Video 1: Qubit Rotation

The Singularity

January 2020

1 Introduction

In this document we will use the operators listed below in circuit diagrams in order to learn to convert quantum circuit diagrams into linear algebra operations.

1.1 Pauli Gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.2 The "CNOT" Gate

$$\mathbf{CNOT} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

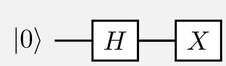
1.3 The Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2 Linear Algebra from Circuit Diagrams

Convert the following diagrams into linear algebra, i.e. change the circuit diagrams to matrices operating on vectors.

2.1 Circuit Diagram 1



Solution.

$$\begin{aligned}
X \cdot H \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} && (1) \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (2) \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (3) \\
&= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} && (4)
\end{aligned}$$

■

2.2 Circuit Diagram 2

Solution.

$$\begin{aligned}
Z \cdot Y \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} && (5) \\
&= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} && (6) \\
&= \begin{pmatrix} 0 \\ i \end{pmatrix} && (7) \\
& && (8)
\end{aligned}$$

■

2.3 Circuit Diagram 3

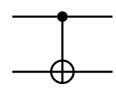
Solution. This one is a little tricky if you don't know about tensor products so let's go through it very carefully. First, operate on each qubit with the Hadamard gate to turn the "ket-0" into a superposition:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now, take the two qubits in superposition (two copies of the above computation), and take their tensor product:

$$\begin{aligned}
H|0\rangle \otimes H|0\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} && (9)\\
&= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} && (10)\\
&= \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} && (11)
\end{aligned}$$

Next, operate on this by the **CNOT**-gate corresponding to the part of the diagram that looks like



$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$$

We operate using the Pauli-Y operator on the first two components of this vector:

$$\begin{aligned}
Y\begin{pmatrix}1/2\\1/2\end{pmatrix} &= \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1/2\\1/2\end{pmatrix} && (12)\\
&= \begin{pmatrix}-i/2\\i/2\end{pmatrix} && (13)
\end{aligned}$$

Finally, operate on the second two components with the Pauli-Z gate:

$$\begin{aligned}
Z\begin{pmatrix}1/2\\1/2\end{pmatrix} &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1/2\\1/2\end{pmatrix} && (14)\\
&= \begin{pmatrix}1/2\\-1/2\end{pmatrix} && (15)
\end{aligned}$$

■