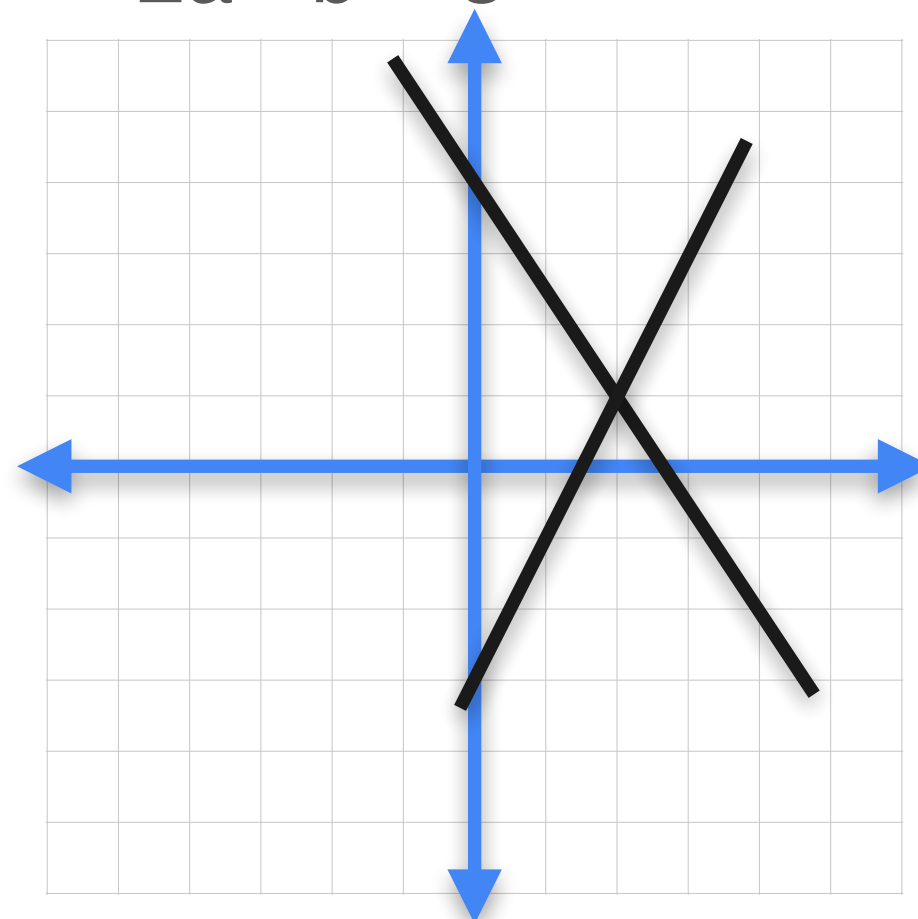
# Quiz

## Problem 1

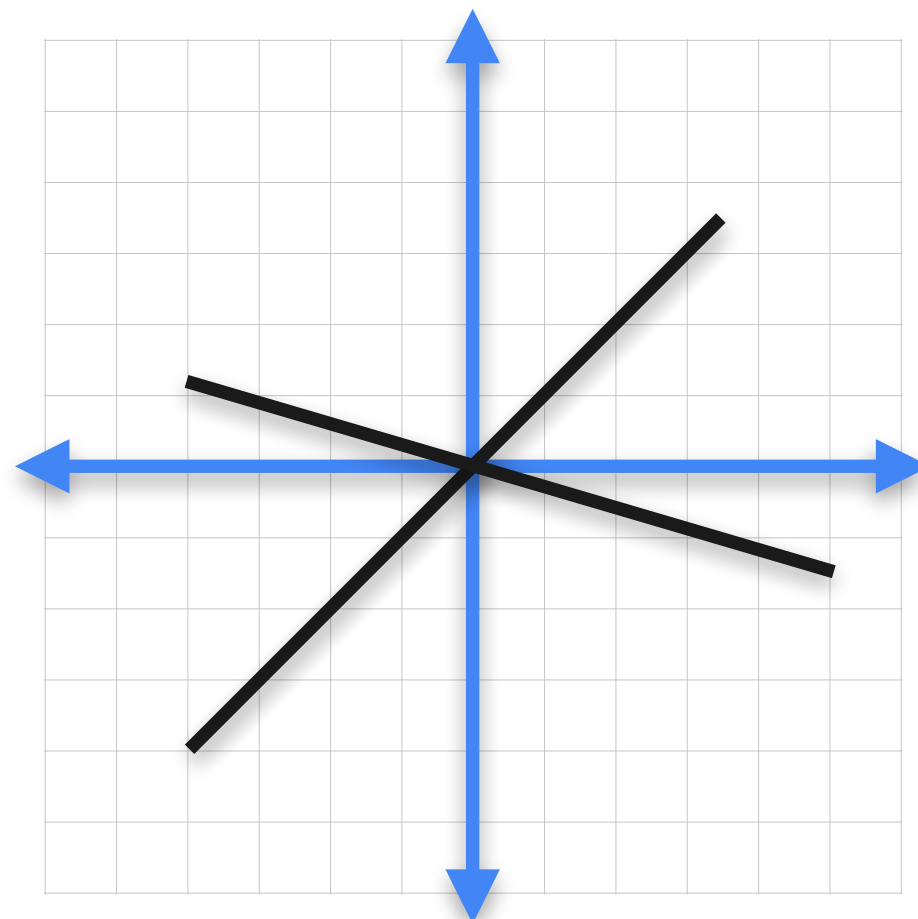
Which of the following plots corresponds to the system of equations:

- $3a + 2b = 8$
- $2a - b = 3$

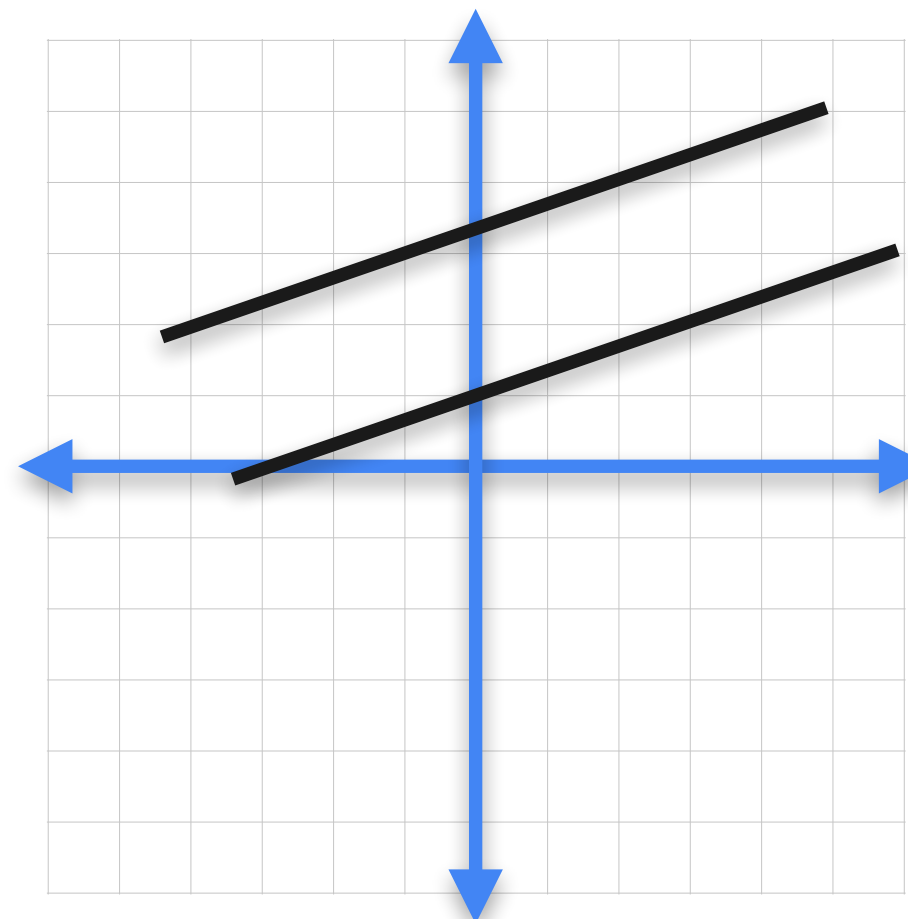
a)



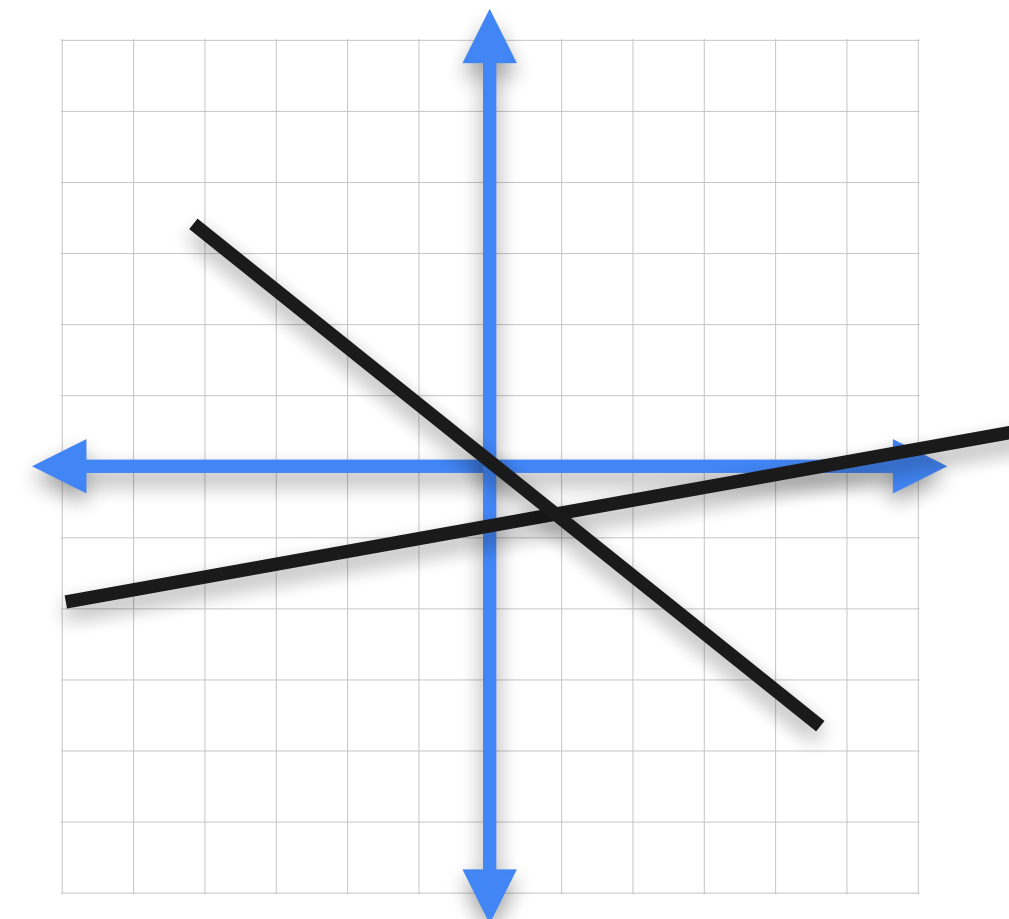
b)



c)



d)



## Problem 2

Is this system singular or non-singular?

# Solution

## Problem 1



## Problem 2

Since the lines cross at a unique point, the system is non-singular.

# Linear equation in 3 variables as a plane

$$a + b + c = 1$$

$$1 + 0 + 0 = 1$$

$$0 + 1 + 0 = 1$$

$$0 + 0 + 1 = 1$$



# Linear equation in 3 variables as a plane

$$3a - 5b + 2c = 0$$

$$3(0) + 5(0) + 2(0) = 0$$

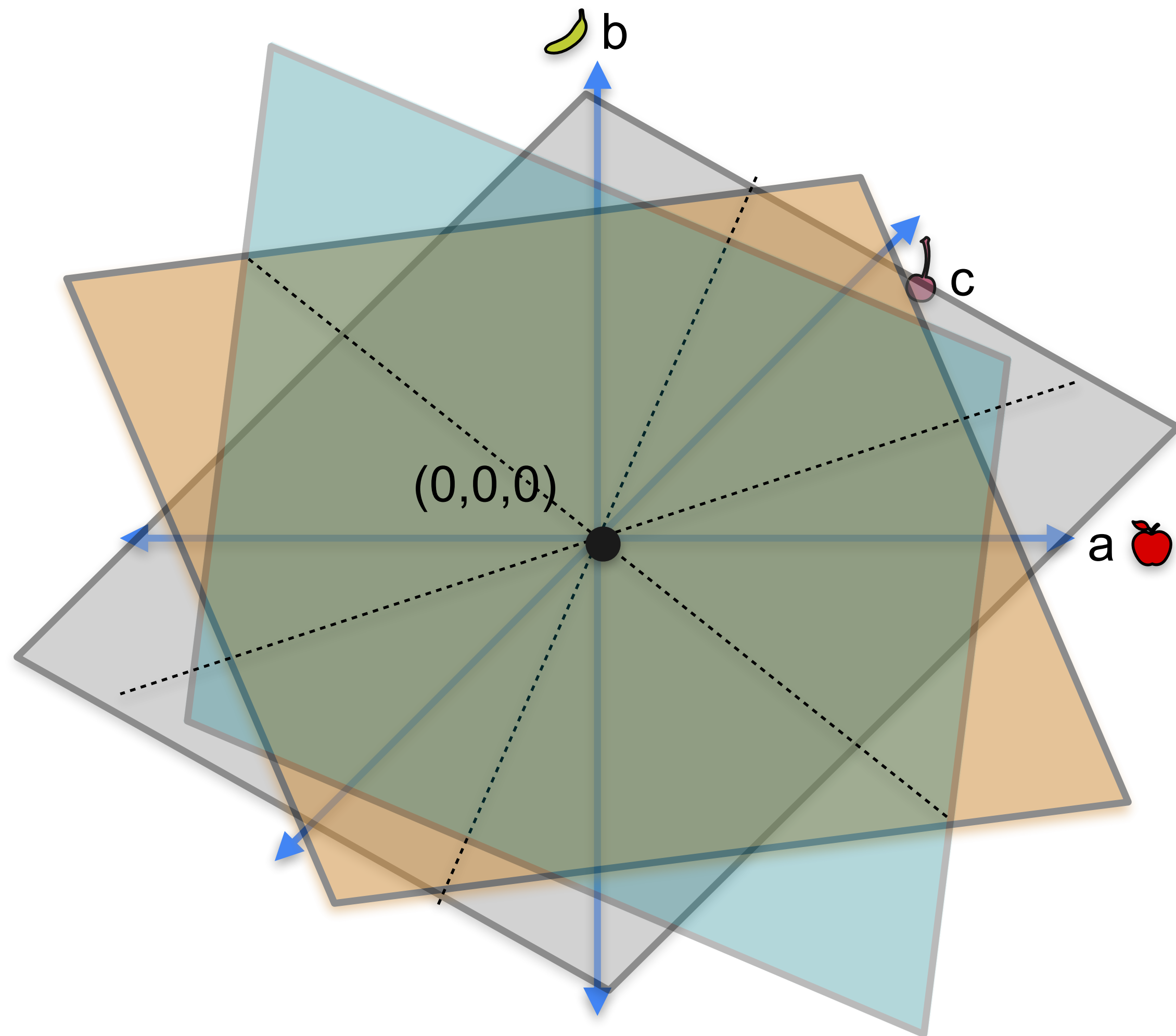
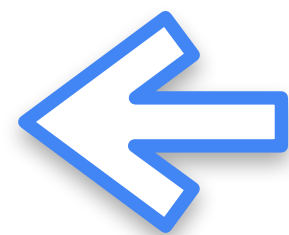




# System 1

## System 1

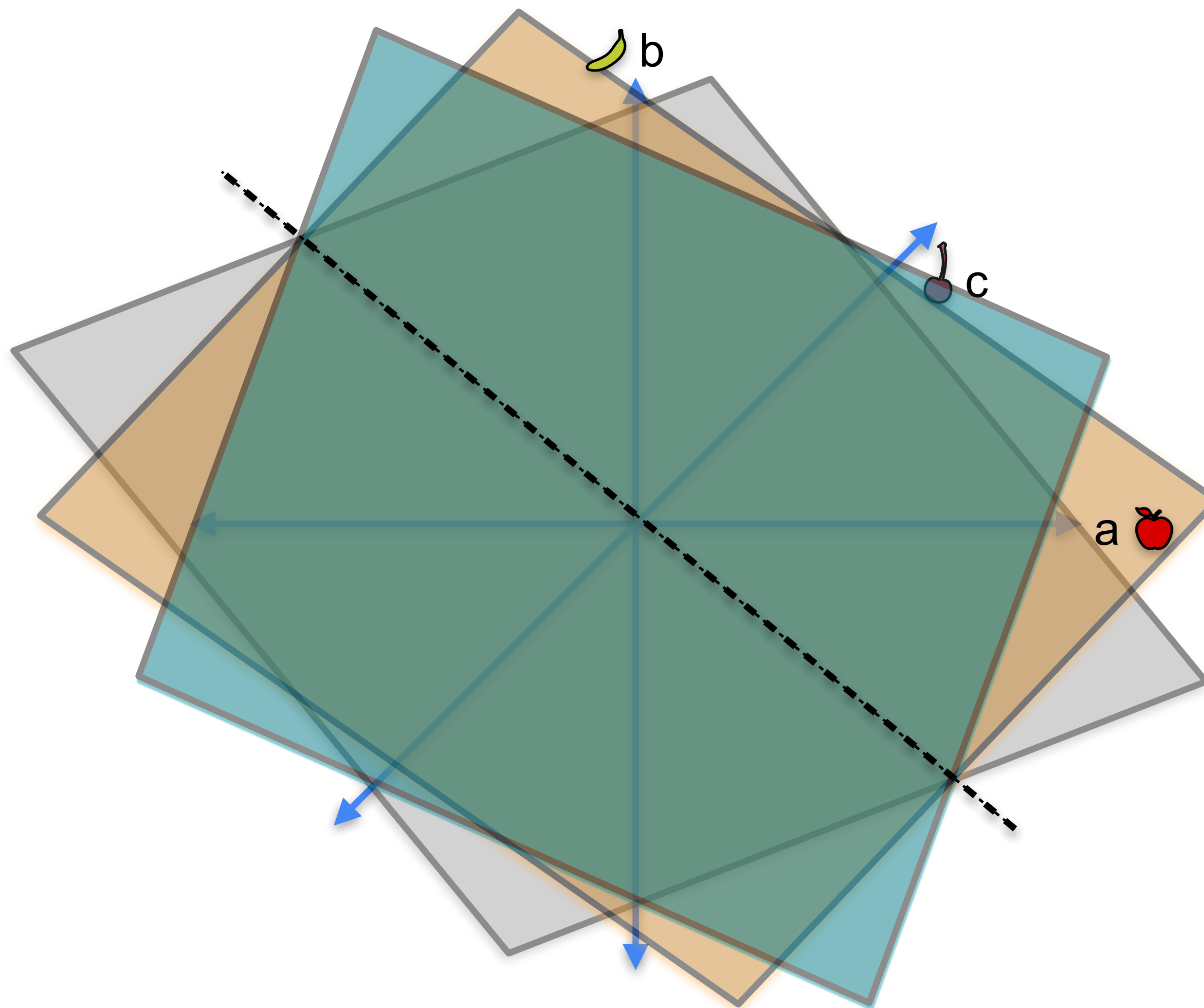
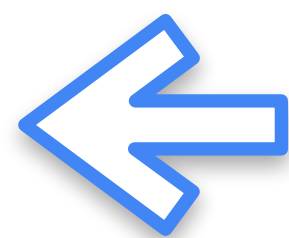
- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$



# System 2

## System 2

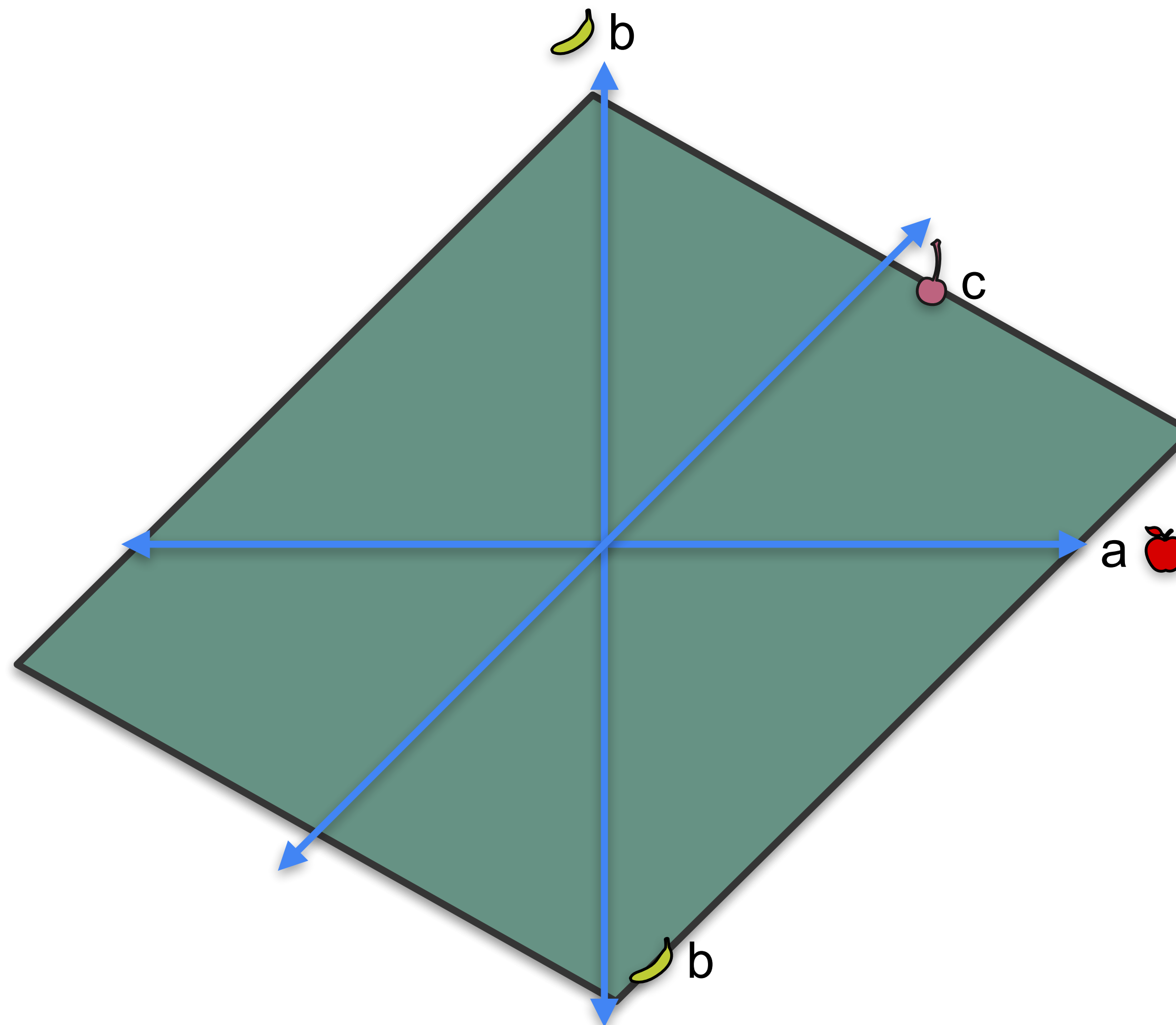
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



# System 3

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$





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# System of Linear Equations

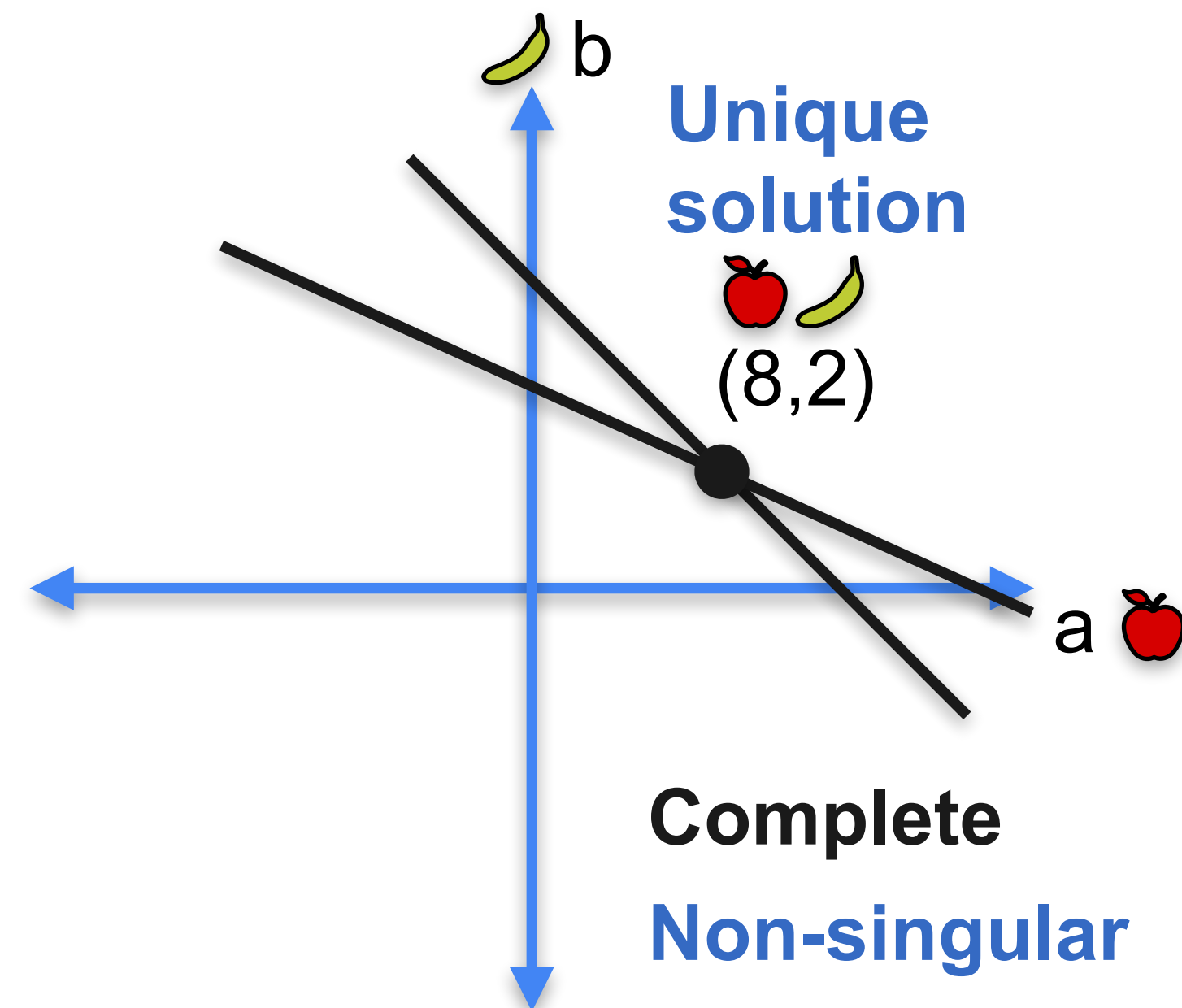
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**A geometric notion of  
singularity**

# Systems of equations as lines

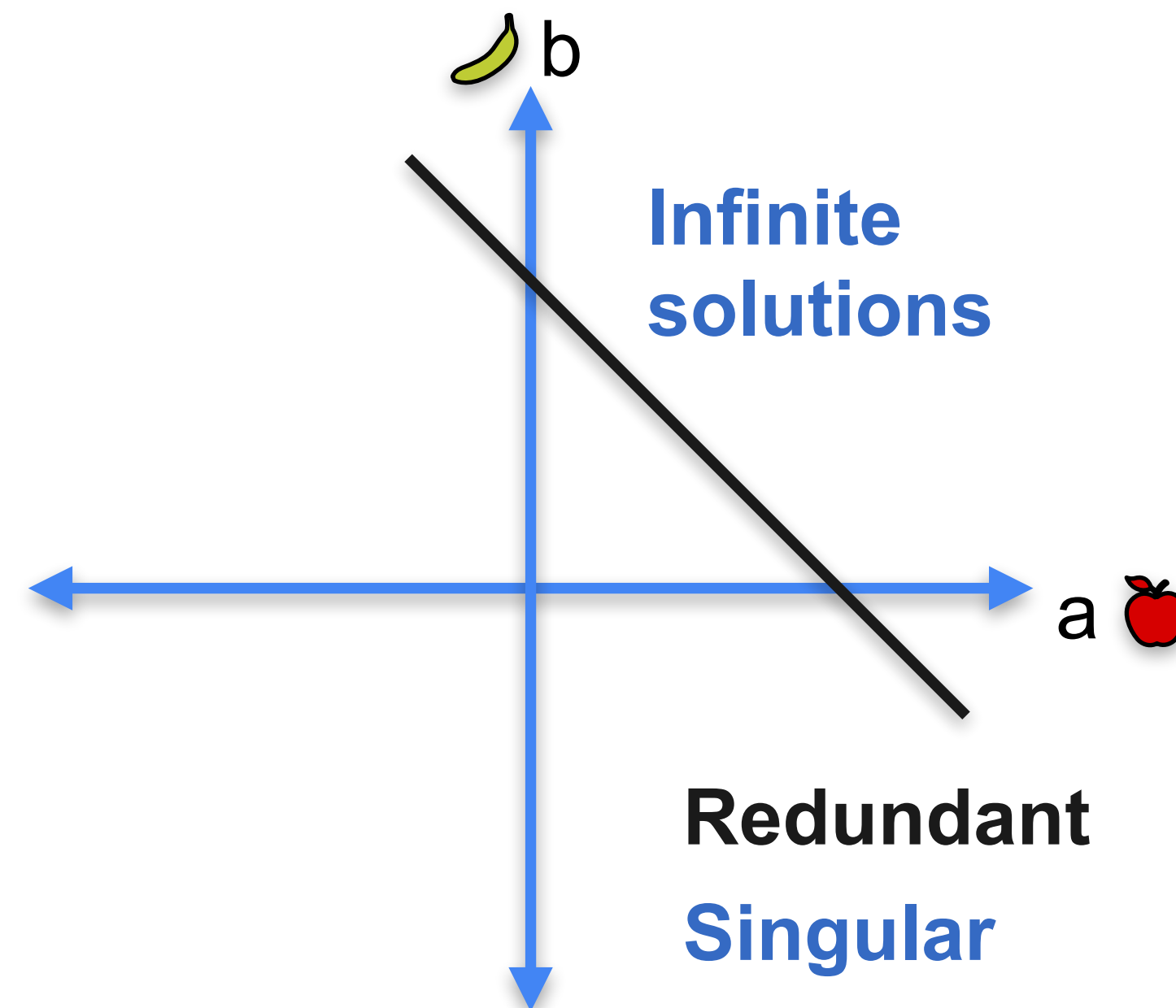
System 1

$$\begin{array}{l} a + b = 10 \\ a + 2b = 12 \end{array}$$



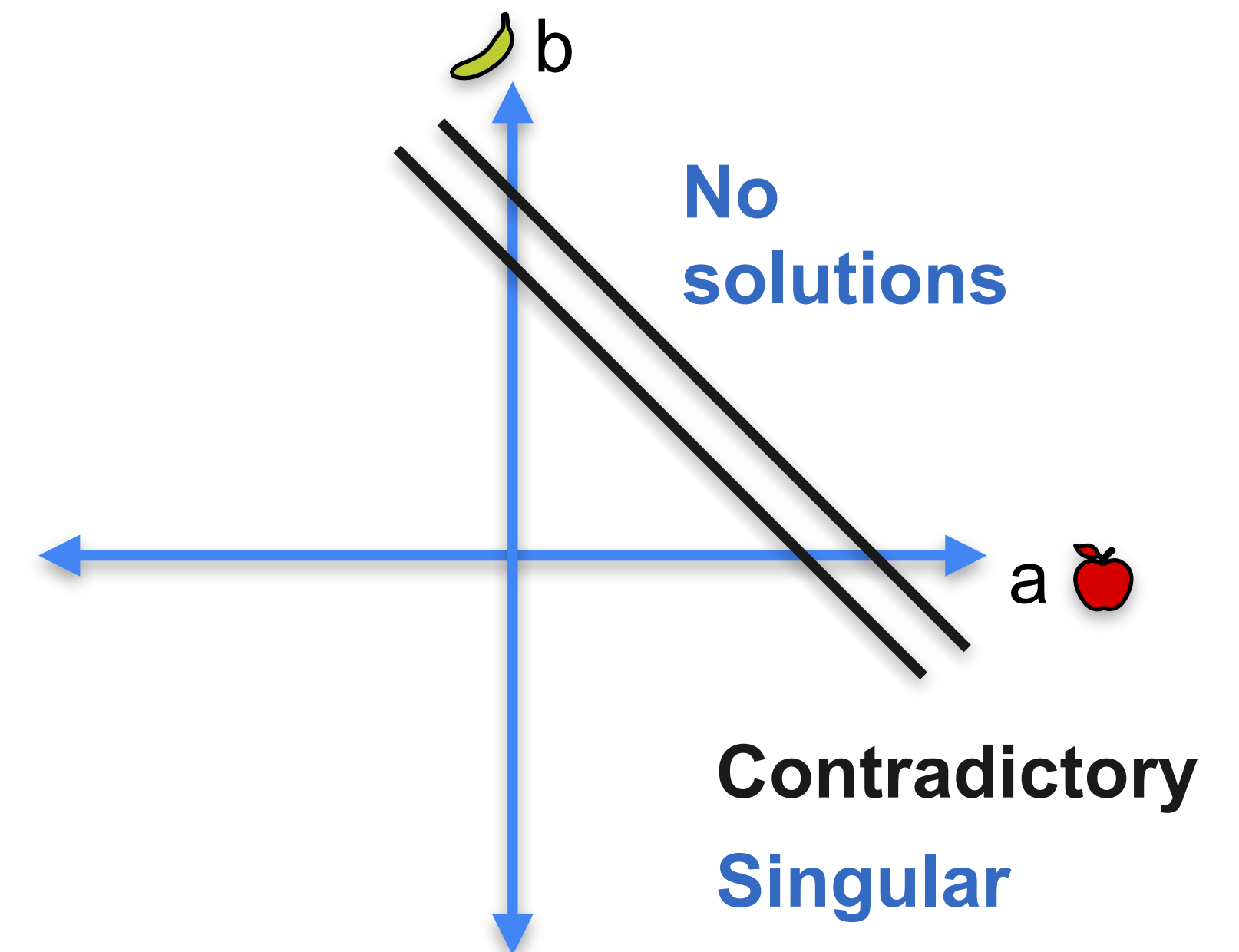
System 2

$$\begin{array}{l} a + b = 10 \\ 2a + 2b = 20 \end{array}$$



System 3

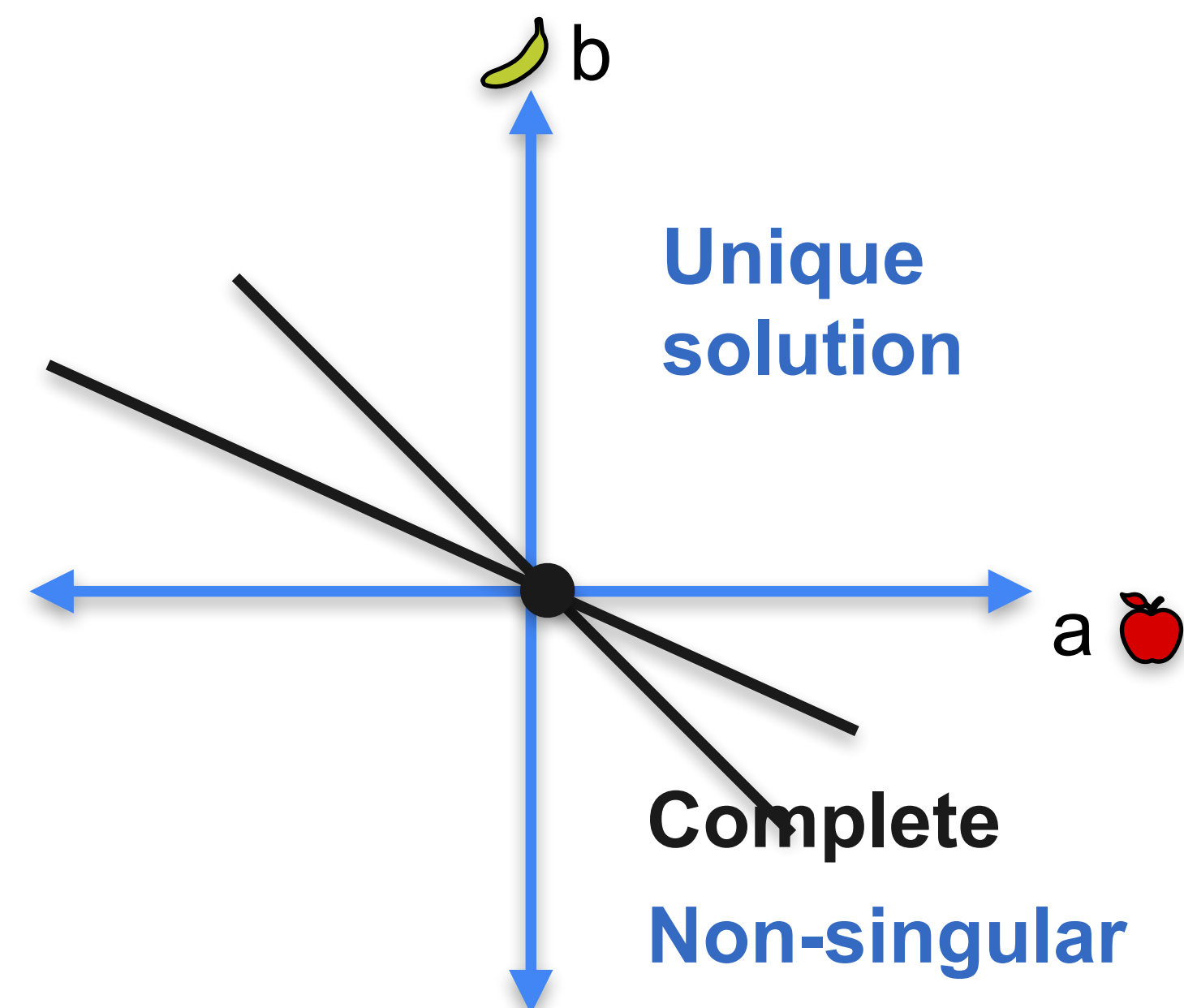
$$\begin{array}{l} a + b = 10 \\ 2a + 2b = 24 \end{array}$$



# Systems of equations as lines

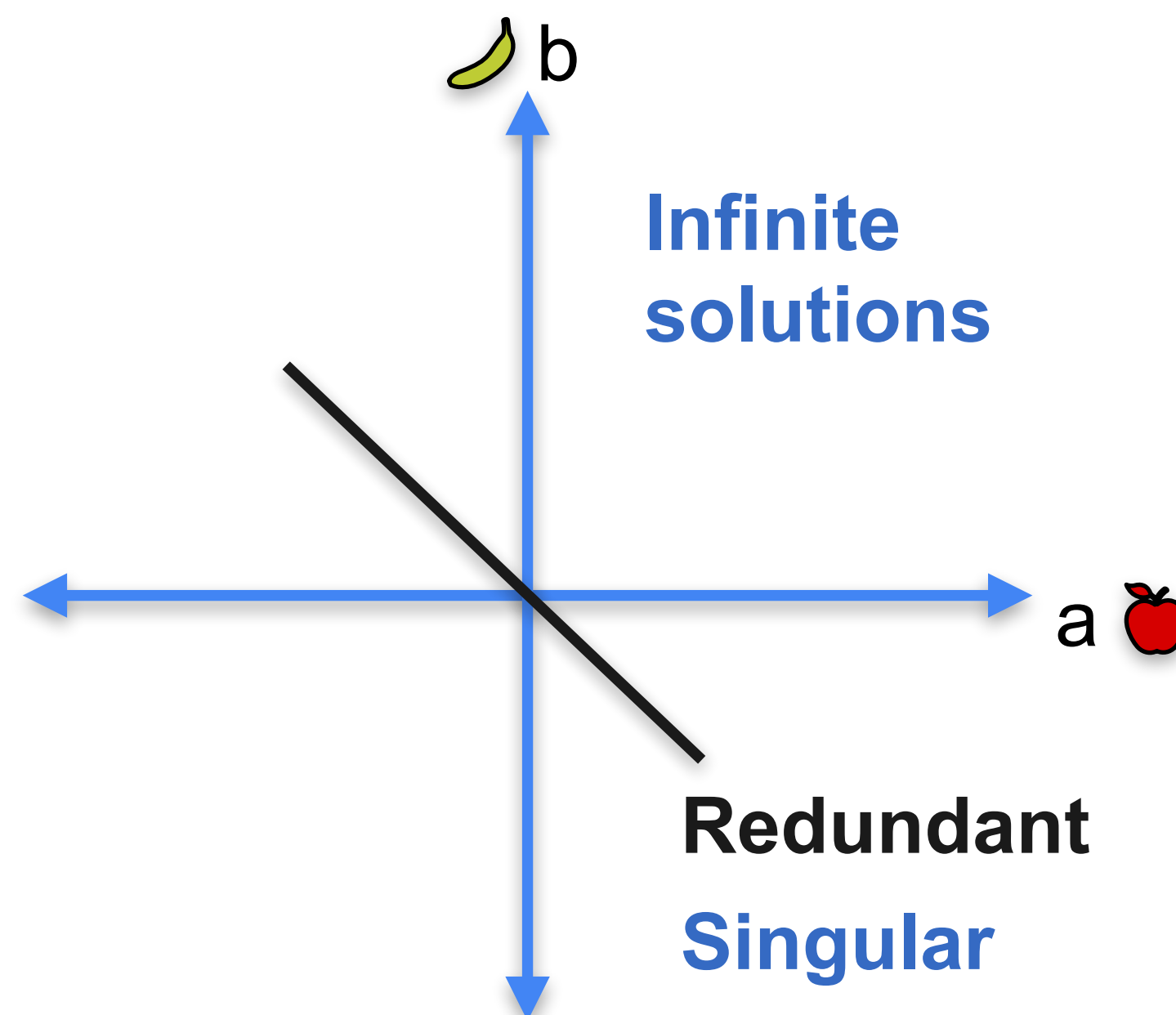
System 1

$$\begin{array}{l} a + b = 0 \\ a + 2b = 0 \end{array}$$



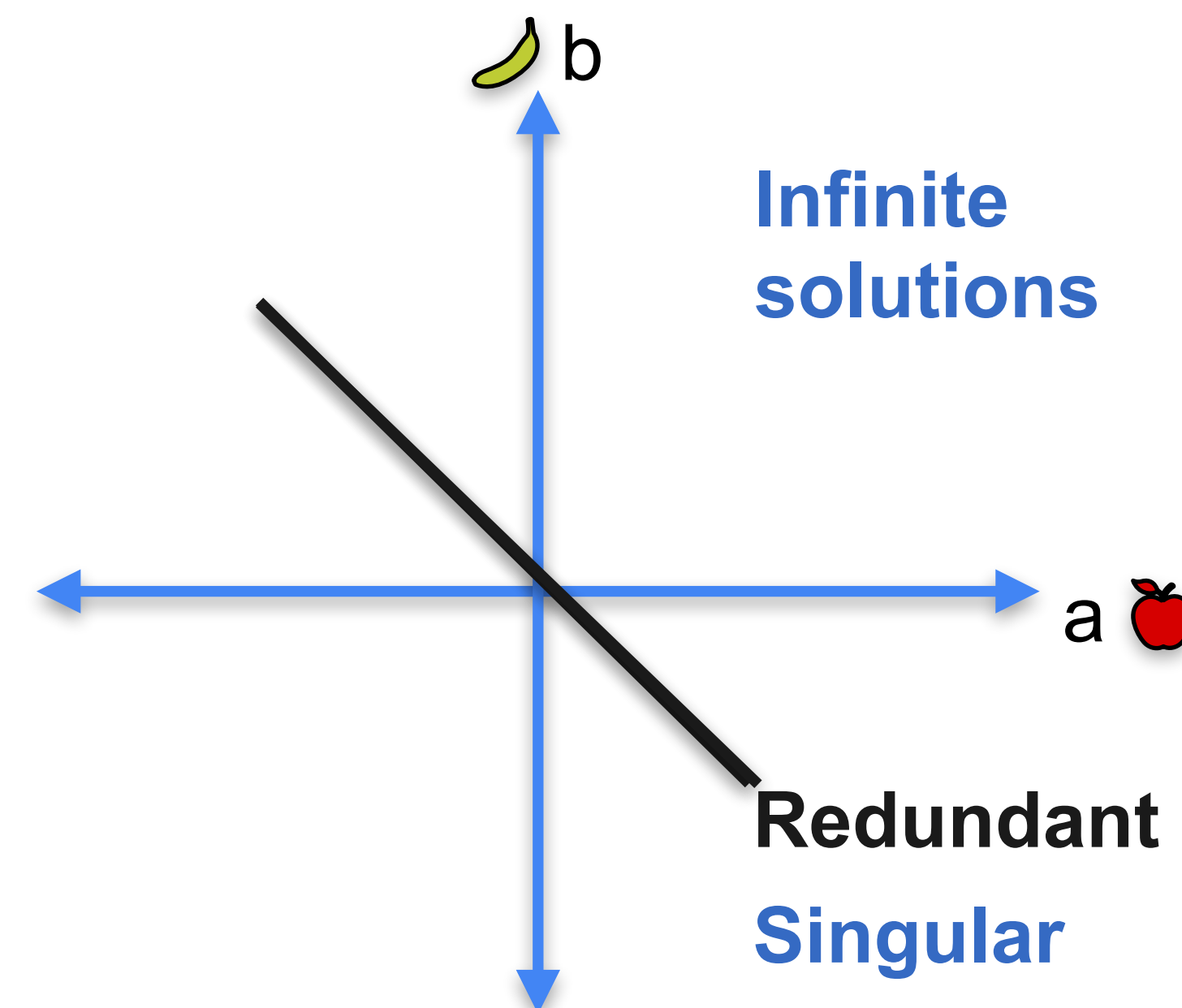
System 2

$$\begin{array}{l} a + b = 0 \\ 2a + 2b = 0 \end{array}$$



System 3

$$\begin{array}{l} a + b = 0 \\ 2a + 2b = 0 \end{array}$$





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# System of Linear Equations

---

**Singular vs non-singular  
matrices**

# Systems of equations as matrices

## System 1

$$\begin{array}{c} a + b = 0 \\ \text{🍏} + \text{🍌} \end{array}$$

$$\begin{array}{c} a + 2b = 0 \\ \text{🍏} + \text{🍌🍌} \end{array}$$

**Non-singular  
system**

| 🍏 | 🍌 |
|---|---|
| 1 | 1 |
| 1 | 2 |

**Non-singular  
matrix**

**(Unique solution)**

## System 2

$$\begin{array}{c} a + b = 0 \\ \text{🍏} + \text{🍌} \end{array}$$

$$\begin{array}{c} 2a + 2b = 0 \\ \text{🍏🍏} + \text{🍌🍌} \end{array}$$

**Singular  
system**

| 🍏 | 🍌 |
|---|---|
| 1 | 1 |
| 2 | 2 |

**Singular  
matrix**

**(Infinitely many solutions)**



# Constants don't matter for singularity

## System 1

$$\begin{aligned}a + b + c &= 10 \\a + 2b + c &= 15 \\a + b + 2c &= 12\end{aligned}$$

Unique solution

Complete

Non-singular

## System 2

$$\begin{aligned}a + b + c &= 10 \\a + b + 2c &= 15 \\a + b + 3c &= 20\end{aligned}$$

Infinite solutions

Redundant

Singular

## System 3

$$\begin{aligned}a + b + c &= 10 \\a + b + 2c &= 15 \\a + b + 3c &= 18\end{aligned}$$

No solutions

Contradictory

Singular

## System 4

$$\begin{aligned}a + b + c &= 10 \\2a + 2b + 2c &= 15 \\3a + 3b + 3c &= 20\end{aligned}$$

Infinite solutions

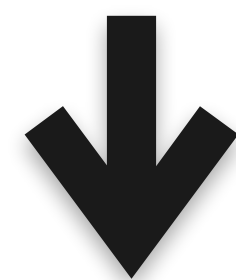
Redundant

Singular

# Constants don't matter for singularity

## System 1

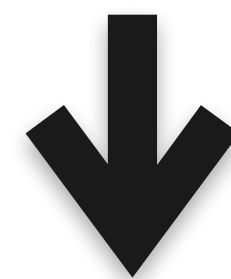
$$\begin{aligned}a + b + c &= 10 \\ a + 2b + c &= 15 \\ a + b + 2c &= 12\end{aligned}$$



$$\begin{aligned}a + b + c &= 0 \\ a + 2b + c &= 0 \\ a + b + 2c &= 0\end{aligned}$$

## System 2

$$\begin{aligned}a + b + c &= 10 \\ a + b + 2c &= 15 \\ a + b + 3c &= 20\end{aligned}$$



$$\begin{aligned}a + b + c &= 0 \\ a + b + 2c &= 0 \\ a + b + 3c &= 0\end{aligned}$$

## System 3

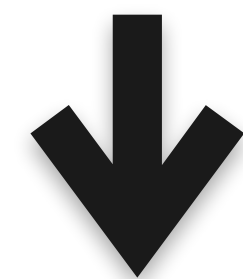
$$\begin{aligned}a + b + c &= 10 \\ a + b + 2c &= 15 \\ a + b + 3c &= 18\end{aligned}$$



$$\begin{aligned}a + b + c &= 0 \\ a + b + 2c &= 0 \\ a + b + 3c &= 0\end{aligned}$$

## System 4

$$\begin{aligned}a + b + c &= 10 \\ 2a + 2b + 2c &= 20 \\ 3a + 3b + 3c &= 30\end{aligned}$$



$$\begin{aligned}a + b + c &= 0 \\ 2a + 2b + 2c &= 0 \\ 3a + 3b + 3c &= 0\end{aligned}$$

# Constants don't matter for singularity

## System 1

$$a + b + c = 0$$

$$a + 2b + c = 0$$

$$a + b + 2c = 0$$

## System 2

$$a + b + c = 0$$

$$a + b + 2c = 0$$

$$a + b + 3c = 0$$

## System 3

$$a + b + c = 0$$

$$a + b + 2c = 0$$

$$a + b + 3c = 0$$

## System 4

$$a + b + c = 0$$

$$2a + 2b + 2c = 0$$

$$3a + 3b + 3c = 0$$

# Constants don't matter for singularity

System 1

$$\begin{aligned}a + b + c &= 0 \\a + 2b + c &= 0 \\a + b + 2c &= 0\end{aligned}$$

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |

Non-singular

System 2

$$\begin{aligned}a + b + c &= 0 \\a + b + 2c &= 0 \\a + b + 3c &= 0\end{aligned}$$

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 1 | 2 |
| 1 | 1 | 3 |

Singular

System 3

$$\begin{aligned}a + b + c &= 0 \\a + b + 2c &= 0 \\a + b + 3c &= 0\end{aligned}$$

System 4

$$\begin{aligned}a + b + c &= 0 \\2a + 2b + 2c &= 0 \\3a + 3b + 3c &= 0\end{aligned}$$

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 3 | 3 |

Singular



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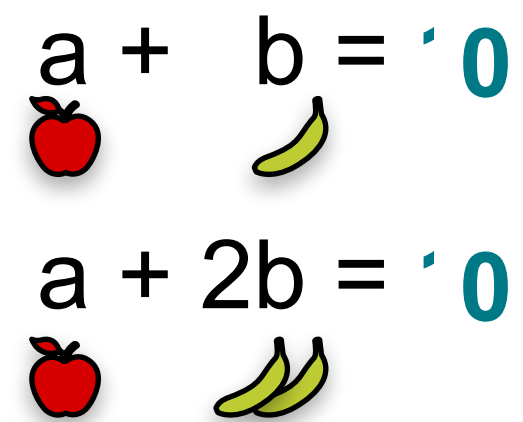
# System of Linear Equations

---

**Linear dependence and  
independence**

# Linear dependence between rows

## Non-singular

$$\begin{array}{l} a + b = 0 \\ a + 2b = 0 \end{array}$$


No equation is a multiple of the other one

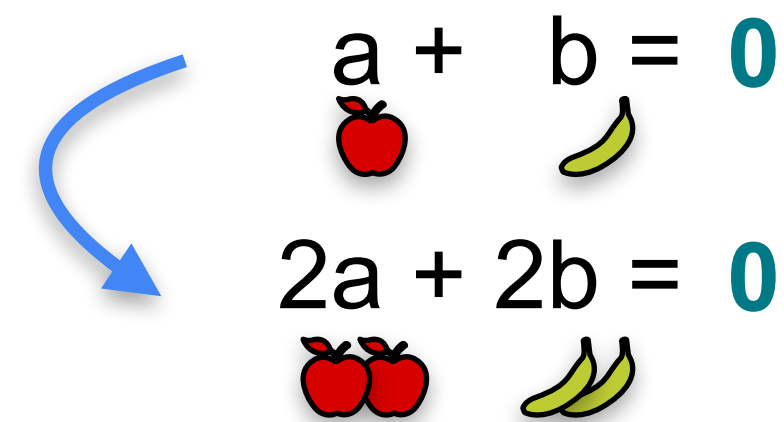
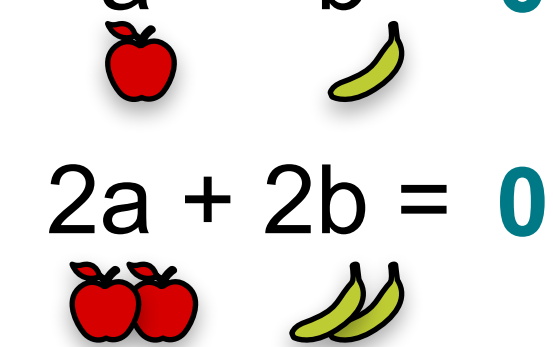


|   |   |
|---|---|
| 1 | 1 |
| 1 | 2 |


No row is a multiple of the other one

Rows are  
*linearly independent*

## Singular system


$$\begin{array}{l} a + b = 0 \\ 2a + 2b = 0 \end{array}$$


Second equation is a multiple of the first one



|   |   |
|---|---|
| 1 | 1 |
| 2 | 2 |

Second row is a multiple of the first row

Rows are  
*linearly dependent*

# Linear dependence and independence

$$\begin{array}{l} a = 1 \\ b = 2 \\ a + b = 3 \end{array} \quad \begin{array}{l} \xrightarrow{\quad} a + 0b + 0c = 1 \\ \xrightarrow{\quad} 0a + b + 0c = 2 \\ \xrightarrow{\quad} \hline a + b + 0c = 3 \end{array}$$

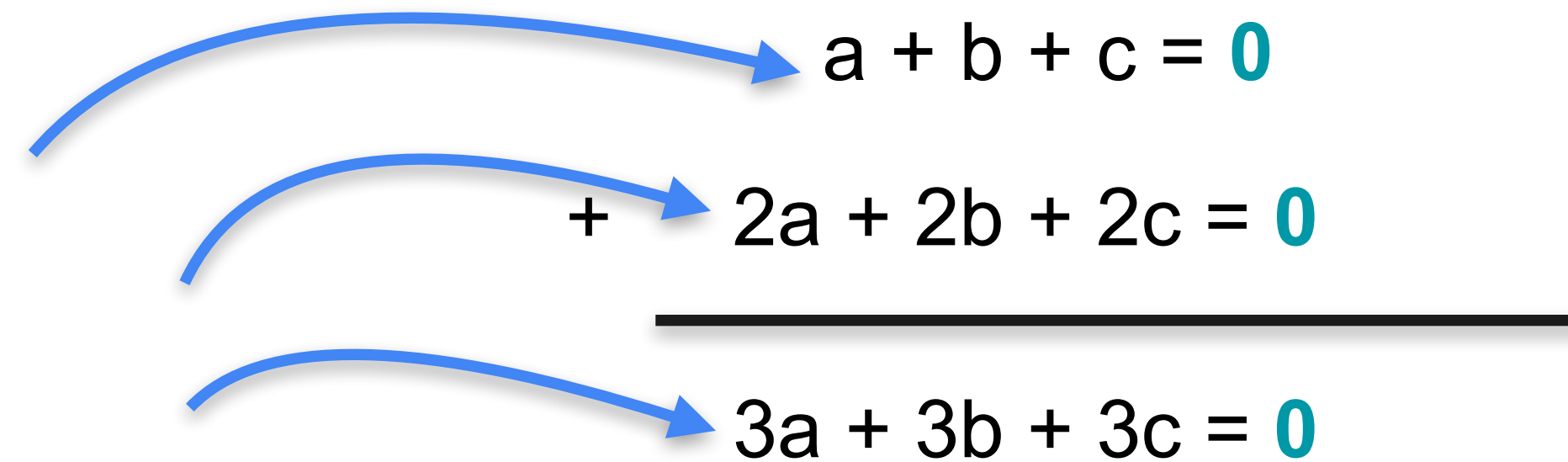
|   |   |   |
|---|---|---|
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 1 | 0 |

Row 1 + Row 2 = Row 3

Row 3 **depends** on rows 1 and 2

Rows are **linearly dependent**

# Linear dependence and independence

$$\begin{array}{l} a + b + c = 0 \\ 2a + 2b + 2c = 0 \\ 3a + 3b + 3c = 0 \end{array}$$

$$\begin{array}{r} a + b + c = 0 \\ + \quad 2a + 2b + 2c = 0 \\ \hline 3a + 3b + 3c = 0 \end{array}$$

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 3 | 3 |

Row 1 + Row 2 = Row 3

Row 3 **depends** on rows 1 and 2

Rows are **linearly dependent**



# Linear dependence and independence

The diagram illustrates the relationship between three linear equations and their corresponding matrix rows. On the left, three equations are listed:

$$\begin{aligned}a + b + c &= 0 \\ a + b + 2c &= 0 \\ a + b + 3c &= 0\end{aligned}$$

Below these equations is a 3x3 matrix representing the coefficients:

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 1 | 2 |
| 1 | 1 | 3 |

On the right, the equations are manipulated to show that the second equation is the average of the first and third:

$$\begin{aligned}a + b + c &= 0 \\ + \quad a + b + 3c &= 0 \\ \hline 2a + 2b + 4c &= 0 \\ \downarrow \div 2 \\ a + b + 2c &= 0\end{aligned}$$

Blue arrows indicate the flow of information: one arrow points from the first equation to the first row of the matrix, another from the third equation to the third row, and a third from the derived equation  $a + b + 2c = 0$  to the second row of the matrix.

Average of Row 1 and Row 3 is Row 2  
Row 2 **depends** on rows 1 and 3

Rows are **linearly dependent**

# Linear dependence and independence

$$a + b + c = 0$$

$$a + 2b + c = 0$$

$$a + b + 2c = 0$$



No relations between equations

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |

No relations between rows

Rows are **linearly independent**

# Quiz: Linear dependence and independence

**Problem:** Determine if the following matrices have linearly dependent or independent rows

|   |   |   |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 3 | 2 | 3 |

|   |   |    |
|---|---|----|
| 1 | 1 | 1  |
| 1 | 1 | 2  |
| 0 | 0 | -1 |

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 2 | 2 |
| 0 | 0 | 3 |

|   |   |    |
|---|---|----|
| 1 | 2 | 5  |
| 0 | 3 | -2 |
| 2 | 4 | 10 |

# Solution: Linear dependence and independence

**Problem:** Determine if the following matrices have linear dependent or independent rows

|   |   |   |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 3 | 2 | 3 |

$$3\text{Row1} + 2\text{Row2} = \text{Row3}$$

**Dependent (singular)**

|   |   |    |
|---|---|----|
| 1 | 1 | 1  |
| 1 | 1 | 2  |
| 0 | 0 | -1 |

$$\text{Row1} - \text{Row2} = \text{Row3}$$

**Dependent (singular)**

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 2 | 2 |
| 0 | 0 | 3 |

No relations

**Independent  
(Non-singular)**

|   |   |    |
|---|---|----|
| 1 | 2 | 5  |
| 0 | 3 | -2 |
| 2 | 4 | 10 |

$$2\text{Row1} = \text{Row3}$$

**Dependent (singular)**



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# System of Linear Equations

---

## **The determinant**

# Linear dependence between rows

Non-singular matrix

|  |  |
|---|---|
| 1   | 1   |
| 1   | 2   |

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \times ? = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Rows linearly independent


Singular matrix

|  |  |
|---|---|
| 1   | 1   |
| 2   | 2   |

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \times 2 = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

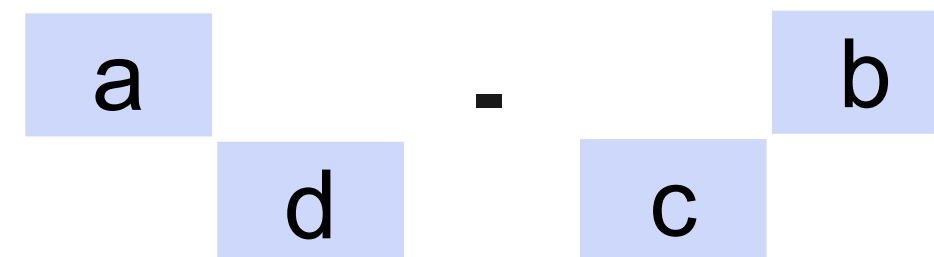
Rows linearly dependent

# Determinant



|   |   |
|---|---|
| a | b |
| c | d |

**Determinant**  $= ad - bc$



$$ak = c$$

$$bk = d$$

$$\frac{c}{a} = \frac{d}{b} = k$$

**Matrix is singular if**

|   |   |
|---|---|
| a | b |
|---|---|

 $\cdot k =$ 

|   |   |
|---|---|
| c | d |
|---|---|

**Determinant**

$$ad = bc$$

$$ad - bc = 0$$

# Determinant

Non-singular matrix



|   |   |
|---|---|
| 1 | 1 |
| 1 | 2 |

Determinant

$$\begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array} -$$

$$1 \cdot 2 - 1 \cdot 1 = 1$$



Singular matrix

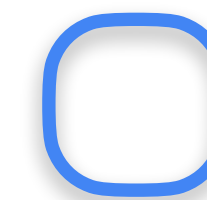


|   |   |
|---|---|
| 1 | 1 |
| 2 | 2 |

Determinant

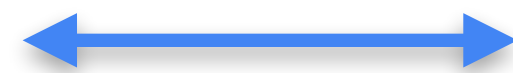
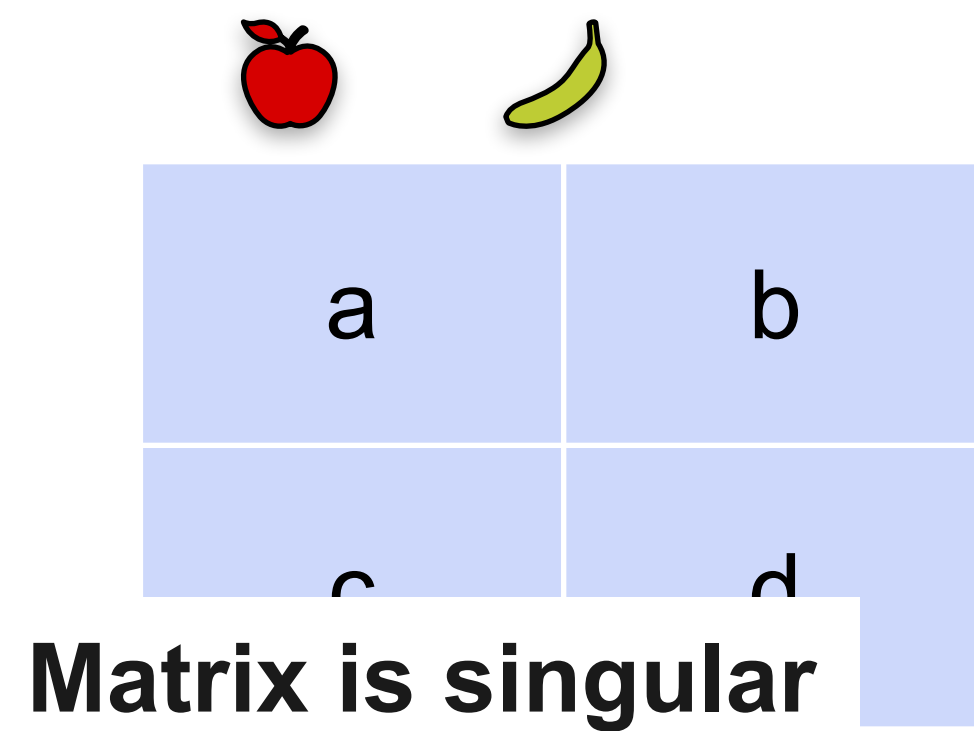
$$\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} -$$

$$1 \cdot 2 - 2 \cdot 1 = 0$$





# Determinant and singularity



$$ad - bc$$

**Determinant is zero**

# Quiz: Determinant

**Problem 1:** Find the determinant of the following matrices

**Matrix 1**

|    |   |
|----|---|
| 5  | 1 |
| -1 | 3 |

|    |    |
|----|----|
| 2  | -1 |
| -6 | 3  |

Are these matrices singular or non-singular?

# Solutions: Determinant

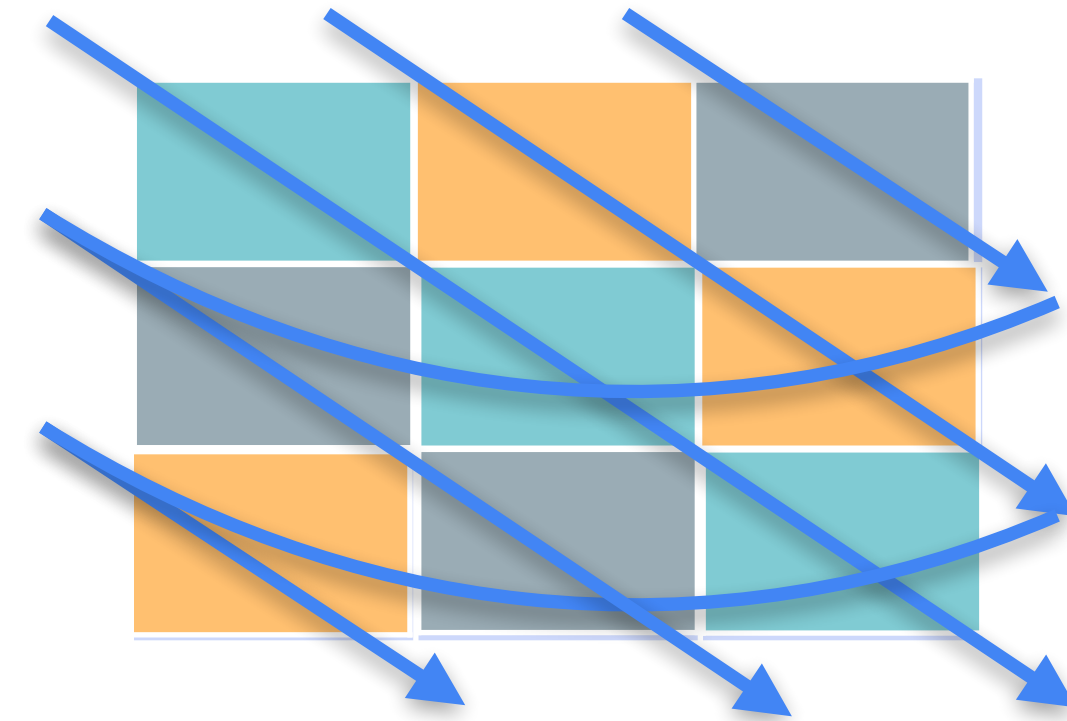
**Matrix 1:**  $\det = 5 \cdot 3 - 1 \cdot (-1) = 15 + 1 = 16$

|    |   |                     |
|----|---|---------------------|
| 5  | 1 | <b>Non-singular</b> |
| -1 | 3 |                     |

**Matrix 2:**  $\det = 2 \cdot 3 - (-1) \cdot (-6) = 6 - 6 = 0$

|    |    |                 |
|----|----|-----------------|
| 2  | -1 | <b>Singular</b> |
| -6 | 3  |                 |

# Diagonals in a 3x3 matrix

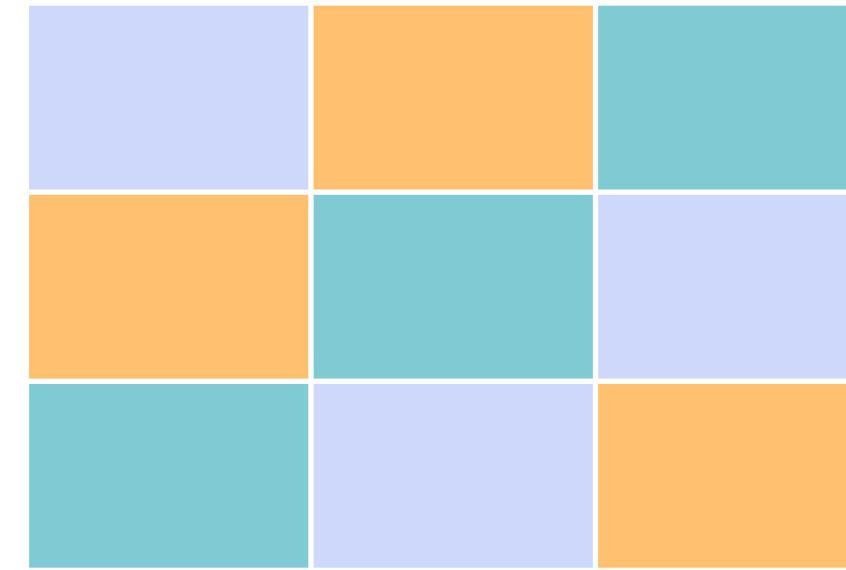


# Determinant

Add



Subtract



# The determinant

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |

# The determinant

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |

|   |   |   |
|---|---|---|
| 1 |   |   |
|   | 2 |   |
|   |   | 2 |

$$+ 1 \cdot 2 \cdot 2$$

# The determinant

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |



$$+ 1 \cdot 2 \cdot 2$$

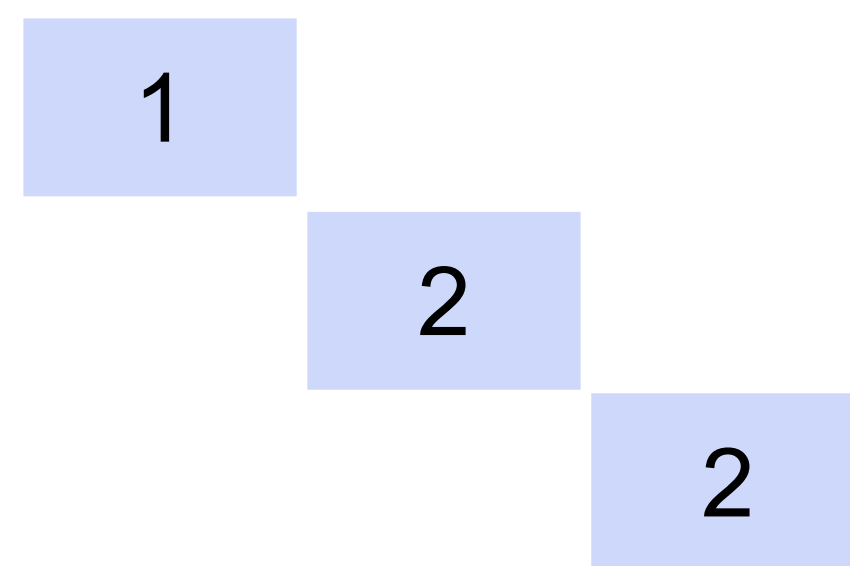


$$+ 1 \cdot 1 \cdot 1$$

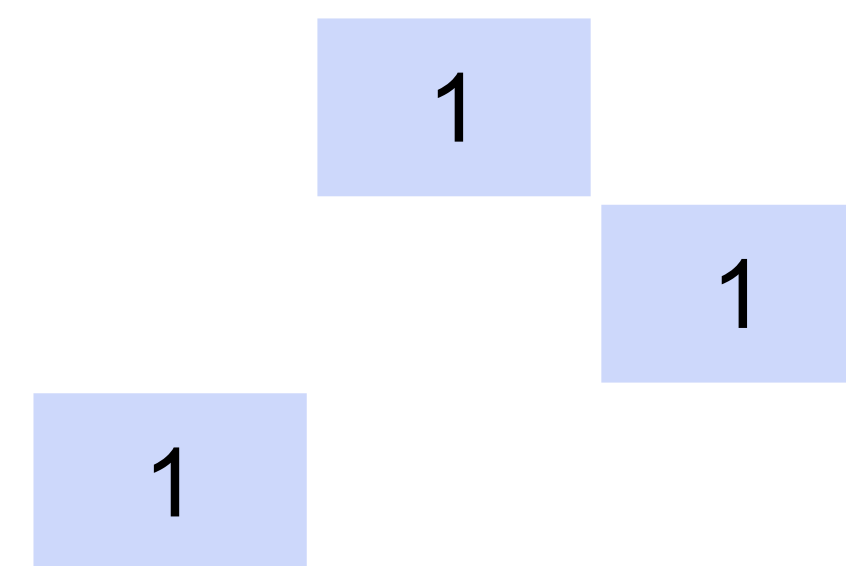


# The determinant

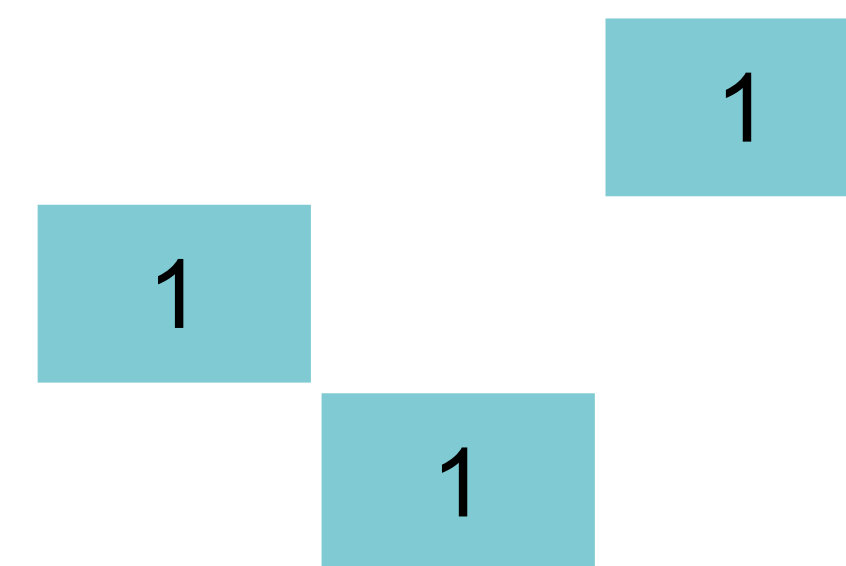
|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |



$$+ 1 \cdot 2 \cdot 2$$



$$+ 1 \cdot 1 \cdot 1$$



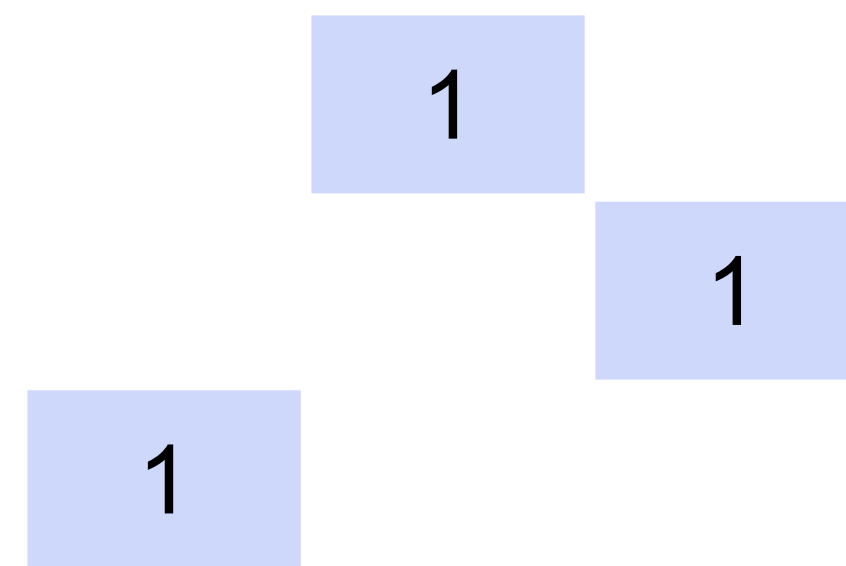
$$+ 1 \cdot 1 \cdot 1$$

# The determinant

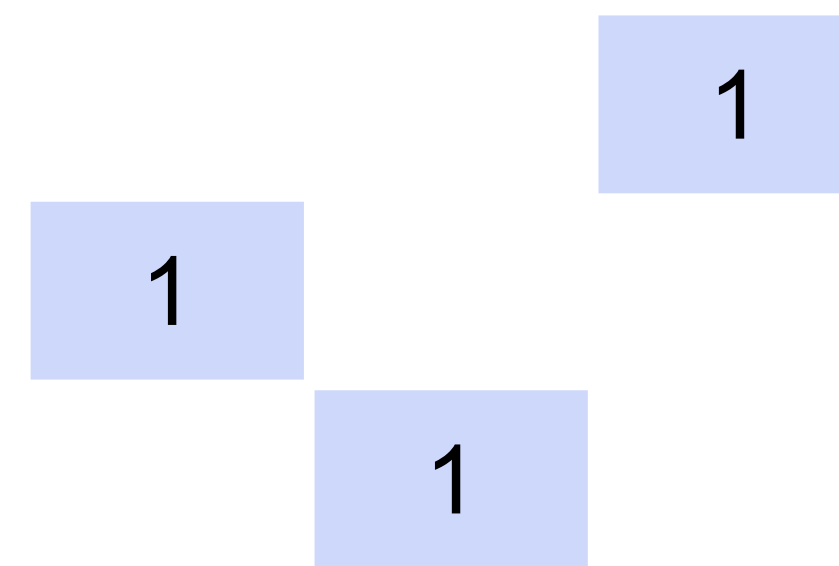
|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |



$$+ 1 \cdot 2 \cdot 2$$



$$+ 1 \cdot 1 \cdot 1$$



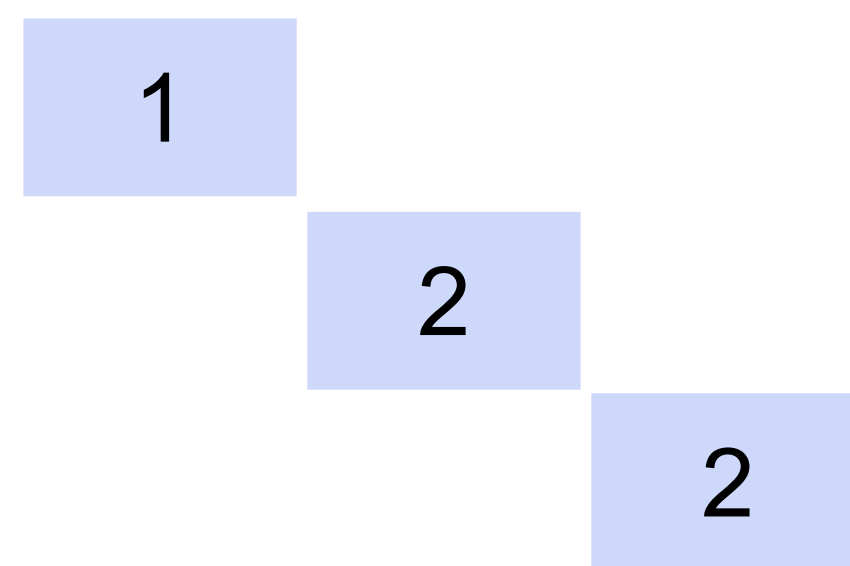
$$+ 1 \cdot 1 \cdot 1$$



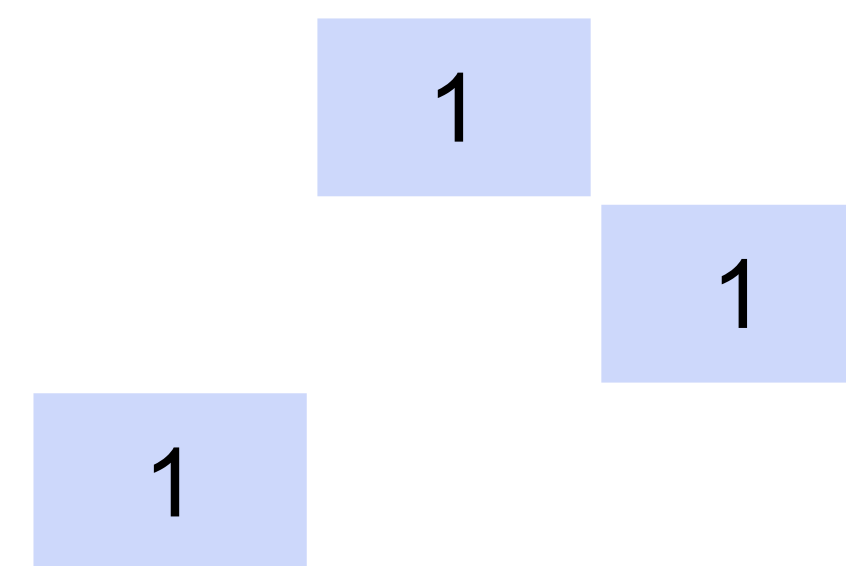
$$- 1 \cdot 2 \cdot 1$$

# The determinant

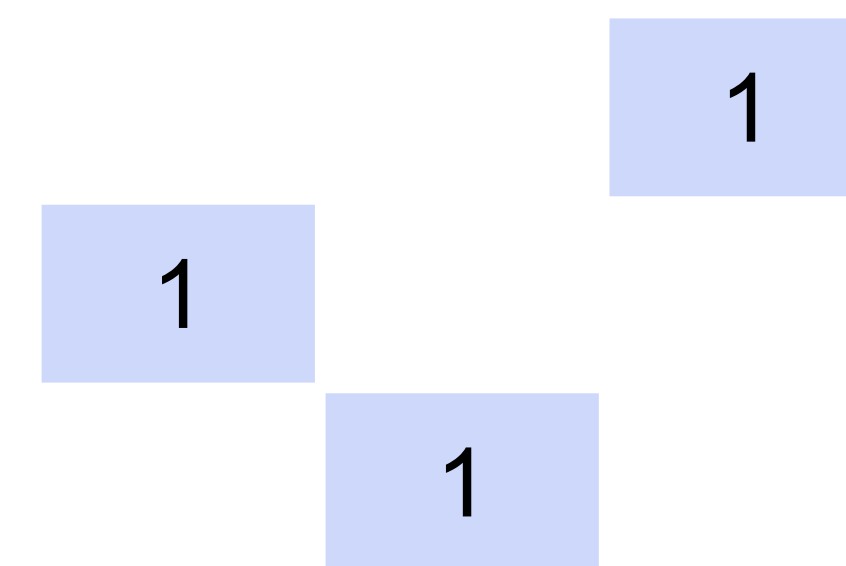
|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |



$$+ 1 \cdot 2 \cdot 2$$



$$+ 1 \cdot 1 \cdot 1$$



$$+ 1 \cdot 1 \cdot 1$$



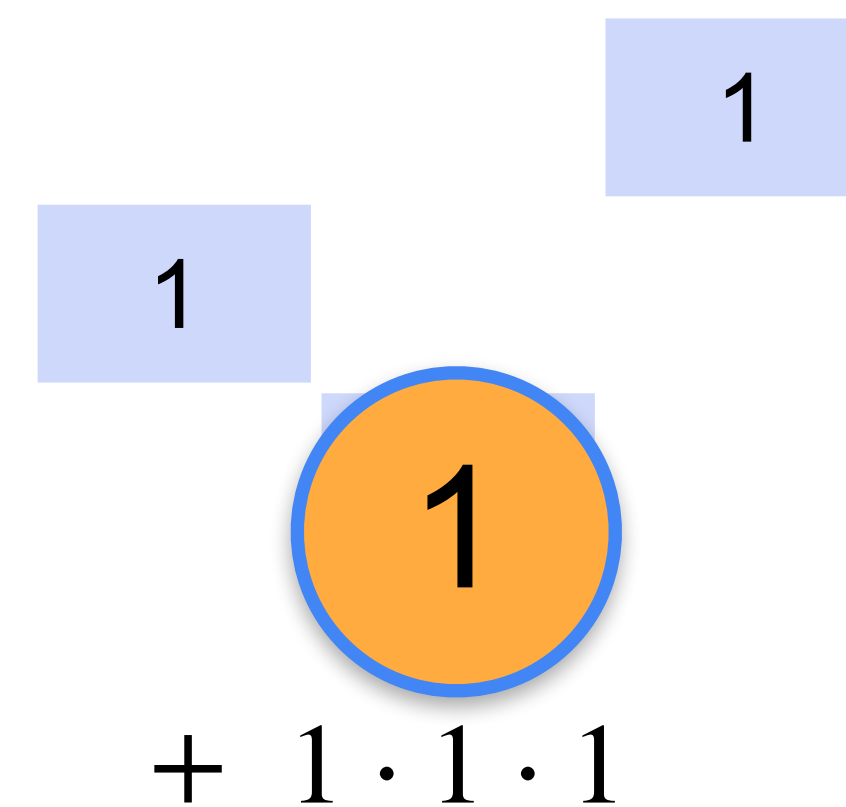
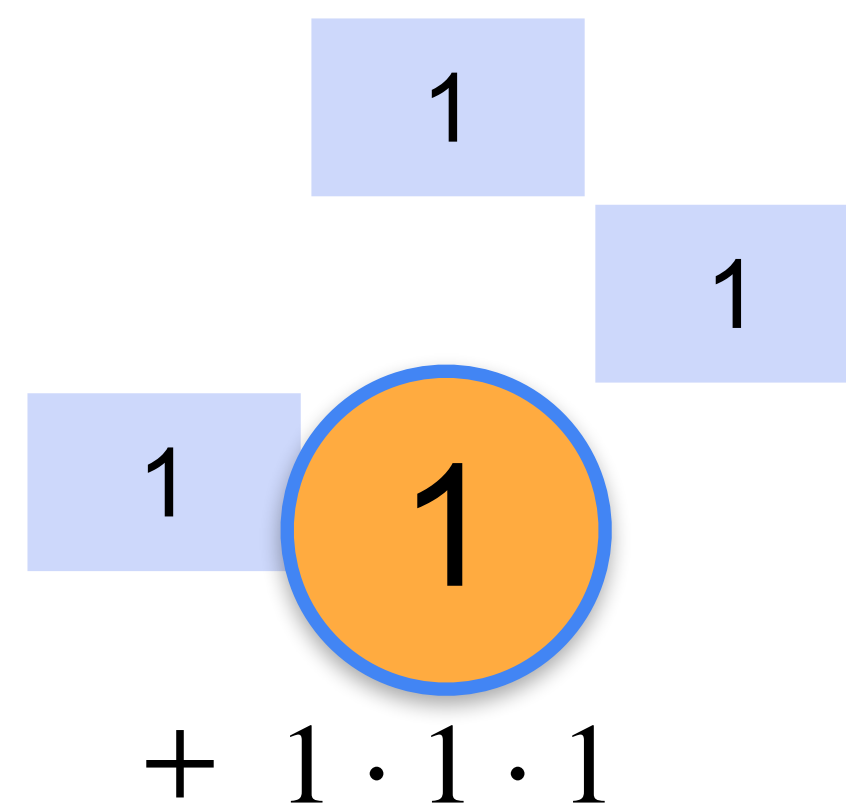
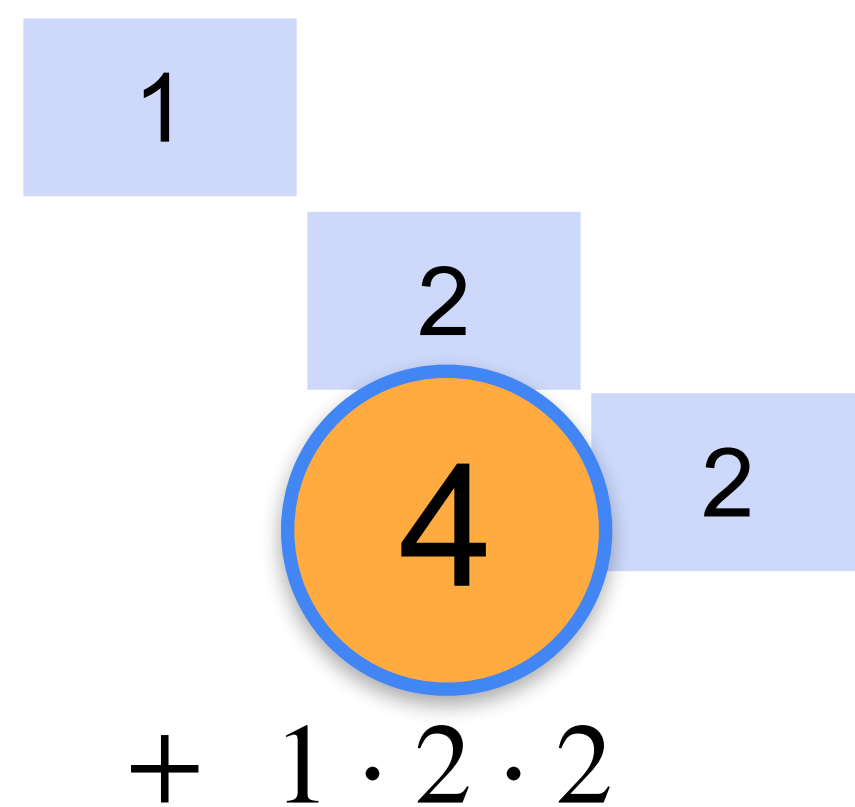
$$- 1 \cdot 2 \cdot 1$$



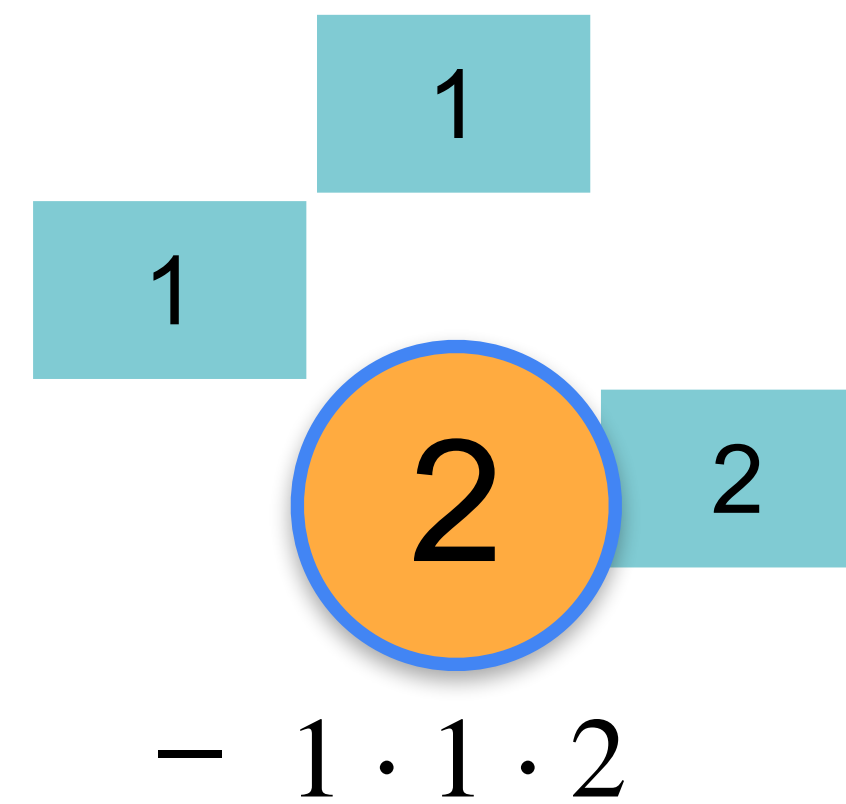
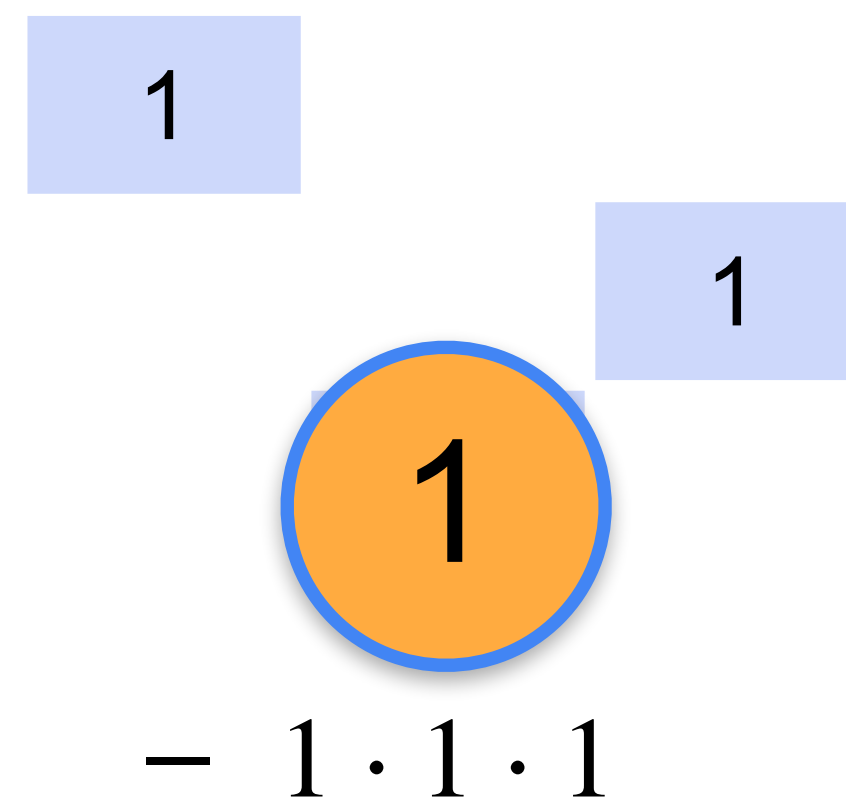
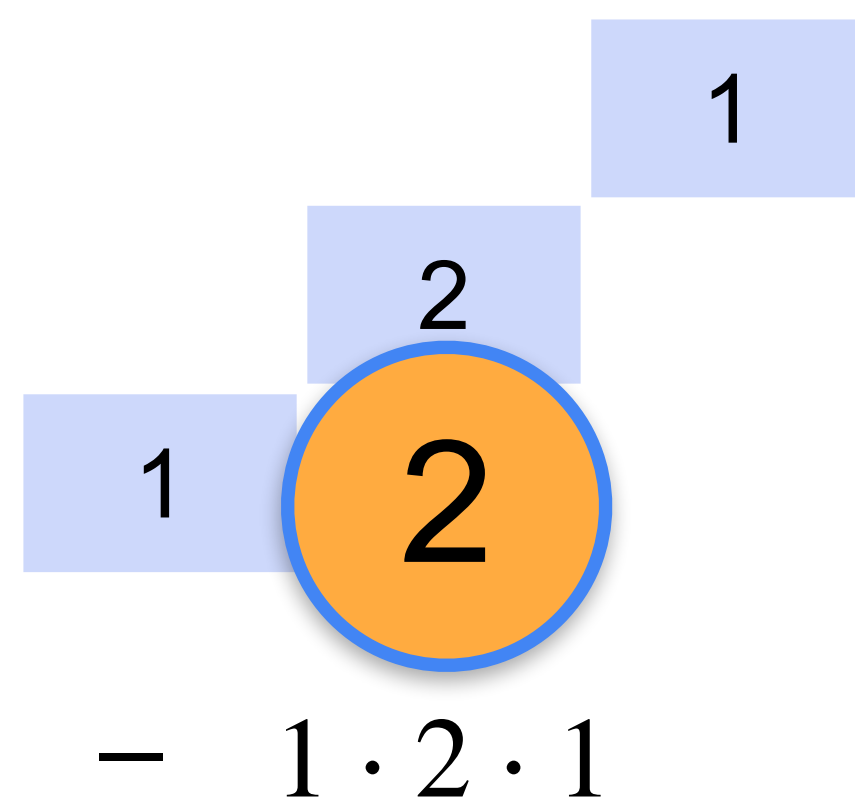
$$- 1 \cdot 1 \cdot 1$$

# The determinant

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |



$$\begin{aligned} \text{Det} &= 4 + 1 + 1 \\ &\quad - 2 - 1 - 2 \\ &= 1 \end{aligned}$$



# Quiz: Determinants

**Problem:** Find the determinant of the following matrices (from the previous quiz). Verify that those with determinant 0 are precisely the singular matrices.

|   |   |   |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 3 | 3 | 3 |

|   |   |    |
|---|---|----|
| 1 | 1 | 1  |
| 1 | 1 | 2  |
| 0 | 0 | -1 |

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 2 | 2 |
| 0 | 0 | 3 |

|   |   |    |
|---|---|----|
| 1 | 2 | 5  |
| 0 | 3 | -2 |
| 2 | 4 | 10 |

# Solution: Determinants

**Problem:** Find the determinant of the following matrices (from the previous quiz). Verify that those with determinant 0 are precisely the singular matrices.

|   |   |   |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 3 | 3 | 3 |

Determinant = 0

**Singular**

|   |   |    |
|---|---|----|
| 1 | 1 | 1  |
| 1 | 1 | 2  |
| 0 | 0 | -1 |

Determinant = 0

**Singular**

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 2 | 2 |
| 0 | 0 | 3 |

Determinant = 6

**Non-singular**

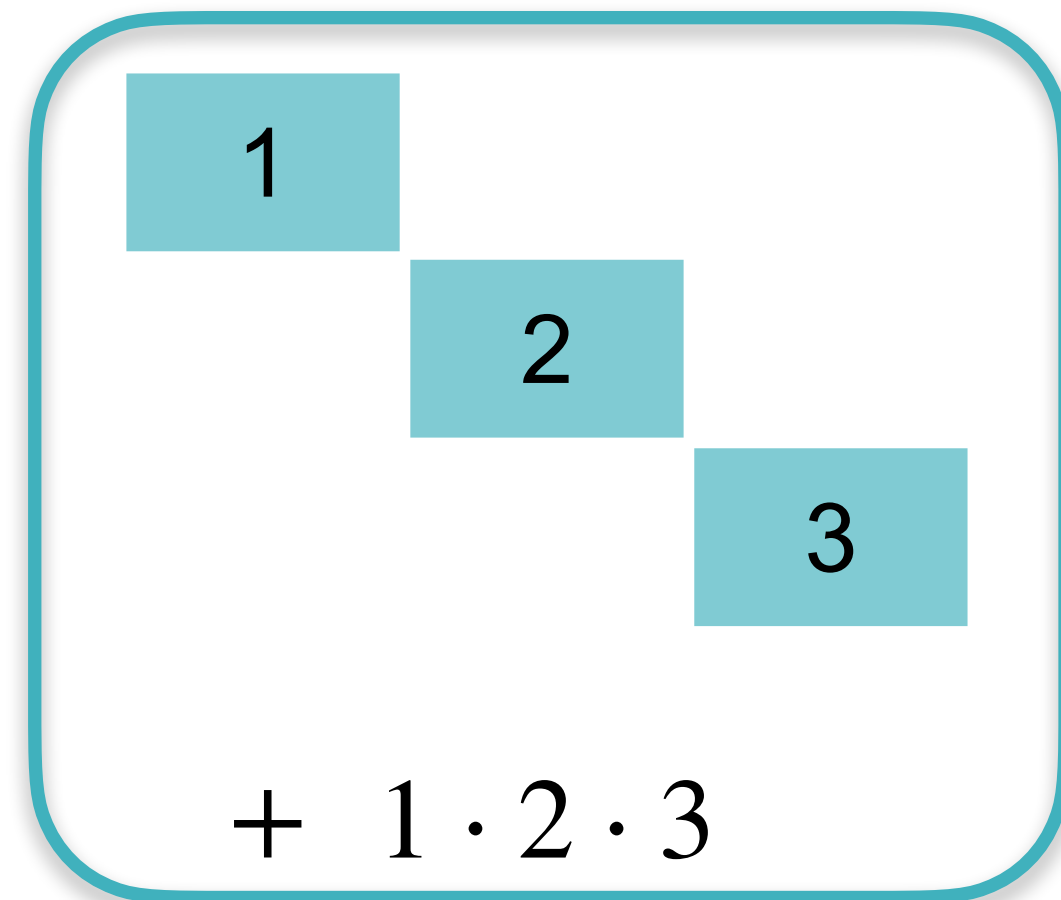
|   |   |    |
|---|---|----|
| 1 | 2 | 5  |
| 0 | 3 | -2 |
| 2 | 4 | 10 |

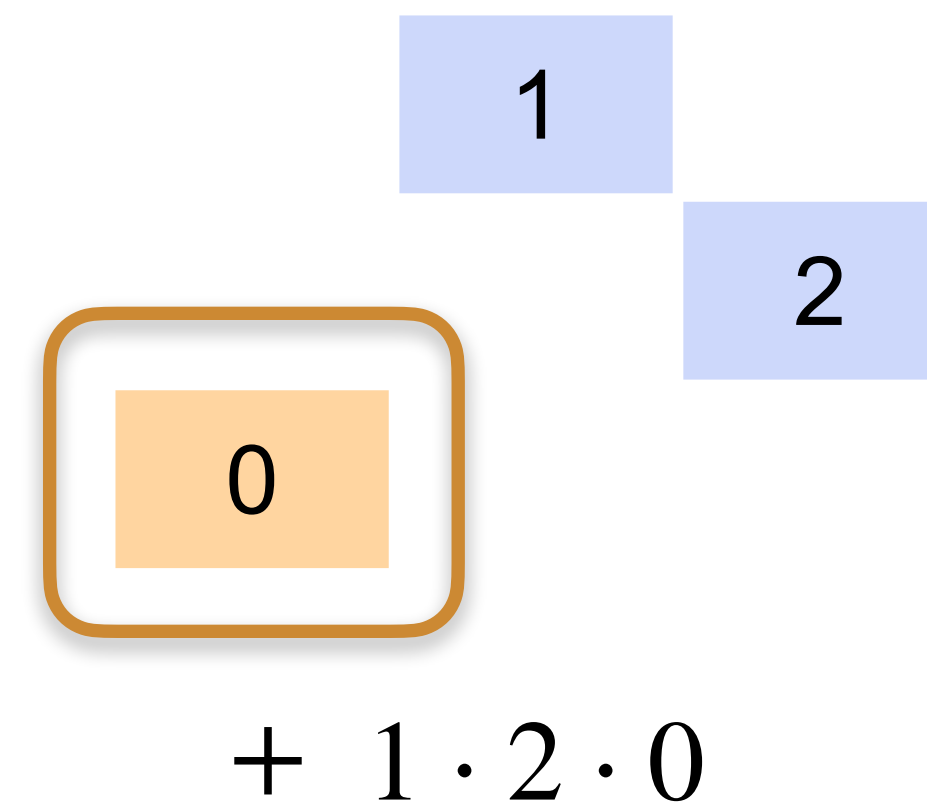
Determinant = 0

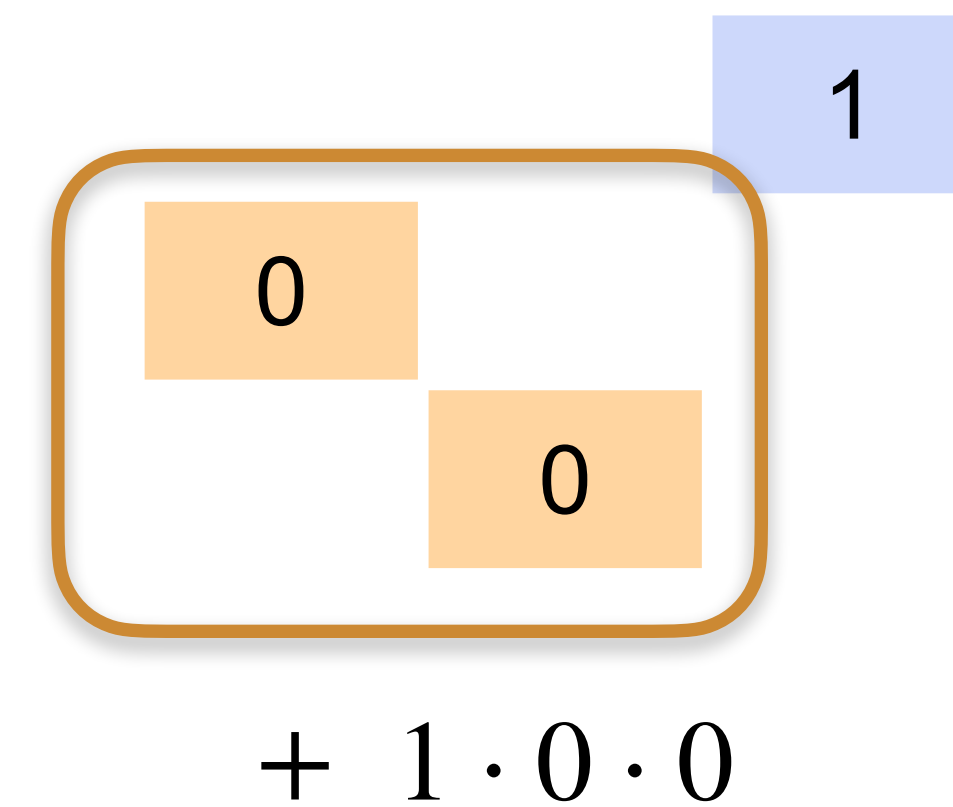
**Singular**

# The determinant

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 2 | 2 |
| 0 | 0 | 3 |


$$+ 1 \cdot 2 \cdot 3$$

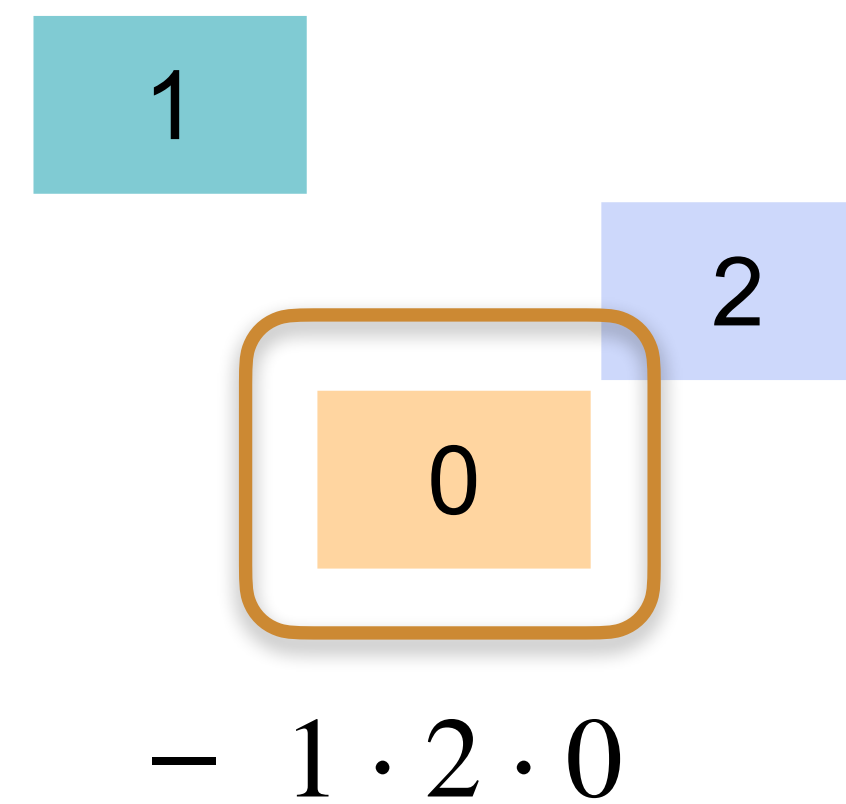

$$+ 1 \cdot 2 \cdot 0$$

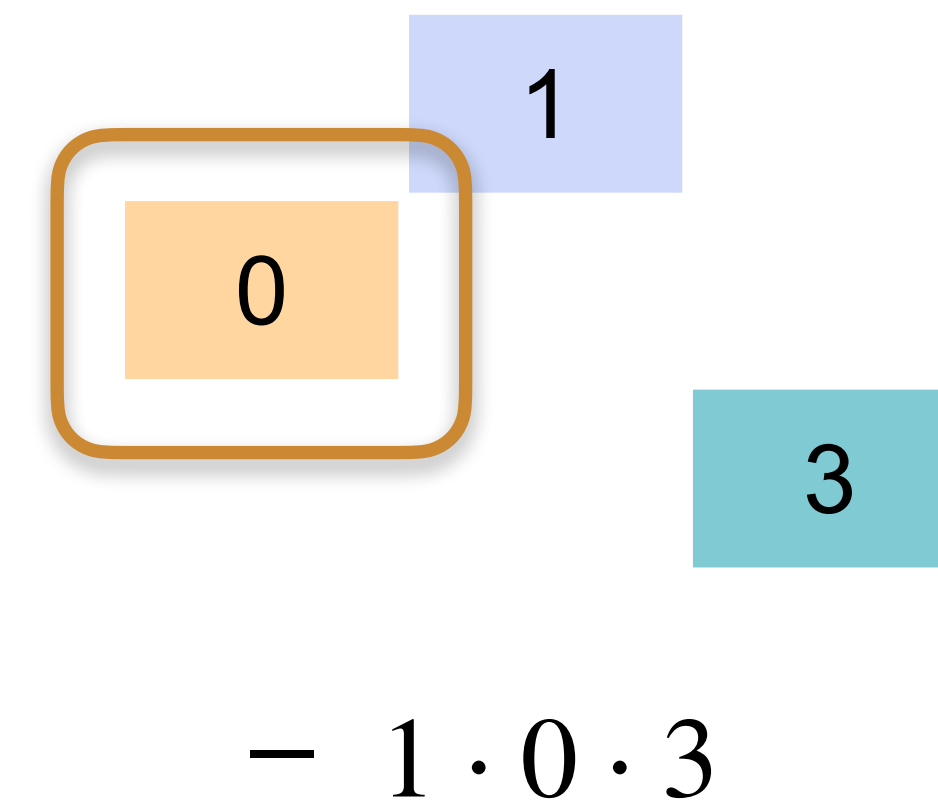

$$+ 1 \cdot 0 \cdot 0$$

$$\text{Det} = 6 + 0 + 0 - 0 - 0 - 0$$

$$= 6$$


$$- 1 \cdot 2 \cdot 0$$


$$- 1 \cdot 2 \cdot 0$$


$$- 1 \cdot 0 \cdot 3$$

# The determinant

|   |   |   |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 2 | 2 |
| 0 | 0 | 0 |

$$+ 1 \cdot 2 \cdot 0$$

$$\text{Det} = 0+0+0-0-0-0$$

$$= 0$$

$$- 1 \cdot 2 \cdot 0$$

$$+ 1 \cdot 2 \cdot 0$$

$$- 1 \cdot 2 \cdot 0$$

$$+ 1 \cdot 0 \cdot 0$$

$$- 1 \cdot 0 \cdot 0$$





DeepLearning.AI

# System of Linear Equations

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## Conclusion