Video 1: Qubit Rotation

The Singularity

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1 Introduction

In this document we will use the operators listed below in circuit diagrams in order to learn to convert quantum circuit diagrams into linear algebra operations.

1.1 Pauli Gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.2 The "CNOT" Gate

$$\textbf{CNOT} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1.3 The Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2 Linear Algebra from Circuit Diagrams

Convert the following diagrams into linear algebra, i.e. change the circuit diagrams to matrices operating on vectors.

2.1 Circuit Diagram 1

$$|0\rangle \; - \boxed{H} - \boxed{X}$$

Solution.

$$\begin{aligned}
X \cdot H \cdot \begin{pmatrix}1\\0\end{pmatrix} &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} && (1)\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} && (2)\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} && (3)\\
&= \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} && (4)
\end{aligned}$$

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2.2 Circuit Diagram 2

Solution.

$$\begin{aligned}
Z \cdot Y \cdot \begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} && (5)\\
&= \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} && (6)\\
&= \begin{pmatrix}0\\i\end{pmatrix} && (7)\\
& && (8)
\end{aligned}$$

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2.3 Circuit Diagram 3

Solution. This one is a little tricky if you don't know about tensor products so let's go through it very carefully. First, operate on each qubit with the Hadamard gate to turn the "ket-0" into a superposition:

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

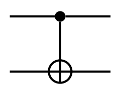
Now, take the two qubits in superposition (two copies of the above computation), and take their tensor product:

$$H|0\rangle \otimes H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{9}$$

$$= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \tag{10}$$

$$= \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} \tag{11}$$

Next, operate on this by the **CNOT**-gate corresponding to the part of the diagram that looks like



$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$$

Next, we operate on the tensor product vector with the operator $Y \otimes Z$:

$$Y \otimes Z\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} \tag{12}$$

$$= \begin{pmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{pmatrix}\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} \tag{13}$$

$$= \begin{pmatrix}-i/2\\i/2\\i/2\\-i/2\end{pmatrix} \tag{14}$$

$\blacksquare$

3 Measurements and Expectation Values

When we see something that looks like

$$\langle\psi_1|\sigma_z|\psi_1\rangle$$

we need to know how to interpret it in terms of linear algebra. The "bra-vector" $\langle\psi_1|$ is just the complex conjugate transpose of the "ket-vector" $|\psi_1\rangle$. So for example, if

$$|\psi_1\rangle = \begin{pmatrix}-i/\sqrt{2}\\i/\sqrt{2}\end{pmatrix}$$

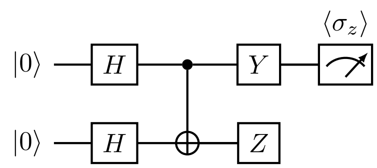
then the bra-vector would be its conjugate transpose:

$$\langle\psi_1| = \begin{pmatrix} i/\sqrt{2}, & -i/\sqrt{2} \end{pmatrix}$$

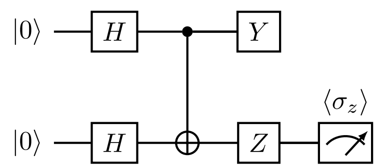
You should think of this as an inner product (or dot-product). However, in this case, we have jammed a matrix in between the two vectors, the Pauli-Z operator:

$$\langle\psi_1|\sigma_z|\psi_1\rangle = \begin{pmatrix} i/\sqrt{2}, & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -i/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

This is often added at the end of a circuit to indicate a measurement:



Now, try working out what the measurement applied to the second wire would be. The diagram is as follows:



The corresponding ket-vector is:

$$|\psi_2\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

So you are trying to compute:

$$\langle\psi_2|\sigma_z|\psi_2\rangle.$$