From Bernoulli to L'Hospital: Unveiling the Origin of the Famous Rule in Calculus

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1 Introduction

L'Hôpital's Rule is a fundamental mathematical tool widely employed in calculus to evaluate indeterminate limits. It provides an elegant solution to the evaluation of limits involving ratios of functions, particularly those that lead to the "0/0" or "∞/∞" indeterminate forms. Named after its creator, Guillaume François Antoine, Marquis de l'Hôpital, this rule has become an indispensable concept in higher mathematics.

The significance of L'Hôpital's Rule extends beyond its immediate applications in calculus. It has found widespread use in various fields, including Computer Science. In Computer Science, the ability to analyze and compute limits is crucial in algorithmic analysis and the study of computational complexity. L'Hôpital's Rule provides a powerful tool for tackling complex limit problems that arise in these areas, enabling the analysis and optimization of algorithms and the estimation of their runtime behavior.

While many students and mathematicians are familiar with the practical application of L'Hôpital's Rule, it is essential to delve into its historical journey. Exploring the history behind this rule allows us to appreciate the intellectual contributions of its creator, as well as the collaborations and influences that shaped its development. By understanding the context in which L'Hôpital's Rule emerged, we gain deeper insights into the motivations and challenges faced by mathematicians of the time, fostering a more comprehensive understanding of the rule itself.

2 Historical Background

The origins of L'Hôpital's Rule can be traced back to the $17th$ century, a period marked by significant advancements in mathematics and the emergence of calculus. During this time, the field was still in its formative stages, with scholars grappling to develop rigorous methods to solve complex problems related to limits and derivatives. Amidst this mathematical revolution, L'Hôpital's Rule emerged as a breakthrough technique that would forever transform the way mathematicians approached limit problems.

2.1 Marquis de l'Hôpital (1661-1704)

Guillaume-François-Antoine Marquis de l'Hôpital, Marquis de Sainte-Mesme, Comte d'Entremont, and Seigneur d'Ouques-la-Chaise was a mathematician and nobleman born into a prominent French family with a long lineage dating back to the 12*th* century. His father, Anne-Alexandre de l'Hôpital, held a high-ranking position as Lieutenant-general in the King's Army, and his mother, Elisabeth Gobelin, was the daughter of Claude Gobelin, an esteemed Intendant in the King's Army and a Councillor of State. The family's association with the house of Orléans and their connections to influential individuals in the royal court bestowed upon them a notable reputation and afforded them a level of protection akin to that of the king himself.

During his early years, l'Hôpital did not exhibit much aptitude for subjects like Latin but demonstrated exceptional mathematical abilities and a genuine passion for the field. At the age of fifteen, he impressed the Duke of Roannès and a Mr Arnaud by solving a difficult problem on the cycloid that had been proposed by Blaise Pascal. Recognizing his talent, it was expected that l'Hôpital would pursue a military career, following in his father's footsteps. He served as a captain in a cavalry regiment while maintaining his interest in mathematics. However, due to nearsightedness that limited his vision to ten paces, he eventually resigned from the army.

Following his departure from the military, l'Hôpital devoted himself entirely to mathematics. It was a fortuitous meeting with Johann Bernoulli in 1691 that would shape the trajectory of his mathematical career. At the time, Bernoulli had just arrived in Paris after lecturing on Leibniz's differential calculus. Intrigued by Bernoulli's expertise, l'Hôpital became his most enthusiastic student and sought private lessons from him. Bernoulli, impressed by l'Hôpital's knowledge and passion, agreed to teach him and formed a close intellectual relationship with him.

2.2 Johann Bernoulli (1667-1748)

The renowned mathematician from Basel, Switzerland, made significant contributions to various branches of mathematics during the 17*th* and 18*th* centuries. Coming from a family of mathematicians, Johann's talent and achievements surpassed even those of his elder brother, Jacob Bernoulli.

In 1691, Johann traveled to Geneva, where he lectured on differential calculus, showcasing his profound understanding of the newly published methods by Leibniz. During his stay in Geneva, Johann engaged in mathematical conversations with influential mathematicians in Malebranche's circle, a leading group in French mathematics at the time. It was there that Johann encountered Guillaume François Antoine, Marquis de l'Hôpital, who would play a crucial role in the development of one of Johann's most influential contributions to mathematics.

Recognizing his profound grasp of calculus, de l'Hôpital sought his assistance in learning the new methods. Johann agreed to teach de l'Hôpital calculus techniques, conducting lessons in both Paris and de l'Hôpital's country house in Oucques. The instruction proved invaluable, and de l'Hôpital compensated Johann generously for his teachings. However, Johann was disheartened when de l'Hôpital published his seminal calculus book, *"Analyse des infiniment petits pour l'intelligence des lignes courbes"*. in 1696, without acknowledging Johann's significant contributions. Only in 1922, with the discovery of a copy of Johann's course notes written by his nephew Nicolaus(I) Bernoulli, was proof of Johann's authorship established. Johann's course closely resembled de l'Hôpital's book, although de l'Hôpital had made several corrections and improvements.

Bernoulli's time in Paris was marked by fruitful collaborations and correspondence with other mathematicians. He developed a strong friendship with Varignon and engaged in extensive correspondence with Leibniz, yielding highly productive results. During this period, Johann achieved significant mathematical milestones, including his independent solution to the catenary problem posed by Jacob Bernoulli, showcasing his prowess as an independent mathematician.

In 1695, Johann received offers for chairs in both Halle and Groningen. Influenced by Huygens' recommendation, he accepted the chair of mathematics in Groningen, where he would have equal status to his brother Jacob. This decision intensified the competitive nature of their relationship, leading to bitter personal conflicts. Johann faced challenges during his time in Groningen, including religious disputes and opposition to his physics experiments from Cartesian scientists and Calvinists.

Johann's contributions to mathematics extended throughout his career. He made significant advancements in the calculus of variations, explored the function $y = x^{1/x}$, investigated series using integration by parts, and derived addition theorems for trigonometric and hyperbolic functions. His work gained international recognition, and he was elected a fellow of prestigious academies in Paris, Berlin, London, St. Petersburg, and Bologna.

3 The origins of I'Hôpital's rule

L'Hôpital's rule, which establishes that when $f(a) = g(a) = 0$ and $f'(a) \neq 0$, the following limit holds:

$$\lim_{x \to a} \frac{g(x)}{f(x)} = \frac{g'(a)}{f'(a)}$$

was first published by the French mathematician in his work *Analyse des infiniment petits (Paris, 1696).* The encounter between Marquis de l'Hôpital and Johann Bernoulli in the late 17th century marked a pivotal moment in l'Hôpital's mathematical journey. At the time, Bernoulli, a young and brilliant mathematician, had just arrived in Paris with extensive knowledge of Leibniz's differential calculus. L'Hôpital, eager to learn, recognized Bernoulli's expertise and invited him to deliver lectures to a circle of mathematicians and scientists in Paris. Impressed by Bernoulli's revelation of the general formula for the radius of curvature of a curve, l'Hôpital sought private lessons from him at his estate in Ouques. Their correspondence and collaboration continued even after Bernoulli's return to Basel, with l'Hôpital acknowledging his indebtedness to both Leibniz and Bernoulli in his own publications.

The extent of l'Hôpital's dependence on Bernoulli became a subject of debate and speculation, with Bernoulli privately claiming a significant contribution to l'Hôpital's work. However, it was only after l'Hôpital's death that Bernoulli publicly claimed a specific section of l'Hôpital's book, fueling further discussions on priority. The situation gained clarity when Bernoulli's manuscript on differential calculus, dating back to their collaboration period, was published in 1922, revealing substantial overlap with l'Hôpital's work. The true nature of their relationship was fully disclosed in 1955 when Bernoulli's early correspondence unveiled a formal agreement between the two. L'Hôpital offered financial support to Bernoulli in exchange for his mathematical collaboration, including working on problems and sharing discoveries exclusively with l'Hôpital. This agreement resolved the question of priority and shed light on their interconnectedness:

"I shall give you with pleasure a pension of three hundred livres, which will begin on the first of January of the present year, and I shall send two hundred livres for the first half of the year because of the journals you have sent, and it will be one hundred and fifty livres for the other half of the year, and so on in the future. I promise to increase this pension soon, since I know it to be very moderate, and I shall do this as soon as my affairs are a little less confused... I am not so unreasonable as to ask for all your time, but I shall ask you to give me some hours occasionally to work on what I shall ask you—and also to share with me your discoveries, with the request not to mention them to others. I also ask you not to send copies of the notes that you let me have to M. Varignon or others, for it would not please me if they were made public. Send me your response to all this and believe me, Monsieur, yours entirely." LE M. DE LHÔPITAL

The nature and duration of the collaboration between l'Hôpital and Bernoulli became clearer with the discovery of letters and documents. A letter from Bernoulli dated July 22, 1694, confirms his acceptance of l'Hôpital's proposal for collaboration and financial support. This arrangement proved beneficial for Bernoulli, who was facing financial difficulties after his recent marriage and sought stability in his career. However, as Bernoulli's financial situation improved, l'Hôpital's circumstances did not follow suit, and their association likely came to an end around 1695.

Published letters from Bernoulli to l'Hôpital contain answers to mathematical questions, including the famous rule for 0/0. The formulation of the rule closely resembles the one found in l'Hôpital's book, "Analyse des infiniment petits," and the examples provided by Bernoulli align with those used by l'Hôpital. Initially bound by the promise of confidentiality, Bernoulli could only express his claim privately. After l'Hôpital's death, Bernoulli felt free to assert his contributions publicly, specifically claiming the rule for 0/0. However, he was unable to provide proof at the time. Subsequently, through the publication of letters and documents, Bernoulli's assertions have been validated.

4 Proof

4.1 Geometric Insight

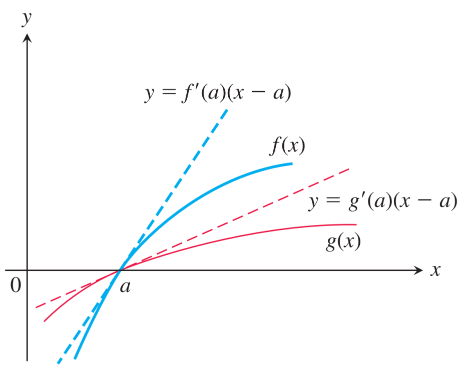

Figure 1: **The two functions in l'Hôpital's Rule, graphed with their linear approximations at x = a**

Before proving L'Hôpital's Rule, we consider a special case to provide some geometric insight for its reasonableness. Let $f(x)$ and $g(x)$ be two functions with continuous derivatives satisfying $f(a) = g(a) = 0$ and $g'(a) \neq 0$.

Figure 1 illustrates the graphs of $f(x)$ and $g(x)$ along with their linearizations $y = f'(a)(x-a)$ and $y = g'(a)(x-a)$. In the vicinity of $x = a$, the linearizations provide good approximations to the functions. More precisely, we have

$$f(x) = f'(a)(x-a) + P_1(x-a) \quad \textit{and} \quad g(x) = g'(a)(x-a) + P_2(x-a),$$

where $P_1 \to 0$ and $P_2 \to 0$ as $x \to a$. Consequently, we obtain

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a) + P_1(x-a)}{g'(a)(x-a) + P_2(x-a)} = \lim_{x\to a} \frac{f'(a) + P_1}{g'(a) + P_2} = \frac{f'(a)}{g'(a)},$$

where $g'(a) \neq 0$. This result is in accordance with L'Hôpital's Rule.

In the subsequent sections, we provide a rigorous proof of L'Hôpital's Rule based on the more general assumptions stated in Cauchy's Mean Value Theorem. These assumptions do not require $g'(a) \neq 0$ or that the two functions have continuous derivatives.

4.2 Proof of l'Hospital's Rule

We begin by establishing Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem involving two functions.

Theorem *Suppose functions f and g are continuous on $[a,b]$ and differentiable throughout (a,b), and also suppose $g'(x) \neq 0$ throughout (a,b). Then, there exists a number c in (a,b) such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a because it always lies between a and x. Therefore

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{c\to a^+} \frac{f'(c)}{g'(c)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x,a]$, $x < a$.

5 Conclusion

L'Hôpital's Rule, despite its name, owes its mathematical development to Johann Bernoulli. The business deal struck between L'Hôpital and Bernoulli resulted in the inclusion of the rule in L'Hôpital's book. L'Hôpital acknowledged Bernoulli's contribution by presenting the rule as "given to me by Mr. Bernoulli." L'Hôpital's Rule has since become an essential tool in calculus, enabling the evaluation of various limits involving indeterminate forms, and remains an enduring testament to the collaborative nature of mathematical progress.

References

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[2] George B. Thomas, Jr. (2014) *Thomas' Calculus Early Transcendentals Thirteenth Edition* . Boston: PErason, pp. 255-261.