

# Video 1: Qubit Rotation

The Singularity

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## 1 Introduction

In this document we will use the operators listed below in circuit diagrams in order to learn to convert quantum circuit diagrams into linear algebra operations.

### 1.1 Pauli Gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

### 1.2 The "CNOT" Gate

$$\text{CNOT} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

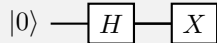
### 1.3 The Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

## 2 Linear Algebra from Circuit Diagrams

Convert the following diagrams into linear algebra, i.e. change the circuit diagrams to matrices operating on vectors.

### 2.1 Circuit Diagram 1



*Solution.*

$$X \cdot H \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1)$$

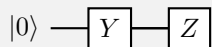
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad (4)$$

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## 2.2 Circuit Diagram 2



*Solution.*

$$Z \cdot Y \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (5)$$

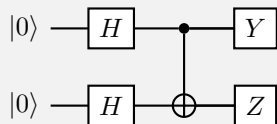
$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6)$$

$$= \begin{pmatrix} 0 \\ i \end{pmatrix} \quad (7)$$

$$(8)$$

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## 2.3 Circuit Diagram 3



*Solution.* This one is a little tricky if you don't know about tensor products so let's go through it very carefully. First, operate on each qubit with the Hadamard gate to turn the "ket-0" into a superposition:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

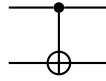
Now, take the two qubits in superposition (two copies of the above computation), and take their tensor product:

$$H|0\rangle \otimes H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (9)$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (10)$$

$$= \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad (11)$$

Next, operate on this by the **CNOT**-gate corresponding to the part of the diagram that looks like



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

We operate using the Pauli-Y operator on the first two components of this vector:

$$Y \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \quad (12)$$

$$= \begin{pmatrix} -i/2 \\ i/2 \end{pmatrix} \quad (13)$$

Finally, operate on the second two components with the Pauli-Z gate:

$$Z \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \quad (14)$$

$$= \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \quad (15)$$

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### 3 Measurements and Expectation Values

When we see something that looks like

$$\langle \psi_1 | \sigma_z | \psi_1 \rangle$$

we need to know how to interpret it in terms of linear algebra. The "bra-vector"  $\langle \psi_1 |$  is just the complex conjugate transpose of the "ket-vector"  $|\psi_1\rangle$ . So for example, if

$$|\psi_1\rangle = \begin{pmatrix} -i/2 \\ i/2 \end{pmatrix}$$

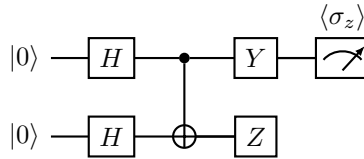
which was one of the vectors we obtained in the previous example (Circuit Diagram 3), then the bra-vector would be its conjugate transpose:

$$\langle\psi_1| = (i/2, \quad -i/2)$$

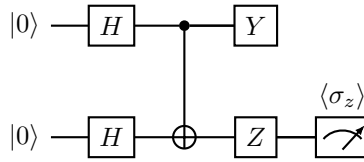
You should think of this as an inner product (or dot-product). However, in this case, we have jammed a matrix in between the two vectors, the Pauli-Z operator:

$$\langle\psi_1|\sigma_z|\psi_1\rangle = (i/2, \quad -i/2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -i/2 \\ i/2 \end{pmatrix}$$

This is often added at the end of a circuit to indicate a measurement:



Now, try working out what the measurement applied to the second wire would be. The diagram is as follows:



The corresponding ket-vector from the previous section (Circuit Diagram 3) was:

$$|\psi_2\rangle = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

So you are trying to compute:

$$\langle\psi_2|\sigma_z|\psi_2\rangle.$$