

Cryptographic Primitives of the Swiss Post Voting System

Pseudo-code Specification

Swiss Post

Version 0.9.9

Abstract

Cryptographic algorithms play a pivotal role in the Swiss Post Voting System: ensuring their faithful implementation is crucially important. This document provides a mathematically precise and unambiguous specification of some cryptographic primitives underpinning the Swiss Post Voting System. It focuses on the elements common to the system and its verifier, such as the verifiable mix net and non-interactive zero-knowledge proofs. We provide technical details about encoding methods between basic data types and describe each algorithm in pseudo-code format.

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Revision chart

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Symbols

\mathbb{A}_{10}	Decimal numbers
\mathbb{A}_{Base16}	Base16 (Hex) alphabet [17]
\mathbb{A}_{Base32}	Base32 alphabet [17]
\mathbb{A}_{Base64}	Base64 alphabet [17]
\mathbb{A}_{UCS}	Alphabet of the Universal Coded Character Set (UCS) according to ISO/IEC10646
\mathcal{B}	Set of possible values for a byte
\mathcal{B}^*	Set of byte arrays of arbitrary length
\mathbb{B}^n	Set of bit arrays of length n
\mathbb{N}	Set of positive integer numbers including 0
\mathbb{N}^+	Set of strictly positive integer numbers
\mathbb{P}	Set of prime numbers
\mathbb{Z}_p	Set of integers modulo p
\mathbb{Z}_q	Set of integers modulo q
\mathbb{G}_q	Set of quadratic residues modulo p , which forms a group of order q
\mathbb{H}_ℓ	Ciphertext domain $(= \underbrace{\mathbb{G}_q \times \dots \times \mathbb{G}_q}_{\ell+1 \text{ times}})$
p	Encryption group modulus, a large safe prime (exact bitlength defined in the relevant section)
q	Encryption group cardinality s.t. $p = 2 \cdot q + 1$. A large prime (exact bit length defined in the relevant section)
g	Generator of the encryption group
$ x $	Bit length of the integer x
\top	Truth value true or successful termination
\perp	Truth value false or unsuccessful termination

1 Introduction

Switzerland has a longstanding tradition of direct democracy, allowing Swiss citizens to vote approximately four times a year on elections and referendums. In recent years, voter turnout hovered below 40 percent [8].

The vast majority of voters in Switzerland fill out their paper ballots at home and send them back to the municipality by postal mail, usually days or weeks ahead of the actual election date. Remote online voting (referred to as e-voting in this document) would provide voters with some advantages. First, it would guarantee the timely arrival of return envelopes at the municipality (especially for Swiss citizens living abroad). Second, it would improve accessibility for people with disabilities. Third, it would eliminate the possibility of an invalid ballot when inadvertently filling out the ballot incorrectly.

In the past, multiple cantons offered e-voting to a part of their electorate. Many voters would welcome the option to vote online - provided the e-voting system protects the integrity and privacy of their vote [9].

State-of-the-art e-voting systems alleviate the practical concerns of mail-in voting and, at the same time, provide a high level of security. Above all, they must display three properties [27]:

- Individual verifiability: allow a voter to convince herself that the system correctly registered her vote
- Universal verifiability: allow an auditor to check that the election outcome corresponds to the registered votes
- Vote secrecy: do not reveal a voter's vote to anyone

Following these principles, the Federal Chancellery defined stringent requirements for e-voting systems. The current draft of the Ordinance on Electronic Voting (VEleS - Verordnung über die elektronische Stimmabgabe) and its technical annex (VEleS annex)[5] describes these requirements.

Swiss democracy deserves an e-voting system with excellent security properties. In response to the release of an earlier version of the documentation and the source code of the e-voting system, multiple researchers published attacks, highlighted vulnerabilities, and suggested improvements[11, 15, 21, 28]. Swiss Post is thankful to all security researchers for their contributions and the opportunity to improve the system's security guarantees. We look forward to actively engaging with academic experts and the hacker community to maximize public scrutiny of the Swiss Post Voting System.

1.1 The Specification of Cryptographic Primitives

A vital element of a trustworthy and robust e-voting system is the description of the cryptographic algorithms in a form that leaves no room for interpretation and minimizes implementation errors [12].

Our pseudo-code description of the cryptographic algorithms—inspired by [13]—follows a consistent pattern:

- we prefix deterministic algorithms with **Get*** and probabilistic algorithms with **Gen***;
- we designate values that do not change between runs as **Context** and variable values as **Input**;
- we ensure that each algorithm does only one thing (single responsibility principle);
- we explicit domains and ranges of input and output values;
- we use the range notation for loops such as $i \in [0, n)$. We include the lower bound but exclude the upper bound, i.e. $0 \leq i < n$;
- we use 0-based indexing to close the representational gap between mathematics and code;
- we use **Require** for preconditions and **Ensure** for post-conditions;
- we use **return** to indicate a potentially early termination of the algorithm with the succeeding variable as the returned value; we use **Output** to describe the values that the algorithm produces;

Furthermore, we believe that a specification encompassing the common elements between the Swiss Post Voting System and its verifier (an open-source software verifying the correct establishment of the election result) benefits both systems.

1.2 Validating the Cryptographic Algorithm’s Correctness

We augment our specification with test values obtained from an independent implementation of the pseudo-code algorithms: our code validates against these test values to increase our confidence in the implementation’s correctness. The specification embeds the test values as JSON files within the document.

2 Security Level

Table 2 describes a *testing-only*, *default*, and an *extended* security level and the associated security parameter selection. Some standards deem the *default* security level sufficient, while other standards advocate for the *extended* security level.

Security Level Name	<i>testing-only</i>	<i>default</i>	<i>extended</i>
Security Level	-	112 bits	128 bits
Group Parameters	$ p = 8n, n \in \mathbb{N}^{+*}$ $ q = p - 1$ $r = 1$	$ p = 2048$ bits $ q = 2047$ bits ^a $r = 112$	$ p = 3072$ bits $ q = 3071$ bits ^b $r = 128$
Symmetric Algorithm	AES-GCM-256	AES-GCM-256	AES-GCM-256
KDF Hash Function	SHA-256	SHA-256	SHA-256

^aBased on NIST [2], ANSSI [1]

^bBased on ECRYPT [26]

Table 2: Security levels.

3 Basic Data Types

We build upon basic data types such as bytes, integers, strings, and arrays. Moreover, we require algorithms to concatenate and truncate strings and byte arrays, to test primality and to sort arrays.

3.1 Byte Arrays

We denote a byte array B of length n as $\langle b_0, b_1, \dots, b_{n-1} \rangle$ where b_i denotes the $i + 1$ -th byte of the array. Byte arrays can be encoded as strings, and, conversely, decoded from strings using Base16, Base32, and Base64 encodings according to RFC4648 [17]. Table 3 shows different examples of byte arrays.

Byte Array	Byte Array (binary form)	Base64	Base32
$\langle 0xF3, 0x01, 0xA3 \rangle$	11110011 00000001 10100011	"8wGj"	"6MA2G==="
$\langle 0xAC \rangle$	10101100	"rA=="	"VQ=====
$\langle 0x1F, 0x7F, 0x9D, 0x15, 0x12 \rangle$	00011111 01111111 10011101 00010101 00010010	"H3+dFRI="	"D57Z2FIS"

Table 3: Example representations of different byte arrays

We indicate concatenation of byte arrays with the \parallel operator.
 $\langle 0xF3, 0x01, 0xA3 \rangle \parallel \langle 0xAC \rangle = \langle 0xF3, 0x01, 0xA3, 0xAC \rangle$

In some cases, we need to cut a byte array to a given bit length, for instance to limit the number of iterations for algorithms such as algorithm 4.1. This is achieved in algorithm 3.1 by taking the low bytes of the given byte array, and applying a bitmask to the first byte taken so that the necessary leading bits are zeroed.

Algorithm 3.1 CutToBitLength: Cuts the given byte array to the requested bit length

Input:

Byte array $B \in \mathcal{B}^N$ s.t. $N \in \mathbb{N}^+$
 Requested length in bits $n \in \mathbb{N}^+$

Require: $n \leq N \cdot 8$

\triangleright This should only be used to cut leading bits

Operation:

```

1: length  $\leftarrow \lceil \frac{n}{8} \rceil$ 
2: offset  $\leftarrow N - \text{length}$ 
3: if  $n \bmod 8 \neq 0$  then
4:    $B'_0 \leftarrow B_{\text{offset}} \wedge (2^{(n \bmod 8)} - 1)$   $\triangleright$  Apply the bitwise-AND operator to mask out
      excess bits in the first byte
5: else
6:    $B'_0 \leftarrow B_{\text{offset}}$ 
7: end if
8: for  $i \in [1, \text{length})$  do
9:    $B'_i \leftarrow B_{\text{offset}+i}$ 
10: end for
11: return  $(B'_0, \dots, B'_{\text{length}-1})$ 
    
```

Output:

$(B'_0, \dots, B'_{\text{length}-1}) \in \mathcal{B}^*$

Test values for algorithm 3.1 are provided in the attached [cut-to-bit-length.json](#) file.

Algorithms 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 encode and decode byte arrays to and from Base16, Base32 and Base64 encodings. We refer to "standard" Base32 and Base64 encoding; we do *not* use Base64 with URL and filename safe alphabet and Base32 with extended hex alphabet. Potentially, decoding Base32 and Base64 may fail since the encoding is not bijective (only injective). For instance, one cannot decode the string "==TEOD8=" even though it is within the required alphabet.

Algorithm 3.2 Base16Encode

Input:

Byte array $B \in \mathcal{B}^*$

Operation:

1: $S \leftarrow \text{Base16}(B)$

Output:

String $S \in \mathbb{A}_{\text{Base16}}^*$

▷ According to RFC4648 [17]

Algorithm 3.3 Base16Decode

Input:

String $S \in \mathbb{A}_{\text{Base16}}^*$

▷ According to RFC4648 [17]

Operation:

1: $B \leftarrow \text{Base16}^{-1}(S)$

Output:

Byte array $B \in \mathcal{B}^*$

Algorithm 3.4 Base32Encode

Input:

Byte array $B \in \mathcal{B}^*$

Operation:

1: $S \leftarrow \text{Base32}(B)$

Output:

String $S \in \mathbb{A}_{\text{Base32}}^*$

▷ According to RFC4648 [17]

Algorithm 3.5 Base32Decode

Input:

String $S \in \mathbb{A}_{Base32}^*$

▷ According to RFC4648 [17]

Operation:

- 1: **if** S is not valid Base32 **then**
 - 2: **return** \perp
 - 3: **end if**
 - 4: $B \leftarrow \text{Base32}^{-1}(S)$
-

Output:

Byte array $B \in \mathcal{B}^*$ or \perp if S is not valid Base32

Algorithm 3.6 Base64Encode

Input:

Byte array $B \in \mathcal{B}^*$

Operation:

- 1: $S \leftarrow \text{Base64}(B)$
-

Output:

String $S \in \mathbb{A}_{Base64}^*$

▷ According to RFC4648 [17]

Algorithm 3.7 Base64Decode

Input:

String $S \in \mathbb{A}_{Base64}^*$

Operation:

- 1: **if** S is not valid Base64 **then**
 - 2: **return** \perp
 - 3: **end if**
 - 4: $B \leftarrow \text{Base64}^{-1}(S)$
-

Output:

Byte array $B \in \mathcal{B}^*$ or \perp if S is not valid Base64

3.2 Integers

When converting integers to byte array, we represent them in big-endian byte order. Since we only work with non-negative integers, we treat them as unsigned integers. Table 4 provides some example integers.

Integer	Byte Array (Hex)
0	<0x00>
3	<0x03>
128	<0x80>
23 591	<0x5C, 0x27>
23 592	<0x5C, 0x28>
4 294 967 295	<0xFF, 0xFF, 0xFF, 0xFF>
4 294 967 296	<0x01, 0x00, 0x00, 0x00, 0x00>

Table 4: Example representations of different integers. We use spaces to separate thousands groups.

Therefore, we ignore leading zeros (with an exception for the value 0) and define algorithm 3.8 to convert byte arrays to integers and algorithm 3.9 to convert integers to byte arrays. We avoid the empty byte array $\langle \rangle$ and represent 0 as $\langle 0x00 \rangle$.

$|x|$ derives the minimal bit length of an integer, e.g. $|4\,294\,967\,295| = 32$ and $|4\,294\,967\,296| = 33$.

Algorithm 3.8 ByteArrayToInteger

Input:

Byte array $B = \langle b_0, b_1, \dots, b_{n-1} \rangle \in \mathcal{B}^n$ of length $n \in \mathbb{N}^+$

Operation:

- 1: $x \leftarrow 0$
 - 2: **for** $i \in [0, n)$ **do**
 - 3: $x \leftarrow 256 \cdot x + b_i$
 - 4: **end for**
-

Output:

$x \in \mathbb{N}$

Algorithm 3.9 IntegerToByteArray

Input:

Positive integer $x \in \mathbb{N}$

Operation:

- 1: $n \leftarrow \text{ByteLength}(x)$ \triangleright Derive minimal length n of byte array; See algorithm 3.10
 - 2: $n \leftarrow \max(n, 1)$ \triangleright Ensure that 0 does not result in the empty byte array
 - 3: **for** $i \in [0, n)$ **do**
 - 4: $b_{n-i-1} \leftarrow x \bmod 256$
 - 5: $x \leftarrow \lfloor \frac{x}{256} \rfloor$
 - 6: **end for**
 - 7: $B \leftarrow \langle b_0, b_1, \dots, b_{n-1} \rangle$
-

Output:

Byte array $B \in \mathcal{B}^n$

We define algorithm 3.10 to compute the byte length of an integer.

Algorithm 3.10 ByteLength: calculate the byte length of an integer

Input:

Integer $n \in \mathbb{N}$

Operation:

- 1: $b \leftarrow \left\lceil \frac{|n|}{8} \right\rceil$
-

Output:

Byte length $b \in \mathbb{N}^+$

3.3 Strings

We encode strings in the universal coded character set (UCS) as defined in ISO/IEC10646, which is used by the encoding format UTF-8 (see RFC3629 [31]). Table 5 highlights some examples.

String	Byte Array (UCS)
"ABC"	$\langle 0x41, 0x42, 0x43 \rangle$
"Ä"	$\langle 0xC3, 0x84 \rangle$
"1001"	$\langle 0x31, 0x30, 0x30, 0x31 \rangle$
"1A"	$\langle 0x31, 0x41 \rangle$

Table 5: Example representations of different strings

Algorithms 3.11 and 3.12 convert byte arrays to strings and vice versa. Potentially, the

ByteArrayToString method can fail since not every byte array maps to a valid sequence of UCS code points.

Algorithm 3.11 StringToByteArray

Input:

String $S \in \mathbb{A}_{UCS}^*$

Operation:

1: $B \leftarrow \text{UCS}(S)$ \triangleright Convert S to its UCS representation

Output:

Byte array $B \in \mathcal{B}^*$

Algorithm 3.12 ByteArrayToString

Input:

Byte array $B = \langle b_0, b_1, \dots, b_{n-1} \rangle \in \mathcal{B}^n$ of length $n \in \mathbb{N}^+$

Operation:

1: **if** B does not correspond to a valid sequence of UCS code points **then**
 2: **return** \perp
 3: **end if**
 4: $S \leftarrow \text{UCS}^{-1}(B)$

Output:

String $S \in \mathbb{A}_{UCS}^*$

Moreover, we specify a method **StringToInteger** that translates a decimal String representation to an integer. Beware that the method **StringToInteger**(String) yields a different result than the conversion **ByteArrayToInteger**(**StringToByteArray**(String)).

Table 6 highlights some examples.

String	Integer
"0"	0
"1"	1
"1001"	1001
"0021"	21
"1A"	\perp

Table 6: Example Conversions of Strings to Integers

Algorithm 3.13 StringToInteger

Input:

String $S \in \mathbb{A}_{10}^*$

Operation:

- 1: **if** S is not valid decimal representation **then**
 - 2: **return** \perp
 - 3: **end if**
 - 4: $x \leftarrow \text{Decimal}(S)$ \triangleright Convert the String into its decimal representation (radix = 10)
-

Output:

Positive integer $x \in \mathbb{N}$

Conversely, the algorithm 3.14 converts integers to Strings.

Algorithm 3.14 IntegerToString

Input:

Positive integer $x \in \mathbb{N}$

Operation:

- 1: $S \leftarrow \text{Decimal}^{-1}(x) \triangleright$ Convert the integers' decimal representation (radix = 10) into a String
-

Output:

String $S \in \mathbb{A}_{10}^*$

4 Basic Algorithms

4.1 Randomness

Several algorithms draw a value at random from a given domain and rely on a primitive providing the requested number of independent random bytes. Standard implementations for generating cryptographically secure random bytes¹ are available in most programming languages; therefore, we omit the pseudo-code for this primitive and call it `RandomBytes(length)`, where `length` $\in \mathbb{N}$ is the required number of bytes, and the output is in \mathcal{B}^{length} .

Algorithm 4.1 `GenRandomInteger`: provide a random integer between 0 (incl.) and m (excl.)

Input:

Upper bound $m \in \mathbb{N}^+$

Operation:

- 1: `length` $\leftarrow \text{ByteLength}(m - 1)$ ▷ See algorithm 3.10
 - 2: `bitLength` $\leftarrow \lceil m - 1 \rceil$
 - 3: `rBytes` $\leftarrow \text{CutToBitLength}(\text{RandomBytes}(\text{length}), \text{bitLength})$ ▷ See algorithm 3.1
 - 4: `r` $\leftarrow \text{ByteArrayToInteger}(\text{rBytes})$ ▷ See algorithm 3.8
 - 5: **if** `r` $\geq m$ **then**
 - 6: go back to step 3
 - 7: **end if**
-

Output:

Random integer $r \in [0, m)$

Algorithm 4.2 `GenRandomVector`: generate a random vector from \mathbb{Z}_q^n

Input:

Exclusive upper bound $q \in \mathbb{N}^+$
Length $n \in \mathbb{N}^+$

Operation:

- 1: **for** $i \in [0, n)$ **do**
 - 2: $r_i \leftarrow \text{GenRandomInteger}(q)$ ▷ See algorithm 4.1
 - 3: **end for**
-

Output:

Random vector $(r_0, \dots, r_{n-1}) \in \mathbb{Z}_q^n$

¹A cryptographically secure random bytes generator has the following characteristics: it is designed for cryptographic use, generates independent, unbiased (i.e. uniform) bytes and relies on a high-quality entropy source[18]

Algorithms 4.3, 4.4, and 4.5 generate random strings using the Base16, Base32, and Base64 alphabet. Their typical use case is to generate random IDs of a specific alphabet; the method does not expect the output to be decodable.

Algorithm 4.3 GenRandomBase16String

Input:

Desired length of string: $\ell \in \mathbb{N}^+$

Operation:

- 1: $\ell_{bytes} = \lceil \frac{4 \cdot \ell}{8} \rceil$
 - 2: $b = \text{RandomBytes}(\ell_{bytes})$
 - 3: $S = \text{Truncate}(\text{Base16Encode}(b), \ell)$ ▷ See algorithms 3.2 and 4.6
-

Output:

$S \in (\mathbb{A}_{Base16})^\ell$ ▷ A random string of the Base16 alphabet [17]

Algorithm 4.4 GenRandomBase32String

Input:

Desired length of string: $\ell \in \mathbb{N}^+$

Operation:

- 1: $\ell_{bytes} = \lceil \frac{5 \cdot \ell}{8} \rceil$
 - 2: $b = \text{RandomBytes}(\ell_{bytes})$
 - 3: $S = \text{Truncate}(\text{Base32Encode}(b), \ell)$ ▷ See algorithms 3.4 and 4.6
-

Output:

$S \in (\mathbb{A}_{Base32})^\ell$ ▷ A random string of the Base32 alphabet [17]

Algorithm 4.5 GenRandomBase64String

Input:

Desired length of string: $\ell \in \mathbb{N}^+$

Operation:

- $\ell_{bytes} = \lceil \frac{6 \cdot \ell}{8} \rceil$
 - $b = \text{RandomBytes}(\ell_{bytes})$
 - $S = \text{Truncate}(\text{Base64Encode}(b), \ell)$ ▷ See algorithms 3.6 and 4.6
-

Output:

$S \in (\mathbb{A}_{Base64})^\ell$ ▷ A random string of the Base64 alphabet [17]

Algorithm 4.6 Truncate

Input:

Character array $S \in \mathbb{A}_x^u, u \in \mathbb{N}^+$

Desired length of string: $\ell \in \mathbb{N}^+$

Ensure: $\ell \leq u$

Operation:

- 1: **for** $i \in [0, \ell)$ **do**
 - 2: $S'_i \leftarrow S_i$
 - 3: **end for**
-

Output:

The truncated string $S' \in \mathbb{A}_x^\ell$

4.2 Recursive Hash

Our recursive hash function—inspired by CHVote[13]—ensures that different inputs to the hash function result in different outputs. In particular, the recursive hash function provides domain-separation: hashing ("A", "B") does not yield the same result as hashing ("AB").

To prevent collisions across the different possible input domains, we prepend a single byte to the scalar input values, according to their type. This implies that `RecursiveHash` will give a different result for the input string "A" than for byte array `<0x41>`.

The recursive definition of the domain implies that infinite inputs are possible in theory (such as self-referencing inputs), in which case the algorithm does not terminate. It is the callers' responsibility to ensure only finite inputs are provided in practice.

Algorithm 4.7 RecursiveHash: Computes the hash value of multiple inputs

Context:

Cryptographic hash function $\text{Hash} : \mathcal{B}^* \mapsto \mathcal{B}^L$, $L \in \mathbb{N}^+$ \triangleright Outputs a byte array of length L

Input:

Values (v_0, \dots, v_{k-1}) . Each value v_i is in domain \mathcal{V} , recursively defined as the union of:

- the set of byte arrays \mathcal{B}^*
- the set of valid UCS strings \mathbb{A}_{UCS}
- the set of non-negative integers \mathbb{N}
- the set of vectors \mathcal{V}^*

Require: $k > 0$, $L > 0$

Operation:

```

1: if  $k > 1$  then                                 $\triangleright$  Avoid computing  $\text{Hash}(\text{Hash}(v_0))$  when  $k = 1$ 
2:    $\mathbf{v} \leftarrow (v_0, \dots, v_{k-1})$ 
3:    $d \leftarrow \text{RecursiveHash}(\mathbf{v})$ 
4: else
5:    $w \leftarrow v_0$ 
6:   if  $w \in \mathcal{B}^*$  then
7:      $d \leftarrow \text{Hash}(\text{<0x00>} || w)$ 
8:   else if  $w \in \mathbb{N}$  then
9:      $d \leftarrow \text{Hash}(\text{<0x01>} || \text{IntegerToByteArray}(w))$            $\triangleright$  See algorithm 3.9
10:  else if  $w \in \mathbb{A}_{UCS}$  then
11:     $d \leftarrow \text{Hash}(\text{<0x02>} || \text{StringToByteArray}(w))$            $\triangleright$  See algorithm 3.11
12:  else if  $w = (w_0, \dots, w_j)$  then
13:    if  $j = 0$  then
14:       $d \leftarrow \text{RecursiveHash}(w_0)$ 
15:    else
16:       $d \leftarrow \text{Hash}(\text{RecursiveHash}(w_0) || \dots || \text{RecursiveHash}(w_j))$ 
17:    end if
18:  else
19:    return  $\perp$ 
20: end if

```

Output:

The digest $d \in \mathcal{B}^L$

Test values for algorithm 4.7 are provided in the attached [recursive-hash-sha3-256.json](#) file.

All test files provided in the current document for the algorithms relying on this algorithm assume that the SHA3-256 hash function is used.

In some case, the output of a hash needs to be uniformly distributed across \mathbb{Z}_q . To achieve this, we use the extendable output function **SHAKE-256**, limiting the output to the number of bits of q , and drawing a new value until the result is within the target domain, in order to avoid modulo-bias. Algorithm 4.8 deterministically draws new values, relying on algorithm 4.9 to perform the actual hashing.

Algorithm 4.8 RecursiveHashToZq: Computes the hash value of multiple inputs uniformly into \mathbb{Z}_q

Input:

Exclusive upper bound $q \in \mathbb{N}^+$

Values $\mathbf{v} = (v_0, \dots, v_{k-1})$. Each value v_i is in domain \mathcal{V} , recursively defined as the union of:

- the set of byte arrays \mathcal{B}^*
- the set of valid UCS strings \mathcal{A}_{UCS}
- the set of non-negative integers \mathbb{N}
- the set of vectors \mathcal{V}^*

Require: $k > 0$, $|q| \geq 512$

Operation:

- 1: $h \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHashOfLength}(|q|, \mathbf{v}))$ ▷ See algorithms 3.8 and 4.9
 - 2: **while** $h \geq q$ **do**
 - 3: $h \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHashOfLength}(|q|, h || \mathbf{v}))$ ▷ Prepend h to \mathbf{v}
 - 4: **end while**
 - 5: **return** h
-

Output:

$h \in \mathbb{Z}_q$

Test values for algorithm 4.8 are provided in the attached [recursive-hash-to-zq.json](#) file.

For algorithm 4.9 we rely on **SHAKE-256**[6], which provides 256-bits of security if and only if at least 64 bytes of output are used. Therefore the algorithm fails if fewer than 512 bits are requested. Without loss of generality, we assume a signature for **SHAKE-256** that takes the requested byte length as first parameter, and the input byte array as second parameter.

Algorithm 4.9 RecursiveHashOfLength: Computes the hash value of multiple inputs to a given bit length

Input:

Requested bit length $\ell \in \mathbb{N}^+$

Values (v_0, \dots, v_{k-1}) . Each value v_i is in domain \mathcal{V} , recursively defined as the union of:

- the set of byte arrays \mathcal{B}^*
- the set of valid UCS strings \mathbb{A}_{UCS}
- the set of non-negative integers \mathbb{N}
- the set of vectors \mathcal{V}^*

Require: $k > 0, \ell \geq 512$

Operation:

```

1:  $L \leftarrow \lceil \ell/8 \rceil$ 
2: if  $k > 1$  then                                ▷ Avoid computing Hash(Hash( $v_0$ )) when  $k = 1$ 
3:    $\mathbf{v} \leftarrow (v_0, \dots, v_{k-1})$ 
4:    $d \leftarrow \text{RecursiveHashOfLength}(\ell, \mathbf{v})$ 
5: else
6:    $w \leftarrow v_0$ 
7:   if  $w \in \mathcal{B}^*$  then
8:      $d \leftarrow \text{CutToBitLength}(\text{SHAKE-256}(L, <0x00> || w), \ell)$       ▷ See algorithm 3.1
9:   else if  $w \in \mathbb{N}$  then
10:     $d \leftarrow \text{CutToBitLength}(\text{SHAKE-256}(L, <0x01> || \text{IntegerToByteArray}(w)), \ell)$ 
                                                                    ▷ See algorithm 3.9
11:   else if  $w \in \mathbb{A}_{UCS}$  then
12:     $d \leftarrow \text{CutToBitLength}(\text{SHAKE-256}(L, <0x02> || \text{StringToByteArray}(w)), \ell)$ 
                                                                    ▷ See algorithm 3.11
13:   else if  $w = (w_0, \dots, w_j)$  then
14:     if  $j = 0$  then
15:        $d \leftarrow \text{RecursiveHashOfLength}(\ell, w_0)$ 
16:     else
17:       for  $i \in [0, j]$  do
18:          $h_i \leftarrow \text{RecursiveHashOfLength}(\ell, w_i)$ 
19:       end for
20:        $d \leftarrow \text{CutToBitLength}(\text{SHAKE-256}(L, h_0 || \dots || h_j), \ell)$ 
21:     end if
22:   else
23:     return  $\perp$ 
24: end if

```

Output:

The digest $d \in \mathcal{B}^L$

▷ Where the $8L - \ell$ first bits are set to 0

4.3 Hash and Square

At various places in the protocol, we break the homomorphic properties of certain operations by hashing and squaring values.² Given our choice of group parameters, modular squaring the hash's output ensures that the resulting value is a mathematical group member.

Algorithm 4.10 HashAndSquare: Hashes a value and squares the result

Context:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

$x \in \mathbb{N}$

Operation:

- 1: $x_h \leftarrow \text{RecursiveHashToZq}(q-1, x) + 1$ \triangleright See algorithm 4.8, avoiding the case $x_h = 0$
 - 2: $y \leftarrow x_h^2 \bmod p$
-

Output:

$y \in \mathbb{G}_q$

²The exponentiation function is a pseudo-random function if the input is randomly distributed.

4.4 KDF

The algorithms below rely on RFC5869[20] to describe a HMAC-based key derivation function (HKDF) to produce key material from a high-entropy source. The pseudo-code algorithms below only use the **HKDF-expand** part of the RFC, because the system only uses this function with cryptographically strong keys.

We note the function specified in section 2.3 of the RFC as **HKDF-Expand**, using the **Hash** function defined in the context of the pseudo-code and giving it the following inputs, in this order:

- a pseudo-random key, of length at least equal to the block size of the **Hash** function;
- additional context **info**, a byte array of arbitrary length;
- and the required length.

Section 2 defines the hash function to use in the implementation.

Algorithm 4.11 KDF: Key derivation function using HKDF-expand

Context:

Cryptographic hash function $\text{Hash} : \mathcal{B}^* \mapsto \mathcal{B}^L, L \in \mathbb{N}^+$
 \triangleright Outputs a byte array of length L

Input:

The cryptographically strong pseudo-random key $\text{PRK} \in \mathcal{B}^l$
 Additional context information $(\text{info}_0, \dots, \text{info}_{n-1}) \in (\mathbb{A}_{UCS}^*)^n$, s.t. $n \in \mathbb{N}$
 The required byte length $\ell \in \mathbb{N}^+$

Require: $l \geq L$

Require: $\ell \leq 255 \cdot L$

Operation:

- 1: $\text{info} \leftarrow \text{StringToByteArray}(\text{info}_0) || \dots || \text{StringToByteArray}(\text{info}_{n-1})$
 \triangleright See algorithm 3.11
 \triangleright I.e. the empty byte array if $n = 0$
 - 2: $\text{OKM} \leftarrow \text{HKDF-Expand}(\text{PRK}, \text{info}, \ell)$
 \triangleright As per RFC5869[20], section 2.3
 - 3: **return** OKM
-

Output:

The output keying material $\text{OKM} \in \mathcal{B}^\ell$

Test values for algorithm 4.11 are provided in the attached [hkdf-expand.json](#) file.

The algorithm below is used when we need the resulting key material to be in \mathbb{Z}_q .

Algorithm 4.12 KDFToZq: Use the KDF function to generate a value in \mathbb{Z}_q

Input:

The cryptographically strong pseudo-random key $\text{PRK} \in \mathcal{B}^l$
Additional context information $\text{info} \in (\mathbb{A}_{UCS^*})^n$, s.t. $n \in \mathbb{N}$
The requested exclusive upper bound $q \in \mathbb{N}^+$

Require: $l \geq L$

Require: $\text{ByteLength}(q) \geq L$

▷ See algorithm 3.10

Operation:

```
1:  $\ell \leftarrow \text{ByteLength}(q)$ 
2:  $h \leftarrow \text{KDF}(\text{PRK}, \text{info}, \ell)$ 
3:  $u \leftarrow \text{ByteArrayToInteger}(\text{CutToBitLength}(h, |q|))$ 
4: while  $u \geq q$  do
5:    $h \leftarrow \text{KDF}(h, \text{info}, \ell)$ 
6:    $u \leftarrow \text{ByteArrayToInteger}(\text{CutToBitLength}(h, |q|))$ 
7: end while
8: return  $u$ 
```

▷ See algorithm 3.10

Output:

The value $u \in \mathbb{Z}_q$

Test values for algorithm 4.12 are provided in the attached [hkdf-expand-to-zq.json](#) file.

5 Symmetric Authenticated Encryption

We define a symmetric authenticated encryption scheme based on Authenticated Encryption with Associated Data (AEAD), as defined in RFC5116[23].

We denote the function specified in section 2.1 as **AuthenticatedEncryption** with the following inputs in this order:

- a **secret key**, a byte array of length k ;
- a **nonce**, a byte array of length n ;
- a **plaintext** with the data to be encrypted, a byte array of length p which may be zero;
- the **associated data** with the data to be authenticated but not encrypted, a byte array which may be of length zero;

It produces a single output **ciphertext**, a byte array at least as long as the plaintext.

We denote the function specified in section 2.2 as **AuthenticatedDecryption** with the following inputs (as defined above) in this order: **secret key**, **nonce**, **associated data**, **ciphertext**. It produces a single output, either **plaintext** or \perp .

For the implementation we use AES-GCM-256 as the AEAD algorithm (see section 2) with $n = 12$ bytes (as described in [23] and recommended in [16]) and $p \leq 64 \times 10^9$ bytes ([7] section 3). We use a randomized nonce as described in [7] section 8.2.2, with the given limitation that "the total number of invocations of the authenticated encryption function shall not exceed 2^{32} " for a given key. Moreover, it is critical that nonces should not be reused, otherwise the security properties break down. In particular, algorithm 5.1 should not be used in a virtualized environment, as a rollback could lead to a nonce reuse. Callers of these algorithms should make sure that the preceding properties are respected.

Algorithm 5.1 GenCiphertextSymmetric: Symmetric authenticated encryption

Context:

Authenticated encryption function $\text{AuthenticatedEncryption} : (\mathcal{B}^k, \mathcal{B}^n, \mathcal{B}^p, \mathcal{B}^*) \mapsto \mathcal{B}^c$ s.t. $k \in \mathbb{N}^+, n \in \mathbb{N}^+, p \in \mathbb{N}, c \in \mathbb{N}^+$ with the constraints on k, n, p of the specific algorithm used

Input:

The encryption key $K \in \mathcal{B}^k$
The plaintext $P \in \mathcal{B}^p$
Associated data $(\text{associated}_0, \dots, \text{associated}_{d-1}) \in (\mathbb{A}_{UCS}^*)^d$, s.t. $d \in \mathbb{N}$

Operation:

- 1: $\text{nonce} \leftarrow \text{RandomBytes}(n)$
 - 2: $\text{associated} \leftarrow \text{StringToByteArray}(\text{associated}_0) \parallel \dots \parallel \text{StringToByteArray}(\text{associated}_{d-1})$
 \triangleright See algorithm 3.11
 \triangleright I.e. the empty byte array if $d = 0$
 - 3: $C \leftarrow \text{AuthenticatedEncryption}(K, \text{nonce}, P, \text{associated})$ \triangleright As per [23], section 2.1
-

Output:

The authenticated ciphertext $C \in \mathcal{B}^c$
The nonce $\text{nonce} \in \mathcal{B}^n$

Algorithm 5.2 GetPlaintextSymmetric: Symmetric authenticated decryption

Context:

Authenticated decryption function $\text{AuthenticatedDecryption} : (\mathcal{B}^k, \mathcal{B}^n, \mathcal{B}^*, \mathcal{B}^c) \mapsto \mathcal{B}^p$, $k \in \mathbb{N}^+, n \in \mathbb{N}^+, p \in \mathbb{N}, c \in \mathbb{N}^+$ with the constraints on k, n of the specific algorithm used

Input:

The encryption key $K \in \mathcal{B}^k$
The ciphertext $C \in \mathcal{B}^c$
The nonce $\text{nonce} \in \mathcal{B}^n$
Associated data $(\text{associated}_0, \dots, \text{associated}_{d-1}) \in (\mathbb{A}_{UCS}^*)^d$, s.t. $d \in \mathbb{N}$

Operation:

- 1: $\text{associated} \leftarrow \text{StringToByteArray}(\text{associated}_0) \parallel \dots \parallel \text{StringToByteArray}(\text{associated}_{d-1})$
 \triangleright See algorithm 3.11
 \triangleright I.e. the empty byte array if $d = 0$
 - 2: $P \leftarrow \text{AuthenticatedDecryption}(K, \text{nonce}, \text{associated}, C)$ \triangleright As per [23], section 2.2
-

Output:

The authenticated plaintext $P \in \mathcal{B}^p$
Or \perp if the ciphertext does not authenticate

6 ElGamal Cryptosystem

The computational proof [25] describes the security properties of the ElGamal encryption scheme. Moreover, it explains that we can share the randomness when encrypting multiple messages (using different public keys). Optimizing the encryption scheme in this way is called multi-recipient ElGamal encryption and prevents us from repeatedly computing the left-hand side of the ciphertext.

6.1 Parameters Generation

We instantiate the ElGamal encryption scheme over the group of nonzero quadratic residues $\mathbb{G}_q \subset \mathbb{Z}_p$, defined by the following *public* parameters: modulus p , cardinality (order) q , and generator g . We pick p and q prime, which implies through quadratic residuosity that $p = 2 \cdot q + 1$. Section 2 defines the bit length of p and q . We define the smallest integer $x > 1$ s.t. x is a generator of \mathbb{G}_q as g .

We pick all the group parameters *verifiably* to demonstrate that they are devoid of hidden properties or back doors. Algorithm 6.1 details the verifiable selection of group parameters. The method takes a seed—the name of the election event—as an input. It leverages the SHAKE128 algorithm which produces a variable length digest [6]. Most implementations of SHAKE128 require as input a byte array and a length in bytes and the method returns a byte array. The initial candidate value for q , however, must be in the interval $[2^{|q|-1}, 2^{|q|}]$. Therefore, we prepend the byte `<0x01>` to the digest’s output (see section 3.2 on how we represent integers) and perform a subsequent bitwise right-shift operation.

Algorithm 6.1 GetEncryptionParameters

Context:

The security level λ defining the bit length of $|p|$, the bit length of $|q|$ and the number of rounds r for primality testing, according to table 2.

Input:

$seed \in \mathbb{A}_{UCS}^*$ ▷ The name of the election event

Require: $|p| \bmod 8 = 0$ ▷ The algorithm below assumes the bit length of p is a multiple of 8

Operation:

```

1:  $i \leftarrow 0$ 
2: do
3:    $\hat{q}_b \leftarrow \text{SHAKE128}((\text{StringToByteArray}(seed) \parallel \text{IntegerToByteArray}(i)), \frac{|p|}{8})$  ▷ See
   algorithm 3.11, 3.9, and FIPS PUB 202 [6]
4:    $q_b \leftarrow \text{<0x01>} \parallel \hat{q}_b$  ▷ Byte array concatenation
5:    $q \leftarrow \text{ByteArrayToInteger}(q_b) \gg 2$  ▷ See algorithm 3.8 ▷ bit-wise right shift
6:    $q \leftarrow q + 1 - (q \bmod 2)$  ▷ Ensuring that q is odd
7:    $i \leftarrow i + 1$ 
8: while ( $\neg \text{isProbablePrime}(q, r)$  or  $\neg \text{isProbablePrime}(2 \cdot q + 1, r)$ )
9:  $p \leftarrow 2 \cdot q + 1$ 
10: for  $i \in [2, 4]$  do ▷ necessarily  $4 \in \mathbb{G}_q$  since every quadratic residue is a group
   element ( $2^2 = 4$ )
11:   if  $i \in \mathbb{G}_q$  then
12:     return  $g \leftarrow i$ 
13:   end if
14: end for

```

Output:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Group generator $g \in \mathbb{G}_q$

Test values for the algorithm 6.1 are provided in the attached [get-encryption-parameters.json](#) file.

6.2 Prime Selection

The Swiss Post Voting System encodes voting options using small primes. Given a mathematical group, we require an algorithm that returns a list of small prime numbers (excluding the generator g) of this mathematical group.

Algorithm 6.2 GetSmallPrimeGroupMembers

Input:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- Desired number of prime group members $r \in \mathbb{N}^+$

Require:

- $g \in [2, 4]$
 - $r \leq q - 4$
 - $r < 10,000$ ▷ For efficiency reasons
-

Operation:

- 1: $\text{current} \leftarrow 5$
 - 2: $\mathbf{p} \leftarrow ()$
 - 3: $\text{count} \leftarrow 0$
 - 4: **while** $\text{count} < r \wedge \text{current} < p$ **do**
 - 5: **if** $\text{current} \in \mathbb{G}_q \wedge \text{IsPrime}(\text{current})$ **then** ▷ See algorithm 6.3
 - 6: $\mathbf{p}[\text{count}] \leftarrow \text{current}$
 - 7: $\text{count} \leftarrow \text{count} + 1$
 - 8: **end if**
 - 9: $\text{current} \leftarrow \text{current} + 2$
 - 10: **end while**
 - 11: **if** $\text{count} \neq r$ **then**
 - 12: **return** \perp
 - 13: **end if**
-

Output:

- The small prime group members in ascending order $\mathbf{p} = (p_0, \dots, p_{r-1})$, $p_i \in (\mathbb{G}_q \cap \mathbb{P}) \setminus \{2, 3\}$
 - \perp if r is bigger than the number of primes in the \mathbb{G}_q group.
-

We define a deterministic primality test that is efficient for small primes.

Algorithm 6.3 lsPrime

Context:

Input:

Number $n \in \mathbb{N}^+$

Operation:

```
1: if  $n = 1$  then
2:   return  $\perp$ 
3: else if  $n = 2$  then
4:   return  $\top$ 
5: else
6:   for  $i \in [2, \lceil \sqrt{n} \rceil]$  do  $\triangleright$  We use ceil to take into account floating point arithmetic
    limitations
7:     if  $n \bmod i = 0$  then
8:       return  $\perp$ 
9:     end if
10:  end for
11:  return  $\top$ 
12: end if
```

Output:

\top if n is prime, \perp otherwise.

6.3 Key Pair Generation

Algorithm 6.4 describes the generation of a multi-recipient ElGamal key pair. We include 0 and 1, in the secret and public key ranges respectively, since the ElGamal encryption scheme is correct and semantically secure, even for those edge cases [4, 30]. However, one must be careful when using the key pair for purposes other than ElGamal encryption, especially when an adversary might maliciously choose the key pair. For instance, imagine if one successively exponentiates a value x by the keys k_1, \dots, k_n :

$$(((x^{k_1})^{k_2}) \dots)^{k_n}$$

Here, a single key $k_i = 0$ would cancel the contributions of all other keys since the result is guaranteed to be 1 — potentially leading to undesired consequences. In that case, the cryptographic protocol must draw the secret keys from \mathbb{Z}_q^+ (to exclude 0) and the public keys from the generators of \mathbb{G}_q (to exclude 1).

Algorithm 6.4 GenKeyPair: Generate a multi-recipient key pair

Input:

- Group modulus $p \in \mathbb{P}$
 - Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 - Group generator $g \in \mathbb{G}_q$
 - Number of key elements $N \in \mathbb{N}^+$
-

Operation:

- 1: **for** $i \in [0, N)$ **do**
 - 2: $sk_i \leftarrow \text{GenRandomInteger}(q)$ ▷ See algorithm 4.1
 - 3: $pk_i \leftarrow g^{sk_i} \bmod p$
 - 4: **end for**
-

Output:

A pair of secret and public keys $\{(sk_i, pk_i)\}_{i=0}^{N-1}, sk_i \in \mathbb{Z}_q, pk_i \in \mathbb{G}_q$

6.4 Encryption

We consider a "multi-recipient message", in which different public keys encrypt different messages. If there are more public keys than messages ($\ell < k$), we drop the excess public keys.

Algorithm 6.5 GetCiphertext: Compute a ciphertext with provided randomness

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
Group generator $g \in \mathbb{G}_q$

Input:

A multi-recipient message $\mathbf{m} \in \mathbb{G}_q^\ell$
The random exponent to use $r \in \mathbb{Z}_q$
A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$

Require: $0 < \ell \leq k$

Operation:

```
1:  $\gamma \leftarrow g^r \bmod p$ 
2: for  $i \in [0, \ell)$  do
3:    $\phi_i \leftarrow \mathbf{pk}_i^r \cdot m_i \bmod p$ 
4: end for
```

Output:

The ciphertext $(\gamma, \phi_0, \dots, \phi_{\ell-1}) \in \mathbb{H}_\ell$

Test values for the algorithm 6.5 are provided in [get-ciphertext.json](#).

6.5 Ciphertext Operations

Algorithm 6.6 GetCiphertextExponentiation: Exponentiate each ciphertext element by an exponent a

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$

Input:

- A multi-recipient ciphertext $C_a = (\gamma, \phi_0, \dots, \phi_{\ell-1}) \in \mathbb{H}_\ell$
 - An exponent $a \in \mathbb{Z}_q$
-

Operation:

- 1: $\gamma \leftarrow \gamma^a \bmod p$
 - 2: **for** $i \in [0, \ell)$ **do**
 - 3: $\phi_i \leftarrow \phi_i^a \bmod p$
 - 4: **end for**
-

Output:

- $(\gamma, \phi_0, \dots, \phi_{\ell-1}) \in \mathbb{H}_\ell$
-

Algorithm 6.7 GetCiphertextVectorExponentiation: Exponentiate a vector of ciphertexts and take the product

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$

Input:

- A vector of ciphertexts $\vec{C} = (C_0, \dots, C_{n-1}) \in (\mathbb{H}_\ell)^n$
 - A vector of exponents $\vec{a} = (a_0, \dots, a_{n-1}) \in \mathbb{Z}_q^n$
-

Operation:

- 1: **product** \leftarrow GetCiphertext($\vec{1}, 0, \mathbf{pk}$) \triangleright Neutral element of ciphertext multiplication
 - 2: **for** $i \in [0, n)$ **do**
 - 3: **product** \leftarrow GetCiphertextProduct(**product**, GetCiphertextExponentiation(C_i, a_i))
 \triangleright See algorithm 6.8 and algorithm 6.6
 - 4: **end for**
-

Output:

- The resulting **product** $\in \mathbb{H}_\ell$
-

Algorithm 6.8 GetCiphertextProduct: Multiply two ciphertexts

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
Group generator $g \in \mathbb{G}_q$

Input:

A multi-recipient ciphertext $C_a = (\gamma_a, \phi_{a,0}, \dots, \phi_{a,\ell-1}) \in \mathbb{H}_\ell$
Another multi-recipient ciphertext $C_b = (\gamma_b, \phi_{b,0}, \dots, \phi_{b,\ell-1}) \in \mathbb{H}_\ell$

Operation:

```
1:  $\gamma \leftarrow \gamma_a \cdot \gamma_b \mod p$ 
2: for  $i \in [0, \ell)$  do
3:    $\phi_i \leftarrow \phi_{a,i} \cdot \phi_{b,i} \mod p$ 
4: end for
```

Output:

$(\gamma, \phi_0, \dots, \phi_{\ell-1}) \in \mathbb{H}_\ell$

Test values for the algorithm 6.8 are provided in [get-ciphertext-product.json](#).

6.6 Decryption

Algorithm 6.9 GetMessage: Retrieve the message from a ciphertext

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
Group generator $g \in \mathbb{G}_q$

Input:

A multi-recipient ciphertext $\mathbf{c} \in \mathbb{H}_\ell$
A multi-recipient secret key $\mathbf{sk} \in \mathbb{Z}_q^k, 0 < \ell \leq k$

Operation:

```
1: for  $i \in [0, \ell)$  do
2:    $m_i \leftarrow \phi_i \cdot \gamma^{-\mathbf{sk}_i} \mod p$ 
3: end for
```

Output:

The multi-recipient message $(m_0, \dots, m_{\ell-1}) \in \mathbb{G}_q^\ell$

Since the system uses a multi-party re-encryption/decryption mixnet, in which the combined public key of the parties is used for encryption, each party actually performs a partial decryption. This entails that the actual output of the decryption phase for each party is actually still a ciphertext and the value for γ needs to be preserved to allow decryption by the following parties. This gives us the partial decryption algorithm in algorithm 6.10.

Algorithm 6.10 GetPartialDecryption: Partially decrypt a provided ciphertext

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
Group generator $g \in \mathbb{G}_q$

Input:

A multi-recipient ciphertext $\mathbf{c} = (\gamma, \phi_0, \dots, \phi_{\ell-1}) \in \mathbb{H}_\ell$
A multi-recipient secret key $\mathbf{sk} \in \mathbb{Z}_q^k, 0 < \ell \leq k$

Operation:

1: $(m_0, \dots, m_{\ell-1}) \leftarrow \text{GetMessage}(\mathbf{c}, \mathbf{sk})$ ▷ See algorithm 6.9
2: **return** $(\gamma, m_0, \dots, m_{\ell-1})$

Output:

The multi-recipient ciphertext $(\gamma, m_0, \dots, m_{\ell-1}) \in \mathbb{H}_\ell$

Algorithm 6.11 GenVerifiableDecryptions: Provide a verifiable partial decryption of a vector of ciphertexts.

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$

Input:

- A vector of ciphertexts $\mathbf{C} = (\mathbf{c}_0, \dots, \mathbf{c}_{N-1}) \in (\mathbb{H}_\ell)^N$
- A multi-recipient key pair $(\mathbf{pk}, \mathbf{sk}) \in \mathbb{G}_q^k \times \mathbb{Z}_q^k$
- An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS^*})^*$

Require: $0 < \ell \leq k$

Operation:

- 1: **for** $i \in [0, N)$ **do**
 - 2: $\mathbf{c}'_i \leftarrow \text{GetPartialDecryption}(\mathbf{c}_i, \mathbf{sk})$ ▷ See algorithm 6.10
 - 3: $(\gamma', \phi'_0, \dots, \phi'_{\ell-1}) \leftarrow \mathbf{c}'_i$
 - 4: $\pi_{\text{dec}, i} \leftarrow \text{GenDecryptionProof}(\mathbf{c}_i, (\mathbf{pk}, \mathbf{sk}), (\phi'_0, \dots, \phi'_{\ell-1}), \mathbf{i}_{\text{aux}})$ ▷ See algorithm 8.2
 - 5: **end for**
 - 6: $\mathbf{C}' \leftarrow (\mathbf{c}'_0, \dots, \mathbf{c}'_{N-1})$
 - 7: $\pi_{\text{dec}} \leftarrow (\pi_{\text{dec}, 0}, \dots, \pi_{\text{dec}, N-1})$
 - 8: **return** $(\mathbf{C}', \pi_{\text{dec}})$
-

Output:

- A vector of partially decrypted ciphertexts $\mathbf{C}' = (\mathbf{c}'_0, \dots, \mathbf{c}'_{N-1}) \in (\mathbb{H}_\ell)^N$
 - A vector of decryption proofs $\pi_{\text{dec}} \leftarrow (\pi_{\text{dec}, 0}, \dots, \pi_{\text{dec}, N-1}) \in (\mathbb{Z}_q \times \mathbb{Z}_q^\ell)^N$
-

Algorithm 6.12 VerifyDecryptions: Verify the decryptions of a vector of ciphertexts.

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$

Input:

A vector of ciphertexts $\mathbf{C} = (\mathbf{c}_0, \dots, \mathbf{c}_{N-1}) \in (\mathbb{H}_\ell)^N$
 A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
 A vector of partially decrypted ciphertexts $\mathbf{C}' = (\mathbf{c}'_0, \dots, \mathbf{c}'_{N-1}) \in (\mathbb{H}_\ell)^N$
 A vector of decryption proofs $\pi_{\text{dec}} = (\pi_{\text{dec},0}, \dots, \pi_{\text{dec},N-1}) \in (\mathbb{Z}_q \times \mathbb{Z}_q^\ell)^N$
 An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS}^*)^*$

Require: $N > 0$

Require: $0 < \ell \leq k$

Operation:

```

1: for  $i \in [0, N)$  do
2:    $(\gamma, \phi_0, \dots, \phi_{\ell-1}) \leftarrow \mathbf{c}_i$ 
3:    $(\gamma', \phi'_0, \dots, \phi'_{\ell-1}) \leftarrow \mathbf{c}'_i$ 
4:   if  $\gamma \neq \gamma'$  then
5:     return  $\perp$ 
6:   end if
7:    $\mathbf{m} \leftarrow (\phi'_0, \dots, \phi'_{\ell-1})$ 
8:    $\text{ok} \leftarrow \text{VerifyDecryption}(\mathbf{c}_i, \mathbf{pk}, \mathbf{m}, \pi_{\text{dec},i}, \mathbf{i}_{\text{aux}})$  ▷ See algorithm 8.3
9:   if  $\neg \text{ok}$  then
10:    return  $\perp$ 
11:  end if
12: end for
13: return  $\top$  ▷ Succeed if and only if all verifications above succeeded

```

Output:

The result of the verification: \top if **all** the verifications are successful, \perp otherwise.

7 Mix Net

Verifiable mix nets underpin most modern e-voting schemes with non-trivial tallying methods since they hide the relationship between encrypted votes (potentially linked to the voter’s identifier) and decrypted votes [14]. A re-encryption mix net consists of a sequence of mixers, each of which shuffles and re-encrypts an input ciphertext list and returns a different ciphertext list containing the same plaintexts. Each mixer proves knowledge of the permutation and the randomness (without revealing them to the verifier). Verifying these proofs guarantees that no mixer added, deleted, or modified a vote. The most widely used verifiable mix nets are the ones from Terelius-Wikström [29] and Bayer-Groth [3]. The Swiss Post Voting System uses the Bayer-Groth mix net, which we describe in this section. The computational proof [25] discusses the security properties of the non-interactive version of the Bayer-Groth mix net. Please note that each control component in the Swiss Post Voting System combines a verifiable shuffle with a subsequent, verifiable decryption step. This section details only the verifiable shuffle.

Our implementation exposes the following two public methods:

Algorithm 7.1 **GenVerifiableShuffle**: Shuffle (including re-encryption), and provide a Bayer-Groth proof of the shuffle

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$

Input:

- A vector of ciphertexts $\mathbf{C} \in (\mathbb{H}_\ell)^N$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$ \triangleright This public key is passed as context to all sub-arguments

Require: $0 < \ell \leq k$

Require: $2 \leq N \leq q - 3$

Operation:

- 1: $(\mathbf{C}', \pi, \mathbf{r}) \leftarrow \text{GenShuffle}(\mathbf{C}, \mathbf{pk})$ \triangleright See algorithm 7.3
 - 2: $(m, n) \leftarrow \text{GetMatrixDimensions}(N)$ \triangleright See algorithm 7.5
 - 3: $\mathbf{ck} \leftarrow \text{GetVerifiableCommitmentKey}(n)$ \triangleright See algorithm 7.6 \triangleright This commitment key is passed as context to all sub-arguments
 - 4: $\text{shuffleStatement} \leftarrow (\mathbf{C}, \mathbf{C}')$
 - 5: $\text{shuffleWitness} \leftarrow (\pi, \mathbf{r})$
 - 6: $\text{shuffleArgument} \leftarrow \text{GetShuffleArgument}(\text{shuffleStatement}, \text{shuffleWitness}, m, n)$ \triangleright See algorithm 7.11
-

Output:

- $\mathbf{C}' \in (\mathbb{H}_\ell)^N$
 - shuffleArgument
-

Algorithm 7.2 VerifyShuffle: Verify the output of a previously generated verifiable shuffle

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$

Input:

- A vector of unshuffled ciphertexts $\mathbf{C} \in (\mathbb{H}_\ell)^N$
- A vector of shuffled, re-encrypted ciphertexts $\mathbf{C}' \in (\mathbb{H}_\ell)^N$
- A Bayer-Groth `shuffleArgument` \triangleright See algorithm 7.11 for the domain
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$ \triangleright This public key is passed as context to all sub-arguments

Require: $0 < \ell \leq k$

Require: $2 \leq N \leq q - 3$

Operation:

- 1: $(m, n) \leftarrow \text{GetMatrixDimensions}(N)$ \triangleright See algorithm 7.5
 - 2: $\mathbf{ck} \leftarrow \text{GetVerifiableCommitmentKey}(n)$ \triangleright See algorithm 7.6 \triangleright This commitment key is passed as context to all sub-arguments
 - 3: $\text{shuffleStatement} \leftarrow (\mathbf{C}, \mathbf{C}')$
 - 4: **return** $\text{VerifyShuffleArgument}(\text{shuffleStatement}, \text{shuffleArgument}, m, n)$ \triangleright See algorithm 7.12
-

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

7.1 Pre-Requisites

7.1.1 Shuffle

Algorithm 7.3 shuffles a list of ciphertexts. We require the shuffled list of ciphertexts, the permutation, and the list of random exponents to prove the shuffle's correctness.

Algorithm 7.3 GenShuffle: Re-encrypting shuffle

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
Group generator $g \in \mathbb{G}_q$

Input:

A vector of ciphertexts $\mathbf{C} \in (\mathbb{H}_\ell)^N$
A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$

Require: $0 < \ell \leq k$

Operation:

```

1:  $(\pi_0, \dots, \pi_{N-1}) \leftarrow \text{GenPermutation}(N)$  ▷ See algorithm 7.4
2: for  $i \in [0, N)$  do
3:    $r_i \leftarrow \text{GenRandomInteger}(q)$  ▷ See algorithm 4.1
4:    $e \leftarrow \text{GetCiphertext}(\vec{1}, r_i, \mathbf{pk})$  ▷ Ciphertext for a vector of  $\ell$  1s. See algorithm 6.5
5:    $\mathbf{C}'_i \leftarrow e \cdot \mathbf{C}_{\pi_i}$  ▷ See algorithm 6.8 for ciphertext multiplication
6: end for

```

Output:

$\mathbf{C}' = (\mathbf{C}'_0, \dots, \mathbf{C}'_{N-1}) \in (\mathbb{H}_\ell)^N$ ▷ The result of the shuffle
 $\pi \in \Sigma_N$ ▷ The permutation used
 $\mathbf{r} = (r_0, \dots, r_{\ell-1}) \in \mathbb{Z}_q^N$ ▷ The exponents used for re-encryption

Algorithm 7.4 provides a way to generate a random permutation of indices for a list of size N . It uses the algorithm formalized by Knuth in [19]. The pseudo-code below assumes 0-based indexing, and as such deviates from standard mathematical notation in favor of closer proximity to the implementation.

Algorithm 7.4 GenPermutation: Permutation of indices up to N

Input:

Permutation size $N \in \mathbb{N}^+$

Operation:

```

1:  $\pi \leftarrow (0, \dots, N - 1)$ 
2: for  $i \in [0, N)$  do
3:    $\text{offset} \leftarrow \text{GenRandomInteger}(N - i)$  ▷ See algorithm 4.1
4:    $\text{tmp} \leftarrow \pi_i$ 
5:    $\pi_i \leftarrow \pi_{i+\text{offset}}$ 
6:    $\pi_{i+\text{offset}} \leftarrow \text{tmp}$ 
7: end for

```

Output:

π ▷ A permutation of the values between 0 and $N - 1$

Ensure: $\forall j \in [0, N) \rightarrow j \in \pi$

Ensure: $\pi \in \mathbb{Z}_{N-1}^N$ ▷ Those two elements combined ensure that $\pi \in \Sigma_N$

7.1.2 Matrix Dimensions

The Bayer-Groth mix net is memory optimal, when the ciphertexts can be arranged into matrices with an equal number of rows and columns. In the worst case, when the number of ciphertexts is prime, the resulting matrix has dimensions $1 \times N$. The below algorithm yields the optimal matrix size for a given number of ciphertexts. As an example, $N = 12$ results in $m = 3, n = 4$, $N = 18$ results in $m = 3, n = 6$, and $N = 23$ results in $m = 1, n = 23$,

Algorithm 7.5 GetMatrixDimensions: Return the optimal dimensions for the ciphertext matrix

Input:

Number of ciphertexts $N \in \mathbb{N}^+ \setminus \{1\}$

Operation:

```
1:  $m \leftarrow 1$ 
2:  $n \leftarrow N$ 
3: for  $i \in [\lfloor \sqrt{N} \rfloor, 1)$  do
4:   if  $i \mid N$  then
5:      $m \leftarrow i$ 
6:      $n \leftarrow \frac{N}{i}$ 
7:   return  $m, n$ 
8: end if
9: end for
```

Output:

$m \in \mathbb{N}^+$
 $n \in \mathbb{N}^+ \setminus \{1\}$

7.2 Commitments

A cryptographic commitment allows a party to commit to a value (the opening), to keep the opening hidden from others, and to reveal it later [10].

We use the Pedersen commitment scheme[24] with a commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu)$ that was generated in a verifiable manner.

The Pedersen commitment scheme satisfies three properties that the Bayer-Groth mix net requires [3].

- *Perfectly hiding*: The commitment is uniformly distributed in \mathbb{G}_q .
- *Computationally binding*: It is computationally infeasible to find two different values producing the same commitment.
- *Homomorphic*: It holds that $\text{GetCommitment}(a + b; r + s) = \text{GetCommitment}(a; r) \text{GetCommitment}(b; s)$ for messages a, b , a commitment key \mathbf{ck} and random values r, s .

The Pedersen commitment scheme is *computationally binding* only if the commitment keys are generated independently and verifiably at random:

Algorithm 7.6 GetVerifiableCommitmentKey: Generates a verifiable commitment key

Context:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

The desired number of elements of the commitment key $\nu \in \mathbb{N}^+$

Require: $\nu \leq q - 3$

Operation:

```

1:  $count = 0$ 
2:  $i = 0$ 
3:  $v \leftarrow \{\}$ 
4: while  $count \leq \nu$  do
5:    $u \leftarrow \text{RecursiveHashToZq}(q, (q, \text{"commitmentKey"}, i, count))$ 
       $\triangleright$  See algorithm 4.8.  $q$  is both the upper bound and the first value to hash.
6:    $w \leftarrow u^2 \bmod p$ 
7:   if  $w \notin \{0, 1, g\} \cup v$  then
8:      $g_i \leftarrow w$ 
9:      $v \leftarrow v \cup g_i$ 
10:     $count = count + 1$ 
11:   end if
12:    $i = i + 1$ 
13: end while
14:  $h \leftarrow g_0$ 

```

\triangleright By convention, we designate g_0 as h

Output:

A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Test values for the algorithm 7.6 are provided in the attached [get-verifiable-commitment-key.json](#) file.

Algorithm 7.7 GetCommitment: Computes a commitment to a value

Context:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

The values to commit to $\mathbf{a} \in \mathbb{Z}_q^\ell$

A random value $r \in \mathbb{Z}_q$

A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$ s.t. $\nu \geq l$

Require: $\ell > 0$

Operation:

1: $c \leftarrow h^r \prod_{i=1}^{\ell} g_i^{a_{i-1}} \bmod p$

Output:

The commitment $c \in \mathbb{G}_q$

Algorithm 7.8 GetCommitmentMatrix: Computes the commitment for a matrix

Context:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

The values to be committed $A \in \mathbb{Z}_q^{n \times m} \triangleright$ We note the columns of A as $\vec{a}_0, \dots, \vec{a}_{m-1}$

The random values to use $\mathbf{r} \in \mathbb{Z}_q^m$

A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$ s.t. $\nu \geq n$

Require: $m, n > 0$

Operation:

1: **for** $i \in [0, m)$ **do**

2: $c_i \leftarrow \text{GetCommitment}(\vec{a}_i, r_i, \mathbf{ck})$

\triangleright See algorithm 7.7

3: **end for**

Output:

The commitments $(c_0, \dots, c_{m-1}) \in \mathbb{G}_q^m$

This algorithm is consistent with the notation defined in [3], in section 2.3 Homomorphic Encryption:

For a matrix $A \in \mathbb{Z}_q^{n \times m}$ with columns $\vec{a}_1, \dots, \vec{a}_m$ we shorten notation by defining $\text{com}_{\mathbf{ck}}(A; \vec{r}) = (\text{com}_{\mathbf{ck}}(\vec{a}_1; r_1), \dots, \text{com}_{\mathbf{ck}}(\vec{a}_m; r_m))$

Algorithm 7.9 GetCommitmentVector: Compute the commitment for a transposed vector. This is only used in the algorithm 7.22, hence the specific indices

Context:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

The values to be committed $\mathbf{d} = (d_0, \dots, d_{2 \cdot m}) \in \mathbb{Z}_q^{2 \cdot m+1}$

The random values to use $\mathbf{t} = (t_0, \dots, t_{2 \cdot m}) \in \mathbb{Z}_q^{2 \cdot m+1}$

A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$ s.t. $\nu \geq 1$

Operation:

- 1: $(c_0, \dots, c_{2 \cdot m}) \leftarrow \text{GetCommitmentMatrix}(\mathbf{d}, \mathbf{t}, \mathbf{ck})$ \triangleright See algorithm 7.8, with a single row of $2 \cdot m + 1$ columns
 - 2: **return** $(c_0, \dots, c_{2 \cdot m})$
-

Output:

The commitments $(c_0, \dots, c_{2 \cdot m}) \in \mathbb{G}_q^{2 \cdot m+1}$

7.3 Arguments

Conceptually, the Bayer-Groth proof of a shuffle consists of six arguments. Figure 1 highlights the hierarchy of these arguments.

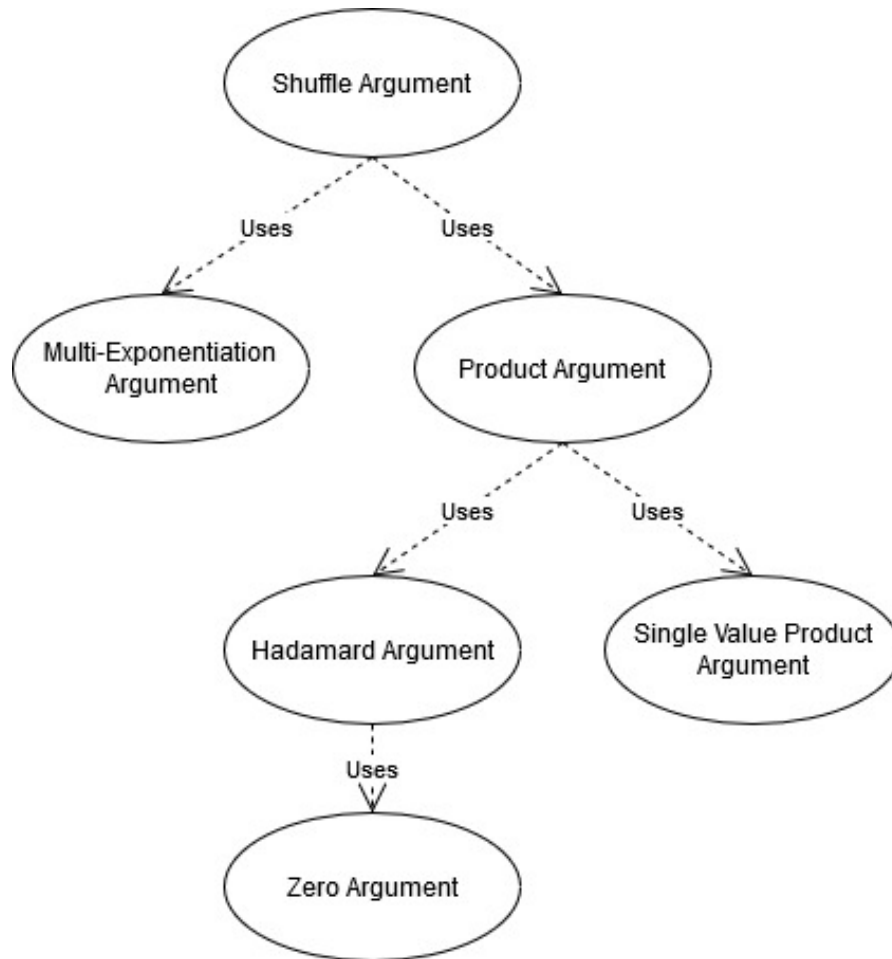


Figure 1: Bayer-Groth Argument for the Correctness of a Shuffle

The shuffle argument invokes a multi-exponentiation and a product argument. The product argument, in turn, uses a Hadamard and a single value product argument. Finally, the Hadamard argument calls a zero argument.

In all knowledge arguments, we will use the following terminology:

statement the public information for which we assert that a property holds

witness the private information we use to make arguments on the validity of the statement

argument the information we provide to a third party which allows them to verify the validity of our statement

We prove and verify the Bayer-Groth mix net in the *non-interactive* setting: the prover and the verifier recursively hash various elements to generate the argument's challenge messages. Since the mathematical group's rank q is much larger than the output domain, converting the hash function's output to an integer of byte size L is sound ($\text{ByteLength}(q) \gg L$).

In some algorithms, we will use the bilinear algorithm $\star : \mathbb{Z}_q^n \times \mathbb{Z}_q^n \mapsto \mathbb{Z}_q$ defined by the value y as:

$$(a_0, \dots, a_{n-1}) \star (b_0, \dots, b_{n-1}) = \sum_{j=0}^{n-1} a_j \cdot b_j \cdot y^{j+1}$$

Where all multiplications are performed modulo q .

Let us formalize it with the following pseudocode algorithm.

Algorithm 7.10 StarMap: Defines the \star bilinear map

Context:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

Value $y \in \mathbb{Z}_q$

First vector $\mathbf{a} = (a_0, \dots, a_{n-1}) \in \mathbb{Z}_q^n$

Second vector $\mathbf{b} = (b_0, \dots, b_{n-1}) \in \mathbb{Z}_q^n$

Operation:

1: $s \leftarrow 0$

2: **for** $j \in [0, n)$ **do**

3: $s \leftarrow s + a_j \cdot b_j \cdot y^{j+1}$

4: **end for**

▷ All operations are performed modulo q

Output:

$s \in \mathbb{Z}_q$

Test values for the algorithm 7.10 are provided in the attached [bilinearMap.json](#) file.

7.3.1 Shuffle Argument

In the following pseudo-code algorithm, we will generate an argument of knowledge of a permutation $\pi \in \Sigma_N$ and randomness $\rho \in \mathbb{Z}_q^N$ such that for given ciphertexts $\vec{C} \in (\mathbb{H}_\ell)^N$ and $\vec{C}' \in (\mathbb{H}_\ell)^N$ it holds that for all $i \in [0, N)$:

$$\vec{C}'_i = \text{GetCiphertextProduct}(\text{GetCiphertext}(\vec{1}, \rho_i, \mathbf{pk}), \vec{C}_{\pi(i)})$$

Algorithm 7.11 GetShuffleArgument: compute a cryptographic argument for the validity of the shuffle

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$
 A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
 A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

The statement composed of
 - The incoming list of ciphertexts $\vec{C} \in (\mathbb{H}_\ell)^N$ s.t. $0 < \ell \leq k$
 - The shuffled and re-encrypted list of ciphertexts $\vec{C}' \in (\mathbb{H}_\ell)^N$
 The witness composed of
 - permutation $\pi \in \Sigma_N$
 - randomness $\vec{\rho} \in \mathbb{Z}_q^N$

The number of rows to use for ciphertext matrices $m \in \mathbb{N}^+$

The number of columns to use for ciphertext matrices $n \in \mathbb{N}^+$ s.t. $2 \leq n \leq \nu$

Require: $\forall i \in [0, N) : \vec{C}'_i = \text{GetCiphertextProduct}(\text{GetCiphertext}(\vec{1}, \rho_i, \mathbf{pk}), \vec{C}_{\pi(i)})$

Require: $N = mn$

Operation:

```

1:  $\mathbf{r} \leftarrow \text{GenRandomVector}(q, m)$  ▷ See algorithm 4.2
2:  $A \leftarrow \text{Transpose}(\text{ToMatrix}(\{\pi(i)\}_{i=0}^{N-1}, m, n))$  ▷ Create a  $n \times m$  matrix. See algorithm 7.14 and algorithm 7.13
3:  $\mathbf{c}_A \leftarrow \text{GetCommitmentMatrix}(A, \mathbf{r}, \mathbf{ck})$  ▷ See algorithm 7.8
4:  $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, \vec{C}, \vec{C}', \mathbf{c}_A))$ 
5:  $\mathbf{s} \leftarrow \text{GenRandomVector}(q, m)$ 
6:  $\mathbf{b} \leftarrow \{x^{\pi(i)}\}_{i=0}^{N-1}$ 
7:  $B \leftarrow \text{Transpose}(\text{ToMatrix}(\mathbf{b}, m, n))$ 
8:  $\mathbf{c}_B \leftarrow \text{GetCommitmentMatrix}(B, \mathbf{s}, \mathbf{ck})$ 
9:  $y \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(\mathbf{c}_B, p, q, \mathbf{pk}, \mathbf{ck}, \vec{C}, \vec{C}', \mathbf{c}_A))$ 
10:  $z \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}("1", \mathbf{c}_B, p, q, \mathbf{pk}, \mathbf{ck}, \vec{C}, \vec{C}', \mathbf{c}_A))$ 
▷ Both  $\vec{C}$  and  $\vec{C}'$  are passed in the vector forms here
11:  $\text{Zneg} \leftarrow \text{Transpose}(\text{ToMatrix}(\{-z\}_{i=1}^N, m, n))$  ▷ Vector of length  $N$ , with all values being  $q - z$ 
12:  $\mathbf{c}_{-z} \leftarrow \text{GetCommitmentMatrix}(\text{Zneg}, \vec{0}, \mathbf{ck})$  ▷ A vector of length  $m$ , with all 0 values
13:  $\mathbf{c}_D \leftarrow \mathbf{c}_A^y \mathbf{c}_B$  ▷ Entry-wise product
14:  $D \leftarrow yA + B$ 
15:  $\mathbf{t} \leftarrow y\mathbf{r} + \mathbf{s}$ 
16:  $b \leftarrow \prod_{i=0}^{N-1} (yi + x^i - z)$ 
17:  $\mathbf{pStatement} \leftarrow (\mathbf{c}_D \mathbf{c}_{-z}, b)$ 
18:  $\mathbf{pWitness} \leftarrow (D + \text{Zneg}, \mathbf{t})$ 
19:  $\text{productArgument} \leftarrow \text{GetProductArgument}(\mathbf{pStatement}, \mathbf{pWitness})$  ▷ See algorithm 7.18
20:  $\rho \leftarrow q - (\vec{\rho} \cdot \mathbf{b})$  ▷ Standard inner product  $\sum_{i=0}^{N-1} \rho_i b_i$ 
21:  $\vec{x} \leftarrow \{x^i\}_{i=0}^{N-1}$ 
22:  $C \leftarrow \text{GetCiphertextVectorExponentiation}(\vec{C}, \vec{x})$  ▷ See algorithm 6.7
23:  $\mathbf{mStatement} \leftarrow (\text{ToMatrix}(\vec{C}', m, n), C, \mathbf{c}_B)$  ▷ See algorithm 7.13
24:  $\mathbf{mWitness} \leftarrow (B, \mathbf{s}, \rho)$ 
25:  $\text{multiExponentiationArgument} \leftarrow \text{GetMultiExponentiationArgument}(\mathbf{mStatement}, \mathbf{mWitness})$  ▷ See algorithm 7.15

```

Output:

shuffleArgument $(\mathbf{c}_A, \mathbf{c}_B, \text{productArgument}, \text{multiExponentiationArgument}) \in \mathbb{G}_q^m \times \mathbb{G}_q^m \times \dots \times \dots$
▷ See algorithm 7.18 and algorithm 7.15 for their respective domains

In the following pseudo-code algorithm, we verify that a provided Shuffle argument adequately supports the corresponding statement.

Algorithm 7.12 VerifyShuffleArgument: Verify a cryptographic argument for the validity of the shuffle

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The statement composed of
 - The incoming list of ciphertexts $\vec{C} \in (\mathbb{H}_\ell)^N$ s.t. $0 < \ell \leq k$
 - The shuffled and re-encrypted list of ciphertexts $\vec{C}' \in (\mathbb{H}_\ell)^N$
- The argument composed of
 - the commitment vector $\mathbf{c}_A \in \mathbb{G}_q^m$
 - the commitment vector $\mathbf{c}_B \in \mathbb{G}_q^m$
 - a productArgument ▷ See algorithm 7.18
 - a multiExponentiationArgument ▷ See algorithm 7.15
- The number of rows to use for ciphertext matrices $m \in \mathbb{N}^+$
- The number of columns to use for ciphertext matrices $n \in \mathbb{N}^+$ s.t. $2 \leq n \leq \nu$

Require: $N = mn$

Operation:

- 1: $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, \vec{C}, \vec{C}', \mathbf{c}_A))$
 - 2: $y \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(\mathbf{c}_B, p, q, \mathbf{pk}, \mathbf{ck}, \vec{C}, \vec{C}', \mathbf{c}_A))$
 - 3: $z \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}("1", \mathbf{c}_B, p, q, \mathbf{pk}, \mathbf{ck}, \vec{C}, \vec{C}', \mathbf{c}_A))$ ▷ Both \vec{C} and \vec{C}' are passed as vectors
 - 4: $\text{Zneg} \leftarrow \text{Transpose}(\text{ToMatrix}(\{-z\}_{i=1}^N, m, n))$ ▷ Vector of length N , with all values being $q - z$, see algorithm 7.14 and algorithm 7.13
 - 5: $\mathbf{c}_{-z} \leftarrow \text{GetCommitmentMatrix}(\text{Zneg}, \vec{0}, \mathbf{ck})$ ▷ See algorithm 7.8
 - 6: $\mathbf{c}_D \leftarrow \mathbf{c}_A^y \mathbf{c}_B$
 - 7: $b \leftarrow \prod_{i=1}^N yi + x^i - z$
 - 8: $\mathbf{pStatement} \leftarrow (\mathbf{c}_D \mathbf{c}_{-z}, b)$
 - 9: $\text{productVerif} \leftarrow \text{VerifyProductArgument}(\mathbf{pStatement}, \text{productArgument})$ ▷ See algorithm 7.19
 - 10: $\vec{x} \leftarrow \{x^i\}_{i=0}^{N-1}$
 - 11: $C \leftarrow \text{GetCiphertextVectorExponentiation}(\vec{C}, \vec{x})$ ▷ See algorithm 6.7
 - 12: $\mathbf{mStatement} \leftarrow (\text{ToMatrix}(\vec{C}', m, n), C, \mathbf{c}_B)$ ▷ See algorithm 7.13
 - 13: $\text{multiVerif} \leftarrow \text{VerifyMultiExponentiationArgument}(\mathbf{mStatement}, \text{multiExponentiationArgument})$ ▷ See algorithm 7.16
 - 14: **if** $\text{productVerif} \wedge \text{multiVerif}$ **then**
 - 15: **return** \top
 - 16: **else**
 - 17: **return** \perp
 - 18: **end if**
-

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 7.12 are provided in the attached [verify-shuffle-argument.json](#) file.

One of the key features of Bayer-Groth's[3] minimal shuffle argument is the transformation of a vector of ciphertexts into a $m \times n$ matrix, by means of which a prover's computation can be optimized. Therefore, the ciphertexts, received as a vector, need to be organized into a matrix. This will be achieved by setting $M_{i,j} = \vec{v}_{ni+j}$. Similarly, the exponents in the matrices A and B need to be arranged into matrices so that the other algorithms get the expected values. However, since exponents' matrices have their dimensions transposed with respect to the ciphertexts, we obtain $M_{i,j} = \vec{v}_{i+nj}$.

For completeness, we describe below the operations of organizing the elements of a vector into a matrix and the transposition of a matrix.

Algorithm 7.13 ToMatrix: convert a vector of elements to a $m \times n$ matrix

Input:

- a vector of elements $\vec{v} \in \mathbb{D}^N$
- the number of requested rows $m \in \mathbb{N}^+$
- the number of requested columns $n \in \mathbb{N}^+$

Require: $N = m \cdot n$

Operation:

- 1: **for** $i \in [0, m)$ **do**
 - 2: **for** $j \in [0, n)$ **do**
 - 3: $M_{i,j} \leftarrow \vec{v}_{ni+j}$
 - 4: **end for**
 - 5: **end for**
-

Output:

The matrix $M = (M_{i,j})_{i,j=0}^{m,n} \in \mathbb{D}^{m \times n}$

Algorithm 7.14 Transpose: transpose a $m \times n$ matrix to a $n \times m$ matrix

Input:

- a matrix of elements $M \in \mathbb{D}^{m \times n}$ s.t. $m, n > 0$
-

Operation:

- 1: **for** $i \in [0, n)$ **do**
 - 2: **for** $j \in [0, m)$ **do**
 - 3: $N_{i,j} \leftarrow M_{j,i}$
 - 4: **end for**
 - 5: **end for**
-

Output:

The matrix $N = (N_{i,j})_{i,j=0}^{n,m} \in \mathbb{D}^{n \times m}$

7.3.2 Multi-Exponentiation Argument

Given ciphertexts $C_{0,0}, \dots, C_{m-1,n-1}$ and C , each $\in \mathbb{H}_\ell$, the algorithm below generates an argument of knowledge of the openings to the commitments \vec{c}_A to $A = \{a_{i,j}\}_{i,j=1}^{n,m}$ such that

$$C = \text{GetCiphertext}(\vec{1}, \rho, \mathbf{pk}) \cdot \prod_{i=0} \vec{C}_i^{\vec{a}_{i+1}}$$

$$\vec{c}_A = \text{GetCommitmentMatrix}(A, \vec{r}, \mathbf{ck})$$

where $\vec{C}_i = (C_{i,0}, \dots, C_{i,n-1})$ and $\vec{a}_j = (a_{1,j}, \dots, a_{n,j})^T$, that is \vec{C}_i refers to the i^{th} row of the matrix, whereas \vec{a}_j refers to the j^{th} column. Furthermore, we use 0-based indexing for C here, which is consistent with the rest of this document, but 1-based indexing for a above, as well as \vec{a} and \mathbf{r} below, allowing for the generation of \vec{a}_0 and r_0 within the pseudo-code.

Algorithm 7.15 GetMultiExponentiationArgument: Compute a multi-exponentiation argument

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$
 A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
 A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

The statement composed of
 - ciphertext matrix $(\vec{C}_0, \dots, \vec{C}_{m-1}) \in (\mathbb{H}_\ell)^{m \times n}$ s.t. $0 < \ell \leq k$ ▷ \vec{C}_i refers to the i^{th} row
 - ciphertext $C \in \mathbb{H}_\ell$
 - commitment vector $\vec{c}_A \in \mathbb{G}_q^m$
 The witness composed of
 - matrix $A = (\vec{a}_1, \dots, \vec{a}_m) \in \mathbb{Z}_q^{n \times m}$ s.t. $n \leq \nu$ ▷ \vec{a}_j refers to the j^{th} column
 - exponents $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}_q^m$
 - exponent $\rho \in \mathbb{Z}_q$

Require: $C = \text{GetCiphertext}(\vec{1}, \rho, \mathbf{pk}) \cdot \prod_{i=0}^{m-1} \vec{C}_i^{\vec{a}_{i+1}}$ ▷ Vector of 1s of length ℓ ▷ See algorithm 6.5, algorithm 6.7, algorithm 6.8

Require: $\vec{c}_A = \text{GetCommitmentMatrix}(A, \vec{r}, \mathbf{ck})$

Require: $n, m > 0$

Operation:

```

1:  $\vec{a}_0 \leftarrow \text{GenRandomVector}(q, n)$  ▷ See algorithm 4.2
2:  $r_0 \leftarrow \text{GenRandomInteger}(q)$  ▷ See algorithm 4.1
3:  $(b_0, \dots, b_{2 \cdot m-1}) \leftarrow \text{GenRandomVector}(q, 2 \cdot m)$ 
4:  $(s_0, \dots, s_{2 \cdot m-1}) \leftarrow \text{GenRandomVector}(q, 2 \cdot m)$ 
5:  $(\tau_0, \dots, \tau_{2 \cdot m-1}) \leftarrow \text{GenRandomVector}(q, 2 \cdot m)$ 
6:  $b_m \leftarrow 0$ 
7:  $s_m \leftarrow 0$ 
8:  $\tau_m \leftarrow \rho$  ▷ Ensuring  $c_{B_m} = \text{GetCommitment}(0, 0, \mathbf{ck})$  and
    $\text{GetCiphertext}(\vec{g}^{b_m}, \tau_m, \mathbf{pk}) = \text{GetCiphertext}(\vec{1}, \rho, \mathbf{pk})$ 
9:  $c_{A_0} \leftarrow \text{GetCommitment}(\vec{a}_0, r_0, \mathbf{ck})$  ▷ See algorithm 7.7
10:  $(D_0, \dots, D_{2 \cdot m-1}) \leftarrow \text{GetDiagonalProducts}((\vec{C}_0, \dots, \vec{C}_{m-1}), (\vec{a}_0, \dots, \vec{a}_m))$  ▷ See algorithm 7.17
11: for  $k \in [0, 2 \cdot m)$  do
12:    $c_{B_k} \leftarrow \text{GetCommitment}((b_k), s_k, \mathbf{ck})$ 
13:    $E_k \leftarrow \text{GetCiphertext}(\vec{g}^{b_k}, \tau_k, \mathbf{pk}) \cdot D_k$  ▷ See algorithm 6.5, we take a vector of messages of length  $\ell$  each having value  $g^{b_k}$ 
14: end for
15:  $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, (\vec{C}_i)_{i=0}^{m-1}, C, \vec{c}_A, c_{A_0}, (c_{B_k})_{k=0}^{2 \cdot m-1}, (E_k)_{k=0}^{2 \cdot m-1}))$ 
   ▷ All operations below are performed modulo  $q$ 
16:  $\vec{a} \leftarrow \vec{a}_0 + \sum_{i=1}^m x^i \vec{a}_i$ 
17:  $r \leftarrow r_0 + \sum_{i=1}^m x^i r_i$ 
18:  $b \leftarrow b_0 + \sum_{k=1}^{2 \cdot m-1} x^k b_k$ 
19:  $s \leftarrow s_0 + \sum_{k=1}^{2 \cdot m-1} x^k s_k$ 
20:  $\tau \leftarrow \tau_0 + \sum_{k=1}^{2 \cdot m-1} x^k \tau_k$ 

```

Output:

The argument $(c_{A_0}, (c_{B_k})_{k=0}^{2 \cdot m-1}, (E_k)_{k=0}^{2 \cdot m-1}, \vec{a}, r, b, s, \tau) \in \mathbb{G}_q \times \mathbb{G}_q^{2 \cdot m} \times \mathbb{H}_\ell^{2 \cdot m} \times \mathbb{Z}_q^n \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$

In the following pseudo-code algorithm, we verify that a provided Multi-Exponentiation argument adequately supports the corresponding statement.

Algorithm 7.16 VerifyMultiExponentiationArgument: Verify a multi-exponentiation argument

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The statement composed of
 - ciphertext matrix $(\vec{C}_0, \dots, \vec{C}_{m-1}) \in (\mathbb{H}_\ell)^{m \times n}$ $\triangleright \vec{C}_i$ refers to the i^{th} row
 - ciphertext $C \in \mathbb{H}_\ell$
 - commitment vector $\vec{c}_A = (c_{A_1}, \dots, c_{A_m}) \in \mathbb{G}_q^m$
- The argument composed of
 - the commitment $c_{A_0} \in \mathbb{G}_q$
 - the commitment vector $\mathbf{c}_B = (c_{B_0}, \dots, c_{B_{2 \cdot m-1}}) \in \mathbb{G}_q^{2 \cdot m}$
 - the ciphertext vector $\mathbf{E} = (E_0, \dots, E_{2 \cdot m-1}) \in \mathbb{H}_\ell^{2 \cdot m}$
 - the vector of exponents $\vec{a} = (a_0, \dots, a_{n-1}) \in \mathbb{Z}_q^n$
 - the exponent $r \in \mathbb{Z}_q$
 - the exponent $b \in \mathbb{Z}_q$
 - the exponent $s \in \mathbb{Z}_q$
 - the exponent $\tau \in \mathbb{Z}_q$

Require: $n, m > 0$

Operation:

- 1: $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, \{\vec{C}_i\}_{i=0}^{m-1}, C, \vec{c}_A, c_{A_0}, \{c_{B_k}\}_{k=0}^{2 \cdot m-1}, \{E_k\}_{k=0}^{2 \cdot m-1}))$
 - 2: $\text{verifCbm} \leftarrow c_{B_m} = 1$
 - 3: $\text{verifEm} \leftarrow E_m = C$
 - 4: $\text{prodCa} \leftarrow c_{A_0} \prod_{i=1}^m c_{A_i}^{x^i}$
 - 5: $\text{commA} \leftarrow \text{GetCommitment}(\vec{a}, r, \mathbf{ck})$ \triangleright See algorithm 7.7
 - 6: $\text{verifA} \leftarrow \text{prodCa} = \text{commA}$
 - 7: $\text{prodCb} \leftarrow c_{B_0} \prod_{k=1}^{2 \cdot m-1} c_{B_k}^{x^k}$
 - 8: $\text{commB} \leftarrow \text{GetCommitment}((b), s, \mathbf{ck})$
 - 9: $\text{verifB} \leftarrow \text{prodCb} = \text{commB}$
 - 10: $\text{prodE} \leftarrow E_0 \prod_{k=1}^{2 \cdot m-1} E_k^{x^k}$
 - 11: $\text{encryptedGb} \leftarrow \text{GetCiphertext}(\vec{g}^b, \tau, \mathbf{pk})$ \triangleright See algorithm 6.5, we take a vector of messages of length ℓ each having value g^b
 - 12: $\text{prodC} \leftarrow \prod_{i=0}^{m-1} \text{GetCiphertextVectorExponentiation}(\vec{C}_i, x^{(m-i)-1} \vec{a})$ \triangleright See algorithm 6.7
 - 13: $\text{verifEC} \leftarrow \text{prodE} = \text{GetCiphertextProduct}(\text{encryptedGb}, \text{prodC})$ \triangleright See algorithm 6.8
 - 14: **if** $\text{verifCbm} \wedge \text{verifEm} \wedge \text{verifA} \wedge \text{verifB} \wedge \text{verifEC}$ **then**
 - 15: **return** \top
 - 16: **else**
 - 17: **return** \perp
 - 18: **end if**
-

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 7.16 are provided in the attached [verify-multiexp-argument.json](#) file.

Algorithm 7.17 GetDiagonalProducts: Compute the products of the diagonals of a ciphertext matrix

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$

Input:

- Ciphertext matrix $C = (\vec{C}_0, \dots, \vec{C}_{m-1}) \in (\mathbb{H}_\ell)^{m \times n}$ $\triangleright \vec{C}_i$ refers to the i^{th} row
 - Exponent matrix $A = (\vec{a}_0, \dots, \vec{a}_m) \in \mathbb{Z}_q^{n \times (m+1)}$ $\triangleright \vec{a}_j$ refers to the j^{th} column
-

Operation:

- 1: **for** $k \in [0, 2 \cdot m)$ **do**
 - 2: $d_k \leftarrow \text{GetCiphertext}(\vec{1}, 0, \mathbf{pk})$ \triangleright Neutral element of ciphertext multiplication
 - 3: **if** $k < m$ **then**
 - 4: lowerbound $\leftarrow m - k - 1$
 - 5: upperbound $\leftarrow m$
 - 6: **else**
 - 7: lowerbound $\leftarrow 0$
 - 8: upperbound $\leftarrow 2 \cdot m - k$
 - 9: **end if**
 - 10: **for** $i \in [\text{lowerbound}, \text{upperbound})$ **do**
 - 11: $j \leftarrow k - m + i + 1$
 - 12: $d_k \leftarrow \text{GetCiphertextProduct}(d_k, \text{GetCiphertextVectorExponentiation}(\vec{C}_i, \vec{a}_j))$ \triangleright
 - 13: See algorithm 6.8 and algorithm 6.7
 - 14: **end for**
 - 15: **end for**
-

Output:

- Diagonal products $D = (d_0, \dots, d_{2 \cdot m - 1}) \in \mathbb{H}_\ell^{2 \cdot m}$
-

7.3.3 Product Argument

The following algorithm provides an argument that a set of committed values have a particular product.

More precisely, given commitments $\vec{c}_A = (c_{A_0}, \dots, c_{A_m})$ to $A = \{a_{i,j}\}_{i,j=0}^{n-1,m-1}$ and a value b , we want to give an argument of knowledge for $\prod_{i=0}^{n-1} \prod_{j=0}^{m-1} a_{i,j} = b$.

We will first compute a commitment c_b as follows:

$$c_b = \text{GetCommitment} \left(\left(\prod_{j=0}^{m-1} a_{0,j}, \dots, \prod_{j=0}^{m-1} a_{n-1,j} \right), s, \mathbf{ck} \right)$$

We will then give an argument that c_b is correct, using a Hadamard argument (see section 7.3.4), showing that the values committed in c_b are indeed the result of the Hadamard product of the values committed in c_A . Additionally, we will show that the value b is the product of the values committed in c_b , using a Single Value Product Argument (see section 7.3.6).

If the number of ciphertexts to be shuffled is prime and they cannot be arranged into a matrix, $m = 1$ and $n = N$, the Hadamard Product is trivially equal to the first (and only) column of the matrix and we can omit the Hadamard argument, calling the Single Value Product argument directly.

Algorithm 7.18 GetProductArgument: Computes a Product Argument

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$
 A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
 A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

The statement composed of
 - commitments $\vec{c}_A = (c_{A_1}, \dots, c_{A_m}) \in \mathbb{G}_q^m$
 - the product $b \in \mathbb{Z}_q$
 The witness composed of
 - the matrix $A \in \mathbb{Z}_q^{n \times m}$
 - the exponents $\vec{r} \in \mathbb{Z}_q^m$

Require: $2 \leq n \leq \nu$

Require: $m > 0$

Require: $\vec{c}_A = \text{GetCommitmentMatrix}(A, \vec{r}, \mathbf{ck})$

▷ See algorithm 7.8

Require: $b = \prod_{i=0}^{n-1} \prod_{j=0}^{m-1} a_{i,j} \mod q$

Operation:

```

1: if  $m > 1$  then
2:    $s \leftarrow \text{GenRandomInteger}(q)$ 
3:   for  $i \in [0, n)$  do
4:      $b_i \leftarrow \prod_{j=0}^{m-1} a_{i,j}$ 
5:   end for
6:    $c_b \leftarrow \text{GetCommitment}((b_0, \dots, b_{n-1}), s, \mathbf{ck})$ 
7:    $\text{hStatement} \leftarrow (\vec{c}_A, c_b)$ 
8:    $\text{hWitness} \leftarrow (A, (b_0, \dots, b_{n-1}), \vec{r}, s)$ 
9:    $\text{hadamardArg} \leftarrow \text{GetHadamardArgument}(\text{hStatement}, \text{hWitness})$  ▷ See algorithm 7.20
10:   $\text{sStatement} \leftarrow (c_b, b)$ 
11:   $\text{sWitness} \leftarrow ((b_0, \dots, b_{n-1}), s)$ 
12:   $\text{singleValueProdArg} \leftarrow \text{GetSingleValueProductArgument}(\text{sStatement}, \text{sWitness})$  ▷ See algorithm 7.25
13: else
14:    $\text{sStatement} \leftarrow (c_{A_1}, b)$ 
15:    $\text{sWitness} \leftarrow (\vec{a}_0, r_0)$ 
16:    $\text{singleValueProdArg} \leftarrow \text{GetSingleValueProductArgument}(\text{sStatement}, \text{sWitness})$  ▷ See algorithm 7.25
17: end if
    
```

Output:

```

18: if  $m > 1$  then
19:    $(c_b, \text{hadamardArg}, \text{singleValueProdArg}) \in \mathbb{G}_q \times \text{HadamardArgument} \times \text{SingleValueProductArgument}$ 
20:    $(c_b, \text{hadamardArg}, \text{singleValueProdArg}) \in \mathbb{G}_q \times \text{HadamardArgument} \times \text{SingleValueProductArgument}$  ▷ See algorithm 7.20 and algorithm 7.25 for the domains
19: else
20:    $\text{singleValueProdArg}$  ▷ See algorithm 7.25 for the domain
20: end if
    
```

In the following pseudo-code algorithm, we verify if a provided Product argument supports the corresponding statement.

Algorithm 7.19 VerifyProductArgument: Verify a Product argument

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The statement composed of
 - commitments $\vec{c}_A = (c_{A_1}, \dots, c_{A_m}) \in \mathbb{G}_q^m$
 - the product $b \in \mathbb{Z}_q$
- The argument composed of
 - the commitment $c_b \in \mathbb{G}_q$ ▷ omitted if $m = 1$
 - a hadamardArg $\in \mathbb{G}_q^{m+1} \times (\mathbb{G}_q \times \mathbb{G}_q \times \mathbb{G}_q^{2 \cdot m+1} \times \mathbb{Z}_q^n \times \mathbb{Z}_q^n \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q)$ ▷ omitted if $m = 1$
 - a singleValueProductArg $\in \mathbb{G}_q \times \mathbb{G}_q \times \mathbb{G}_q \times \mathbb{Z}_q^n \times \mathbb{Z}_q^n \times \mathbb{Z}_q \times \mathbb{Z}_q$

Require: $n \leq \nu$

Operation:

```

1: if  $m > 1$  then
2:   hStatement  $\leftarrow (\vec{c}_A, c_b)$ 
3:   sStatement  $\leftarrow (c_b, b)$ 
4:   if VerifyHadamardArgument(hStatement, hadamardArg)  $\wedge$ 
5:   VerifySingleValueProductArgument(sStatement, singleValueProductArg) then
6:     return  $\top$ 
7:   else
8:     return  $\perp$ 
9:   end if
10: else
11:   sStatement  $\leftarrow (c_{A_1}, b)$ 
12:   if VerifySingleValueProductArgument(sStatement, singleValueProductArg) then
13:     return  $\top$ 
14:   else
15:     return  $\perp$ 
16:   end if
17: end if

```

▷ See algorithm 7.21 and algorithm 7.26

▷ See algorithm 7.26

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 7.19 are provided in the attached [verify-p-argument.json](#) file.

7.3.4 Hadamard Argument

The operations given in the algorithm below are more readable using vector notation. That is, we note \vec{a} for the vector $(a_0, \dots, a_{n-1}) \in \mathbb{Z}_q^n$. By extension, we denote matrix $a_{0,0}, \dots, a_{n-1,m-1}$ as $\vec{a}_0, \dots, \vec{a}_{m-1}$ where each vector corresponds to a column of the matrix.

In the following algorithm, we generate an argument of knowledge of the openings $\vec{a}_0, \dots, \vec{a}_{m-1}$ and \vec{b} to the commitments \mathbf{c}_A and c_b , such that:

$$\begin{aligned} \mathbf{c}_A &= \text{GetCommitmentMatrix}((\vec{a}_0, \dots, \vec{a}_{m-1}), \mathbf{r}, \mathbf{ck}) \\ c_b &= \text{GetCommitment}((b_0, \dots, b_{n-1}), s, \mathbf{ck}) \\ b_i &= \prod_{j=0}^{m-1} a_{i,j} \text{ for } i = 0, \dots, n-1 \end{aligned}$$

where the product in the last line matches the entry-wise product, also known as Hadamard product.

The subsequent pseudo-code algorithm verifies if a provided Hadamard argument supports the corresponding statement.

Algorithm 7.20 GetHadamardArgument: Computes a Hadamard Argument

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$
 A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
 A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

The statement composed of
 - commitment $\mathbf{c}_A = (c_{A_0}, \dots, c_{A_{m-1}}) \in \mathbb{G}_q^m$
 - commitment $c_b \in \mathbb{G}_q$
 The witness composed of
 - matrix $A = (\vec{a}_0, \dots, \vec{a}_{m-1}) \in \mathbb{Z}_q^{n \times m}$
 - vector $\mathbf{b} \in \mathbb{Z}_q^n$
 - exponents $\mathbf{r} = (r_0, \dots, r_{m-1}) \in \mathbb{Z}_q^m$
 - exponent $s \in \mathbb{Z}_q$

Require: $m \geq 2$

▷ Hadamard product only makes sense for $m \geq 2$

Require: $0 < n \leq \nu$

Require: $\mathbf{c}_A = \text{GetCommitmentMatrix}(A, \mathbf{r}, \mathbf{ck})$

▷ See algorithm 7.8

Require: $c_b = \text{GetCommitment}(\mathbf{b}, s, \mathbf{ck})$

▷ See algorithm 7.7

Require: $\vec{b} = \prod_{j=0}^{m-1} \vec{a}_j$

▷ Uses the Hadamard product, ie $b_i = \prod_{j=0}^{m-1} a_{i,j}$

Operation:

```

1: for  $j \in [0, m)$  do
2:    $\vec{b}_j \leftarrow \prod_{i=0}^j \vec{a}_i$ 
3: end for
4:  $s_0 \leftarrow r_0$ 
5: if  $m > 2$  then
6:    $(s_1, \dots, s_{m-2}) \leftarrow \text{GenRandomVector}(q, m-2)$ 
7: end if
8:  $s_{m-1} \leftarrow s$ 
9:  $c_{B_0} \leftarrow c_{A_0}$ 
10: for  $j \in [1, m-1)$  do
11:    $c_{B_j} \leftarrow \text{GetCommitment}(\vec{b}_j, s_j, \mathbf{ck})$ 
12: end for
13:  $c_{B_{m-1}} \leftarrow c_b$ 
14:  $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, \mathbf{c}_A, c_b, (c_{B_0}, \dots, c_{B_{m-1}})))$ 
15:  $y \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}("1", p, q, \mathbf{pk}, \mathbf{ck}, \mathbf{c}_A, c_b, (c_{B_0}, \dots, c_{B_{m-1}})))$ 
    ▷ Use  $y$  to define  $\star : \mathbb{Z}_q^n \times \mathbb{Z}_q^n \mapsto \mathbb{Z}_q$ , see algorithm 7.10
    ▷ All exponentiations of  $x$  below are performed modulo  $q$ 

16: for  $i \in [0, m-1)$  do
17:    $\vec{d}_i = x^{i+1} \vec{b}_i$ 
18:    $c_{D_i} = c_{B_i}^{x^{i+1}}$ 
19:    $t_i = x^{i+1} s_i$ 
20: end for
21:  $\vec{d} \leftarrow \sum_{i=1}^{m-1} x^i \vec{b}_i$ 
22:  $c_D \leftarrow \prod_{i=1}^{m-1} c_{B_i}^{x^i}$ 
23:  $t \leftarrow \sum_{i=1}^{m-1} x^i s_i$ 
24:  $-\vec{1} \leftarrow (q-1, \dots, q-1) \in \mathbb{Z}_q^n$ 
25:  $c_{-1} \leftarrow \text{GetCommitment}(-\vec{1}, 0, \mathbf{ck})$ 
26:  $\text{statement} \leftarrow ((c_{A_1}, \dots, c_{A_{m-1}}, c_{-1}), (c_{D_0}, \dots, c_{D_{m-2}}, c_D), y)$ 
27:  $\text{witness} \leftarrow ((\vec{a}_1, \dots, \vec{a}_{m-1}, -\vec{1}), (\vec{d}_0, \dots, \vec{d}_{m-2}, \vec{d}), (r_1, \dots, r_{m-1}, 0), (t_0, \dots, t_{m-2}, t))$ 
28:  $\text{zeroArg} \leftarrow \text{GetZeroArgument}(\text{statement}, \text{witness})$ 
    ▷ See algorithm 7.22
    ▷ Provide an argument that  $\sum_{i=0}^{m-2} \vec{a}_{i+1} \star \vec{d}_i - \vec{1} \star \vec{d} = 0$ 

```

Output:

$((c_{B_0}, \dots, c_{B_{m-1}}), \text{zeroArg}) \in \mathbb{G}_q^m \times (\mathbb{G}_q \times \mathbb{G}_q \times \mathbb{G}_q^{2 \cdot m+1} \times \mathbb{Z}_q^n \times \mathbb{Z}_q^n \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q)$

Algorithm 7.21 VerifyHadamardArgument: Verifies a Hadamard Argument

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The statement composed of
 - commitment $\mathbf{c}_A = (c_{A_0}, \dots, c_{A_{m-1}}) \in \mathbb{G}_q^m$
 - commitment $c_b \in \mathbb{G}_q$
- The argument composed of
 - commitment vector $\mathbf{c}_B = (c_{B_0}, \dots, c_{B_{m-1}}) \in \mathbb{G}_q^m$
 - a zero argument, composed of
 - commitment $c_{A_0} \in \mathbb{G}_q$
 - commitment $c_{B_m} \in \mathbb{G}_q$
 - commitment vector $\mathbf{c}_d \in \mathbb{G}_q^{2 \cdot m+1}$
 - exponent vector $\mathbf{a}' \in \mathbb{Z}_q^n$
 - exponent vector $\mathbf{b}' \in \mathbb{Z}_q^n$
 - exponent $r' \in \mathbb{Z}_q$
 - exponent $s' \in \mathbb{Z}_q$
 - exponent $t' \in \mathbb{Z}_q$

Require: $n > 0$

Operation:

- 1: $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, \mathbf{c}_A, c_b, (c_{B_0}, \dots, c_{B_{m-1}})))$
 - 2: $y \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}("1", p, q, \mathbf{pk}, \mathbf{ck}, \mathbf{c}_A, c_b, (c_{B_0}, \dots, c_{B_{m-1}})))$
 - 3: **for** $i \in [0, m-1)$ **do**
 - 4: $c_{D_i} \leftarrow c_{B_i}^{x^{i+1}}$
 - 5: **end for**
 - 6: $c_D \leftarrow \prod_{i=1}^{m-1} c_{B_i}^{x^i}$
 - 7: $-\vec{1} \leftarrow (q-1, \dots, q-1) \in \mathbb{Z}_q^n$
 - 8: $c_{-1} \leftarrow \text{GetCommitment}(-\vec{1}, 0, \mathbf{ck})$
 - 9: $\text{zeroStatement} \leftarrow ((c_{A_1}, \dots, c_{A_{m-1}}, c_{-1}), (c_{D_0}, \dots, c_{D_{m-2}}, c_D), y)$
 - 10: $\text{zeroArgument} \leftarrow (c_{A_0}, c_{B_m}, \mathbf{c}_d, \mathbf{a}', \mathbf{b}', r', s', t')$
 - 11: **if** $c_{B_0} = c_{A_0} \wedge c_{B_{m-1}} = c_b \wedge \text{VerifyZeroArgument}(\text{zeroStatement}, \text{zeroArgument})$ **then**
▷ See algorithm 7.23
 - return** \top
 - 12: **else**
 - return** \perp
 - 13: **end if**
-

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 7.21 are provided in the attached [verify-h-argument.json](#) file.

7.3.5 Zero Argument

In the following algorithm, we generate an argument of knowledge of the values $\mathbf{a}_1, \mathbf{b}_0, \dots, \mathbf{a}_m, \mathbf{b}_{m-1}$ such that $\sum_{i=1}^m \mathbf{a}_i \star \mathbf{b}_{i-1} = 0$.

Algorithm 7.22 GetZeroArgument: Computes a Zero Argument

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The statement composed of
 - commitments $\mathbf{c}_A \in \mathbb{G}_q^m$
 - commitments $\mathbf{c}_B \in \mathbb{G}_q^m$
 - the value $y \in \mathbb{Z}_q$ defining the bilinear mapping \star ▷ See algorithm 7.10
- The witness composed of
 - matrix $A = (\vec{a}_1, \dots, \vec{a}_m) \in \mathbb{Z}_q^{n \times m}$ ▷ The \vec{a}_i values correspond to the columns of A
 - matrix $B = (\vec{b}_0, \dots, \vec{b}_{m-1}) \in \mathbb{Z}_q^{n \times m}$ ▷ The \vec{b}_i values correspond to the columns of B
 - vector of exponents $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}_q^m$
 - vector of exponents $\mathbf{s} = (s_0, \dots, s_{m-1}) \in \mathbb{Z}_q^m$

Require: $\mathbf{c}_A = \text{GetCommitmentMatrix}(A, \mathbf{r}, \mathbf{ck})$ ▷ See algorithm 7.8

Require: $\mathbf{c}_B = \text{GetCommitmentMatrix}(B, \mathbf{s}, \mathbf{ck})$ ▷ See algorithm 7.8

Require: $\sum_{i=1}^m \mathbf{a}_i \star \mathbf{b}_{i-1} = 0$

Require: $n, m > 0$

Operation:

- 1: $\vec{a}_0 \leftarrow \text{GenRandomVector}(q, n)$ ▷ See algorithm 4.2
 - 2: $\vec{b}_m \leftarrow \text{GenRandomVector}(q, n)$
 - 3: $r_0 \leftarrow \text{GenRandomInteger}(q)$
 - 4: $s_m \leftarrow \text{GenRandomInteger}(q)$
 - 5: $c_{A_0} \leftarrow \text{GetCommitment}(\vec{a}_0, r_0, \mathbf{ck})$
 - 6: $c_{B_m} \leftarrow \text{GetCommitment}(\vec{b}_m, s_m, \mathbf{ck})$
 - 7: $\mathbf{d} = (d_0, \dots, d_{2 \cdot m}) \leftarrow \text{ComputeDVector}((\vec{a}_0, \dots, \vec{a}_m), (\vec{b}_0, \dots, \vec{b}_m), y)$ ▷ See algorithm 7.24
 - 8: $\mathbf{t} \leftarrow \text{GenRandomVector}(q, 2 \cdot m + 1)$
 - 9: $t_{m+1} \leftarrow 0$
 - 10: $\mathbf{c}_d \leftarrow \text{GetCommitmentVector}((d_0, \dots, d_{2 \cdot m}), \mathbf{t}, \mathbf{ck})$ ▷ See algorithm 7.9
 - 11: $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, c_{A_0}, c_{B_m}, \mathbf{c}_d, \mathbf{c}_B, \mathbf{c}_A))$ ▷ See algorithm 3.8 and algorithm 4.7
 - ▷ Below this point, all operations are performed modulo q
 - 12: **for** $j \in [0, n)$ **do**
 - 13: $\mathbf{a}'_j \leftarrow \sum_{i=0}^m x^i \cdot \vec{a}_{j,i}$
 - 14: $\mathbf{b}'_j \leftarrow \sum_{i=0}^m x^{m-i} \cdot \vec{b}_{j,i}$
 - 15: **end for**
 - 16: $r' \leftarrow \sum_{i=0}^m x^i \cdot r_i$
 - 17: $s' \leftarrow \sum_{i=0}^m x^{m-i} \cdot s_i$
 - 18: $t' \leftarrow \sum_{i=0}^{2 \cdot m} x^i \cdot t_i$
-

Output:

The argument $(c_{A_0}, c_{B_m}, \mathbf{c}_d, \mathbf{a}', \mathbf{b}', r', s', t') \in \mathbb{G}_q \times \mathbb{G}_q \times \mathbb{G}_q^{2 \cdot m + 1} \times \mathbb{Z}_q^n \times \mathbb{Z}_q^n \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$

In the following algorithm, we verify if a provided zero argument supports the corresponding statement. We conform to the convention of using the symbol \top for true and \perp for false.

Algorithm 7.23 VerifyZeroArgument: Verifies a Zero Argument

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The **statement** composed of
 - commitments $\mathbf{c}_A = (c_{A_1}, \dots, c_{A_m}) \in \mathbb{G}_q^m$
 - commitments $\mathbf{c}_B = (c_{B_0}, \dots, c_{B_{m-1}}) \in \mathbb{G}_q^m$
 - the value $y \in \mathbb{Z}_q$ defining the bilinear mapping \star ▷ See algorithm 7.10
 - The **argument** composed of
 - the commitment $c_{A_0} \in \mathbb{G}_q$
 - the commitment $c_{B_m} \in \mathbb{G}_q$
 - the commitment vector $\mathbf{c}_d = (c_{d_0}, \dots, c_{d_{2m}}) \in \mathbb{G}_q^{2m+1}$
 - the exponent vector $\mathbf{a}' \in \mathbb{Z}_q^n$
 - the exponent vector $\mathbf{b}' \in \mathbb{Z}_q^n$
 - the exponent $r' \in \mathbb{Z}_q$
 - the exponent $s' \in \mathbb{Z}_q$
 - the exponent $t' \in \mathbb{Z}_q$
-

Operation:

- 1: $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, c_{A_0}, c_{B_m}, \mathbf{c}_d, \mathbf{c}_B, \mathbf{c}_A))$
 - 2: $\text{verifCd} \leftarrow c_{d_{m+1}} = 1$
 - 3: $\text{prodCa} \leftarrow \prod_{i=0}^m c_{A_i}^{x^i}$ ▷ The exponentiations of x are computed modulo q , whereas the product and the exponentiations of commitments are computed modulo p
 - 4: $\text{commA} \leftarrow \text{GetCommitment}(\mathbf{a}', r', \mathbf{ck})$
 - 5: $\text{verifA} \leftarrow \text{prodCa} = \text{commA}$
 - 6: $\text{prodCb} \leftarrow \prod_{i=0}^m c_{B_{m-i}}^{x^i}$ ▷ The exponentiations of x are computed modulo q , whereas the product and the exponentiations of commitments are computed modulo p
 - 7: $\text{commB} \leftarrow \text{GetCommitment}(\mathbf{b}', s', \mathbf{ck})$
 - 8: $\text{verifB} \leftarrow \text{prodCb} = \text{commB}$
 - 9: $\text{prodCd} \leftarrow \prod_{i=0}^{2m} c_{d_i}^{x^i}$ ▷ The exponentiations of x are computed modulo q , whereas the product and the exponentiations of commitments are computed modulo p
 - 10: $\text{prod} \leftarrow \mathbf{a}' \star \mathbf{b}'$ ▷ Using algorithm 7.10 with value y
 - 11: $\text{commD} \leftarrow \text{GetCommitment}(\text{prod}, t', \mathbf{ck})$
 - 12: $\text{verifD} \leftarrow \text{prodCd} = \text{commD}$
 - 13: **if** $\text{verifCd} \wedge \text{verifA} \wedge \text{verifB} \wedge \text{verifD}$ **then**
 return \top
 - 14: **else**
 return \perp
 - 15: **end if**
-

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 7.23 are provided in the attached [verify-za-argument.json](#) file.

Algorithm 7.24 ComputeDVector: Compute the vector \mathbf{d} for the algorithm 7.22

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$

Input:

First matrix $A = (\vec{a}_0, \dots, \vec{a}_m) \in \mathbb{Z}_q^{n \times (m+1)}$
 Second matrix $B = (\vec{b}_0, \dots, \vec{b}_m) \in \mathbb{Z}_q^{n \times (m+1)}$
 Value $y \in \mathbb{Z}_q$

Operation:

```

1: for  $k \in [0, 2 \cdot m]$  do
2:    $d_k \leftarrow 0$ 
3:   for  $i \in [\max(0, k - m), m]$  do
4:      $j \leftarrow (m - k) + i$ 
5:     if  $j > m$  then
6:       break from loop and proceed with next  $k$ 
7:     end if
8:      $d_k \leftarrow d_k + \vec{a}_i \star \vec{b}_j$  ▷ See algorithm 7.10, addition is modulo  $q$ 
9:   end for
10: end for
  
```

Output:

$\mathbf{d} = (d_0, \dots, d_{2 \cdot m}) \in \mathbb{Z}_q^{2 \cdot m + 1}$

7.3.6 Single Value Product Argument

In the following algorithm we generate an argument of knowledge of the opening (\mathbf{a}, r) where $\mathbf{a} = (a_0, \dots, a_{n-1})$ s.t. $c_a = \text{GenCommitment}(\mathbf{a}, r)$ and $b = \prod_{i=0}^{n-1} a_i \mod q$.

Algorithm 7.25 GetSingleValueProductArgument: Computes a Single Value Product Argument

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The statement composed of
 - commitment $c_a \in \mathbb{G}_q$
 - the product $b \in \mathbb{Z}_q$
- The witness composed of
 - vector $\mathbf{a} = (a_0, \dots, a_{n-1}) \in \mathbb{Z}_q^n$
 - the randomness $r \in \mathbb{Z}_q$

Require: $n \geq 2$

Require: $c_a = \text{GenCommitment}(\mathbf{a}, r, \mathbf{ck})$

▷ See algorithm 7.7

Require: $b = \prod_{i=0}^{n-1} a_i \mod q$

Operation:

- 1: **for** $k \in [0, n)$ **do**
 - 2: $b_k \leftarrow \prod_{i=0}^k a_i \mod q$
 - 3: **end for**
 - 4: $(d_0, \dots, d_{n-1}) \leftarrow \text{GenRandomVector}(q, n)$ ▷ See algorithm 4.2
 - 5: $r_d \leftarrow \text{GenRandomInteger}(q)$
 - 6: $\delta_0 \leftarrow d_0$
 - 7: **if** $n > 2$ **then**
 - 8: $(\delta_1, \dots, \delta_{n-2}) \leftarrow \text{GenRandomVector}(q, n-2)$
 - 9: **end if**
 - 10: $\delta_{n-1} \leftarrow 0$
 - 11: $s_0 \leftarrow \text{GenRandomInteger}(q)$
 - 12: $s_x \leftarrow \text{GenRandomInteger}(q)$
 - 13: **for** $k \in [0, n-1)$ **do**
 - 14: $\delta'_k \leftarrow -\delta_k d_{k+1} \mod q$
 - 15: $\Delta_k \leftarrow \delta_{k+1} - a_{k+1} \delta_k - b_k d_{k+1} \mod q$
 - 16: **end for**
 - 17: $c_d \leftarrow \text{GetCommitment}((d_0, \dots, d_{n-1}), r_d, \mathbf{ck})$ ▷ See algorithm 7.7
 - 18: $c_\delta \leftarrow \text{GetCommitment}((\delta'_0, \dots, \delta'_{n-2}), s_0, \mathbf{ck})$
 - 19: $c_\Delta \leftarrow \text{GetCommitment}((\Delta_0, \dots, \Delta_{n-2}), s_x, \mathbf{ck})$
 - 20: $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, c_\Delta, c_\delta, c_d, b, c_a))$
 - 21: **for** $k \in [0, n)$ **do**
 - 22: $\tilde{a}_k \leftarrow x \cdot a_k + d_k \mod q$
 - 23: $\tilde{b}_k \leftarrow x \cdot b_k + \delta_k \mod q$
 - 24: **end for**
 - 25: $\tilde{r} \leftarrow x \cdot r + r_d \mod q$
 - 26: $\tilde{s} \leftarrow x \cdot s_x + s_0 \mod q$
-

Output:

$(c_d, c_\delta, c_\Delta, (\tilde{a}_0, \dots, \tilde{a}_{n-1}), (\tilde{b}_0, \dots, \tilde{b}_{n-1}), \tilde{r}, \tilde{s}) \in \mathbb{G}_q \times \mathbb{G}_q \times \mathbb{G}_q \times \mathbb{Z}_q^n \times \mathbb{Z}_q^n \times \mathbb{Z}_q \times \mathbb{Z}_q$

In the following pseudo-code algorithm, we verify if a provided Single Value Product argument supports the corresponding statement.

Algorithm 7.26 VerifySingleValueProductArgument: Verifies a Single Value Product Argument

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$
- A multi-recipient public key $\mathbf{pk} \in \mathbb{G}_q^k$
- A commitment key $\mathbf{ck} = (h, g_1, \dots, g_\nu) \in (\mathbb{G}_q \setminus \{1, g\})^{\nu+1}$

Input:

- The statement composed of
 - commitment $c_a \in \mathbb{G}_q$
 - the product $b \in \mathbb{Z}_q$
 - The argument composed of
 - the commitment $c_d \in \mathbb{G}_q$
 - the commitment $c_\delta \in \mathbb{G}_q$
 - the commitment $c_\Delta \in \mathbb{G}_q$
 - the exponent vector $\tilde{a} = (\tilde{a}_0, \dots, \tilde{a}_{n-1}) \in \mathbb{Z}_q^n, n \geq 2$
 - the exponent vector $\tilde{b} = (\tilde{b}_0, \dots, \tilde{b}_{n-1}) \in \mathbb{Z}_q^n$
 - the exponent $\tilde{r} \in \mathbb{Z}_q$
 - the exponent $\tilde{s} \in \mathbb{Z}_q$
-

Operation:

- 1: $x \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(p, q, \mathbf{pk}, \mathbf{ck}, c_\Delta, c_\delta, c_d, b, c_a))$
 - 2: $\text{prodCa} \leftarrow c_a^x \cdot c_d$
 - 3: $\text{commA} \leftarrow \text{GetCommitment}(\tilde{a}, \tilde{r}, \mathbf{ck})$
 - 4: $\text{verifA} \leftarrow \text{prodCa} = \text{commA}$
 - 5: $\text{prodDelta} \leftarrow c_\Delta^x \cdot c_\delta$
 - 6: **for** $i \in [0, n-1)$ **do**
 - 7: $e_i \leftarrow x \cdot \tilde{b}_{i+1} - \tilde{b}_i \cdot \tilde{a}_{i+1}$
 - 8: **end for**
 - 9: $\text{commDelta} \leftarrow \text{GetCommitment}((e_0, \dots, e_{n-2}), \tilde{s}, \mathbf{ck})$
 - 10: $\text{verifDelta} \leftarrow \text{prodDelta} = \text{commDelta}$
 - 11: $\text{verifB} \leftarrow \tilde{b}_0 = \tilde{a}_0 \wedge \tilde{b}_{n-1} = x \cdot b$
 - 12: **if** $\text{verifA} \wedge \text{verifDelta} \wedge \text{verifB}$ **then**
 - 13: **return** \top
 - 13: **else**
 - 14: **return** \perp
 - 14: **end if**
-

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 7.26 are provided in the attached [verify-svp-argument.json](#) file.

8 Zero-Knowledge Proofs

8.1 Introduction

This section introduces various Zero-Knowledge Proofs of Knowledge, based on the work by Maurer[22]. We extensively document and formalize the zero-knowledge proof system’s security—including the non-interactive case—in the computational proof [25]. In each case, the idea is to make a **statement**, consisting of an homomorphism $\phi : \mathbb{G}_1 \mapsto \mathbb{G}_2$ and an image y and provide a Zero-Knowledge Proof of the Pre-image w such that $y = \phi(w)$. We name that pre-image the **witness**.

While such proofs are usually interactive, we rely on the Fiat-Shamir transform to turn them non-interactive. We use the hash function described in algorithm 4.7. The proof consists of the following steps:

- draw $b \in \mathbb{G}_1$ at random
- compute $c = \phi(b)$
- compute $e = \text{RecursiveHash}(\phi, y, c, \text{auxiliaryData})$
- compute $z = b \star w^e$ (where \star is the group operation for \mathbb{G}_1 , and exponentiation is the repetition of that operation)
- output $\pi = (e, z)$

The verification can be summarized as:

- compute $x = \phi(z)$
- compute $c' = x \otimes y^{-e}$ (where \otimes is the group operation for \mathbb{G}_2 , and exponentiation is the repetition of that operation)
- if and only if $\text{RecursiveHash}(\phi, y, c', \text{auxiliaryData}) = e$, the proof is valid

Each type of proof is a specialization of the generic prove and verify algorithm.

8.2 Decryption Proof

We prove that a decryption matches the message encrypted under the advertised public key. In this case, the phi-function maps our witness—the private key—to the public key and the decryption of the ciphertext. Hence, we define the phi-function as shown in algorithm 8.1.

Algorithm 8.1 ComputePhiDecryption: Compute the phi-function for decryption

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$

Input:

Preimage $(x_0, \dots, x_{\ell-1}) \in \mathbb{Z}_q^\ell$
 Base $\gamma \in \mathbb{G}_q$

Operation:

```

1: for  $i \in [0, \ell)$  do
2:    $y_i \leftarrow g^{x_i}$ 
3:    $y_{\ell+i} \leftarrow \gamma^{x_i}$ 
4: end for

```

$\triangleright y_i = \mathbf{pk}_i$ when $x_i = \mathbf{sk}_i$
 $\triangleright y_{\ell+i} = g^{\mathbf{sk}_i \cdot r} = \frac{\phi_i}{m_i}$ when $\gamma = g^r$ and $x_i = \mathbf{sk}_i$

\triangleright All symbols used in the comments above are aligned with algorithms 6.4 and 6.5

Output:

The image $(y_0, \dots, y_{2\ell-1}) \in \mathbb{G}_q^{2\ell}$

This algorithm implies that for the multi-recipient ElGamal key pair $(\mathbf{pk}, \mathbf{sk})$ and the valid decryption $m = (m_0, \dots, m_{\ell-1})$ of the ciphertext $(\gamma, \phi_0, \dots, \phi_{\ell-1})$, the computation of the $\text{ComputePhiDecryption}(\mathbf{sk}, \gamma)$ would yield $(\mathbf{pk}_0, \dots, \mathbf{pk}_{\ell-1}, \frac{\phi_0}{m_0}, \dots, \frac{\phi_{\ell-1}}{m_{\ell-1}})$.

Generating and verifying decryption proofs The algorithms below are the adaptations of the general case presented in section 8.1, with explicit domains and operations. Our phi-function defined in algorithm 8.1 has domain $(\mathbb{Z}_q^\ell, +)$ and co-domain $(\mathbb{G}_q^{2\ell}, \times)$. Therefore the operations given as \star will be replaced with addition (modulo q), and the "exponentiation" used in the computation of z is actually a multiplication; whereas the operation denoted by \otimes is multiplication (modulo p) and the exponentiation used in the computation of c' is a modular exponentiation in \mathbb{G}_q .

Algorithm 8.2 GenDecryptionProof: Generate a proof of validity for the provided decryption

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
Group generator $g \in \mathbb{G}_q$

Input:

A multi-recipient ciphertext $\mathbf{C} = (\gamma, \phi_0, \dots, \phi_{\ell-1}) \in \mathbb{H}_\ell$
A multi-recipient key pair $(\mathbf{pk}, \mathbf{sk}) \in \mathbb{G}_q^k \times \mathbb{Z}_q^k$
A multi-recipient message $\mathbf{m} = (m_0, \dots, m_{\ell-1}) \in \mathbb{G}_q^\ell$ s.t. $\mathbf{m} = \text{GetMessage}(\mathbf{C}, \mathbf{sk})$
An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS}^*)^*$

Require: $0 < \ell \leq k$

Operation:

```

1:  $\mathbf{b} \leftarrow \text{GenRandomVector}(q, \ell)$  ▷ See algorithm 4.2
2:  $\mathbf{c} \leftarrow \text{ComputePhiDecryption}(\mathbf{b}, \gamma)$  ▷ See algorithm 8.1
3:  $\mathbf{f} \leftarrow (p, q, g, \gamma)$ 
4: for  $i \in [0, \ell)$  do
5:    $y_i \leftarrow \mathbf{pk}_i$ 
6:    $y_{\ell+i} \leftarrow \frac{\phi_i}{m_i}$ 
7: end for
8:  $\mathbf{h}_{\text{aux}} \leftarrow (\text{"DecryptionProof"}, (\phi_0, \dots, \phi_{\ell-1}), \mathbf{m}, \mathbf{i}_{\text{aux}})$  ▷ If  $\mathbf{i}_{\text{aux}}$  is empty, we omit it
9:  $e \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(\mathbf{f}, (y_0, \dots, y_{2 \cdot \ell - 1}), \mathbf{c}, \mathbf{h}_{\text{aux}}))$  ▷ See
   algorithms 3.8 and 4.7
10:  $\mathbf{sk}' \leftarrow (\mathbf{sk}_0, \dots, \mathbf{sk}_{\ell-1})$ 
11:  $\mathbf{z} \leftarrow \mathbf{b} + e \cdot \mathbf{sk}'$ 

```

Output:

Proof $(e, \mathbf{z}) \in \mathbb{Z}_q \times \mathbb{Z}_q^\ell$

Algorithm 8.3 VerifyDecryption: Verifies the validity of a decryption proof

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$

Input:

A multi-recipient ciphertext $C = (\gamma, \phi_0, \dots, \phi_{\ell-1}) \in \mathbb{H}_\ell$
 A multi-recipient public key $\mathbf{pk} = (\mathbf{pk}_0, \dots, \mathbf{pk}_{k-1}) \in \mathbb{G}_q^k$
 A multi-recipient message $\mathbf{m} = (m_0, \dots, m_{\ell-1}) \in \mathbb{G}_q^\ell$ ▷ We expect
 $\mathbf{m} = \text{GetMessage}(\mathbf{C}, \mathbf{sk})$
 The proof $(e, \mathbf{z}) \in \mathbb{Z}_q \times \mathbb{Z}_q^\ell$
 An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS^*})^*$

Require: $0 < \ell \leq k$

Operation:

```

1:  $\mathbf{x} \leftarrow \text{ComputePhiDecryption}(\mathbf{z}, \gamma)$  ▷ See algorithm 8.1
2:  $\mathbf{f} \leftarrow (p, q, g, \gamma)$ 
3: for  $i \in [0, \ell)$  do
4:    $y_i \leftarrow \mathbf{pk}_i$ 
5:    $y_{\ell+i} \leftarrow \frac{\phi_i}{m_i}$ 
6: end for
7: for  $i \in [0, 2 \cdot \ell)$  do
8:    $c'_i \leftarrow x_i y_i^{-e}$ 
9: end for
10:  $\mathbf{h}_{\text{aux}} \leftarrow (\text{"DecryptionProof"}, (\phi_0, \dots, \phi_{\ell-1}), \mathbf{m}, \mathbf{i}_{\text{aux}})$  ▷ If  $\mathbf{i}_{\text{aux}}$  is empty, we omit it
11:  $h \leftarrow \text{RecursiveHash}(\mathbf{f}, (y_0, \dots, y_{2 \cdot \ell-1}), (c'_0, \dots, c'_{2 \cdot \ell-1}), \mathbf{h}_{\text{aux}})$  ▷ See algorithm 4.7
12:  $e' \leftarrow \text{ByteArrayToInteger}(h)$  ▷ See algorithm 3.8
13: if  $e = e'$  then
14:   return  $\top$ 
15: else
16:   return  $\perp$ 
17: end if

```

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 8.3 are provided in the attached [verify-decryption.json](#) file.

8.3 Exponentiation proof

We prove that the same secret exponent is used for a vector of exponentiations. In this case, the phi-function maps our witness—the secret exponent—to the exponentiation of a given vector of bases. We define the phi-function as shown in algorithm 8.4.

Algorithm 8.4 ComputePhiExponentiation: compute the phi-function for exponentiation

Context:

Group modulus $p \in \mathbb{P}$

Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

Preimage $x \in \mathbb{Z}_q$

Bases $(g_0, \dots, g_{n-1}) \in \mathbb{G}_q^n$ s.t. $n \in \mathbb{N}^+$

Operation:

- 1: **for** $i \in [0, n)$ **do**
 - 2: $y_i \leftarrow g_i^x \bmod p$
 - 3: **end for**
 - 4: **return** (y_0, \dots, y_{n-1})
-

Output:

$\mathbf{y} = (y_0, \dots, y_{n-1}) \in \mathbb{G}_q^n$

Generating and verifying exponentiation proofs The algorithms below are the adaptations of the general case presented in section 8.1, with explicit domains and operations. Our phi-function defined in algorithm 8.4 has domain $(\mathbb{Z}_q, +)$ and co-domain (\mathbb{G}_q^n, \times) . Therefore the operations given as \star will be replaced with addition (modulo q), and the "exponentiation" used in the computation of z is a multiplication; whereas the operation denoted by \otimes is multiplication (modulo p) and the exponentiation used in the computation of c' is a modular exponentiation in \mathbb{G}_q .

Algorithm 8.5 GenExponentiationProof: Generate a proof of validity for the provided exponentiation

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

- A vector of bases $\mathbf{g} = (g_0, \dots, g_{n-1}) \in \mathbb{G}_q^n$ s.t. $n \in \mathbb{N}^+$
 - The witness – a secret exponent $x \in \mathbb{Z}_q$
 - The statement – a vector of exponentiations $\mathbf{y} = (y_0, \dots, y_{n-1}) \in \mathbb{G}_q^n$ s.t. $y_i = g_i^x$
 - An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS}^*)^*$
-

Operation:

- 1: $b \leftarrow \text{GenRandomInteger}(q)$ ▷ See algorithm 4.1
 - 2: $\mathbf{c} \leftarrow \text{ComputePhiExponentiation}(b, \mathbf{g})$ ▷ See algorithm 8.4
 - 3: $\mathbf{f} \leftarrow (p, q, \mathbf{g})$
 - 4: $\mathbf{h}_{\text{aux}} \leftarrow (\text{"ExponentiationProof"}, \mathbf{i}_{\text{aux}})$ ▷ If \mathbf{i}_{aux} is empty, we omit it
 - 5: $e \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(\mathbf{f}, \mathbf{y}, \mathbf{c}, \mathbf{h}_{\text{aux}}))$ ▷ See algorithms 3.8 and 4.7
 - 6: $z \leftarrow b + e \cdot x \mod q$
-

Output:

- Proof $(e, z) \in \mathbb{Z}_q \times \mathbb{Z}_q$
-

Algorithm 8.6 VerifyExponentiation: Verifies the validity of an exponentiation proof

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$

Input:

A vector of bases $\mathbf{g} = (g_0, \dots, g_{n-1}) \in \mathbb{G}_q^n$ s.t. $n \in \mathbb{N}^+$
The **statement** – a vector of exponentiations $\mathbf{y} = (y_0, \dots, y_{n-1}) \in \mathbb{G}_q^n$
The **proof** $(e, z) \in \mathbb{Z}_q \times \mathbb{Z}_q$
An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS}^*)^*$

Operation:

```

1:  $\mathbf{x} \leftarrow \text{ComputePhiExponentiation}(z, \mathbf{g})$  ▷ See algorithm 8.4
2:  $\mathbf{f} \leftarrow (p, q, \mathbf{g})$ 
3: for  $i \in [0, n)$  do
4:    $c'_i \leftarrow x_i \cdot y_i^{-e}$ 
5: end for
6:  $\mathbf{h}_{\text{aux}} \leftarrow (\text{"ExponentiationProof"}, \mathbf{i}_{\text{aux}})$  ▷ If  $\mathbf{i}_{\text{aux}}$  is empty, we omit it
7:  $h \leftarrow \text{RecursiveHash}(\mathbf{f}, \mathbf{y}, (c'_0, \dots, c'_{n-1}), \mathbf{h}_{\text{aux}})$  ▷ See algorithm 4.7
8:  $e' \leftarrow \text{ByteArrayToInteger}(h)$  ▷ See algorithm 3.8
9: if  $e = e'$  then
   return  $\top$ 
10: else
   return  $\perp$ 
11: end if

```

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 8.6 are provided in the attached [verify-exponentiation.json](#) file.

8.4 Plaintext equality proof

We prove that two encryptions under different keys correspond to the same plaintext. The ciphertexts are written as $\mathbf{c} = (c_0, c_1) = (g^r, h^r m)$ and $\mathbf{c}' = (c'_0, c'_1) = (g^{r'}, h'^{r'} m)$, where g is the generator, h and h' are the public keys, and m is the same message in both cases. In this case, the phi-function is defined by the primes p and q , defining \mathbb{G}_q , as well as the generator g and the public keys h and h' , as follows:

$$\begin{aligned} \phi_{\text{PlaintextEquality}} : \mathbb{Z}_q^2 &\mapsto \mathbb{G}_q^3 \\ \phi_{\text{PlaintextEquality}}(x, x') &= (g^x, g^{x'}, \frac{h^x}{h'^{x'}}) \end{aligned}$$

This implies that $\phi_{\text{PlaintextEquality}}(r, r') = (c_0, c'_0, \frac{c_1}{c'_1})$, if and only if the message is the same in both encryptions.

Algorithm 8.7 ComputePhiPlaintextEquality: Compute the phi-function for plaintext equality

Context:

Group modulus $p \in \mathbb{P}$
Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
Group generator $g \in \mathbb{G}_q$

Input:

Preimage $(x, x') \in \mathbb{Z}_q^2$
First public key $h \in \mathbb{G}_q$
Second public key $h' \in \mathbb{G}_q$

Operation:

1: **return** $(g^x, g^{x'}, \frac{h^x}{h'^{x'}})$ ▷ All exponentiations performed modulo p

Output:

The image $(g^x, g^{x'}, \frac{h^x}{h'^{x'}}) \in \mathbb{G}_q^3$

Generating and verifying plaintext equality proofs The algorithms below are the adaptations of the general case presented in section 8.1, with explicit domains and operations. Our ϕ -function defined in algorithm 8.7 has domain $(\mathbb{Z}_q^2, +)$ and co-domain (\mathbb{G}_q^3, \times) . Therefore the operations given as \star will be replaced with addition (modulo q), and the "exponentiation" used in the computation of z is a multiplication; whereas the operation denoted by \otimes is multiplication (modulo p) and the exponentiation used in the computation of c' is a modular exponentiation in \mathbb{G}_q .

Algorithm 8.8 GenPlaintextEqualityProof: Generate a proof of equality of the plaintext corresponding to the two provided encryptions

Context:

- Group modulus $p \in \mathbb{P}$
- Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
- Group generator $g \in \mathbb{G}_q$

Input:

- The first ciphertext $\mathbf{C} = (c_0, c_1) \in \mathbb{G}_q^2$
 - The second ciphertext $\mathbf{C}' = (c'_0, c'_1) \in \mathbb{G}_q^2$
 - The first public key $h \in \mathbb{G}_q$
 - The second public key $h' \in \mathbb{G}_q$
 - The witness—the randomness used in the encryptions— $(r, r') \in \mathbb{Z}_q^2$
 - An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS^*})^*$
-

Operation:

- 1: $(b_1, b_2) \leftarrow \text{GenRandomVector}(q, 2)$ ▷ See algorithm 4.2
 - 2: $\mathbf{c} \leftarrow \text{ComputePhiPlaintextEquality}((b_1, b_2), h, h')$ ▷ See algorithm 8.7
 - 3: $\mathbf{f} \leftarrow (p, q, g, h, h')$
 - 4: $\mathbf{y} \leftarrow (c_0, c'_0, \frac{c_1}{c'_1})$
 - 5: $\mathbf{h}_{\text{aux}} \leftarrow (\text{"PlaintextEqualityProof"}, c_1, c'_1, \mathbf{i}_{\text{aux}})$ ▷ If \mathbf{i}_{aux} is empty, we omit it
 - 6: $e \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(\mathbf{f}, \mathbf{y}, \mathbf{c}, \mathbf{h}_{\text{aux}}))$ ▷ See algorithms 3.8 and 4.7
 - 7: $\mathbf{z} \leftarrow (b_1 + e \cdot r, b_2 + e \cdot r')$
-

Output:

- Proof $(e, \mathbf{z}) \in \mathbb{Z}_q \times \mathbb{Z}_q^2$
-

Algorithm 8.9 VerifyPlaintextEquality: Verifies the validity of a plaintext equality proof

Context:

Group modulus $p \in \mathbb{P}$
 Group cardinality $q \in \mathbb{P}$ s.t. $p = 2 \cdot q + 1$
 Group generator $g \in \mathbb{G}_q$

Input:

The first ciphertext $\mathbf{C} = (c_0, c_1) \in \mathbb{G}_q^2$
 The second ciphertext $\mathbf{C}' = (c'_0, c'_1) \in \mathbb{G}_q^2$
 The first public key $h \in \mathbb{G}_q$
 The second public key $h' \in \mathbb{G}_q$
 The proof $(e, \mathbf{z}) \in \mathbb{Z}_q \times \mathbb{Z}_q^2$
 An array of optional additional information $\mathbf{i}_{\text{aux}} \in (\mathbb{A}_{UCS}^*)^*$

Operation:

```

1:  $\mathbf{x} \leftarrow \text{ComputePhiPlaintextEquality}(\mathbf{z}, h, h')$  ▷ See algorithm 8.7
2:  $\mathbf{f} \leftarrow (p, q, g, h, h')$ 
3:  $\mathbf{y} \leftarrow (c_0, c'_0, \frac{c_1}{c'_1})$ 
4:  $\mathbf{c}' \leftarrow \mathbf{x} \cdot \mathbf{y}^{-e}$ 
5:  $\mathbf{h}_{\text{aux}} \leftarrow (\text{"PlaintextEqualityProof"}, c_1, c'_1, \mathbf{i}_{\text{aux}})$  ▷ If  $\mathbf{i}_{\text{aux}}$  is empty, we omit it
6:  $e' \leftarrow \text{ByteArrayToInteger}(\text{RecursiveHash}(\mathbf{f}, \mathbf{y}, \mathbf{c}', \mathbf{h}_{\text{aux}}))$  ▷ See algorithms 3.8 and 4.7
7: if  $e = e'$  then
    return  $\top$ 
8: else
    return  $\perp$ 
9: end if
    
```

Output:

The result of the verification: \top if the verification is successful, \perp otherwise.

Test values for the algorithm 8.9 are provided in the attached [verify-plaintext-equality.json](#) file.

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References

- [1] ANSSI (Agence nationale de la sécurité des systèmes d'informations). Référentiel Général de Sécurité, version 2.0, Annexe B1, Mécanismes cryptographiques, 2014.
- [2] Elaine Barker. NIST Special Publication 800-57 Part 1, Revision 5, Recommendation for Key Management. Special Publication 800-57, National Institute of Standards and Technology, U.S. Department of Commerce, 2020.
- [3] Stephanie Bayer and Jens Groth. Efficient Zero-Knowledge Argument for Correctness of a Shuffle. In David Pointcheval and Thomas Johansson, editors, *Advances in Cryptology – EUROCRYPT 2012*, pages 263–280, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg.
- [4] Mihir Bellare, Alexandra Boldyreva, and Jessica Staddon. Randomness Re-use in Multi-recipient Encryption Schemes. In *Public Key Cryptography - PKC 2003, 6th International Workshop on Theory and Practice in Public Key Cryptography, Miami, FL, USA, January 6-8, 2003, Proceedings*, pages 85–99, 2003.
- [5] Die Schweizerische Bundeskanzlei (BK). Federal Chancellery Ordinance on Electronic Voting (OEV), Draft of 28 April 2021.
- [6] Morris Dworkin. Sha-3 standard: Permutation-based hash and extendable-output functions, 2015-08-04 2015.
- [7] Morris J Dworkin. *Sp 800-38d. recommendation for block cipher modes of operation: Galois/counter mode (gcm) and gmac*. National Institute of Standards & Technology, 2007.
- [8] Eidgenössisches Departement für auswärtige Angelegenheiten EDA. Swiss Political System - Direct Democracy. <https://www.eda.admin.ch/aboutswitzerland/en/home/politik/uebersicht/direkte-demokratie.html/>. Retrieved on 2020-07-15.
- [9] gfs.bern. Vorsichtige Offenheit im Bereich digitale Partizipation - Schlussbericht, 3 2020.
- [10] Oded Goldreich. *Foundations of cryptography: volume 1, basic tools*. Cambridge University Press, 2007.
- [11] Rolf Haenni. Swiss Post Public Intrusion Test: Undetectable attack against vote integrity and secrecy, 2019.
- [12] Rolf Haenni, Eric Dubuis, Reto E Koenig, and Philipp Locher. CHVote: Sixteen Best Practices and Lessons Learned. In *International Joint Conference on Electronic Voting*, pages 95–111. Springer, 2020.

- [13] Rolf Haenni, Reto E. Koenig, Philipp Locher, and Eric Dubuis. CHVote Protocol Specification, Version 3.2. Cryptology ePrint Archive, Report 2017/325, 2020. <https://eprint.iacr.org/2017/325>.
- [14] Thomas Haines, Rageev Goré, and Bhavesh Sharma. Did you mix me? Formally Verifying Verifiable Mix Nets in Electronic Voting.
- [15] Thomas Haines, Sarah Jamie Lewis, Olivier Pereira, and Vanessa Teague. How not to prove your election outcome. In *2020 IEEE Symposium on Security and Privacy (SP)*, pages 644–660. IEEE, 2020.
- [16] Tetsu Iwata, Keisuke Ohashi, and Kazuhiko Minematsu. Breaking and repairing gcm security proofs. In *Annual Cryptology Conference*, pages 31–49. Springer, 2012.
- [17] Simon Josefsson et al. The Base16, Base32, and Base64 Data Encodings. Technical report, RFC 4648, October, 2006.
- [18] Jonathan Katz and Yehuda Lindell. *Introduction to modern cryptography*. CRC press, 2020.
- [19] Donald E. Knuth. *The Art of Computer Programming, Volume 2 (3rd Ed.): Seminumerical Algorithms*. Addison-Wesley Longman Publishing Co., Inc., USA, 1997.
- [20] Hugo Krawczyk and Pasi Eronen. HMAC-based extract-and-expand key derivation function (HKDF). Technical report, RFC 5869, May, 2010.
- [21] Philipp Locher, Rolf Haenni, and Reto E Koenig. Analysis of the cryptographic implementation of the Swiss Post voting protocol, 2019.
- [22] Ueli Maurer. Unifying zero-knowledge proofs of knowledge. In Bart Preneel, editor, *Progress in Cryptology – AFRICACRYPT 2009*, pages 272–286, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg.
- [23] David McGrew. An Interface and Algorithms for Authenticated Encryption. RFC 5116, January 2008.
- [24] Torben P. Pedersen. Non-interactive and information-theoretic secure verifiable secret sharing. In *CRYPTO*, pages 129–140, 1991.
- [25] Swiss Post. Protocol of the Swiss Post Voting System. Computational Proof of Complete Verifiability and Privacy. Version 0.9.11. <https://gitlab.com/swisspost-evoting/e-voting/e-voting-documentation/-/tree/master/Protocol>, Oct 2021.
- [26] Nigel P Smart et al. Algorithms, key size and protocols report. *ECRYPT - CSA, H2020-ICT-2014, Project 645421*, 2018.
- [27] Ben Smyth. A foundation for secret, verifiable elections. *IACR Cryptology ePrint Archive*, 2018:225, 2018.

- [28] Vanessa Teague and Olivier Pereira. Report on the SwissPost-Scytl e-voting system, trusted-server version, 2019.
- [29] Björn Terelius and Douglas Wikström. Proofs of restricted shuffles. In *International Conference on Cryptology in Africa*, pages 100–113. Springer, 2010.
- [30] Yiannis Tsiounis and Moti Yung. On the security of elgamal based encryption. In *Public Key Cryptography*, pages 117–134, 1998.
- [31] Francois Yergeau. RFC3629: UTF-8, a transformation format of ISO 10646, 2003.