

Video 1: Qubit Rotation

The Singularity

January 2020

1 Introduction

In this document we will use the operators listed below in circuit diagrams in order to learn to convert quantum circuit diagrams into linear algebra operations.

1.1 Pauli Gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.2 The "CNOT" Gate

$$\text{CNOT} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

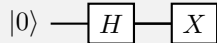
1.3 The Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2 Linear Algebra from Circuit Diagrams

Convert the following diagrams into linear algebra, i.e. change the circuit diagrams to matrices operating on vectors.

2.1 Circuit Diagram 1



Solution.

$$X \cdot H \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1)$$

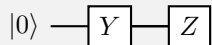
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad (4)$$

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2.2 Circuit Diagram 2



Solution.

$$Z \cdot Y \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (5)$$

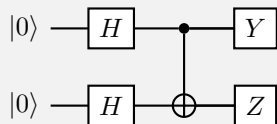
$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6)$$

$$= \begin{pmatrix} 0 \\ i \end{pmatrix} \quad (7)$$

$$(8)$$

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2.3 Circuit Diagram 3



Solution. This one is a little tricky if you don't know about tensor products so let's go through it very carefully. First, operate on each qubit with the Hadamard gate to turn the "ket-0" into a superposition:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

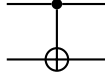
Now, take the two qubits in superposition (two copies of the above computation), and take their tensor product:

$$H|0\rangle \otimes H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (9)$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (10)$$

$$= \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad (11)$$

Next, operate on this by the **CNOT**-gate corresponding to the part of the diagram that looks like



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

Next, we operate on the tensor product vector with the operator $Y \otimes Z$:

$$Y \otimes Z \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad (12)$$

$$= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \quad (13)$$

$$= \begin{pmatrix} -i/2 \\ i/2 \\ i/2 \\ -i/2 \end{pmatrix} \quad (14)$$

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3 Measurements and Expectation Values

When we see something that looks like

$$\langle \psi_1 | \sigma_z | \psi_1 \rangle$$

we need to know how to interpret it in terms of linear algebra. The "bra-vector" $\langle \psi_1 |$ is just the complex conjugate transpose of the "ket-vector" $|\psi_1\rangle$. So for example, if

$$|\psi_1\rangle = \begin{pmatrix} -i/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

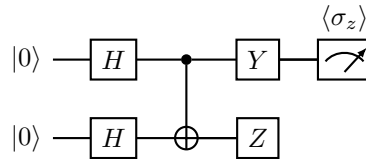
then the bra-vector would be its conjugate transpose:

$$\langle\psi_1| = (i/\sqrt{2}, -i/\sqrt{2})$$

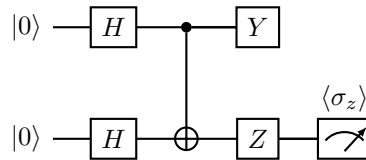
You should think of this as an inner product (or dot-product). However, in this case, we have jammed a matrix in between the two vectors, the Pauli-Z operator:

$$\langle\psi_1|\sigma_z|\psi_1\rangle = (i/\sqrt{2}, -i/\sqrt{2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -i/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

This is often added at the end of a circuit to indicate a measurement:



Now, try working out what the measurement applied to the second wire would be. The diagram is as follows:



The corresponding ket-vector is:

$$|\psi_2\rangle = \begin{pmatrix} i/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

So you are trying to compute:

$$\langle\psi_2|\sigma_z|\psi_2\rangle.$$