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Unicycle Vehicles - Vers. of January 19, 2019 - A. Fagiolini

A unicycle is the schematization of a terrestrial vehicle with one wheel, which can rotate around an axis orthogonal to the motion surface plane and passing through the contact point between the wheel and the plane itself. It can also represent a vehicle with a pair of wheels connected through an axle, whose rotational axis passes through the midpoint of the axle itself.

Referring to Fig. XXX, in case of a single-wheel unicycle, denote with (x, y) the position of the contact point between the wheel and the motion surface, and with ψ the orientation (yaw) angle measured with respect to a reference direction taken as the positive x-axis. In case of two-wheel unicycle, (x, y) indicate the midpoint position of the axle and ψ the orientation (yaw) angle from the x-axis and the orthogonal to the axle itself. Consider the following reference frames:

- $\mathcal{F}_0 = \left\{ p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \hat{i}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{j}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$
- $\mathcal{F}_b = \left\{ p_b = \begin{pmatrix} x \\ y \end{pmatrix}, \hat{i}_b = \begin{pmatrix} c_\psi \\ s_\psi \end{pmatrix}, \hat{j}_b = \begin{pmatrix} -s_\psi \\ c_\psi \end{pmatrix} \right\}.$

1.1 Kinematic Model

Let the vehicle configuration be $q = (x, y, \psi)^T$. The kinematic model of a unicycle vehicle describes, at every configuration q, the set of admissible speed $\dot{q}$ that the vehicle can have as a function of suitable input speed variables. In order to obtain such a model, consider that the wheel or pair of wheels introduces the constraint:

$$\dot{p}_b \cdot \hat{j}_b = 0\,,$$
$$-\dot{x}\, s_\psi + \dot{y}\, c_\psi = 0\,,$$

which can be expressed in so-called Pfaffian form as

$$A(q)\,\dot{q} = \begin{pmatrix} -s_\psi & c_\psi & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{pmatrix} = 0\,.$$

The above constraint tells us that all the vehicle's admissible speeds $\dot{q}$ must belong to the null-column space of $A(q)$, $\ker(A(q))$. Since $\text{rank}(A(q)) = 1$ for all q, then $\dim(\ker(A(q))) = 3 - 1 = 2$ for all q, and thus we need to seek two independent basis vectors to characterize it. It is straightforward to see that the choice of vectors, $(c_\psi, s_\psi, 0)^T$ and $(0, 0, 1)^T$, forms a basis for $\ker(A(q))$. Let $S(q)$ be the matrix formed by such vectors and let $\nu = (v, \omega)^T$ two coefficients, allowing to write every admissible $\dot{q}$ in terms of the vectors of the null-column space A, i.e. $\dot{q} = S(q)\,\nu$, or more explicitly:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} c_\psi & 0 \\ s_\psi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix}, \tag{1.1}$$

where v can be interpreted as the vehicle's forward speed and ω its orientation (yaw) rate. To show this, consider the following:

$$\begin{aligned} ||\dot{p}_b|| &= \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{v^2\, c_\psi^2 + v^2\, s_\psi^2} = |v|\,, \\ \hat{\dot{p}}_b &= \frac{\dot{p}_b}{||\dot{p}_b||} = \frac{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{v}{|v|} \begin{pmatrix} c_\psi \\ s_\psi \end{pmatrix} = \text{sign}(v)\, \hat{i}_b\,, \\ \dot{p}_b &= |v|\, \text{sign}(v)\, \hat{i}_b = v\, \hat{i}_b\,. \end{aligned}$$

and finally

$$\dot{\psi} = \omega\,.$$

The above equations represent the unicycle model expressed with respect to a fixed frame. As we will show below, depending on the specific control objective, it is often more convenient to choose a different (possibly moving) reference frame and derive the corresponding system's model in the new frame.

1.2 Path Following

1.2.1 Alignment to a generic curve

Let us begin by focusing on the problem of aligning a unicycle vehicle, to a generic curve described by the equation $c(q) = 0$. Assume that the forward speed v is an assigned function of time, i.e. $v = \bar{v}(t)$. The control objective is achieved by designing a feedback law for the steering rate input variable ω ensuring that the vehicle's configuration asymptotically tends to the desired curve. It is convenient to recast our problem in terms of a state regulation at the origin, for which we need to introduce the Frenet-Serret frame $\mathcal{F}_c$ (also known as the Tangential-Normal-Binormal (TNB) frame) of the curve $c(q)$:

- $$\mathcal{F}_c = \left\{ p_c = \begin{pmatrix} x_c \\ y_c \\ 0 \end{pmatrix}, \hat{i}_c = \begin{pmatrix} c_{\psi_c} \\ s_{\phi_c} \\ 0 \end{pmatrix}, \hat{j}_c = \begin{pmatrix} -s_{\psi_c} \\ c_{\psi_c} \\ 0 \end{pmatrix}, \hat{k}_c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where p_c is the orthogonal projection of the vehicle's position p_b onto the curve $c(q) = 0$, $\hat{i}_c$ is tangential to the curve and points towards the desired motion direction, $\hat{j}_c$ is orthogonal/normal to curve and lays on the vehicle's motion plane, and $\hat{k}_c$ is orthogonal to the other two unit vectors and chosen so as to form a right-handed tern. Let σ be the curvilinear abscissa representing the distance travelled by the origin of the frame $\mathcal{F}_c$, d be the curvilinear ordinate representing the distance of the origin of $\mathcal{F}_b$ from $\mathcal{F}_c$, and θ the difference between the vehicle's orientation and the desired one. In order to find the required controller, we need to determine the unicycle's kinematic model in terms of the newly defined variables, i.e., the dynamics of the robot as seen from an observer attached to $\mathcal{F}_c$. To achieve this consider the following:

$$
\begin{aligned}
\dot{p}_c &= \dot{p}_b - (\dot{p}_b - \dot{p}_c) = \\
&= \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \left(\begin{pmatrix} 0 & -\dot{\psi}_c \\ \dot{\psi}_c & 0 \end{pmatrix} \begin{pmatrix} c_{\psi_c} & -s_{\psi_c} \\ s_{\psi_c} & c_{\psi_c} \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix} + \begin{pmatrix} c_{\psi_c} & -s_{\psi_c} \\ s_{\psi_c} & c_{\psi_c} \end{pmatrix} \begin{pmatrix} \dot{\sigma} \\ \dot{d} \end{pmatrix} \right) = \\
&= \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \dot{\sigma}(1 - d\,\dot{\psi}_c(\sigma))\,\hat{i}_c - \dot{d}\,\hat{j}_c \,.
\end{aligned}
$$

Having denoted with

$$
\gamma(\sigma) \stackrel{\text{def}}{=} \frac{\partial c(\sigma)}{\partial \sigma},
$$

the *curvature* of the path described by $c(q) = 0$, one can obtain by the rule chain

$$
\dot{\psi}_c = \gamma(\sigma)\,\dot{\sigma} \,.
$$

By substituting the first two equations of the kinematic model in (1.1) into the expression found above, and then projecting it along the axes of $\mathcal{F}_c$, one obtains the following:

$$
\begin{aligned}
\dot{\sigma}(1 - d\,c(\sigma)) = (\dot{x}, \dot{y}) \cdot \hat{i}_c = v(c_\psi c_{\psi_c} + s_\psi s_{\psi_c}) = v\,c_{\psi-\psi_c} &\stackrel{\text{def}}{=} v\,c_\theta \,, \\
\dot{d} = (\dot{x}, \dot{y}) \cdot \hat{j}_c = v(s_\psi c_{\psi_c} - c_\psi s_{\psi_c}) = v\,s_{\psi-\psi_c} &\stackrel{\text{def}}{=} v\,s_\theta \,,
\end{aligned}
$$

where we used $\theta = \psi - \psi_c$. It also holds:

$$
\dot{\theta} = \dot{\psi} - \dot{\psi}_c = \omega - \gamma(\sigma)\,\dot{\sigma} \,.
$$

Therefore, the kinematic model of a unicycle vehicle described w.r.t. an observer moving with frame $\mathcal{F}_c$ is as follows:

$$
\begin{pmatrix} \dot{\sigma} \\ \dot{d} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dfrac{c_\theta}{1 - d\,\gamma(\sigma)} & 0 \\ s_\theta & 0 \\ -\dfrac{\gamma(\sigma)\,c_\theta}{1 - d\,\gamma(\sigma)} & 1 \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix}, \qquad (1 - d\,\gamma(\sigma) \neq 0) \,.
$$

Let us begin by developing a Ляпунòв-based controller allowing the alignment of the vehicle's motion along a desired path. It is commonly assumed that the forward speed v is a preassigned function of time, i.e. $v = \bar{v}(t)$, which may be specified by e.g. a higher-level planner system. The control objective thus consists of finding a feedback law for the orientation (yaw) rate ω, ensuring that the unicycle vehicle approach the desired path and travel along it with the specified velocity $\bar{v}$. With this aim, consider the following candidate Ляпунòв function:

$$V = \frac{1}{2} d^2 + \frac{1}{2} \theta^2 \,.$$

Its time-derivative can be made negative semi-definite as follows:

$$\begin{aligned} \dot{V} &= d\,\bar{v}\,s_\theta + \theta \left(\omega - \frac{\bar{v}\,\gamma(\sigma)\,c_\theta}{1 - d\,\gamma(\sigma)} \right) = \\ &= \theta \left(\omega - \frac{\bar{v}\,\gamma(\sigma)\,c_\theta}{1 - d\,\gamma(\sigma)} + d\,\bar{v}\,\mathrm{sinc}(\theta) \right) := -k\,\theta^2 \,, \end{aligned}$$

where $\mathrm{sinc}(\theta) = s_\theta / \theta$ and the choice below has been made:

$$\boxed{\omega = \frac{\bar{v}\,\gamma(\sigma)\,c_\theta}{1 - d\,\gamma(\sigma)} - d\,\bar{v}\,\mathrm{sinc}(\theta) - k\,\theta} \,.$$

The resulting model of the controlled unicycle vehicle becomes:

$$\begin{pmatrix} \dot{\sigma} \\ \dot{d} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dfrac{\bar{v}\,c_\theta}{1 - d\,\gamma(\sigma)} \\ \bar{v}\,s_\theta \\ -d\,\bar{v}\,\mathrm{sinc}(\theta) - k\,\theta \end{pmatrix} .$$

Ляпунòв's Theorem ensures that the controlled system is at least marginally stable, since the set of points vanishing $\dot{V}$ is $N = \{q \,|\, \dot{V} = 0\} = \{\theta = 0\}$. Furthermore, under the hypothesis that $\bar{v} \neq 0$, by Красòвскии-Lasalle's Theorem, one can also conclude in favor of the asymptotical stability of the system's equilibrium in the origin, as indeed the maximum invariant set, $E = \{q \,|\, \dot{V} \equiv 0\} = \{d = 0\,,\ \theta = 0\}$, only contains the origin itself. This fact can be shown by considering the system's dynamics complying with the constraint specified by $\dot{V} \equiv 0$, telling us also furthermore information on how the vehicle moves after convergence to the desired path. By imposing $\theta \equiv 0$ and thus $\dot{\theta} = 0$ in the model, one obtains:

$$\begin{pmatrix} \dot{\sigma} \\ \dot{d} \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\bar{v}}{1 - d\,\gamma(\sigma)} \\ 0 \\ -d\,\bar{v} \end{pmatrix} .$$

whose unique solution is $d = 0$ and $\dot{\sigma} = \bar{v}$, which tells us that the unicycle vehicle asymptotically aligns to the desired curve and travels along it with the desired speed $\bar{v}$.

Alternatively, let us assume d as the vehicle's output and apply the exact linearization technique to the relation between such output and the input variable ω. The second time derivative of d can be linearized as follows:

$$\ddot{d} = \dot{\bar{v}}\, s_\theta + \bar{v}\, c_\theta \left(\omega - \frac{\bar{v}\, \gamma(\sigma)\, c_\theta}{1 - d\, \gamma(\sigma)} \right) := -k_1 d - k_2 \dot{d}\,,$$

with $k_1, k_2 > 0$, and thus with the input control choice:

$$\omega = -\frac{k_1 d + k_2 \dot{d} + \dot{\bar{v}}\, s_\theta}{\bar{v}\, c_\theta} + \frac{\bar{v}\, \gamma(\sigma)\, c_\theta}{1 - d\, \gamma(\sigma)}\,,$$

which is defined for every state except for $\theta = \pm\pi/2$ and for $1 - d\,\gamma(\sigma) \neq 0$. As usual, one can define the new variables $\xi_1 \stackrel{\text{def}}{=} d$ and $\xi_2 \stackrel{\text{def}}{=} \dot{d}$, and, in order to complete the change of coordinates, one can choose the variable $\eta \stackrel{\text{def}}{=} \sigma$ whose dynamics is

$$\dot{\eta} = \frac{\bar{v}\, c_\theta}{1 - \xi_1\, \gamma(\eta)} = \frac{\sqrt{\bar{v}^2 - \xi_2^2}}{1 - \xi_1\, \gamma(\eta)}\,,$$

where the relation $\cos(\arcsin(x)) = \sqrt{1 - x^2}$ has been used. It is worth noticing that, for any arbitrary initial state conditions of the original variables, it holds $\bar{v}^2 - \xi_2 = \bar{v}^2 - (\bar{v}\, s_\psi)^2 = \bar{v}^2(1 - s_\psi^2) \geq 0$, which ensures that the result of root square in the above equation is always defined. In the new variables, the model of the controlled unicycle is expressed as follows:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \xi_2 \\ -k_1\, \xi_1 - k_2\, \xi_2 \\ \dfrac{\sqrt{\bar{v}^2 - \xi_2^2}}{1 - \xi_1\, \gamma(\eta)} \end{pmatrix}.$$

The controlled system's zero dynamics is described by the conditions $\xi_1 = \xi_2 = 0$ for all t, which implies that $\dot{\eta} = \pm|\bar{v}|$. Therefore, the above found control law ensures that the unicycle system asymptotically reaches the desired curve. Furthermore, depending on its initial state, the unicycle eventually moves towards either the increasing or the decreasing direction of the curvilinear abscissa σ.

1.2.2 Alignment to a straight line

Let us specialize the two control laws found above in the event that the desired curve is a straight line. Observing that the curvature is null, i.e., $\gamma(\sigma) = 0$ for all σ, the kinematics of the unicycle vehicle, expressed in terms of frame $\mathcal{F}_c$ reduces to:

$$\begin{pmatrix} \dot{\sigma} \\ \dot{d} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} c_\theta & 0 \\ s_\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix},$$

showing a quite apparent similarity with kinematics of the unicycle vehicle in the original variables of frame $\mathcal{F}_0$. As stated above, the forward speed v is commonly given by a preassigned function of time, i.e. $v = \bar{v}(t)$. Then the control law found by the application of Ляпунòв's approach specializes to

$$\omega = -d\,\bar{v}\,\mathrm{sinc}(\theta) - k\,\theta \ ,$$

while the control law derived from the application of the exact linearization approach is

$$\omega = -\frac{k_1\,d + k_2\,\dot{d} + \dot{\bar{v}}\,s_\theta}{\bar{v}\,c_\theta}, \qquad (\theta \neq \pm\pi/2) \ .$$

It is worth noticing that the Ляпунòв-based controller is preferable to the one designed through feedback linearization. The reason is twofold. First, the Ляпунòв's controller is globally valid; secondly, it does not require the estimation of $\dot{d}$ from the measures of d, which is inevitably affected by noise.

1.3 Trajectory Tracking

Focus now on the problem of designing feedback control laws for the input variables v and ω, allowing a unicycle vehicle to asymptotically track (reach) a moving reference moving according to a known function of time $\hat{q}(t) = (\hat{x}(t), \hat{y}(t), \hat{\psi}(t))^T$. Consider the coordinate frames defined in the previous section. First of all, one has to take into account that the kinematic constraint, introduced by the wheel or pair of wheels, may prevent the unicycle vehicle to be able to exactly track any arbitrary desired motion $\hat{q}(t)$. In fact, given a generic $\hat{q}(t)$, it is rarely the case when such a function complies with the vehicle's kinematics, i.e. there exist two corresponding smooth functions $\hat{v}(t)$ and $\hat{\omega}(t)$ generating such trajectory $\hat{q}(t)$. This implies that a trajectory planner system has to carefully design $\hat{q}(t)$. To this purpose, it is more convenient to choose $\hat{v}(t)$ and $\hat{\omega}(t)$ and let the corresponding trajectory $\hat{q}(t)$ be generated through a copy of the unicycle vehicle model:

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{y}} \\ \dot{\hat{\psi}} \end{pmatrix} = \begin{pmatrix} c_{\hat{\psi}} & 0 \\ s_{\hat{\psi}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\omega} \end{pmatrix} .$$

The problem we are trying to solve normally arises in applications involving a moving leader robot and a follower robot, which is required to track the motion of the first robot. Consider the following coordinate frames:

- (base frame) $\mathcal{F}_0 = \left\{ p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \hat{i}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{j}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$,
- (follower's body frame) $\mathcal{F}_b = \left\{ p_b = \begin{pmatrix} x \\ y \end{pmatrix}, \hat{i}_b = \begin{pmatrix} c_\psi \\ s_\psi \end{pmatrix}, \hat{j}_b = \begin{pmatrix} -s_\psi \\ c_\psi \end{pmatrix} \right\}$.

- (leader's frame) $\mathcal{F}_t = \left\{ p_t = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}, \hat{i}_t = \begin{pmatrix} c_{\hat{\psi}} \\ s_{\hat{\psi}} \end{pmatrix}, \hat{j}_t = \begin{pmatrix} -s_{\hat{\psi}} \\ c_{\hat{\psi}} \end{pmatrix} \right\}$.

We begin by defining, as usual, the *tracking error* variables, as seen from an observer fixed at the origin of the base frame $\mathcal{F}_0$:

$$e_x \stackrel{\text{def}}{=} x - \hat{x}\,, \quad e_y \stackrel{\text{def}}{=} y - \hat{y}\,, \quad e_\psi \stackrel{\text{def}}{=} \psi - \hat{\psi}\,.$$

By derivation with respect to time the above definitions, one can obtain the error dynamics of the unicycle vehicle:

$$\begin{pmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_\psi \end{pmatrix} = \begin{pmatrix} v\,c_\psi - \dot{\hat{x}} \\ v\,s_\psi - \dot{\hat{y}} \\ \omega - \dot{\hat{\psi}} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} v\,c_\psi - \hat{v}\,c_{\hat{\psi}} \\ v\,s_\psi - \hat{v}\,s_{\hat{\psi}} \\ \omega - \hat{\omega} \end{pmatrix} = \begin{pmatrix} R(\psi)\begin{pmatrix} v \\ 0 \end{pmatrix} - R(\hat{\psi})\begin{pmatrix} \hat{v} \\ 0 \end{pmatrix} \\ \omega - \hat{\omega} \end{pmatrix}.$$

Moreover, since the follower robot must control its own motion, based on measurement of the *relative displacement* between its own configuration q and the one $\hat{q}$ predicted for the leader robot, it is convenient to write the tracking error variables in the follower's body frame $\mathcal{F}_b$, by rotating them of $-\psi$:

$$\begin{pmatrix} e_x^b \\ e_y^b \end{pmatrix} = \begin{pmatrix} c_\psi & s_\psi \\ -s_\psi & c_\psi \end{pmatrix}\begin{pmatrix} e_x \\ e_y \end{pmatrix} = R(-\psi)\begin{pmatrix} e_x \\ e_y \end{pmatrix},$$
$$e_\psi^b = \psi - \hat{\psi}\,.$$

Derivation with respect to time the error variables yields:

$$\begin{aligned} \begin{pmatrix} \dot{e}_x^b \\ \dot{e}_y^b \end{pmatrix} &= R(-\psi)\begin{pmatrix} \dot{e}_x \\ \dot{e}_y \end{pmatrix} + \begin{pmatrix} 0 & \dot{\psi} \\ -\dot{\psi} & 0 \end{pmatrix} R(-\psi)\begin{pmatrix} e_x \\ e_y \end{pmatrix} = \\ &= R(\psi - \psi)\begin{pmatrix} v \\ 0 \end{pmatrix} - R(-(\psi - \hat{\psi}))\begin{pmatrix} \hat{v} \\ 0 \end{pmatrix} + \begin{pmatrix} e_y^b\,\omega \\ -e_x^b\,\omega \end{pmatrix} = \\ &= \begin{pmatrix} v - \hat{v}\,c_{e_\psi^b} + e_y^b\,\omega \\ \hat{v}\,s_{e_\psi^b} - e_x^b\,\omega \end{pmatrix} \end{aligned}$$

In the follower's body frame $\mathcal{F}_b$, let us define the longitudinal and vertical variables σ and d, and the relative angle θ as $(\sigma, d, \theta)^T \stackrel{\text{def}}{=} (e_x^b, e_y^b, e_\psi^b)^T$. The unicycle vehicle kinematics reads:

$$\boxed{\begin{pmatrix} \dot{\sigma} \\ \dot{d} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v - \hat{v}\,c_\theta + d\,\omega \\ \hat{v}\,s_\theta - \sigma\,\omega \\ \omega - \hat{\omega} \end{pmatrix}}\,.$$

We are now ready to address the problem of designing suitable feedback laws, for the two input variables v and ω, stabilizing the origin of the state vector $(\sigma, d, \theta)^T$. A first solution can be found through application of Ляпунòв's method. In this vein, the usual quadratic candidate Ляпунòв function

$$V = \frac{1}{2}\sigma^2 + \frac{1}{2}d^2 + \frac{1}{2}\theta^2\,,$$

has a time derivative

$$\dot{V} = \sigma(v - \hat{v}\,c_\theta) + d\,\hat{v}\,s_\theta + \theta(\omega - \hat{\omega})\,,$$

whose sign is not well defined. As a first step, one can choose the control $v = \hat{v}\,c_\theta - k_v\,\sigma$, yielding to:

$$\dot{V} = -k_v\sigma^2 + d\,\hat{v}\,s_\theta + \theta(\omega - \hat{\omega})\,,$$

A quick look at the obtained expression suggests that a slightly modified version of the candidate function V can be selected:

$$V = \frac{1}{2}\sigma^2 + \frac{1}{2}d^2 + 1 + \int_0^\theta s_\theta\,d\theta\,,$$

whose time derivative can be made negative semi-definite, i.e.,

$$\dot{V} = -k_v\sigma^2 + s_\theta(d\,\hat{v} + \omega - \hat{\omega}) := -k_v\sigma^2 - k_\omega\theta\,s_\theta\,,$$

by choosing the steering rate variable ω as $\omega = \hat{\omega} - d\,\hat{v} - k_\omega\,\theta$. By Ляпунòв's Theorem the equilibrium in the origin is at least asymptotically stable. The vehicle's kinematics complying with the constraint of belonging to the maximum invariant set $E = \{q\,|\,\dot{V} \equiv 0\}$ is

$$\begin{pmatrix} 0 \\ \dot{d} \\ 0 \end{pmatrix} = \begin{pmatrix} d(\hat{\omega} - d\,\hat{v}) \\ 0 \\ -d\,\hat{v} \end{pmatrix}\,,$$

whose unique solution, for $\hat{v} \neq 0$, is the equilibrium in the origin. By Красòвскии-Lasalle's Theorem the state of the controlled unicycle vehicle asymptotically converges to the origin. In conclusion, a follower unicycle vehicle can be driven so as to asymptotically track/reach a leader robot, by the following feedback control laws for the forward speed v and steering rate ω:

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} \hat{v}\,c_\theta - k_v\,\sigma \\ \hat{\omega} - d\,\hat{v} - k_\omega\,\theta \end{pmatrix}\,.$$

Finally observe that, if the leader's forward speed $\hat{v}$ is available in the follower's body frame $\mathcal{F}_b$, then the following quantities $\hat{v}_\sigma = \hat{v}\,c_\theta$ and $\hat{v}_d = \hat{v}\,s_\theta$ must be substituted in the above formula:

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} \hat{v}_\sigma - k_v\,\sigma \\ \hat{\omega} - d\,\sqrt{\hat{v}_\sigma^2 + \hat{v}_d^2} - k_\omega\,\theta \end{pmatrix}\,.$$

2

Car-like Vehicles - Vers. of January 19, 2019 - A. Fagiolini

A car-like is the schematization of a terrestrial vehicle with two wheels, among which the front wheel can rotate around an axis, orthogonal to the motion surface plane and passing through the contact point between the wheel and the plane itself, while the rear wheel is rigidly attached to the longitudinal vehicle axis. It can also represent a vehicle with two pairs of wheels mounted on the same axles, in which the front axle can rotate and the rear one always forms a right angle with the longitudinal vehicle axis.

Referring to Fig. XXX, denote with θ and ψ (steering) the angles formed by the x-axis and the longitudinal vehicle axis, and by the longitudinal vehicle axis and a line orthogonal to the front axle, respectively. Having denoted with L be the wheelbase, i.e. the distance between the front and the rear wheels or axles, consider the following reference frames:

- $\mathcal{F}_0 = \left\{ p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \hat{i}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{j}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$
- $\mathcal{F}_r = \left\{ p_r = \begin{pmatrix} x_r \\ y_r \end{pmatrix}, \hat{i}_r = \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix}, \hat{j}_r = \begin{pmatrix} -s_\theta \\ c_\theta \end{pmatrix} \right\},$
- $\mathcal{F}_f = \left\{ p_f = \begin{pmatrix} x_f \\ y_f \end{pmatrix}, \hat{i}_f = \begin{pmatrix} c_{\theta+\psi} \\ s_{\theta+\psi} \end{pmatrix}, \hat{j}_f = \begin{pmatrix} -s_{\theta+\psi} \\ c_{\theta+\psi} \end{pmatrix} \right\}.$

As we will show below, depending on the specific control objective, it is convenient to express the vehicle's configuration with respect to a different reference point, in terms of which the kinematic model of the vehicle can be found.

2.1 Rear-Reference Kinematic Models

Let us first focus on the models with reference point on the rear wheel or rear axle. Let the vehicle configuration vector be $q_r = (x_r, y_r, \theta, \psi)^T$. Each wheel or pair of wheels introduces in the model a kinematic constraint requiring that the wheel's transversal speed is always null, which is expressed by the following conditions:

$$
\begin{aligned}
\dot{p}_r \cdot \hat{j}_r &= 0\,, \\
\dot{p}_f \cdot \hat{j}_f &= 0\,.
\end{aligned}
$$

After having computed the involved speeds,

$$
\begin{aligned}
\dot{p}_r &= \begin{pmatrix} \dot{x}_r \\ \dot{y}_r \end{pmatrix}, \\
\dot{p}_f &= \dot{p}_r + \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix} \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} L \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{x}_r - L\, s_\theta\, \dot{\theta} \\ \dot{y}_r + L\, c_\theta\, \dot{\theta} \end{pmatrix},
\end{aligned}
$$

one can write the above constraints in Pfaffian form as

$$
A(q_r)\,\dot{q}_r = \begin{pmatrix} -s_\theta & c_\theta & 0 & 0 \\ -s_{\theta+\psi} & c_{\theta+\psi} & L\, c_\psi & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

It is easy to verify that the rank of $A(q_r)$ is 2 for every q_r, and thus its null-column space, $\ker(A(q_r))$, has dimension $4 - 2 = 2$. In order to find a first basis for the null-column space of $A(q_r)$, one can observe that the vector $(0, 0, 0, 1)^T$ obviously belongs to it, and another vector $(c_\theta, s_\theta, a, 0)^T$ is suggested by focusing on the first row of $A(q_r)$, where the entry element a can be found by solving the second equation:

$$
\begin{aligned}
-s_{\theta+\psi}\, c_\theta + c_{\theta+\psi}\, s_\theta + a\, L\, c_\psi &= 0\,, \\
-s_\psi + a\, L\, c_\psi &= 0\,, \\
a = \tfrac{1}{L}\, \tan(\psi) &:= \tfrac{1}{L} t_\psi\,,
\end{aligned}
$$

which is valid for every configuration with $\psi \neq \pm\pi/2$. This choice of basis vectors leads to the following *rear-reference, rear-traction* model of the form $\dot{q}_r = S_1(q_r)\,\nu_r$, with $\nu_r = (v_r, \omega)^T$:

$$
\begin{pmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} c_\theta & 0 \\ s_\theta & 0 \\ \frac{1}{L}\, t_\psi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_r \\ \omega \end{pmatrix}, \qquad (\psi \neq \pm\pi/2)\,, \tag{2.1}
$$

where v_r and ω can be interpreted as the longitudinal speed of the rear reference point and the angular speed of the front wheel or axle, respectively, as it can be verified below:

$$
\begin{aligned}
|\dot{p}_r| = \sqrt{\dot{x}_r^2 + \dot{y}_r^2} = |v_r|\,,\ \ \hat{\dot{p}}_r &= \frac{\begin{pmatrix} \dot{x}_r \\ \dot{y}_r \end{pmatrix}}{\sqrt{\dot{x}_r^2 + \dot{y}_r^2}} = \frac{v_r}{|v_r|} \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix} = \mathrm{sign}(v_r)\, \hat{i}_r\,, \\
\dot{p}_r = |\dot{p}_r|\, \hat{\dot{p}}_r &= |v_r|\, \mathrm{sign}(v_r)\, \hat{i}_r = v_r\, \hat{i}_r\,,
\end{aligned}
$$

$$
\dot{\psi} = \omega\,.
$$

A second basis of the null-column space of $A(q_r)$ can be obtained by multiplying the first column of $S_1(q_r)$ by c_ψ, which leads to the following *rear-reference, front-traction* model of the form $\dot{q}_r = S_2(q_r)\,\nu_f$, with $\nu_f = (v_f, \omega)^T$:

$$\begin{pmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} c_\psi c_\theta & 0 \\ c_\psi s_\theta & 0 \\ \frac{1}{L}\, s_\psi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_f \\ \omega \end{pmatrix},$$

where v_f and ω can be interpreted as the longitudinal speed of the front reference point, as it can be verified below:

$$\begin{aligned}
|\dot{p}_f| &= \sqrt{\dot{x}_f^2 + \dot{y}_f^2} = \sqrt{(\dot{x}_r - L\, s_\theta\, \dot{\theta})^2 + (\dot{y}_r + L\, c_\theta\, \dot{\theta})^2} = \\
&= \sqrt{(c_\psi c_\theta v_f - L s_\theta \tfrac{1}{L} s_\psi v_f)^2 + (c_\psi s_\theta v_f + L c_\theta \tfrac{1}{L} s_\psi v_f)^2} = |v_f|\,, \\
\hat{\dot{p}}_f &= \frac{\begin{pmatrix} \dot{x}_f \\ \dot{y}_f \end{pmatrix}}{\sqrt{\dot{x}_f^2 + \dot{y}_f^2}} = \frac{1}{|v_f|} \begin{pmatrix} \dot{x}_r - L\, s_\theta\, \dot{\theta} \\ \dot{y}_r + L\, c_\theta\, \dot{\theta} \end{pmatrix} = \frac{v_f}{|v_f|} \begin{pmatrix} c_\psi c_\theta - s_\psi s_\theta \\ c_\psi s_\theta + s_\psi c_\theta \end{pmatrix} = \\
&= \frac{v_r}{|v_r|} \begin{pmatrix} c_{\theta+\psi} \\ s_{\theta+\psi} \end{pmatrix} = \operatorname{sign}(v_f)\, \hat{\imath}_f\,, \\
\dot{p}_f &= |\dot{p}_f|\, \hat{\dot{p}}_f = |v_f| \operatorname{sign}(v_f)\, \hat{\imath}_f = v_f\, \hat{\imath}_f\,.
\end{aligned}$$

2.2 Front-Reference Kinematic Models

Let us now consider the models with reference point on the front wheel or front axle. Let the vehicle configuration vector be $q_f = (x_f, y_f, \theta, \psi)^T$. After expressing the following speeds in terms of q_f,

$$\begin{aligned}
\dot{p}_f &= \begin{pmatrix} \dot{x}_f \\ \dot{y}_f \end{pmatrix}, \\
\dot{p}_r &= \dot{p}_f + \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix} \begin{pmatrix} c_\psi & -s_\psi \\ s_\psi & c_\psi \end{pmatrix} \begin{pmatrix} -L \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{x}_f + L\, s_\theta\, \dot{\theta} \\ \dot{y}_f - L\, c_\theta\, \dot{\theta} \end{pmatrix},
\end{aligned}$$

one can write the kinematic constraints due to the two wheels as in the following Pfaffian form:

$$A(q_f)\, \dot{q}_f = \begin{pmatrix} -s_\theta & c_\theta & -L & 0 \\ -s_{\theta+\psi} & c_{\theta+\psi} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As shown in the previous section, one can choose two different basis vectors of the null column space of $A(q_f)$ and consequently obtain models for the two possible configurations of the vehicle's traction. More precisely, one can obtain a *front-reference, rear-traction* model of the form $\dot{q}_f = S_3(q_f)\,\nu_r$, with $\nu_r = (v_r, \omega)^T$,

$$\begin{pmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \dfrac{c_{\theta+\psi}}{c_\psi} & 0 \\ \dfrac{s_{\theta+\psi}}{c_\psi} & 0 \\ \frac{1}{L}\,t_\psi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_r \\ \omega \end{pmatrix}, \qquad (\psi \neq \pm\pi/2)\,,$$

and a *front-reference, front-traction* model of the form $\dot{q}_f = S_4(q_f)\,\nu_f$, with $\nu_f = (v_f, \omega)^T$,

$$\begin{pmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} c_{\theta+\psi} & 0 \\ s_{\theta+\psi} & 0 \\ \frac{1}{L}\,s_\psi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_f \\ \omega \end{pmatrix}. \tag{2.2}$$

2.3 Path following

2.3.1 Alignment to a straight line

Let us begin by addressing the problem of the stabilization, about the straight horizontal line $\hat{y} = 0$, of a car-like vehicle with the *front-reference front-traction* kinematics in Eq. 2.2. Suppose that the longitudinal front speed is an assigned function of time $v_f = \bar{v}_f(t)$. To this purpose, consider the candidate Ляпунòв function

$$V = \frac{1}{2}y_f^2 + \frac{1}{2}(\theta+\psi)^2\,,$$

whose time derivative is

$$\begin{aligned}\dot{V} &= y_f \bar{v}_f s_{\theta+\psi} + (\theta+\psi)(\omega + \tfrac{1}{L}s_\psi \bar{v}_f) = \\ &= (\theta+\psi)(\omega + \tfrac{1}{L}s_\psi \bar{v}_f + y_f \bar{v}_f \operatorname{sinc}(\theta+\psi))\,.\end{aligned}$$

One can try to obtain the negative semi-definite form

$$\dot{V} := -k(\theta+\psi)^2\,,$$

which is achieved through the following control law for the steering speed:

$$\omega = -k(\theta+\psi) - \frac{1}{L}s_\psi \bar{v}_f - y_f \operatorname{sinc}(\theta+\psi)\,\bar{v}_f\,.$$

Through this choice, the kinematics of the controlled system becomes:

$$\begin{aligned}
\dot{x}_f &= c_{\theta+\psi}\,\bar{v}_f\,,\\
\dot{y}_f &= s_{\theta+\psi}\,\bar{v}_f\,,\\
\dot{\theta} &= \tfrac{1}{L}\,s_\psi\,\bar{v}_f\,,\\
\dot{\psi} &= -k(\theta+\psi) - \tfrac{1}{L}s_\psi\bar{v}_f - y_f\,\mathrm{sinc}(\theta+\psi)\,\bar{v}_f\,.
\end{aligned}$$

Based on Ляпунòв's Theorem and Красòвскии-Lasalle's Theorem, the control law found above ensures the convergence to the following conditions:

$$\theta = -\psi\,, \quad \dot{\theta} = -\dot{\psi}\,.$$

After their substitution into the model one obtains:

$$\begin{aligned}
\dot{x}_f &= \bar{v}_f\,,\\
\dot{y}_f &= 0\,,\\
\dot{\theta} &= -\tfrac{1}{L}\,s_\theta\,\bar{v}_f\,,\\
\dot{\psi} &= -\tfrac{1}{L}s_\psi\bar{v}_f - y_f\bar{v}_f\,.
\end{aligned}$$

As long as the longitudinal vehicle speed remains non null, $\bar{v}_f \neq 0$, one can establish, by using the third and the forth equations, that also $y_f = 0$. The dynamics of the two angle variables become:

$$\dot{\theta} = -\frac{\bar{v}_f}{L}s_\theta\,, \quad \dot{\psi} = -\frac{\bar{v}_f}{L}s_\psi$$

The equilibria of the two models are $\theta = \pi k$ and $\psi = \pi k$, respectively. Now, if $\bar{v}_f(t) > v_f^{min} > 0$, the origins $\theta = 0$ and $\psi = 0$ are both asymptotically stable for every state in $-\pi < \theta, \psi < \pi$. Indeed one can use the Ляпунòв functions

$$V_\theta = \frac{1}{2}\theta^2\,, \quad V_\psi = \frac{1}{2}\psi^2\,,$$

whose derivatives are

$$\dot{V}_\theta = -\frac{\bar{v}_f}{L}\theta s_\theta < -\frac{v_f^{min}}{L}\theta s_\theta < 0\,, \quad \dot{V}_\psi = -\frac{\bar{v}_f}{L}\psi s_\psi < -\frac{v_f^{min}}{L}\psi s_\psi < 0\,.$$

On the contrary, if $\bar{v}_f < v_f^{max} < 0$, then the state points $\theta = \pi$ and $\psi = \pi$ are asymptotically stable for every state in $-2\pi < \theta, \psi < 0$. This fact can be proved, as usual, by translating the equilibria in the origins, i.e. by defining

$$\theta^* = \theta - \pi\,, \quad \psi^* = \psi - \pi\,,$$

and then showing that the Ляпунòв functions,

$$V_{\theta^*} = \frac{1}{2}\theta^{*2}\,, \quad V_{\psi^*} = \frac{1}{2}\psi^{*2}\,,$$

possess negative definite time derivatives. Indeed we have:

$$\begin{aligned}
\dot{V}_{\theta^*} = -\frac{\bar{v}_f}{L}\theta^* s_{\theta^*+\pi} = \frac{\bar{v}_f}{L}\theta^* s_{\theta^*} &< -\frac{|v_f^{max}|}{L}\theta^* s_{\theta^*} < 0\,,\\
\dot{V}_{\psi^*} &< -\frac{|v_f^{max}|}{L}\psi^* s_{\psi^*} < 0\,.
\end{aligned}$$

In conclusion, the established steering control ensures that a car-like vehicle asymptotically aligns with a horizontal straight line. Moreover, as desirable, the finally achieved traveling direction depends on the sign of $\bar{v}_f$.

Focus now on the same stabilization problem for a car-like vehicle with the *rear-reference, rear-traction* kinematics in Eq. 2.1. Suppose this time that the longitudinal rear speed is an assigned function time $v_r = \bar{v}_r(t)$. The equations of the model allow for the application of the backstepping technique, assuming as an internal input the derivative $\dot{\theta}$ of the rear angle. We first need to put the model in such a form that the first part to be stabilized is affine with respect to the considered internal input. To this purpose, let us define:

$$\dot{\theta} = \frac{\bar{v}_r}{L}\, t_\psi := \omega_\theta \,,$$

and consequently have

$$\psi = \arctan\left(\frac{L}{\bar{v}_r}\omega_\theta\right) \,,$$

and let us replace the fourth equation in the model with the following one:

$$\dot{\omega}_\theta = \frac{\dot{\bar{v}}_r}{L}\, t_\psi + \frac{\bar{v}_r}{L}\,\frac{\dot{\psi}}{c_\psi^2} = \frac{\dot{\bar{v}}_r}{\bar{v}_r}\,\omega_\theta + \frac{\bar{v}_r^2 + L^2\omega_\theta^2}{L\,\bar{v}_r}\,\omega \,,$$

where the following simplification has been used:

$$c_\psi^2 = \cos\left(\arctan\left(L\,\omega_\theta/\bar{v}_r\right)\right)^2 = \left(\sqrt{1 + L^2\,\omega_\theta^2/\bar{v}_r^2}\right)^{-2} \,.$$

The overall system kinematics has now the desired form:

$$\begin{aligned}
\dot{x}_r &= \bar{v}_r\, c_\theta \,,\\
\dot{y}_r &= \bar{v}_r\, s_\theta \,,\\
\dot{\theta} &= \omega_\theta \,,\\
\dot{\omega}_\theta &= \frac{\dot{\bar{v}}_r}{\bar{v}_r}\omega_\theta + \frac{\bar{v}_r^2 + L^2\omega_\theta^2}{L\,\bar{v}_r}\omega \,.
\end{aligned}$$

We can now exploit the new variable ω_θ as an internal input, in order to stabilize the equilibrium in the origin of the subsystem described by the first three equations, and then "step back" in the dynamics so as to obtain a control law for the external input ω, which stabilizes the equilibrium in the origin of the overall system. The first three equations merely represent the kinematic model of a unicycle vehicle, for which the origin can be made asymptotically stable by applying a control law of the form:

$$\omega_\theta = \boxed{-y_r\,\bar{v}_r\,\mathrm{sinc}(\theta) - k_\theta\,\theta \stackrel{\text{def}}{=} \Omega_\theta(x_r, y_r, \theta)} \,.$$

Observe that $\Omega_\theta(0,0,0) = 0$. A Ляпунòв function for the subsystem and its derivative are:

$$V = \frac{1}{2}y_r^2 + \frac{1}{2}\theta^2 \,, \qquad \dot{V} = -k_\theta \theta^2 \,.$$

By Красòвскии-Lasalle's Theorem, it can be shown that the maximum invariant set only consists of the sought horizontal straight line, and thus one can conclude that any evolution of the subsystem asymptotically tends to such a line. In order to step back, let us define another variable,

$$z \stackrel{\text{def}}{=} \omega_\theta - \Omega_\theta \,,$$

and rewrite the system kinematics in terms of the new state:

$$\begin{aligned}
\dot{x}_r &= \bar{v}_r \, c_\theta \,, \\
\dot{y}_r &= \bar{v}_r \, s_\theta \,, \\
\dot{\theta} &= z + \Omega_\theta \,, \\
\dot{z} &= \frac{\dot{\bar{v}}_r}{\bar{v}_r}(z + \Omega_\theta) + \frac{\bar{v}_r^2 + L^2(z + \Omega_\theta)^2}{L\,\bar{v}_r}\omega - \dot{\Omega}_\theta \,.
\end{aligned}$$

Now consider the following candidate Ляпунòв function:

$$V^* = V + \frac{1}{2}z^2 \,,$$

whose time derivative is given by:

$$\begin{aligned}
\dot{V}^* &= \dot{V} + \left(\frac{\partial V}{\partial x_r}, \frac{\partial V}{\partial y_r}, \frac{\partial V}{\partial \theta}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z + z\,\dot{z} = \\
&= \dot{V} + z\left(\theta + \frac{\dot{\bar{v}}_r}{\bar{v}_r}(z + \Omega_\theta) + \frac{\bar{v}_r^2 + L^2(z + \Omega_\theta)^2}{L\,\bar{v}_r}\omega - \dot{\Omega}_\theta\right) \\
&:= -k_\theta\,\theta^2 - k_\omega\,z^2 \,,
\end{aligned}$$

where the following feedback control law has been applied:

$$\omega = \frac{L\,\bar{v}_r}{\bar{v}_r^2 + L^2(z + \Omega_\theta)^2}\left(\dot{\Omega}_\theta - \theta - \frac{\dot{\bar{v}}_r}{\bar{v}_r}(z + \Omega_\theta) - k_\omega\,z\right) .$$

Analogous conclusions can be made which again exploit Красòвскии-Lasalle's Theorem in order to prove that the equilibrium in the origin is asymptotically stable for the entire model of a car-like system. Furthermore, it is convenient in practice to express the above control law in terms of the original coordinates. To achieve this objective we conveniently carried out the following computations:

$$\begin{gathered}
z = \frac{\bar{v}_r}{L}t_\psi - \Omega_\theta \,, \\
\frac{L\bar{v}_r}{\bar{v}_r^2 + L^2\bar{v}_r^2 t_\psi^2/L^2} = \frac{L}{\bar{v}_r}\frac{1}{1 + t_\psi^2} = \frac{L}{\bar{v}_r}c_\psi^2 \,,
\end{gathered}$$

and also

$$
\begin{aligned}
\dot{\Omega}_\theta &= -\dot{y}_r\,\bar{v}_r\,\mathrm{sinc}(\theta) - y_r\,\dot{\bar{v}}_r\,\mathrm{sinc}(\theta) - y_r\,\bar{v}_r\frac{d}{dt}\mathrm{sinc}(\theta) - k_\theta\,\dot{\theta} = \\
&= -(\bar{v}_r^2 s_\theta + y_r\,\dot{\bar{v}}_r)\,\mathrm{sinc}(\theta) - \left(\frac{y_r\bar{v}_r^2}{L}\,\frac{c_\theta\,\theta - s_\theta}{\theta^2} + k_\theta\,\frac{\bar{v}_r}{L}\right) t_\psi \;.
\end{aligned}
$$

By using De L'Hôpital's Theorem, one can ensure that the term in the above expression divided by θ^2 is well-defined for every θ. Indeed we have:

$$
\lim_{\theta\to 0}\frac{c_\theta\,\theta - s_\theta}{\theta^2} = \lim_{\theta\to 0}\frac{-\,s_\theta\,\theta + c_\theta - c_\theta}{2\,\theta} = -\frac{1}{2}\lim_{\theta\to 0} s_\theta = 0\,.
$$

Putting all together we can finally obtain an input control law, stabilizing the entire state of the kinematic model of a car-like vehicle:

$$
\omega = \frac{L}{\bar{v}_r}c_\psi^2\left(\dot{\Omega}_\theta - \theta - \frac{\dot{\bar{v}}_r}{L}t_\psi - k_\omega\left(\frac{\bar{v}_r}{L}t_\psi - \Omega_\theta\right)\right)\;.
$$