Oscillators

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“But you, Daniel, shut up the words and seal the book, until the time of the end. Many shall run to and fro, and knowledge shall increase.”

Daniel 12:4

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Sumário

1 - Introduction

Oscillators are present in electronic equipment in general with the purpose of generating periodic signals for the most diverse functions, such as clock, modulation, locked phase loop, among many others. Regardless the application, the present material, rather, seeks to introduce basic concepts of physical principles and mathematical modeling, illustrated with simulation examples, so that the reader can experience the concepts presented. The items are organized in the sequence of principle, modeling, and simulation, seeking to cover all the necessary steps to design each topic studied. The subject itself is very broad, and with the natural technological development leading to new topologies and techniques, however, all based on the theory described here. The final goal, after all, is that after the content is properly worked, it is possible not only to carry out small projects with the oscillators presented, but also to easily understand other circuits and literature on the same subject.

2 – The Basic Oscillator

2.1 – Physical Principle

Oscillation is the periodic movement of a body, that is, it repeats itself at a regular period of time. The oscillation of a mechanical system will always start from an initial potential energy applied to it. Taking a simple pendulum as an example, we have that a lossless system will oscillate infinitely always with the same amplitude of motion, Figure 1 (a), alternating between potential energy EP and kinetic energy EC, Table 1, respectively at the moments when it stops and enters into movement. The same occurs in a lossy system, evidently tending to stop, Figure 1 (b).

Figure 1 - Pendulum (a) lossless and (b) lossy.

In an electrical system, exactly the same occurs with the generation of periodic voltage and current, in this case an ideal LC circuit, Figure 2, which can start to oscillate either by a voltage accumulated in the capacitor, or current accumulated in the inductor, or both. Components L and C, with some initial energy, Table 1, when connected to each other, will start a process of alternating voltage and current between them in a sinusoidal way. Figure 4 (a) demonstrates this effect where voltage is maximum when current is minimum and vice versa. In a real system the losses are represented by a resistor, Figure 3, either parallel or series, in this case there is an evanescence of the oscillation, similarly to the pendulum with losses, exemplified in Figure 4 (b).

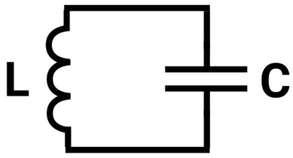


Figure 2 - Ideal LC circuit.

Table 1 - Energy in mechanical and electrical systems.

Pendulum	LC circuit
$E_C = \frac{mv^2}{2}$ $\quad E_P = mgh$	$E_{Cap} = \frac{CV^2}{2}$ $\quad E_I = \frac{LI^2}{2}$
E_C: Kinetic energy. E_P: Potential energy. m: mass. v: speed. g: gravitational force. h: height.	E_{Cap}: Capacitor energy. E_I: Inductor energy. C: capacitance. V: voltage. L: inductance. I: current.

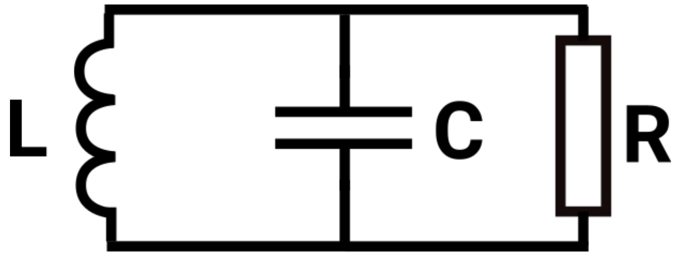


Figura 3 - Parallel RLC circuit.

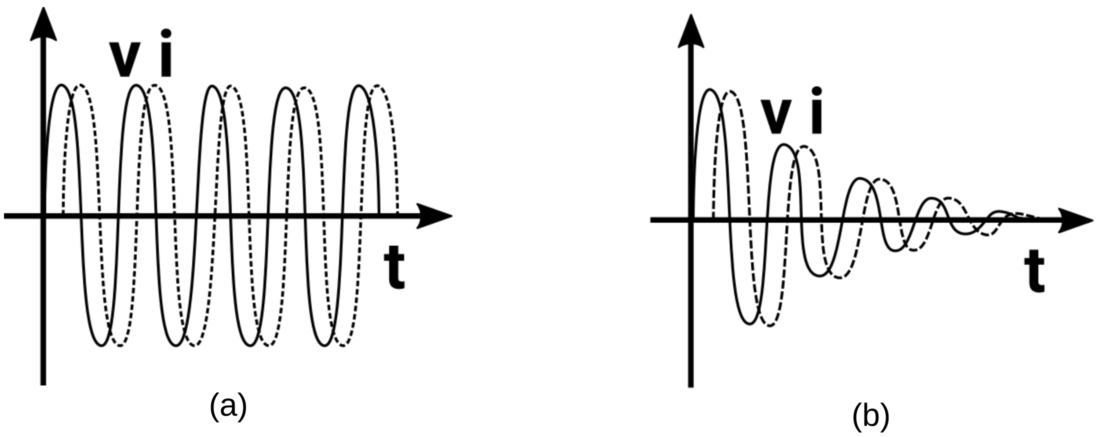


Figura 4 - Voltage in the components and current in the inductor in the circuits (a) LC and (b) parallel RLC.

The RLC circuit generates a sinusoidal oscillation, and for this type of wave it is called a harmonic oscillator, unlike the non-sinusoidal oscillators, or relaxation oscillators, both discussed throughout the text.

2.2 - RLC Circuit and the Oscillation

The modeling of the RLC circuit for the analysis of the oscillatory behavior is based on the capacitor current and inductor voltage as shown in Figure 5, respectively Eqs.1 and 2. The Laplace transform of this current and voltage naturally gives the initial current and voltage conditions, Eqs.3 and 4.

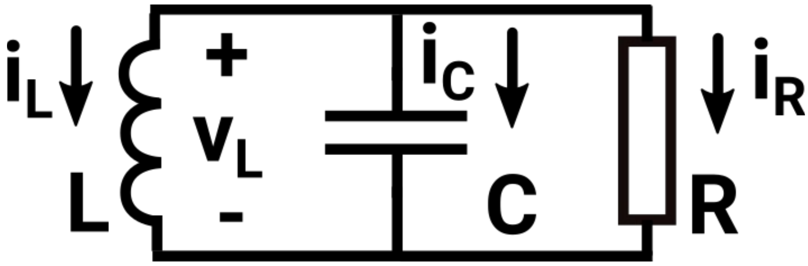


Figure 5 - Voltages and currents in the parallel RLC circuit.

$$i_C = C\frac{dv}{dt} \quad \text{(Eq.1)}$$

$$v_L = L\frac{di}{dt} \quad \text{(Eq.2)}$$

$$I_C = -I_L - I_R = C\left[sV_C - V_C(0)\right] \quad \text{(Eq.3)}$$

$$V_L = V_C = L\left[sI_L - I_L(0)\right] \quad \text{(Eq.4)}$$

Isolating I_L from Eq.4 results in Eq.5, with Eq.6 being the combination of Eqs.3 and 5.

$$I_L = \frac{1}{s}\left[\frac{V_C}{L} + I_L(0)\right] \quad \text{(Eq.5)}$$

$$\frac{1}{s}\left[\frac{V_C}{L} + I_L(0)\right] + \frac{V_C}{R} = -C\left[sV_C - V_C(0)\right] \quad \text{(Eq.6)}$$

We obtain the expression for the voltage of capacitor V_C, Eq.9, by developing Eq.6, with intermediate steps in Eqs.7 and 8. The frequency response of the LC circuit of Figure 2, is found with R tending to infinity, which eliminates the s/RC term from Eq.9, resulting in Eq.10.

$$\frac{V_C}{L} + I_L(0) + s\frac{V_C}{R} = -s^2 C V_C + sCV_C(0) \quad \text{(Eq.7)}$$

$$V_C\left[s^2 C + \frac{s}{R} + \frac{1}{L}\right] = -I_L(0) + sCV_C(0) \quad \text{(Eq.8)}$$

$$V_C = \frac{-\frac{I_L(0)}{C} + sV_C(0)}{s^2 + \frac{s}{RC} + \frac{1}{LC}} \quad \text{(Eq.9)}$$

$$V_C = \frac{-\frac{I_L(0)}{C} + sV_C(0)}{s^2 + \frac{1}{LC}} \quad \text{(Eq.10)}$$

Eq.10 can be developed to fit in the sin and cos formats of Table 2, resulting in the time equations Eq.12. For the LC circuit an initial current $I_L(0)$ results in a sine, while an initial voltage $V_C(0)$ in a cosine.

$$V_C=-\left[I_L(0)\sqrt{\frac{L}{C}}\right]\frac{\sqrt{\frac{1}{LC}}}{s^2+\frac{1}{LC}}+[V_C(0)]\frac{s}{s^2+\frac{1}{LC}} \quad \text{(Eq.11)}$$

$$v_C=-\left[I_L(0)\sqrt{\frac{L}{C}}\right]sen\left(\sqrt{\frac{1}{LC}}t\right)+[V_C(0)]\cos\left(\sqrt{\frac{1}{LC}}t\right) \quad \text{(Eq.12)}$$

From Eqs.5 and 11 we find Eq.13, which when developed for the formats in Table 2 results in Eq.14, with Eq. 15 the result of the transformation into the time domain.

$$I_L=\frac{1}{s}\left[-\frac{1}{L}\left[I_L(0)\sqrt{\frac{L}{C}}\right]\frac{\sqrt{\frac{1}{LC}}}{s^2+\frac{1}{LC}}+\frac{1}{L}[V_C(0)]\frac{s}{s^2+\frac{1}{LC}}+I_L(0)\right] \quad \text{(Eq.13)}$$

$$I_L=-\left[I_L(0)\frac{1}{LC}\right]\frac{1}{s\left(s^2+\frac{1}{LC}\right)}+\frac{1}{L}[V_C(0)]\sqrt{LC}\frac{\sqrt{\frac{1}{LC}}}{s^2+\frac{1}{LC}}+I_L(0)\frac{1}{s} \quad \text{(Eq.14)}$$

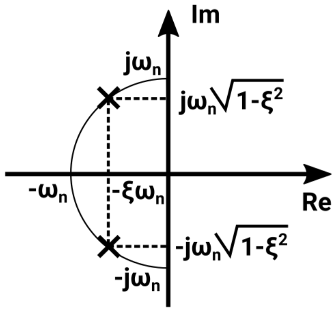
$$i_L=-[I_L(0)]\left[1-\cos\left(\sqrt{\frac{1}{LC}}t\right)\right]+\left[V_C(0)\sqrt{\frac{C}{L}}\right]sen\left(\sqrt{\frac{1}{LC}}t\right)+I_L(0)u(t) \quad \text{(Eq.15)}$$

The RLC circuit, in turn, presents the term with exponent $-\xi\omega_n$ equal to 1/(2RC), that is, the resistance gives rise to the term that dictates the decay of the oscillation, where ξ is the damping factor and ω_n the natural frequency, Table 3. In all cases with amplitudes depending on the initial values.

Figure 7 relates the time response with the pole positioning, Figure 6, where in Figura 7 (a), poles to the left, the exponential is decreasing, while in Figure 7 (b), poles on the complex axis, the system is oscillatory. Figure 7 (c), poles to the right, the unstable system, the latter possible in active electronic circuits.

Table 2 - Laplace Transform.

$\frac{1}{s}$	$u(t)$
$\frac{a\omega}{s^2+\omega^2}$	$a\,sen(\omega t)$
$\frac{as}{s^2+\omega^2}$	$a\cos(\omega t)$
$\frac{a}{s(s^2+\omega^2)}$	$\frac{a}{\omega^2}(1-\cos(\omega t))$
$\frac{a}{s^2+2\xi\omega_n s+\omega_n^2}$	$a\frac{1}{\omega_n\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\,sen\left(\omega_n\sqrt{1-\xi^2}\,t\right)$
$\frac{bs+a}{s^2+2\xi\omega_n s+\omega_n^2}$	$e^{-\xi\omega_n t}\left[\frac{1}{\omega_n\sqrt{1-\xi^2}}(a-b\xi\omega_n)\,sen\left(\omega_n\sqrt{1-\xi^2}\,t\right)+b\cos\left(\omega_n\sqrt{1-\xi^2}\,t\right)\right]$



$$p_{1,2}=-\xi\omega_n\pm j\omega_n\sqrt{1-\xi^2}$$

Figure 6 - Poles in the complex plane.

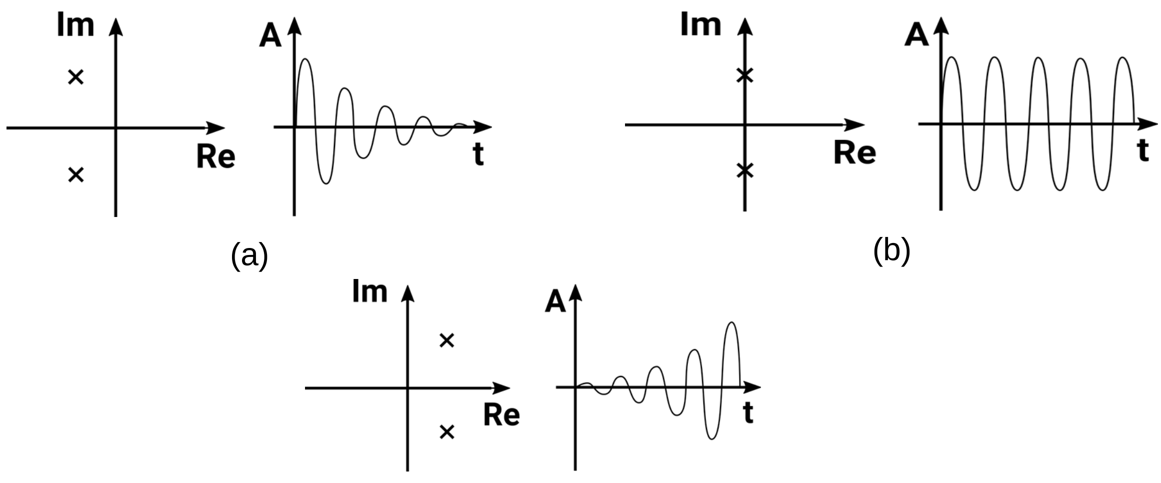


Figure 7 - Poles and time response: (a) stable, (b) oscillating and (c) unstable.

Table 3 - Variables of the frequency response.

ξ	Damping factor
ω_n	Natural frequency: transient oscillation frequency for zero dumping.
ω_p	Self frequency: natural dumping frequency during oscillation.

The lossy model, Eq.9, presents the behavior of Figure 8, now with a decay of the oscillation according to the resistor R. The oscillation frequency is ω_p, Table 3. The time response is obtained from Eq.16 when determining the constants for the respective equation in Table 2.

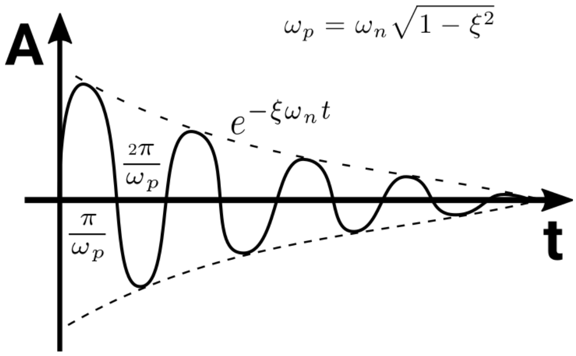


Figure 8 - Oscillation with losses.

$$V_C=\frac{sV_C(0)-\frac{I_L(0)}{C}}{s^2+\frac{s}{RC}+\frac{1}{LC}}=\frac{bs+a}{s^2+2\xi\omega_n s+\omega_n^2} \quad \text{(Eq.16)}$$

The variables a and b are respectively:

$$a=\frac{-I_L(0)}{C} \quad \text{(Eq.17)}$$

$$b=V_C(0) \quad \text{(Eq.18)}$$

The frequency ω_n is easily found by:

$$\omega_n=\sqrt{\frac{1}{LC}} \quad \text{(Eq.19)}$$

The damping factor ξ comes from Eqs. 19 e 20, resulting in Eq.21.

$$2\xi\omega_n = \frac{1}{RC} \quad \text{(Eq.20)}$$

$$\xi = \frac{1}{2R}\sqrt{\frac{L}{C}} \quad \text{(Eq.21)}$$

The variables a, b, ω_n and ξ as a function of the components and initial conditions now give the answer in time of Eq.16 from Table 2.

2.3 - Examples of Basic Oscillation Simulations

The concepts presented here can be implemented in a simulator, here LTSpice. In Figura 9 the RLC circuit, in which we can define initial conditions of voltage in the capacitor and current in the inductor, to start the oscillation in transient simulation. In Figure 10 simulations with initial voltage and current with the loss given by the resistor, and defining the respective attenuation. For initial voltage $V_C(0)$ of 2V it takes the amplitude of the same level, as predicted by Eq.10, as well as the oscillation frequency ω equal to √LC, or a period of 0.628µs. Eq.10 is equivalent to the cos of Table 2. For an initial current of 1A the oscillation frequency is maintained, but with amplitude now given by $I_L(0)\sqrt{L/C}$, from Eq.10 and the sin of Table 2.

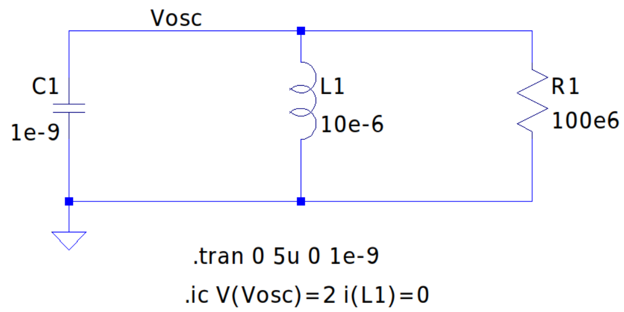


Figure 9 - LTSpice RLC circuit with initial voltage and current defined by initial conditions, file LCOscill.asc.

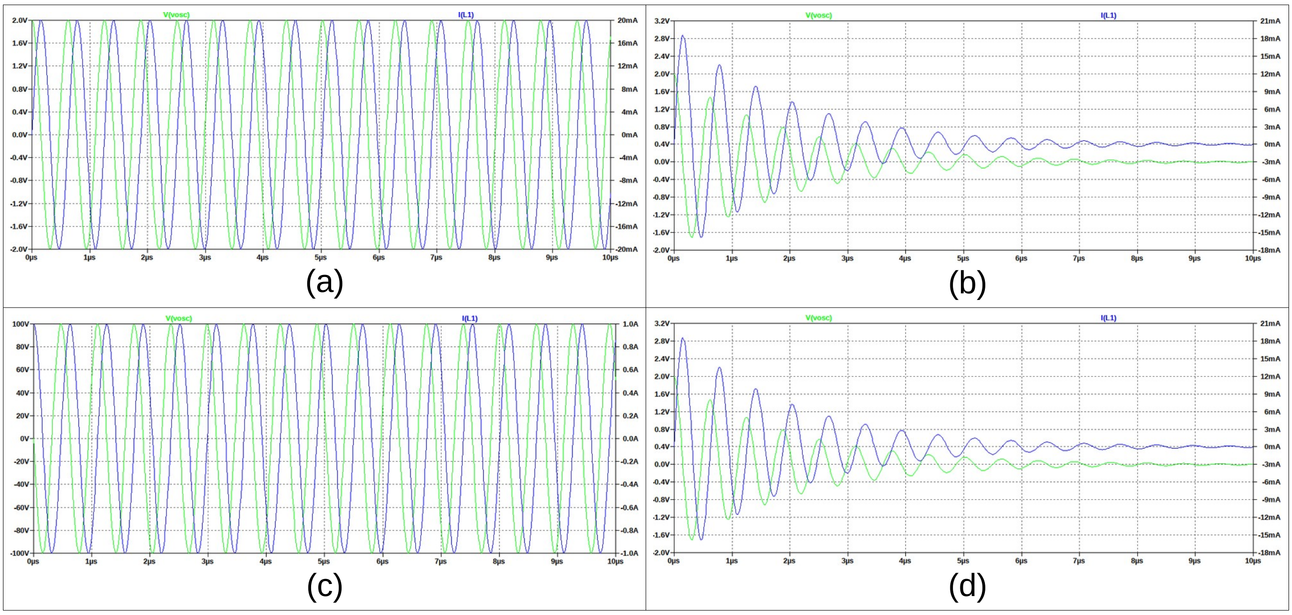


Figure 10 - Capacitor voltage and inductor current for oscillation by initial voltage of 2V across the capacitor with (a) R=100MΩ and (b) R=1kΩ, and initial current of 1A in the inductor with (a) R=1MΩ and (b) R=1kΩ.

Figure 11 shows the circuit with an initial voltage of source V1 of 2V that charges capacitor C1 through switch S1, commanded by source V2, a pulse of 500ns that keeps this switch closed in this time interval. The charged capacitor is connected to the LR set from 1μs when the switch S2 is closed, thus forming the RLC circuit. The resulting oscillations are shown in Figures 13 (a) and (b), the first without visible attenuation due to the use of R of 100MΩ, here considered infinite, and the second with strong attenuation due to R of 1kΩ. Note that the 2V amplitude is the same as the initial voltage, as predicted by Eq.10, and the oscillation frequency ω is equal to √LC, or a period of 0.628μs. Eq.10 is equivalent to the cos of Table 2 as are the results for the circuit of Figure 9.

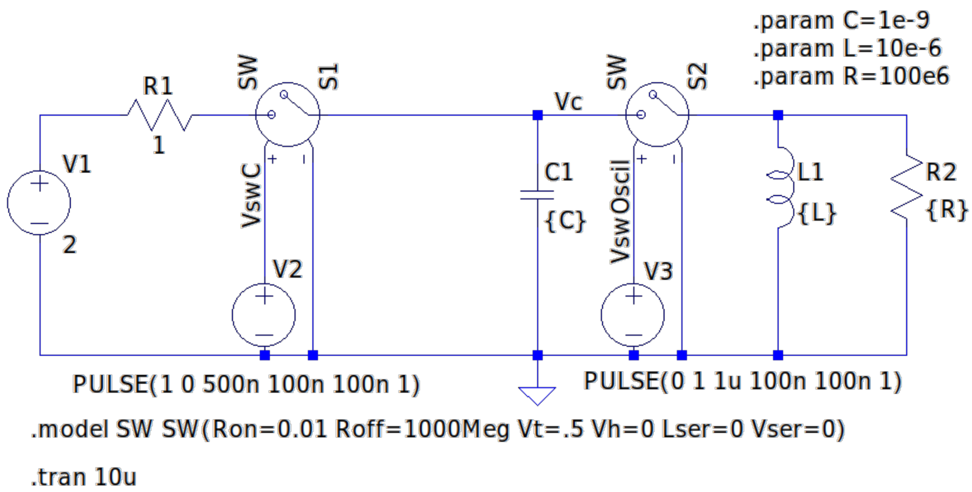


Figure 11 - LTSpice RLC circuit with initial voltage, file LCOscillatorVCInitial.asc.

The circuit with an initial current of 1A in L1 by the source I1 is shown in Figure 12, applied in the first 500ns by the switch S1, closed at this time, adding the capacitor C1 in the time of 1μs when the switch S2 is opened. The oscillations according to the resistors, 1MΩ and 1kΩ, are shown in Figures 13 (a) and (b), respectively, in transient simulation. The first resistor is very large such that there is no visible attenuation, the circuit in this way can be considered only LC, and the second one with a very visible attenuation in the simulation range. The oscillation frequency does not change, with amplitude given by $I_L(0)\sqrt{L/C}$, from Eq.10 and the sin of Table 2, the same as the circuit with initial condition of Figure 9.

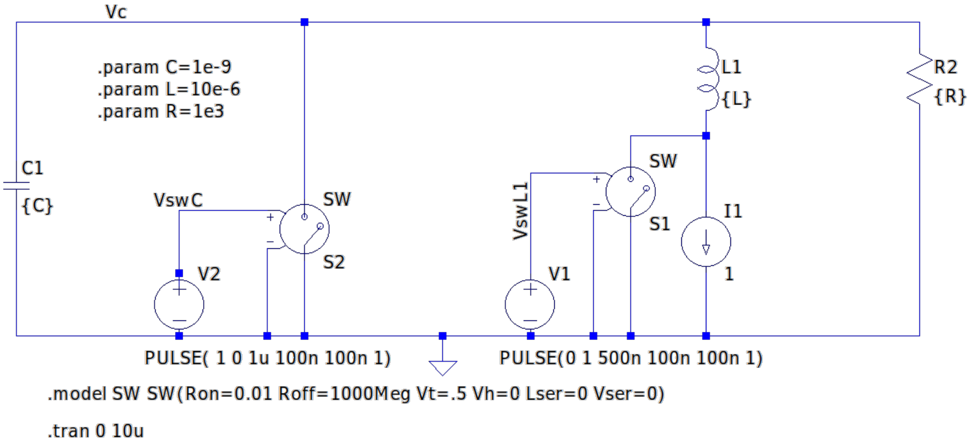


Figure 12 - LTSpice RLC circuit with initial current,
file LCOscillatorILInitial.asc.

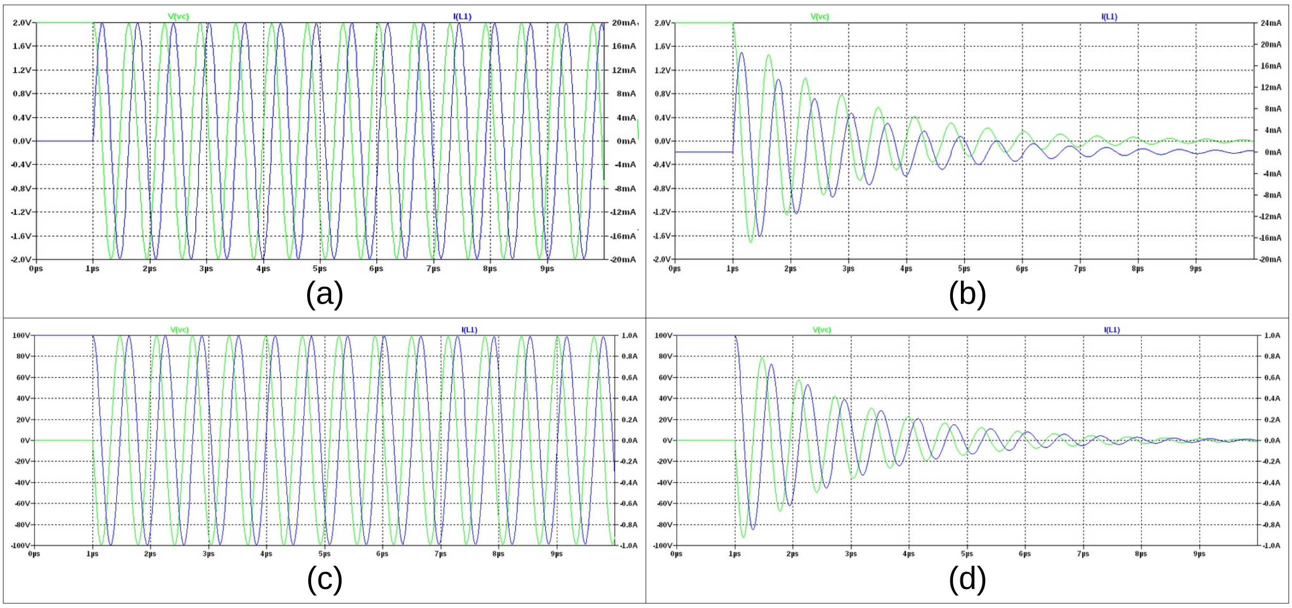


Figure 13 - Capacitor voltage and inductor current based on switches for 2V initial capacitor voltage oscillation with (a) R=100MΩ and (b) R=1kΩ, and initial current of 1A in the inductor with (a) R=1MΩ and (b) R=1kΩ.

3 - Electronic Oscillators

Concept: are electronic circuits that produce an output signal without the need for an input signal.

Types:

a) Feedback oscillators or harmonics – generate sinusoidal signals.

A system remains in oscillation or it has no losses, which does not happen in reality, or energy is injected to compensate for the losses. In a pendulum, a 'poke' is enough in each turn, in electronic circuits the use of an active component playing the role of a negative resistor, Figure 14. Still making a mechanical parallel, the new 'poke' of the pendulum at each cycle cannot be very strong, otherwise there is an increase in the amplitude of oscillation, and not too weak, otherwise it tends to stop, in short, it must be adequate to remain constant. In electronic circuits, the energy injected must have the same character of keeping the maximum amplitude constant, that is, if too much energy the system tends to an infinite amplitude, or diverges, and if too little energy tends to a null signal.

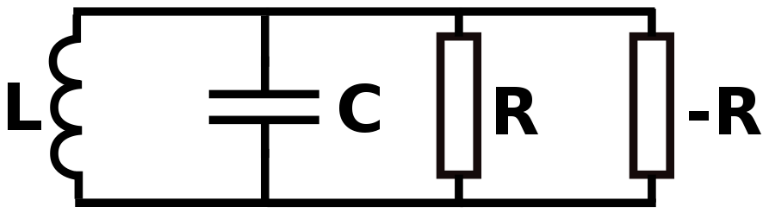


Figure 14 - RLC circuit and negative resistor.

The energy injection to compensate for the losses is done by an amplifier block, implemented in a system according to Figure 15, where we have the amplifier in block A, fed back by an LC network, block B, that is, the LC network responsible for the oscillation. and the amplifier playing the role of the negative resistor. In this type of oscillator, the output is fed back at the input, so that amplifier A reinforces the output, maintaining the oscillation according to the feedback block B. From Figure 15 we see that Vo=VoAB. We can physically understand that to keep the output stable, the gain AB must be equal to 1, Barkhausen's criterion, that is, the output seeks to follow the input so that there is neither attenuation, AB<1, nor infinite increase, AB>1. Therefore, once unity gain and phase 0° are established in the aforementioned feedback system, it will oscillate with constant amplitude. Note, however, that the Barkhausen criterion applies to linear circuits with a feedback loop, it is a necessary but not sufficient condition for oscillation, some circuits satisfy the condition, but do not oscillate. In practice, electronic oscillators have nonlinearities that can affect their correct operation, a subject still under study.

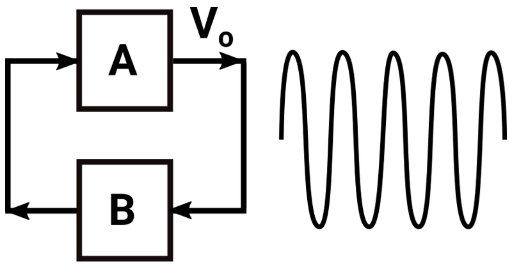


Figure 15 - No input feedback system.

b) Relaxation oscillators - generate non-sinusoidal signals.

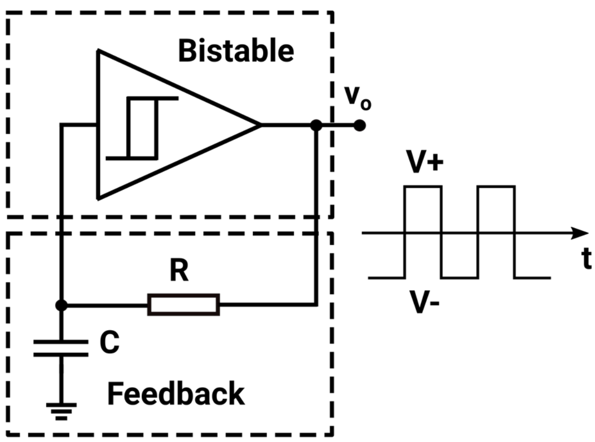


Figure 16 - Relaxation Oscillator.

Here the oscillation is generated through RC circuits, like timers, resulting in square or non-sinusoidal signals. They are usually based on Schmitt triggers, or other circuits, which charge and discharge capacitors in a resistor.

4 - Oscillation Condition

We start from the general analysis of a feedback system to determine the oscillation condition. From the general block diagram of Figure 17 we develop the respective transfer function, Eq.22. Since the oscillator is a system with no input signal, then we have Vi(s)=0, Eq.23, which implies Eq.24, that is, the product A(s)B(s) equals ∓1. This result means the modulus of the product A(s)B(s) equal to 1, Eq.25, and feedback network phase of 180° for a value of -1, Eq. 26, and 0° for 1, Eq. 27, therefore design conditions of electronic circuits to ensure oscillation. Figure 18 illustrates the behavior of the signals in the blocks in the oscillation process where the signal at the input of A(s) is always positive feedback, with the sign of the summing block positive if B(s) is positive and negative if B(s) is also negative. The output voltage Vo(s) is attenuated and phased out according to the feedback, block B(s) for the gain, remembering that for the product A(s)B(s)>1 the signal tends to grow infinitely, in practice saturate, and if A(s)B(s)<1 the oscillation tends to zero. A practical electronic system starts to oscillate when it is turned on and receives energy from the supply, in a process of increasing the amplitude until it stabilizes at the product |A(s)B(s)|=1.

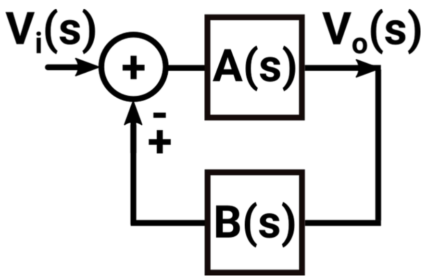


Figure 17 - Feedback system.

$$\frac{Vo(s)}{Vi(s)}=\frac{A(s)}{1\pm A(s)B(s)} \quad \text{(Eq.22)}$$

$$Vi(s)=0 \quad \text{(Eq.23)}$$

$$1\pm A(s)B(s)=0 \quad \text{(Eq.24)}$$

$$|A(s)B(s)|=1 \quad \text{(Eq.25)}$$

$$\phi[A(s)B(s)]=180^o \quad \text{(Eq.26)}$$

$$\phi[A(s)B(s)]=0^o$$ (Eq.27)

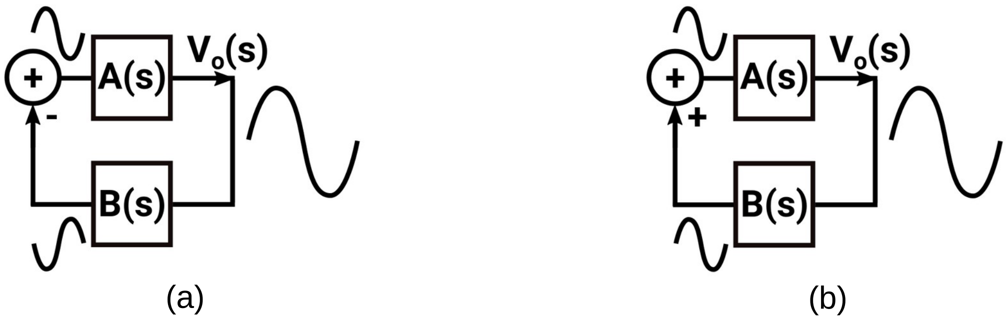


Figure 18 - System signals with always positive feedback on A(s) if the sign of B(s) is (a) negative and (b) positive.

5 - Harmonic Oscillators

Harmonic oscillators can be divided into three main groups according to the feedback network [Gil_UERJ_Oscil]. Table 4 summarizes this classification, where we see, in addition to the network, also the frequency characteristics and some of the circuits of the respective networks.

Table 4 - Types of oscillators according to the feedback network.

Feedback Network	Frequency	Circuit
RC	Low (5 to 1 MHz)	Wien Bridge
LC	High/Variable	Hartley Colpitts Clapp
Cristal*	High/Fixed	Hartley Colpitts Clapp

*The crystal component can be implemented in any of the LC network circuits.

5.1 – RC network - Wien Bridge

5.1.1 – Introduction to the Wien Bridge

Based on the Wheatstone bridge, hence the name Wien, it is a two-stage circuit, amplifier and feedback network, Figure 19, with good resonance frequency stability, low distortion, and easy tuning, so it is very used in commercial audio generators.

The feedback network, Figure 20(a), consists of a high-pass, series RC filter, followed by a low-pass, parallel RC, resulting in a high-quality Q-factor bandpass filter at the resonant frequency f_r. This feedback forms a lead-lag, lead-lag network with 0° phase at resonance as well as 1/3 gain, as shown in Figure 20 (b). The circuit goes into oscillation when the amplifier gain is equal to 3, that is, amplifier product and feedback equal to 1.

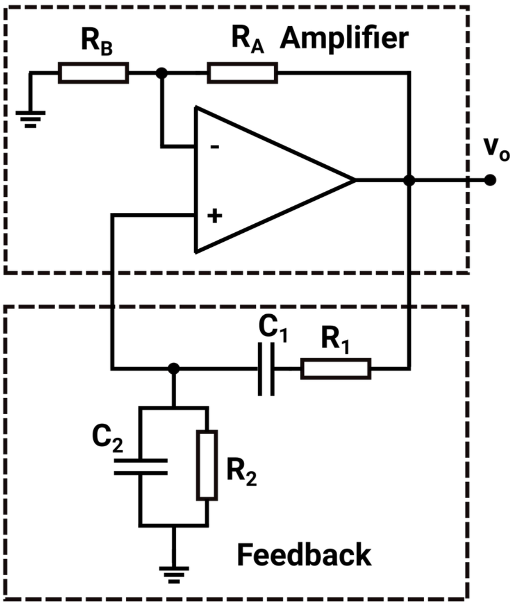
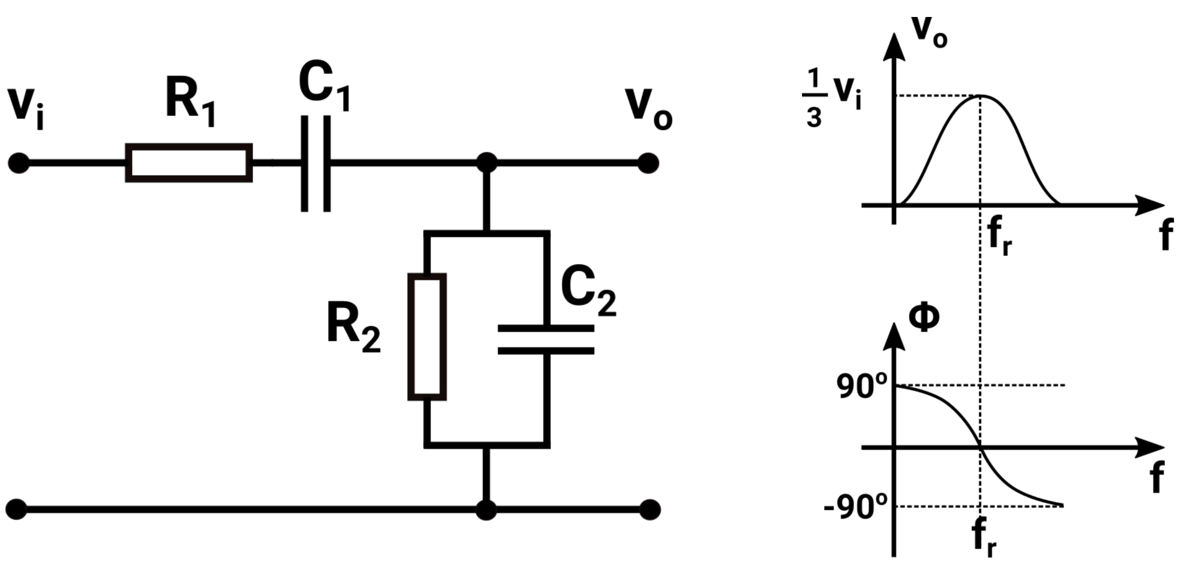


Figure 19 - Wien Bridge.



(a) (b)

Figure 20 -(a) RC feedback network and (b) frequency response.

5.1.2 - Modeling for Project

The resonant frequency and gain of the feedback network can easily be found by its circuit analysis [Wien]. Initially we used a common value for the resistors, Eq.28, as well as for the capacitors, Eq.29, with respective series and parallel impedances by Eqs.30 and 31.

$$R_1 = R_2 = R \quad \text{(Eq.28)}$$

$$C_1 = C_2 = C \quad \text{(Eq.29)}$$

$$Z_1 = R + \frac{1}{sC} \quad \text{(Eq.30)}$$

$$Z_2 = \frac{R}{1 + sRC} \quad \text{(Eq.31)}$$

The voltage gain of the A_{Vnet} network is given by Eq.32, which, when Z1 and Z2 are replaced by the respective expressions, results in Eq.33, whose development with jω instead of s, Eq.34, leads us to the gain of the network as a function of frequency, Eq.35.

$$\frac{V_o}{V_i} = A_{Vnet} = \frac{Z_2}{Z_1 + Z_2} \quad \text{(Eq.32)}$$

$$A_{Vnet} = \frac{\frac{R}{1 + sRC}}{R + \frac{1}{sC} + \frac{R}{1 + sRC}} \quad \text{(Eq.33)}$$

$$s = j\omega \quad \text{(Eq.34)}$$

$$A_{Vnet} = \frac{j\omega RC}{1 + 3j\omega RC - (\omega CR)^2} \quad \text{(Eq.35)}$$

Eq.36 is the condition for the A_{Vnet} phase to be 0°, basically eliminating the real part of the denominator. From this condition appear the resonance frequency, Eq.37 and the gain of 1/3 of the network, Eq.38.

$$1 - (\omega CR)^2 = 0 \quad \text{(Eq.36)}$$

$$\boxed{f_r = \frac{1}{2\pi RC}} \quad \text{(Eq.37)}$$

$$\boxed{A_{Vnet} = \frac{1}{3}} \quad \text{(Eq.38)}$$

The operational amplifier gain for the non-inverting configuration is given by Eq. 39, and since the amplifier and mains gain are equal to 1, we easily find a gain of 3 as necessary for the circuit to oscillate.

$$A_{Vamp} = 1 + \frac{R_A}{R_B} \quad \text{(Eq.39)}$$

$$A_{Vamp} A_{Vnet} = 1 \quad \text{(Eq.40)}$$

$$\boxed{A_{Vamp} = 1 + \frac{R_A}{R_B} = 3} \quad \text{(Eq.41)}$$

Therefore, Eqs.37, 38 and 41 are the ones used for the design of this oscillator.

The oscillation amplitude can be stabilized by non-linear networks, such as diodes, as shown in Figure 21, in this case the Wien Bridge with diodes. With the output amplitude limited by the supply voltage and the resistors RC and RD, according to Eqs. 42 to 48. In these, the gain of 1/3, Eq. 42, is combined with the voltage v_a, Eq. 43, to a superposition approximation of the resistive divider voltages of V_{CC} (v_o=0V) and v_o (V_{CC}=0V), knowing that the difference between vi and v_a is 0.7V of diode drop. Since the conduction of the diode for v_o is negative, we use v_o according to Eq. 45. Developing this system, Eqs. 46 and 47, we arrive at the amplitude expression according to Eq. 48, now no longer dependent on supply voltage only, before adjustable according to R_C and R_D.

$$v_i = \frac{v_o}{3} \quad \text{(Eq.42)}$$

$$v_a = \frac{V_{cc} R_D}{R_C + R_D} + \frac{v_o R_C}{R_C + R_D} \quad \text{(Eq.43)}$$

$$v_i - v_a = 0.7 \quad \text{(Eq.44)}$$

$$v_o = -\hat{v}_o \quad \text{(Eq.45)}$$

$$-\frac{\hat{v}_o}{3}-\frac{V_{cc}R_D}{R_C+R_D}+\frac{\hat{v}_o R_C}{R_C+R_D}=0.7 \quad \text{(Eq.46)}$$

$$-\hat{v}_o(R_C+R_D)-3V_{cc}R_D+3\hat{v}_o R_C=2.1(R_C+R_D) \quad \text{(Eq.47)}$$

$$\hat{v}_o=\frac{3V_{cc}R_D+2.1(R_C+R_D)}{2R_C-R_D} \quad \text{(Eq.48)}$$

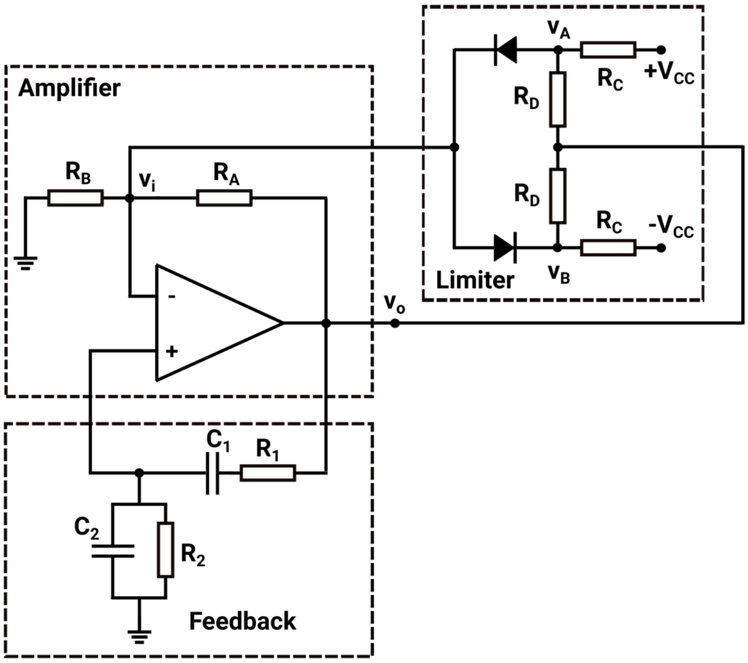


Figure 21 - Wien Bridge Circuit with limiter.

5.1.3 - Exemple of Simulation

In Figure 22 the feedback network and in Figure 23 the respective frequency response in AC simulation. The AC source is 1V resulting in a 1/3V peak and 0° phase at 1kHz, as expected in Figure 23.

Figure 24 shows the design in a simulator of a Wien Bridge oscillator for a frequency of 1kHz. Resistors RA and RB have a 2 to 1 ratio, hence the desired operational block gain of 3. The resistor Rtol (tolerance) is to force a gain just above 1 and ensure the oscillation. A 10nF capacitor C being arbitrated, and consequently the 15.9kΩ resistor R, according to Eq.37. It is important to note that the power supplies are of the step type, starting from 0V, because it is a transient simulation, a DC supply does not work in this case.

For the same Wien bridge, now with limiter, we use RC=400kΩ and RD=100kΩ, with VCC of 15V, which results in vo=7.93V. The transient simulations are shown in Figure 26. The simulated voltage presents vo=6.97V, approximately 1V error due to the approximate model, in this case the superposition approximation.

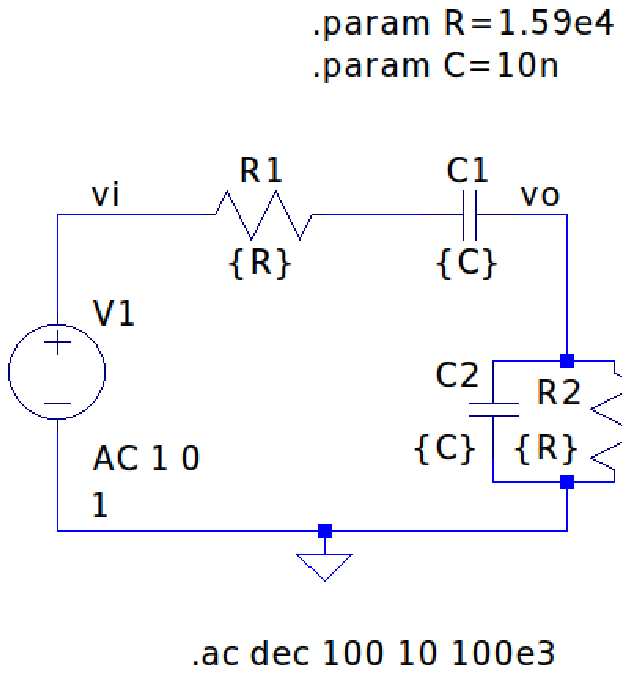


Figure 22 - Wien LTSpice bridge feedback network, file WienBridgeNetwork.asc.

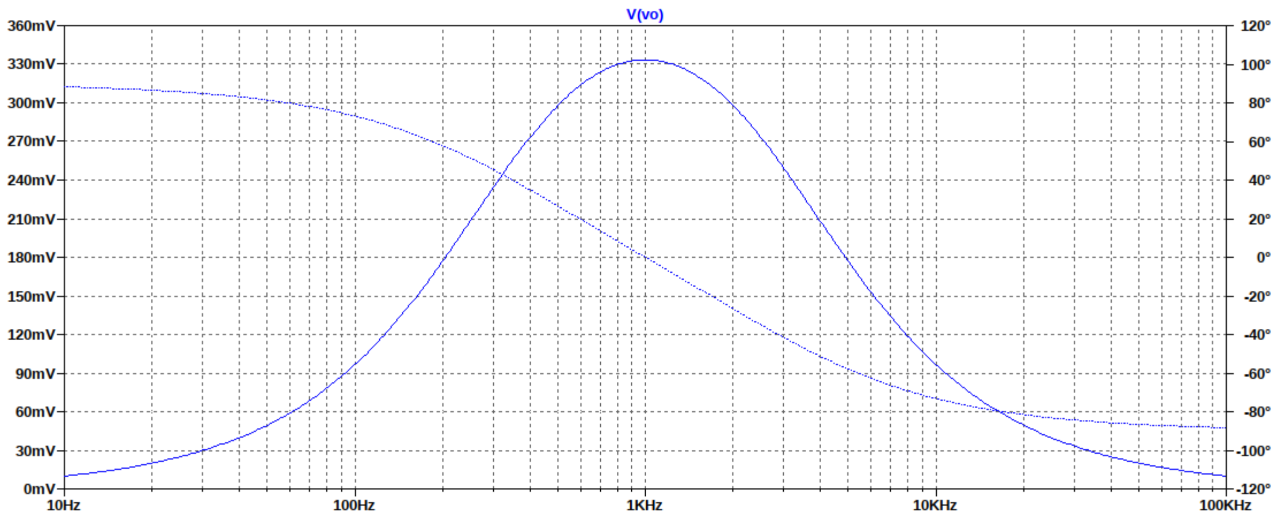


Figure 23 - Frequency response of the Wien LTSpice bridge.

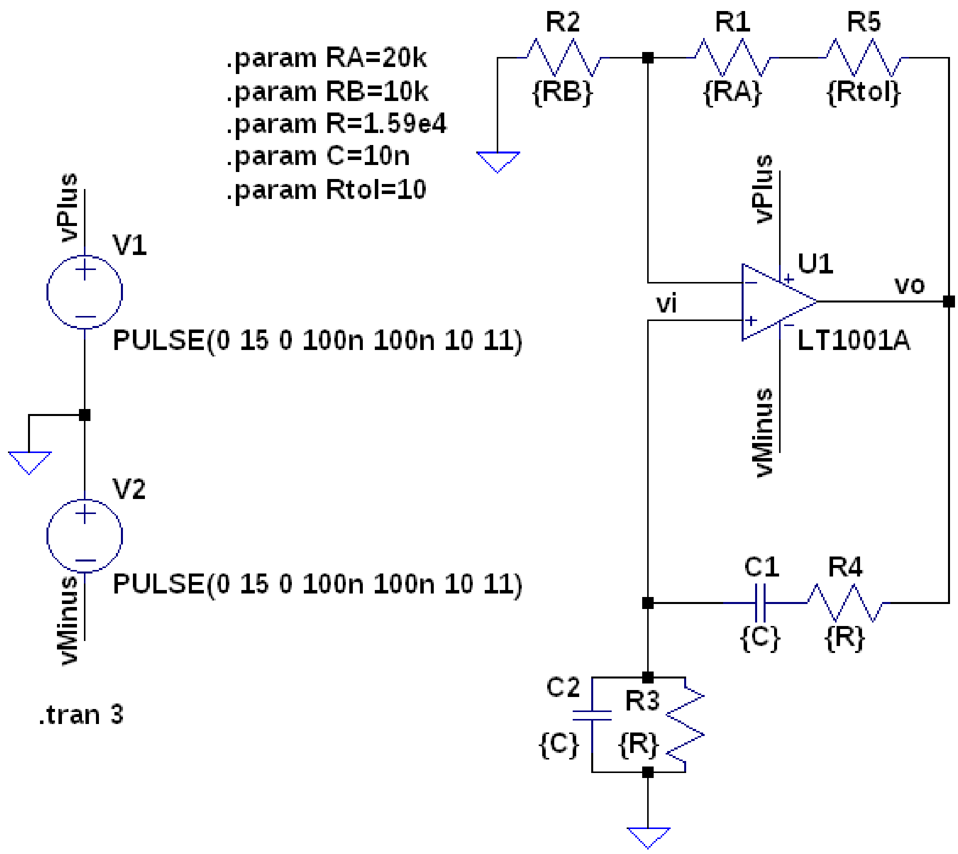


Figure 24 - Wien Bridge LTSpice, file WienBridge.asc.

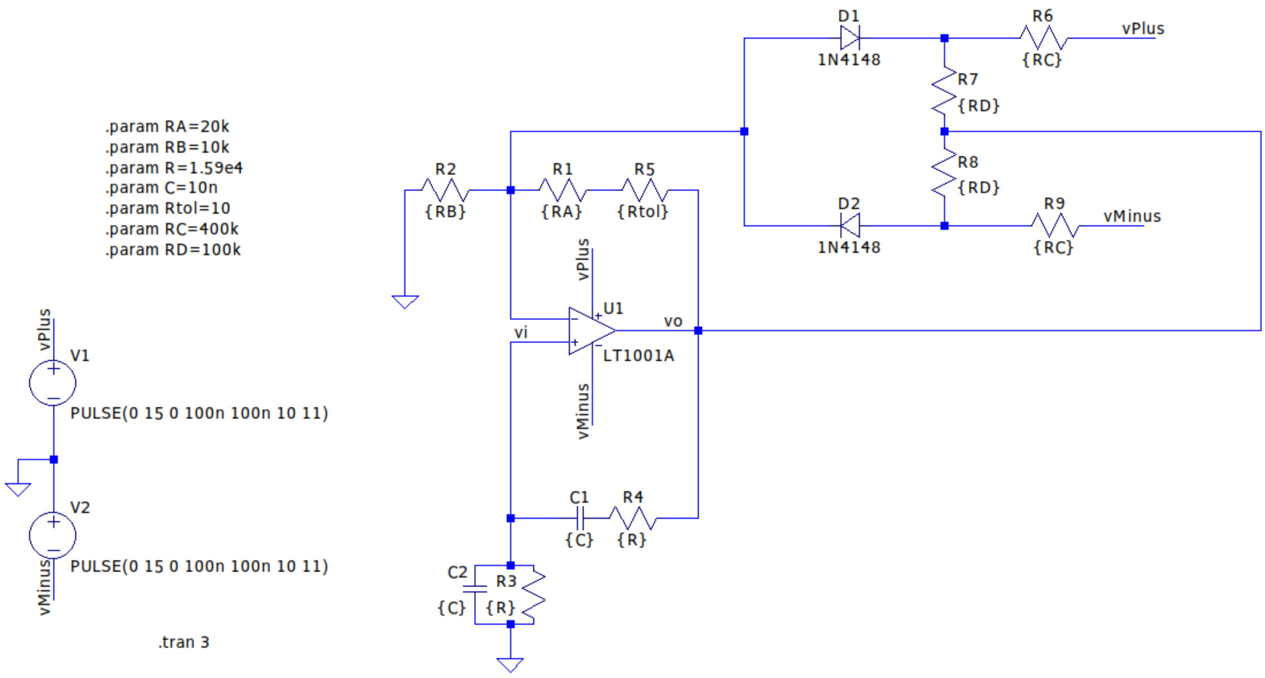
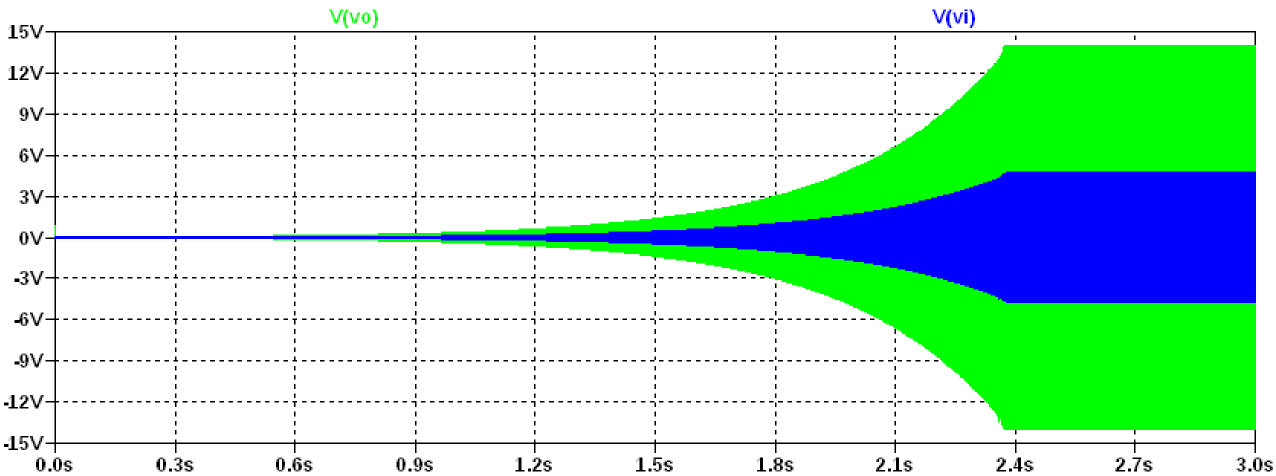
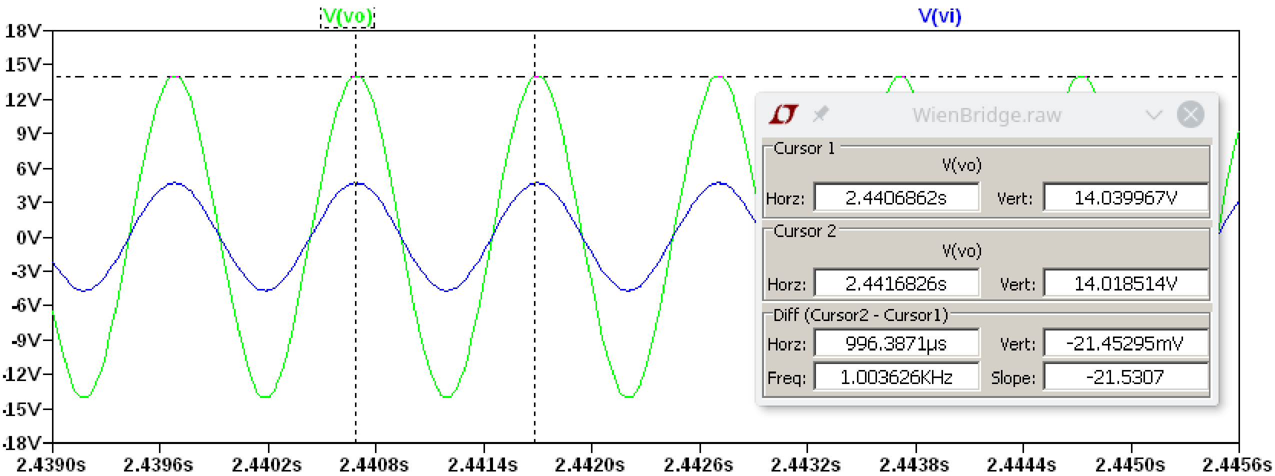


Figure 25 - Wien bridge with LTSpice limiter, file WienBridgeLimiter.asc.

The resulting oscillation of the circuit in Figure 24, without a limiter, is shown in Figure 26 (a), both for the voltage at the output of the operational amplifier, Vo, and at the input, Vi. A zoom view is found in Figure 26 (b) where we verify the projected oscillation frequency of 1kHz.

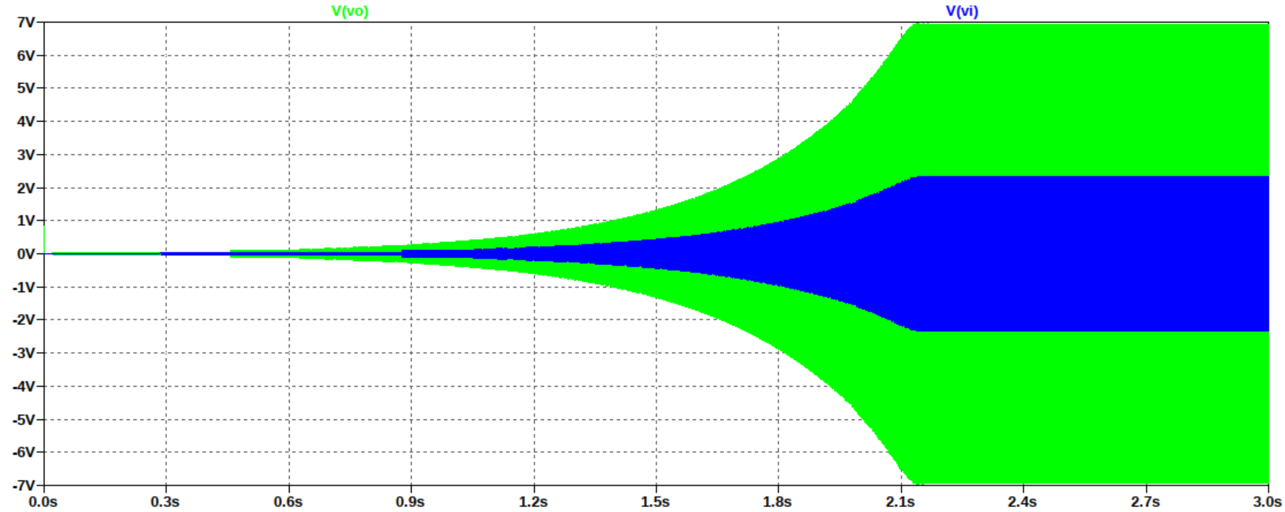


(a)

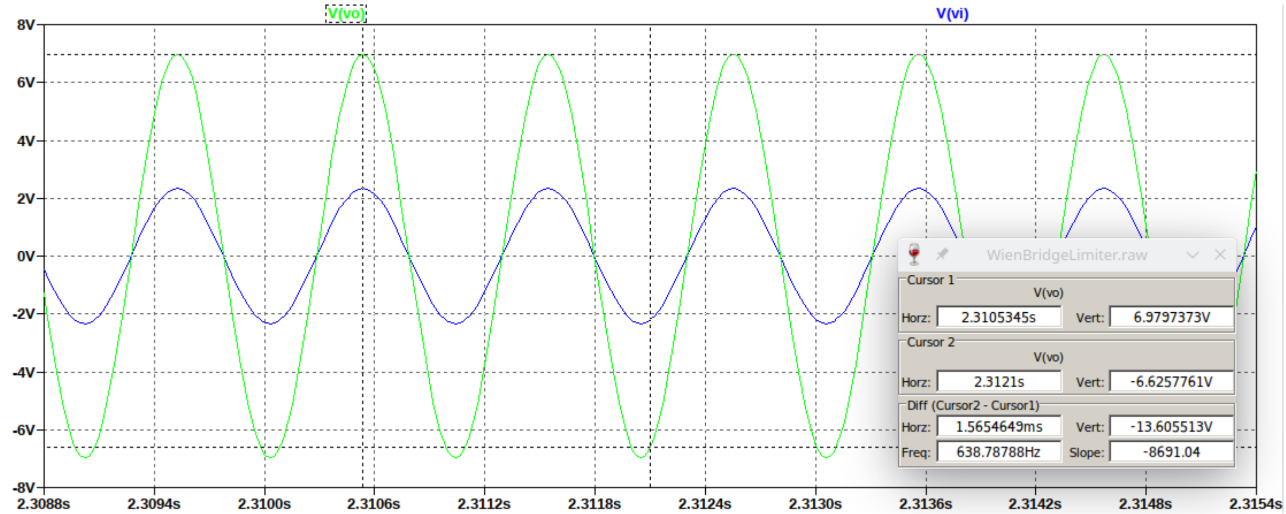


(b)

Figure 26 - Output voltages vo and feedback vi from the Wien bridge:
(a) full simulation and (b) simulation zoom.



(a)



(b)

Figure 27 - Output voltages vo and feedback vi from the Wien bridge:
(a) full simulation and (b) simulation zoom.

5.2 – LC Networks

5.2.1 – Basic Networks

Here we will deal with the three-element feedback networks [Gil_UERJ_Oscil], Figure 28, based on two configurations: Hartley and Colpitts, Figures 29 (a) and (b). The Clapp configuration, Figures 29 (c) is a derivation of Colpitts for variable frequency oscillators, variable frequency oscillator VFO, where the capacitor corresponding to reactance X_{2b} is variable, avoiding using the capacitors relative to X_1 and X_2 for that, since would also cause feedback voltage variation [WikiClapp].

The Hartley configuration consists of two inductors ("Hartlei" with the sound "y=I" for inductor), while the Colpitts configuration for two capacitors ("Colpitts" with C for capacitor). Both networks provide feedback for oscillation, but Colpitts has advantages because it has no mutual inductance problem, as well as a better sinusoidal signal due to the low impedance of the capacitors at high frequency [Colpitts].

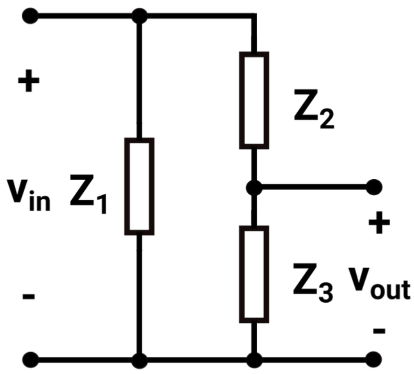


Figure 28 - LC general network.

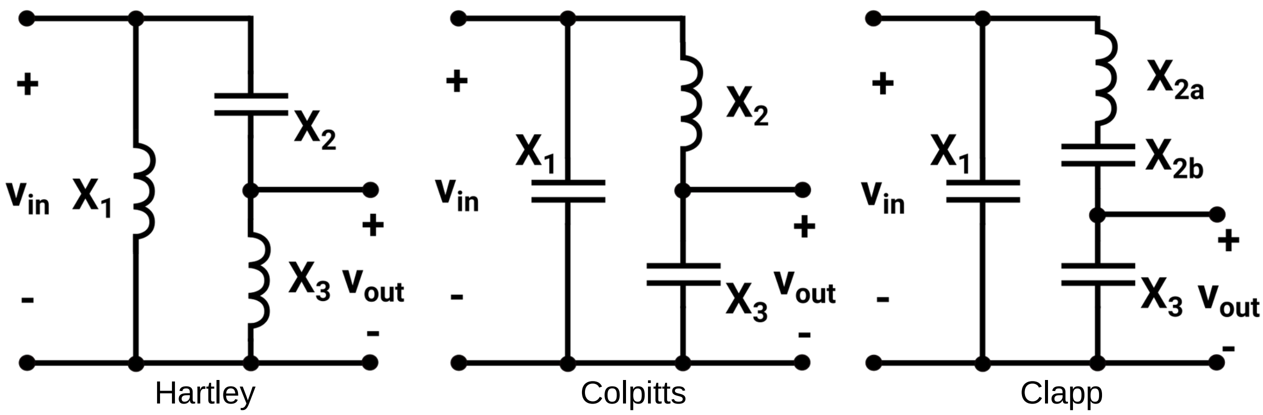


Figure 29 - LC specific network.

Figure 30 shows the general structure where the amplifier can be either a transistor or an operational amplifier. In Figure 31 (a) we have the small signal model of the bipolar transistor, BJT, and in Figure 31 (b) the field effect transistor, FET, for example JFET or MOSFET. For the BJT, in particular, the transconductance model based on Eq. 49 is used. As for the operational amplifier, the inverting configuration can be used, Figure 32 (a), with an equivalent model in Figure 32 (b), from the gain and input impedance equations, Eqs.41 and 42.

$$g_m v_{be} = g_m r_\pi i_b = \beta_o i_b \quad \text{(Eq.49)}$$

$$A_{Vampop} = \frac{-R_A}{R_B} \quad \text{(Eq.50)}$$

$$Z_{Iampop} = R_B \quad \text{(Eq.51)}$$

The implementations of the Hartley and Colpitts oscillators, BJT and operational amplifier, are in Figures 33 and 34, the Clapp in Figure 35. Here are exemplified some versions of these oscillators, since there are a large number of other variations of these same oscillators in the literature.

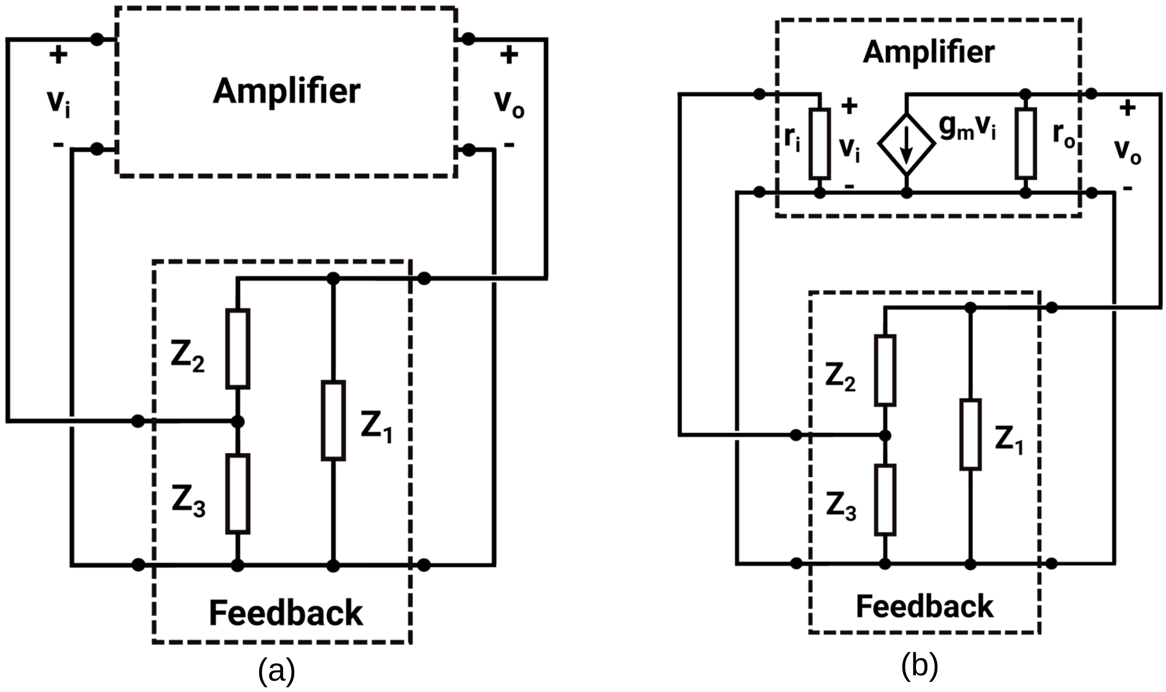


Figure 30 - Feedback system (a) general and (b) transcoductance model.

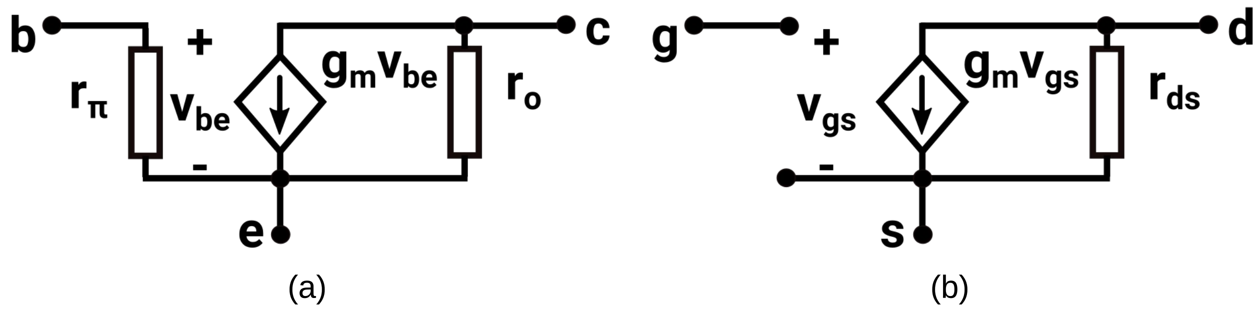
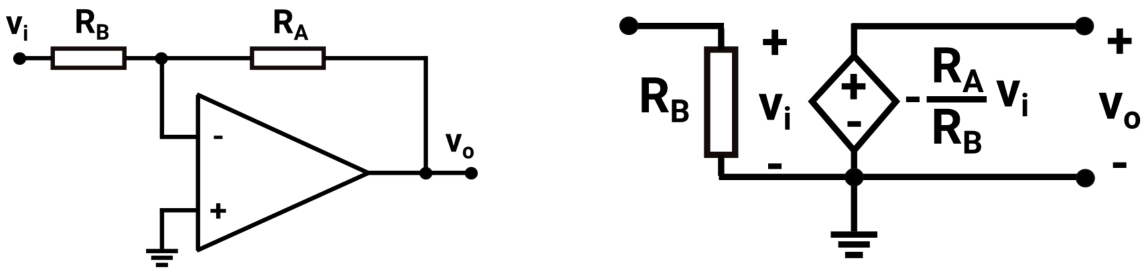
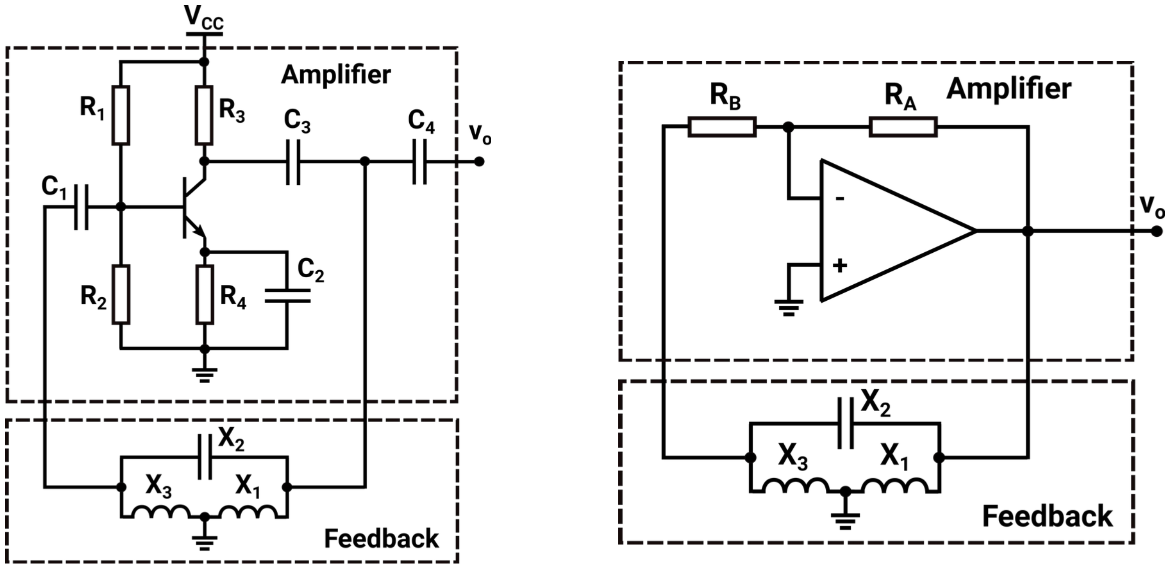


Figure 31 - Small signal model (a)BJT and (b)FET.



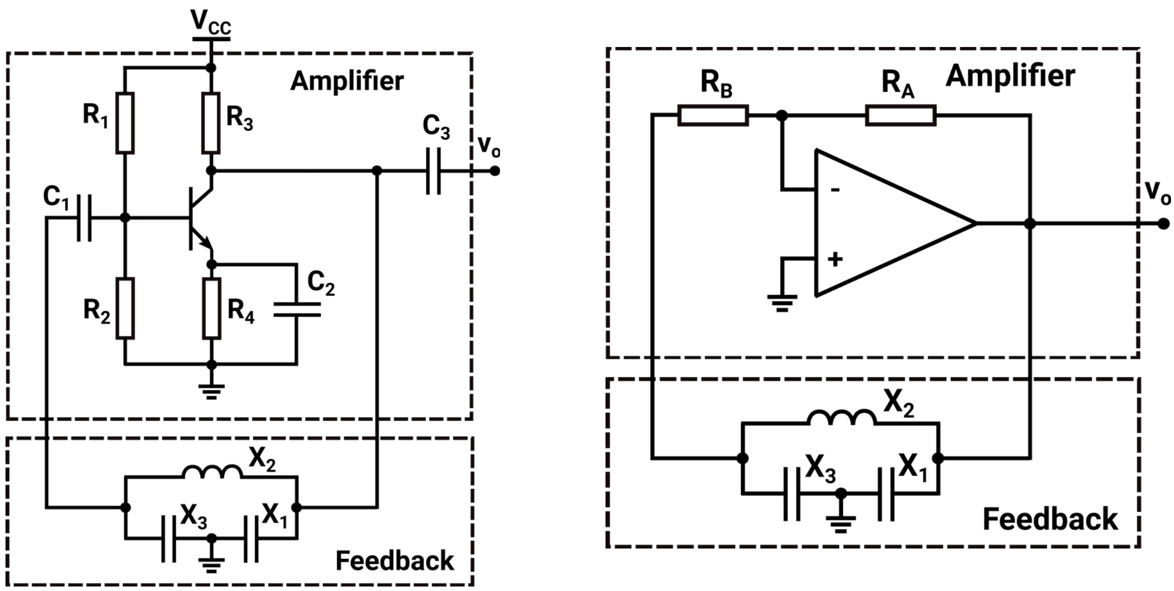
(a) (b)

Figure 32 - (a) Inverting operational amplifier, and (b) equivalent circuit.



(a) (b)

Figure 33 - Hartley Oscillator (a) BJT and (b) operational amplifier.



(a) (b)

Figure 34 - Colpitts Oscillator(a) BJT and (b) operational amplifier.

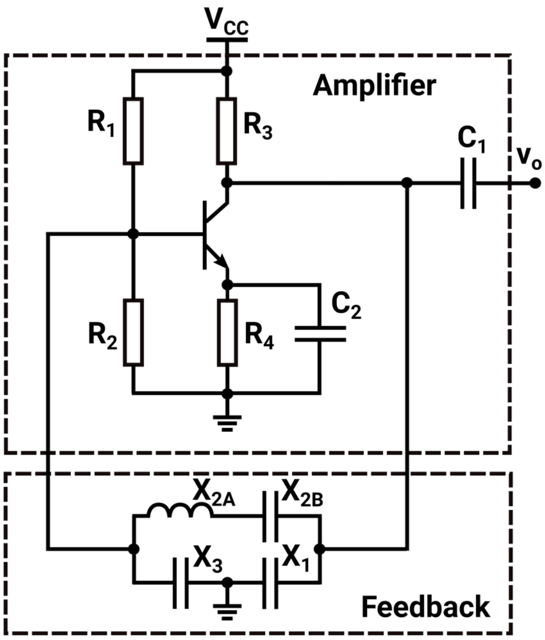


Figure 35 - Oscillator Clapp.

5.2.2 – Gain and Oscillation Frequency

From the analysis of the circuit in Figure 30, we found the gain and oscillation frequency. Here we will consider the resistance r_i of infinite value. The gain and input impedance of the feedback network are given by Eqs.52 and 53. The amplifier gain with the feedback impedance is found in Eq.54. With the unity gain condition for the amplifier and feedback gains, Eq.55, we arrive at Eq.56, whose development, Eqs.57 and 58, results in Eq.59.

$$\frac{V_{\text{out}}}{V_{\text{in}}}=A_{Vrede}=\frac{Z_3}{Z_2+Z_3} \quad \text{(Eq.52)}$$

$$Z_{EQrede}=\frac{Z_1(Z_2+Z_3)}{Z_1+Z_2+Z_3} \quad \text{(Eq.53)}$$

$$\frac{v_o}{v_i}=A_{Vamp}=-g_m\frac{r_o Z_{EQrede}}{r_o+Z_{EQrede}} \quad \text{(Eq.54)}$$

$$A_{Vamp}A_{Vrede}=1 \quad \text{(Eq.55)}$$

$$A_{Vamp}A_{Vrede}=-g_m\frac{r_o\dfrac{Z_1(Z_2+Z_3)}{Z_1+Z_2+Z_3}}{r_o+\dfrac{Z_1(Z_2+Z_3)}{Z_1+Z_2+Z_3}}\frac{Z_3}{Z_2+Z_3} \quad \text{(Eq.56)}$$

$$A_{Vamp}A_{Vrede}=-g_m\frac{r_oZ_1Z_3}{r_o(Z_1+Z_2+Z_3)+Z_1(Z_2+Z_3)} \quad \text{(Eq.57)}$$

$$\begin{aligned} Z_1 &= jX_1 \\ Z_2 &= jX_2 \\ Z_3 &= jX_3 \end{aligned} \quad \text{(Eq.58)}$$

$$A_{Vamp}A_{Vrede}=-g_m\frac{-r_oX_1X_3}{jr_o(X_1+X_2+X_3)-X_1(X_2+X_3)} \quad \text{(Eq.59)}$$

The unit gain condition now passes through the elimination of the imaginary term from the denominator of Eq.59, that is, Eq.60, which leads to Eqs.61 and 62, respectively for the gain condition, Eq.63, and oscillation frequencies, Table 5.

$$\boxed{X_1+X_2+X_3=0} \quad \text{(Eq.60)}$$

$$A_{Vamp}A_{Vrede}=-g_m\frac{r_oX_3}{X_2+X_3} \quad \text{(Eq.61)}$$

$$X_2+X_3=-X_1 \quad \text{(Eq.62)}$$

$$\boxed{A_{Vamp}A_{Vrede}=g_m\frac{r_oX_3}{X_1}} \quad \text{(Eq.63)}$$

From the latter we derive the amplifier and oscillation feedback gains, Eqs. 64 and 65.

$$A_{Vamp}=-g_mr_o \quad \text{(Eq.64)}$$

$$A_{Vrede}=\frac{-X_3}{X_1} \quad \text{(Eq.65)}$$

For the operational amplifier, Figure 34 (a), the gain is given by Eq.66, with the product of the gains by Eq.67.

$$A_{Vampop}=\frac{-R_A}{R_B} \quad \text{(Eq.66)}$$

$$\boxed{A_{Vampop}A_{Vrede}=\frac{R_A X_3}{R_B X_1}} \quad \text{(Eq.67)}$$

From Eq. 60 it is possible to determine the Hartley oscillation frequencies, Eqs. 68 and 69, and from Colpitts, Eqs. 70 and 71.

$$j\omega L_1 - j\frac{1}{\omega C_2} + j\omega L_3 = 0 \quad \text{(Eq.68)}$$

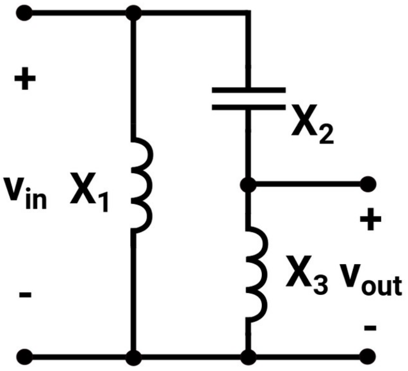
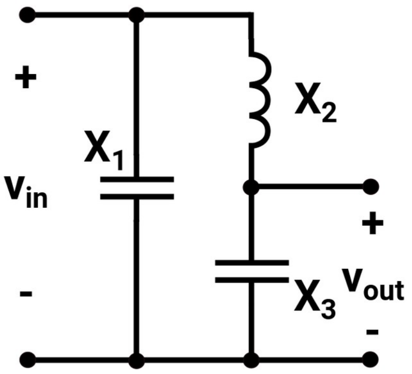
$$f_{osc}=\frac{1}{2\pi\sqrt{(L_1+L_3)C_2}} \quad \text{(Eq.69)}$$

$$-j\frac{1}{\omega C_1} + j\omega L_2 - j\frac{1}{\omega C_3} = 0 \quad \text{(Eq.70)}$$

$$f_{osc}=\frac{1}{2\pi\sqrt{\frac{C_1 C_3}{C_1+C_3}L_2}} \quad \text{(Eq.71)}$$

Table 5 summarizes the characteristics of the Hartley and Colpitts topologies, with the specific reactances for each case, resulting in the expressions of gain and oscillation frequency. The gain must be greater than 1, this occurs because, in practice, the it must be enough to start and maintain the oscillation process, with the system itself guaranteeing saturation in the case of real systems.

Table 5 - Gain and Oscillation Frequency for the Hartley and Colpitts.

Feedback Networks	Reactances	$g_m \frac{r_o X_3}{X_1} > 1$ ou $\frac{R_A X_3}{R_B X_1} > 1$	$X_1 + X_2 + X_3 = 0$
 Hartley	$X_1 = \omega L_1$ $X_2 = \frac{-1}{\omega C_2}$ $X_3 = \omega L_3$	$g_m \frac{r_o L_3}{L_1} > 1$ ou $\frac{R_A L_3}{R_B L_1} > 1$	$f_{osc} = \frac{1}{2\pi\sqrt{(L_1 + L_3) C_2}}$
 Colpitts	$X_1 = \frac{-1}{\omega C_1}$ $X_2 = \omega L_2$ $X_3 = \frac{-1}{\omega C_3}$	$g_m \frac{r_o C_1}{C_3} > 1$ ou $\frac{R_A C_1}{R_B C_3} > 1$	$f_{osc} = \frac{1}{2\pi\sqrt{\frac{C_1 C_3}{C_1 + C_3} L_2}}$

5.2.3 – Ideal Network Simulations

Here, this theoretical basis is verified by simulating the ideal model, Figure 30, for Hartley, Figure 36, and Colpitts, Figure 37. The components used are in Table 6, as well as the simulated gain, and oscillation frequencies calculated and simulated, with almost exact values. Gain, in particular, refers to the case of stable oscillation.

The Hartley oscillator simulation, Figure 36, starts with charging capacitor C2 by switch S1 in the first 300ns, then being opened and the oscillator being started in 1µs by switches S2 and S3. The waveforms for three values of resistor R1 in transient simulation can be seen in Figures 38 (a), (b) and (c), in the case R1 for gain less, equal and greater than 1, decreasing oscillation, constant, and growing.

The Coplitts simulation, Figure 37, is similar to the Hartley one, with switch S1 charging capacitors C1 and C3 up to 300ns, with S2 starting the oscillation process in 1μs. The graphs resulting from the transient simulations also for gain less than, equal to and greater than 1 can be found in Figures 39 (a), (b) and (c).

The ideal models, therefore, support the theory regarding oscillators based on amplifiers with feedback.

Table 6 - Ideal oscillators: calculation and simulation.

Components	$g_m \frac{r_o X_3}{X_1} > 1$	f_{osc} design	f_{osc} simulation
Hartley gm=0.05 S (Given) $g_m \frac{r_o L_3}{L_1} = 1$ e L1=L3 => ro=20 Ω L1=L3=2.2mH (Given) fosc=20kHz (Given) $C_2 = \frac{1}{(L_1 + L_3)(2\pi f_{osc})^2}$ => C2=14.4nF C2=13.8nF (Tuned)	1	20.425 kHz	20.667 kHz
Colpitts gm=0.5 S (Given) C1=13.8nF (Given) L2=2.2mF (Given) fosc=30kHz (Given) $C_3 = \frac{1}{L_2 (2\pi f_{osc})^2 - \frac{1}{C_1}}$ => C3=175.3nF C3=120nF (Tuned) $g_m \frac{r_o C_3}{C_1} = 1$ => ro=17.4 Ω	1.0005	30.5 kHz	31.6 kHz

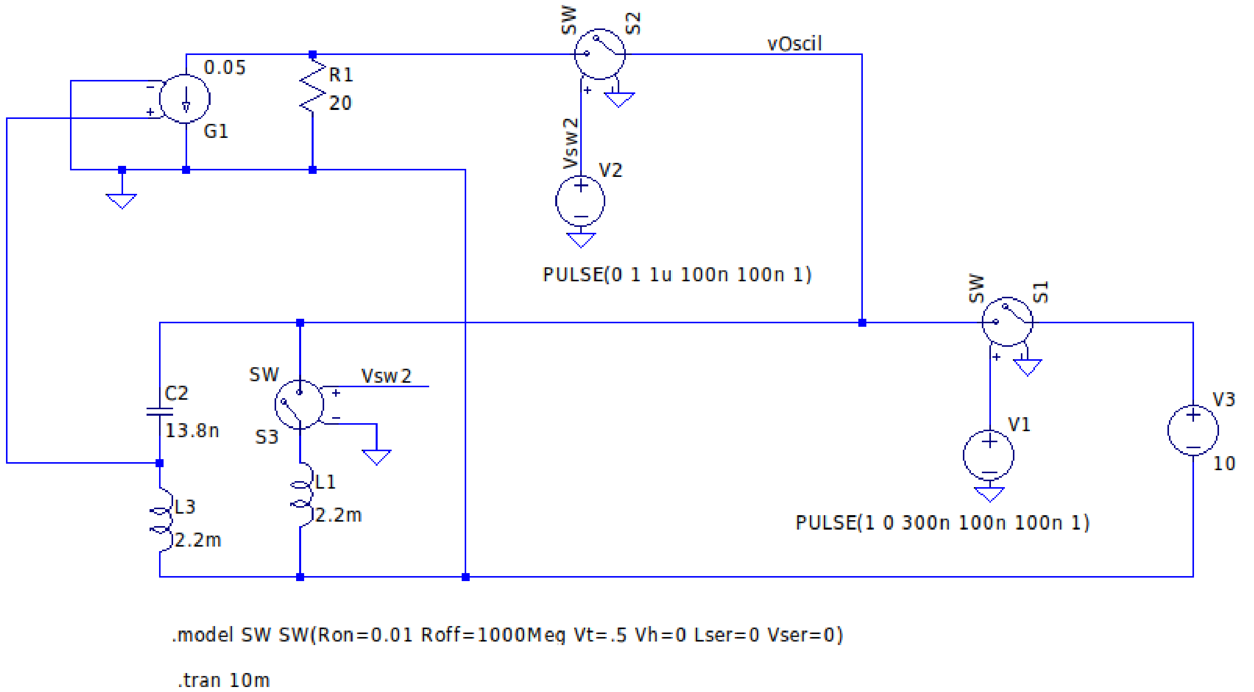


Figure 36 - Hartley ideal LTSpice, file HartleyIdeal.asc.

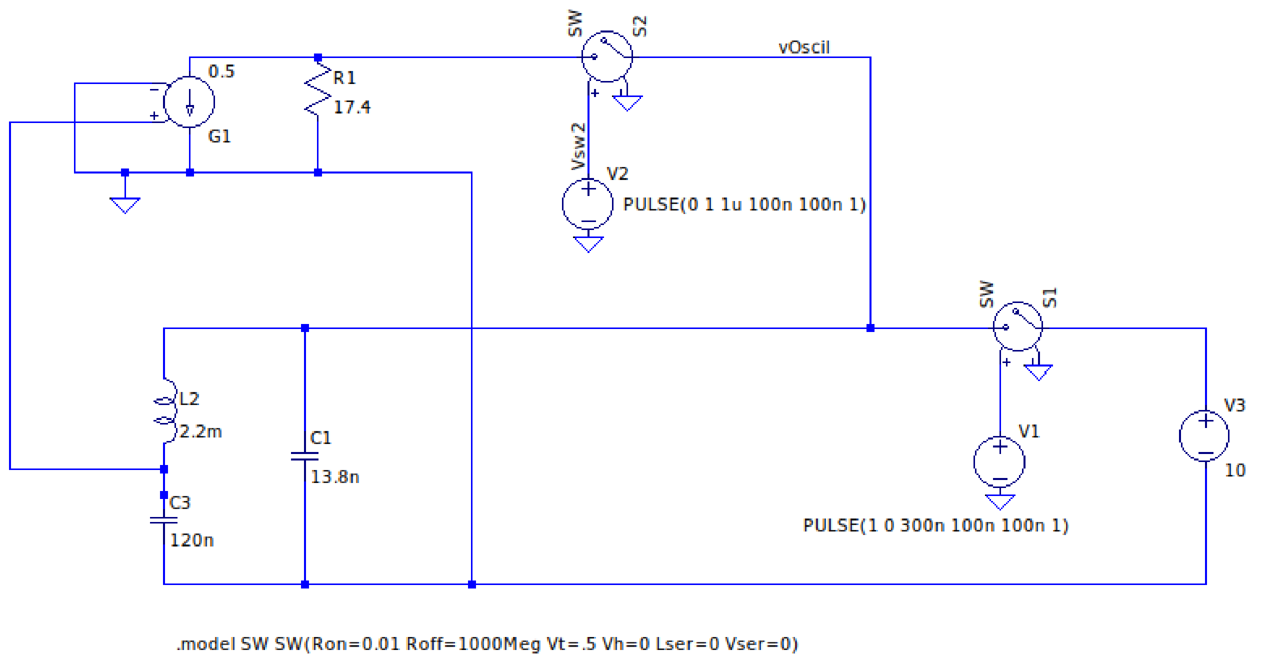


Figure 37 - Colpitts ideal LTSpice, file ColpittsIdeal.asc.

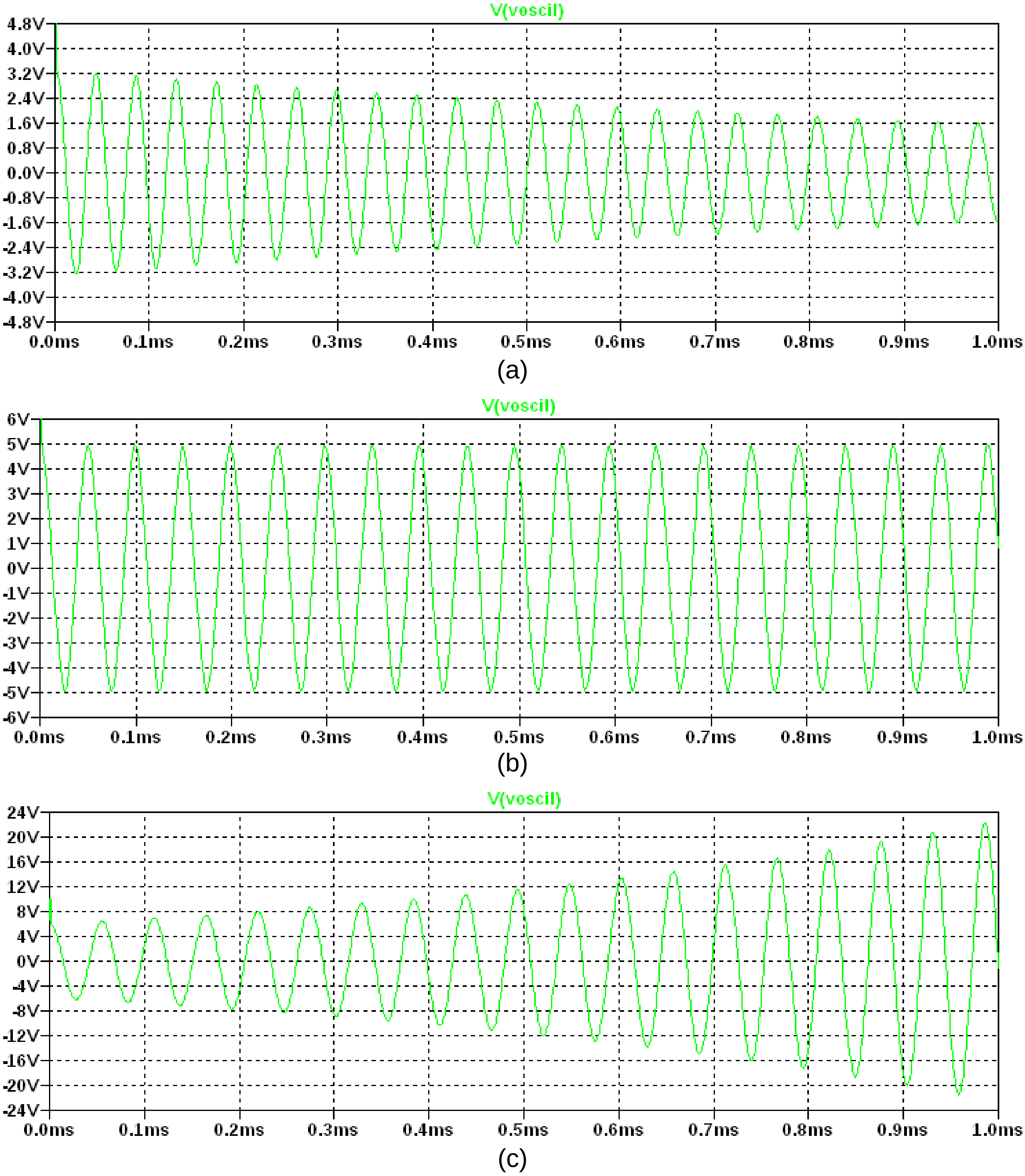
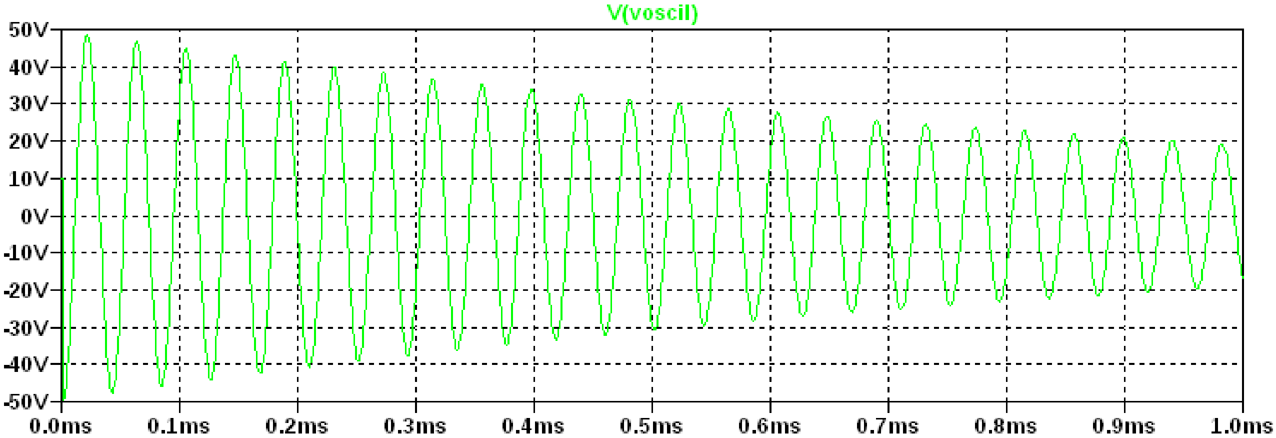
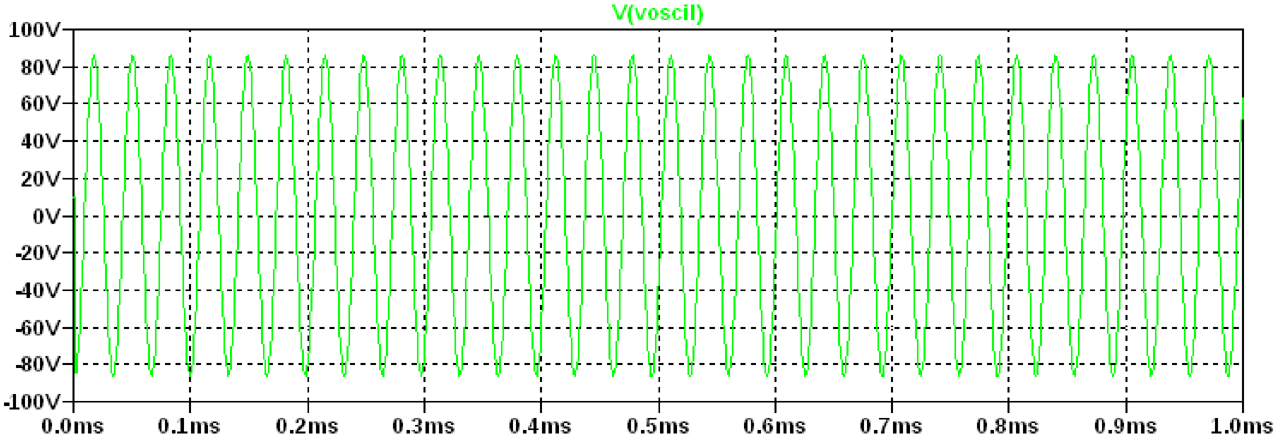


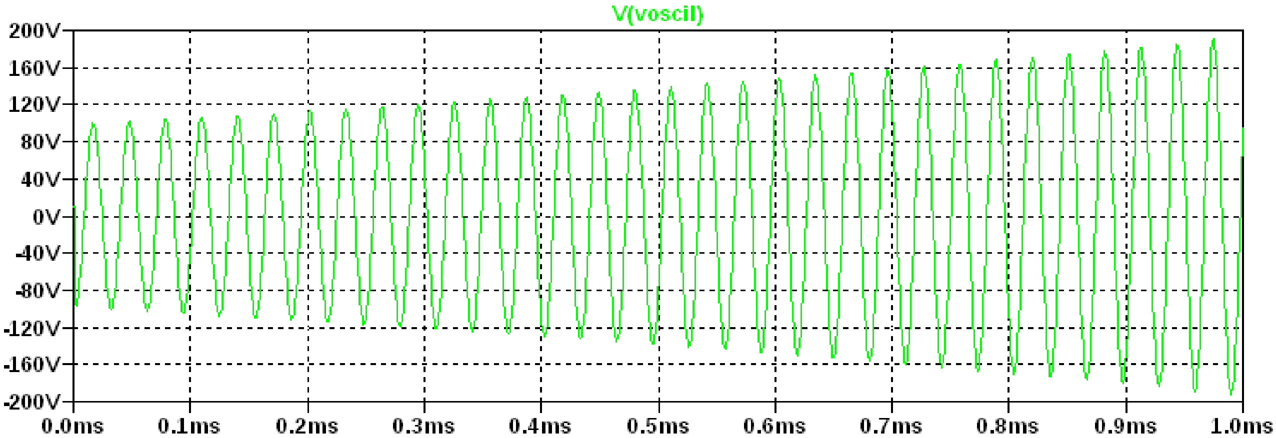
Figure 38 - Hartley ideal (a) R1=10Ω (b) R1=20Ω (c) R1=30Ω.



(a)



(b)



(c)

Figure 39 - Colpitts ideal (a) R1=10Ω (b) R1=17.4Ω (c) R1=20Ω.

5.2.2 – Simulations with Electronic Components

In Figure 40 we can see the Hartley oscillator implemented with an operational amplifier, here adopted this version due to its design simplicity compared to any transistor-based topology, as in Figure 33 (a). In this circuit the power supplies are voltage steps in order to start the oscillation, a single DC supply may not start this process. The operational used is the LT1001A here presenting good results, Figures 41 (a) and (b), transient simulation. Note that this circuit uses the same components as in the ideal case, so the same oscillation frequency is expected, here a little below as shown in Figure 41 (b). The resistor Rtol (tolerance) is used to adjust the distortion level of the waveform, being manually adjusted.

The Colpitts oscillator, Figure 42, also uses the components of the ideal example, therefore with the same oscillation frequency. The LT1022A is now operational to guarantee oscillation at the designed frequency, since the LT1001A does not reach this frequency in this topology. In Figure 43(a) and (b) we have the complete simulation and approximation with reading of the resulting frequency, respectively. The summary of these simulations can be found in Table 7.

In these electronic versions we see the practical issue of the operational amplifier slew rate, which may be under the desired frequency, working either at a lower frequency or generating a triangular wave, since it cannot generate a sinusoidal.

Table 7 - Electronic oscillators: design and simulation.

Components	$\frac{R_A X_3}{R_B X_1} > 1$	f_{osc} - design	f_{osc} - simulation
Hartley L1=2.2mH C2=13.8nF L3=2.2mH	$\frac{R_A L_3}{R_B L_1} = 1.5$ RA=15kΩ RB=10kΩ	20.425 kHz	17.997 kHz
Colpitts C1=13.8nF L2=2.2mF C3=120nF	$\frac{R_A C_3}{R_B C_1} = 1$ RA=87kΩ RB=10kΩ	30.5 kHz	31.667 kHz

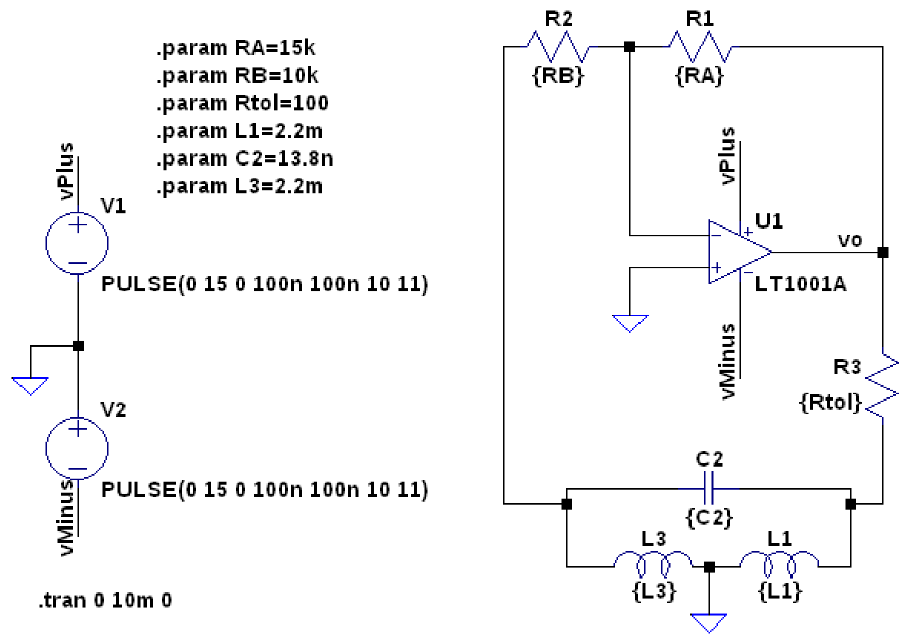
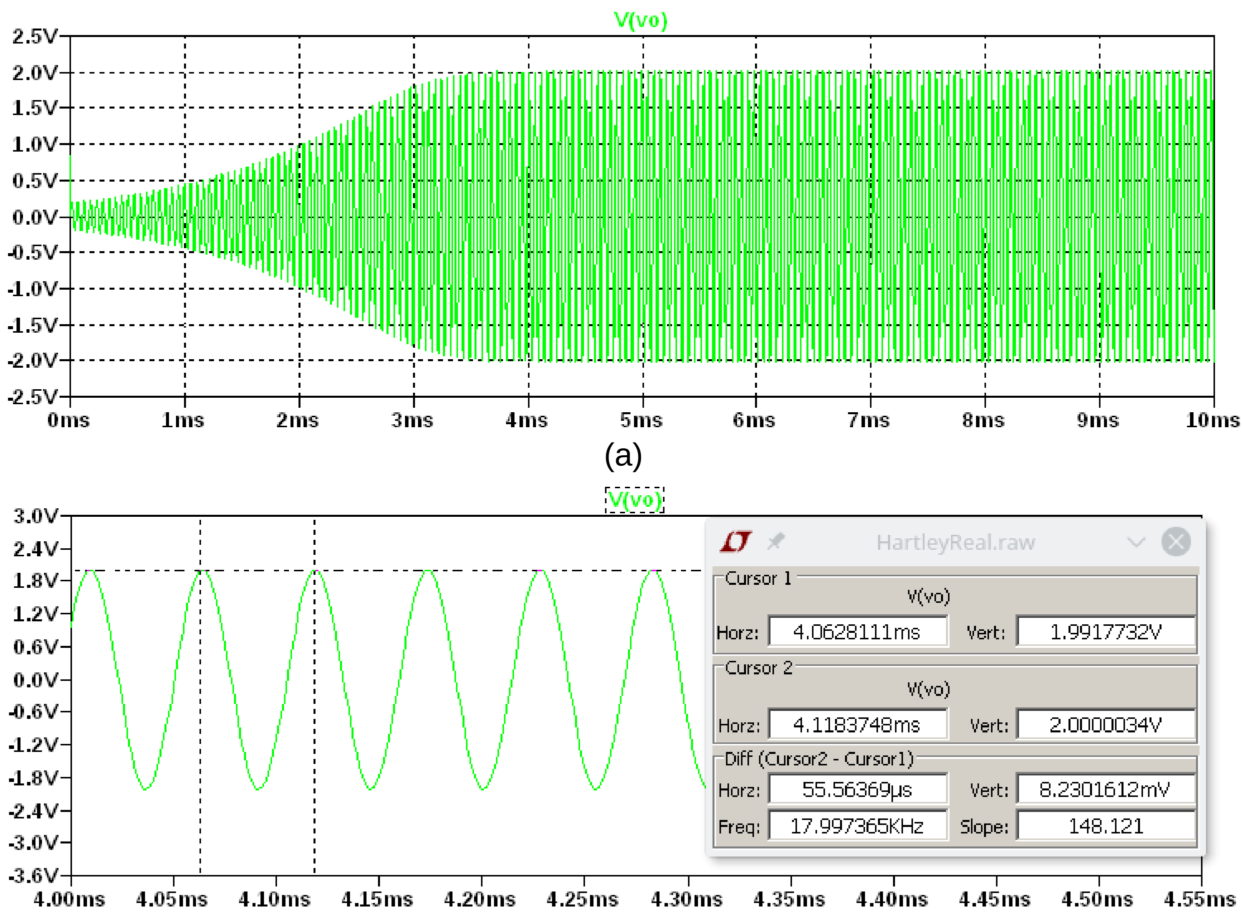


Figure 40 - Electronic Hartley LTSpice, file HartleyReal.asc.



(a)

(b)

Figure 41 - Electronic Hartley: (a) full simulation and (b) simulation zoom.

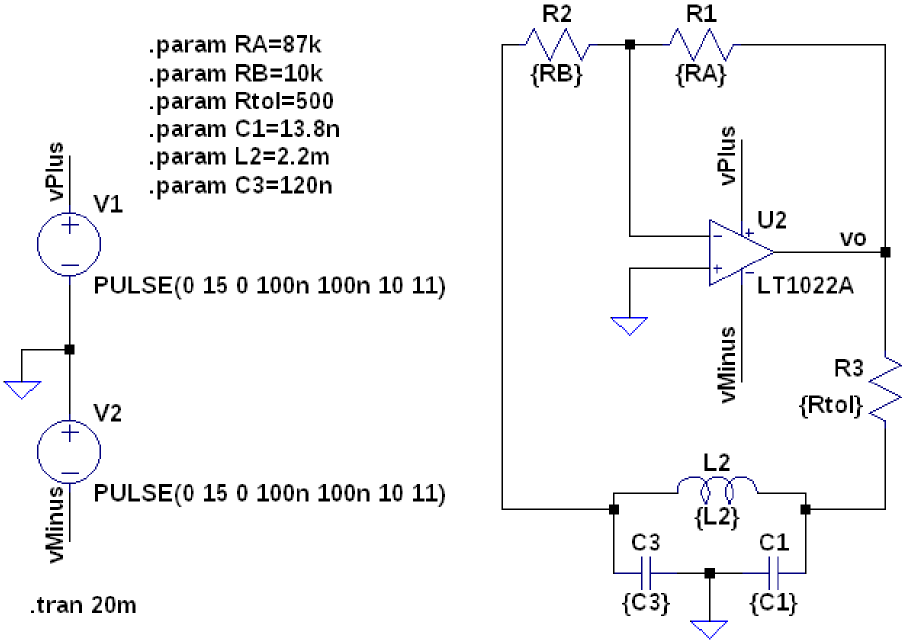


Figura 42 - Colpitts eletrônico LTSpice, arquivo ColpittsReal.asc.

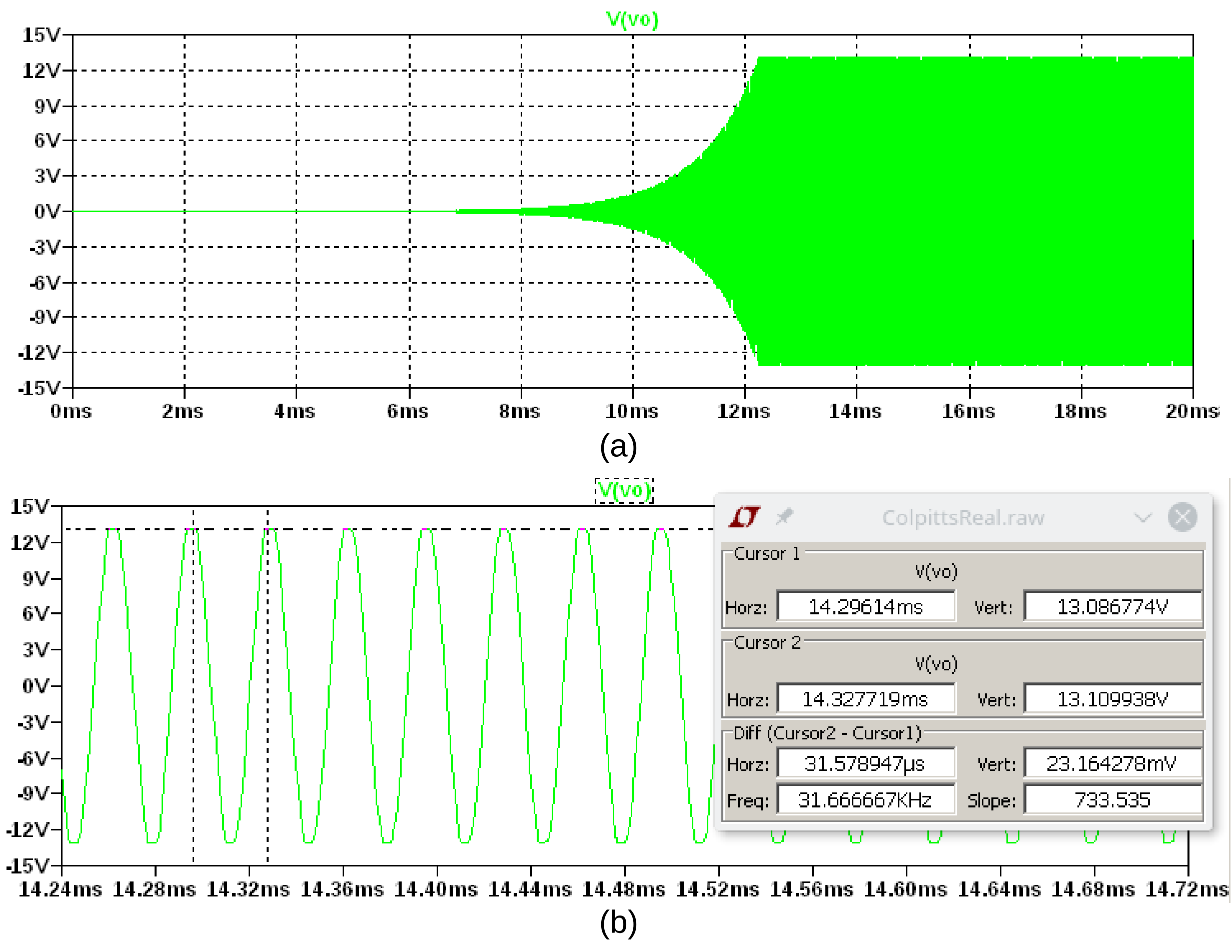
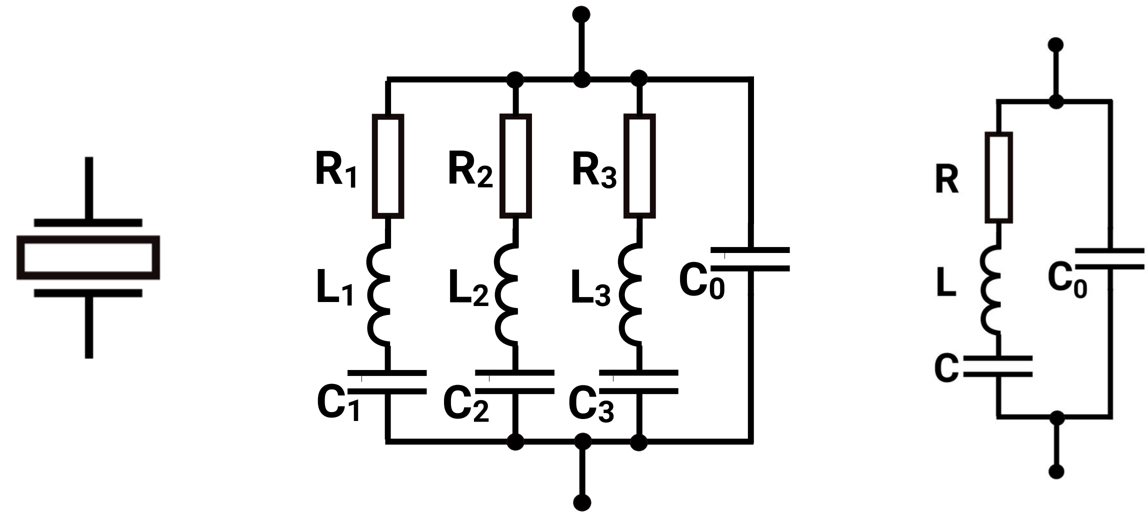


Figure 43 - Electronic Colpitts: (a) full simulation and (b) simulation zoom.

5.4 – Networks Based on Cristal

Piezoelectric crystals such as quartz crystal exhibit very stable electromechanical properties with time and temperature, as well as high selectivity, or very high Q quality factor [SEDRA], hence used when precision and oscillation frequency stability are required. In Figure 44 we see its symbology, complete and reduced models [Gil_UERJ_Oscil], the latter used for design and simulation. Figure 45 shows a Colpitts oscillator with a crystal in place of the inductor, the analysis for design has the same systematics as the conventional Colpitts, but now using the simplified model, Figure 45 (c). The simulations of a crystal circuit also use the simplified model with the respective components parameters, resistance, inductance and capacitances, of the used crystal.



(a) (b) (c)

Figure 44 - (a) Crystal symbology, (b) full model and (c) reduced model (around the frequencies of inductive behavior) [Gil_UERJ_Oscil].

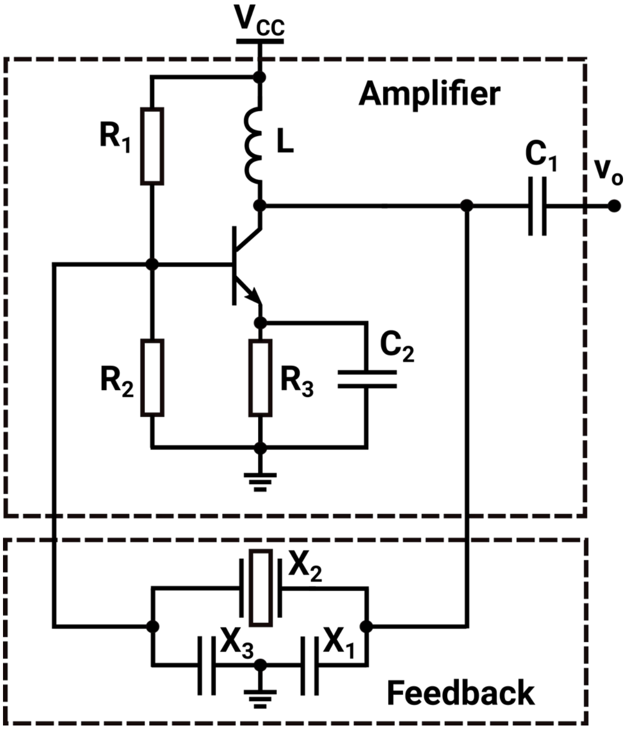


Figure 45 - Oscillator Colpitts based on cristal.

6 - Relaxation Oscillator

6.1 – Principle

Relaxation oscillator or astable multivibrator, it is based on capacitor charging and discharging. The subject of multivibrators, however, is a prerequisite for this type of oscillator.

Multivibrators are a class of circuits used to implement two-state, low and high logic signals. They can be:

-Monostable: It is a circuit that, when triggered, maintains a temporary logic state. For example, timer circuits. It can be compared to a pendulum with losses, which stops swinging after receiving energy.

-Bistable: It only changes its logic state when receiving a trigger pulse, the flip-flop is one case. It would be a pendulum that rests at two ends of its trajectory.

-Astable: Changes state over time without any trigger pulse. As an example:

clock circuits. It is similar to the lossless pendulum.

We can see in Table 8 a summary with the illustration of each of these multivibrators.

Figure 46 shows the implementation of a bistable circuit with an operational amplifier, basically using positive feedback, which results in a transfer curve with hysteresis, Figure 46 next to each amplifier according to the input. V_{TH} and V_{TL} are the threshold voltages, threshold, and V+ and V- the maximum operational output amplitude. Circuit also called Schmitt Trigger. For the inverter circuit Figure 46 (a), the threshold voltage V_T is given by the resistive divider composed of R_A and R_B, Eq.72, voltage that varies as V_O changes between high and low, hence the hysteresis in the transfer graph. For the non-inverting Schmitt Trigger, the transition voltage occurs when v_t reaches v_t=0V from virtual ground, Eq. 75 result of Eqs. 73 and 74, of the voltages in R_A and R_B.

The oscillator, the astable, here is composed of a bistable with a feedback network, that is, when turned on, it assumes one of its maximum states, charging the capacitor C, such that when it reaches the trigger voltage, it changes state, initiating the discharge until the new state change, and so on. We see in Figure 47 the complete circuit with the bistable and astable feedback blocks properly highlighted, as well as the waveforms at the input and output of the operational. The oscillation frequency is determined by Eqs. 76 and 88.

Integrated timers also implement the aforementioned multivibrator functions, one of which is the 555 integrated circuit, capable of operating as monostable, bistable and astable.

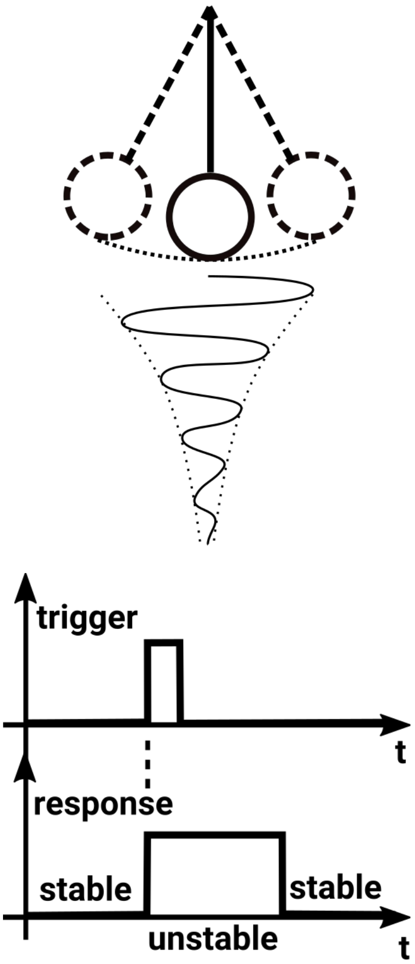
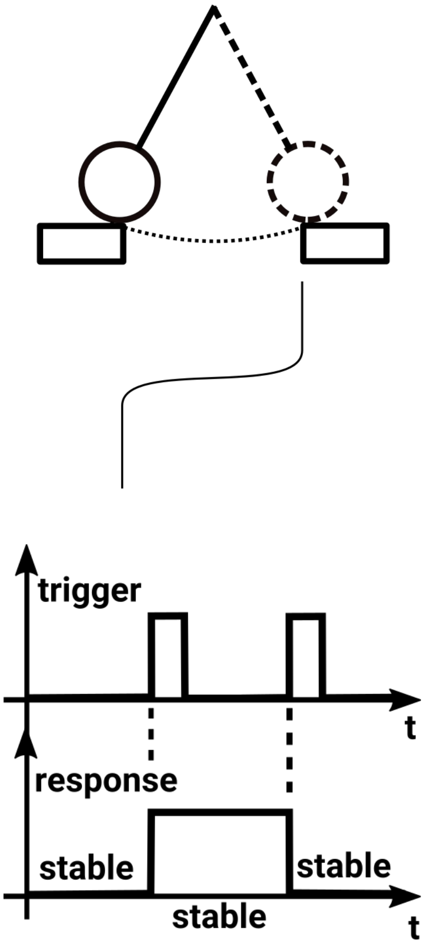
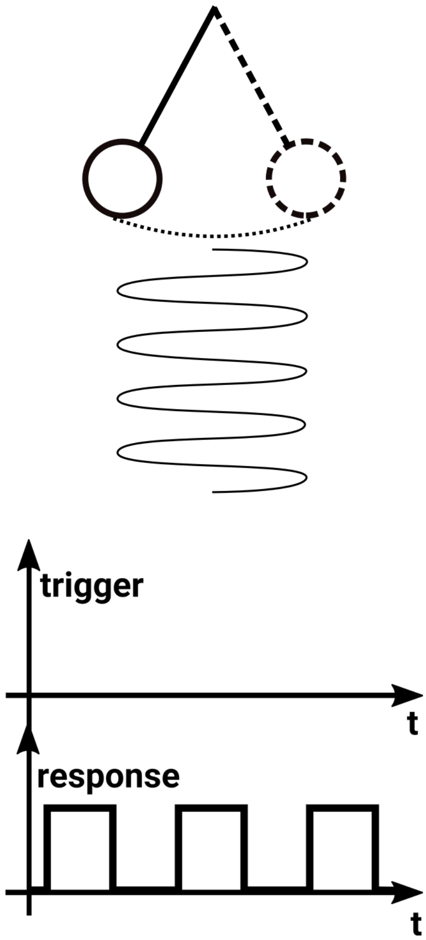
$$V_T = \frac{R_B}{R_A + R_B} \quad \text{(Eq.72)}$$

$$I = \frac{v_o}{R_A} \quad \text{(Eq.73)}$$

$$v_t = R_B I \quad \text{(Eq.74)}$$

$$v_t = v_o \frac{R_B}{R_A} \quad \text{(Eq.75)}$$

Table 8 - Types of multivibrators.

Multivibrators		
Monostable	Bistable	Astable
		

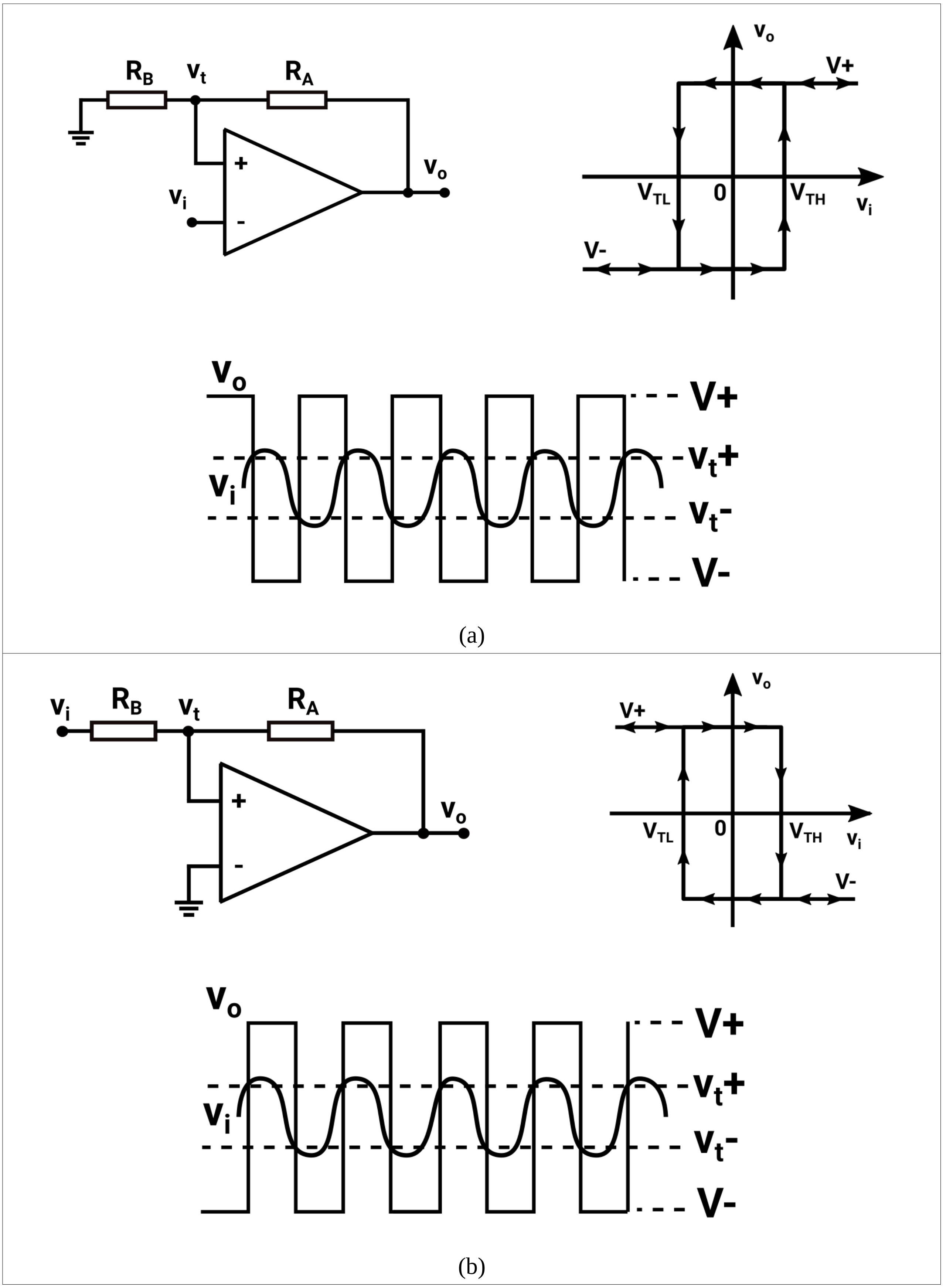


Figure 46 - Operational amplifier as a bistable and its transfer function:
(a) non-inverting and (b) inverting.

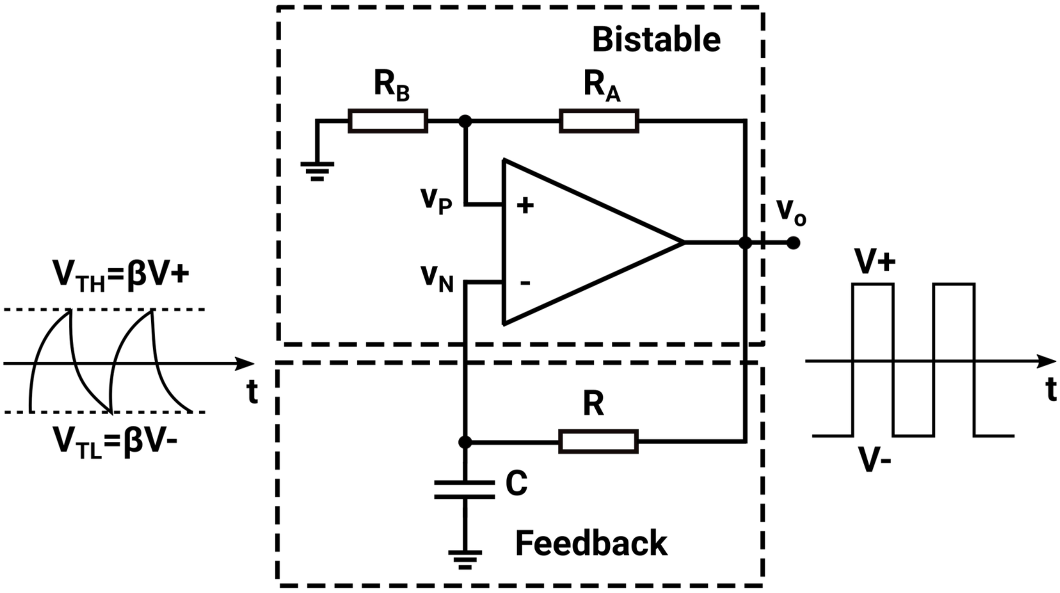


Figure 47 - Square wave oscillator based on astable multivibrator.

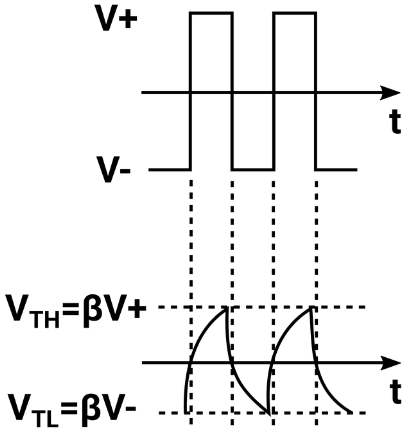


Figure 48 - Waveforms at the operational output and inverting input.

The resistive divider formed by R_A and R_B defines the voltage v_P that will be compared with v_N.

$$\beta=\frac{R_B}{R_A+R_B} \quad \text{(Eq.76)}$$

We developed the expressions for charge, Eq.77, and discharge, Eq.78, which follow Figure 48 [SEDRA], from Eq. 111, appendix.

$$v_n = (\mathrm{V+}) - \left((\mathrm{V+}) - \beta(\mathrm{V-})\right) e^{-t/RC} \quad \text{(Eq.77)}$$

$$v_n = (\mathrm{V-}) - \left((\mathrm{V-}) - \beta(\mathrm{V+})\right) e^{-t/RC} \quad \text{(Eq.78)}$$

Developing with time intervals T1 for charging, and T2 for discharging:

$$\frac{(\mathrm{V+}) - v_n}{(\mathrm{V+}) - \beta(\mathrm{V-})} = e^{-T1/RC} \quad \text{(Eq.79)}$$

$$\frac{(\mathrm{V-}) - v_n}{(\mathrm{V-}) - \beta(\mathrm{V+})} = e^{-T2/RC} \quad \text{(Eq.80)}$$

For charge $v_n = \beta V+$ and discharge $v_n = \beta V-$:

$$T1 = -RC \ln \frac{(\mathrm{V+}) - \beta(\mathrm{V+})}{(\mathrm{V+}) - \beta(\mathrm{V-})} \quad \text{(Eq.81)}$$

$$T2 = -RC \ln \frac{(\mathrm{V-}) - \beta(\mathrm{V-})}{(\mathrm{V-}) - \beta(\mathrm{V+})} \quad \text{(Eq.82)}$$

Dividing by the supply voltages V+ and V-, as well as developing the signs of the logarithms:

$$T1 = RC \ln \frac{1 - \beta(\mathrm{V+/V-})}{1 - \beta} \quad \text{(Eq.83)}$$

$$T2 = RC \ln \frac{1 - \beta(\mathrm{V-/V+})}{1 - \beta} \quad \text{(Eq.84)}$$

With symmetrical supply V+=-(V-) we have:

$$T1 = RC \ln \frac{1+\beta}{1-\beta} \quad \text{(Eq.85)}$$

$$T2 = RC \ln \frac{1+\beta}{1-\beta} \quad \text{(Eq.86)}$$

For the full range T=T1+T2:

$$T = 2RC \ln \frac{1+\beta}{1-\beta} \quad \text{(Eq.87)}$$

The oscillation frequency is simply:

$$f_{osc} = \frac{1}{2RC \ln\left[\frac{1+\beta}{1-\beta}\right]} \quad \text{(Eq.88)}$$

6.2 – Simulation Example

Figure 49 shows the relaxation oscillator in simulator, with the respective waveforms in Figure 50 in transient simulation. A summary of the calculation and simulation can be found in Table 9, with differences due to operational limitations.

Table 9 - Square wave oscillator: calculation and simulation.

	Design	Simulation
f_{osc}	274.24 Hz	311.67 Hz

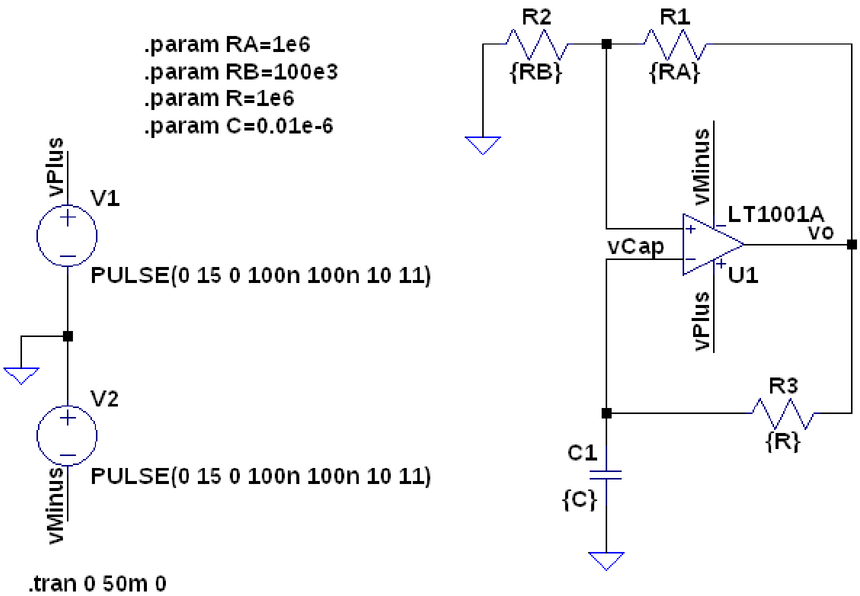


Figure 49 - LTSpice square wave oscillator,
file SquareWaveOscillator.asc.

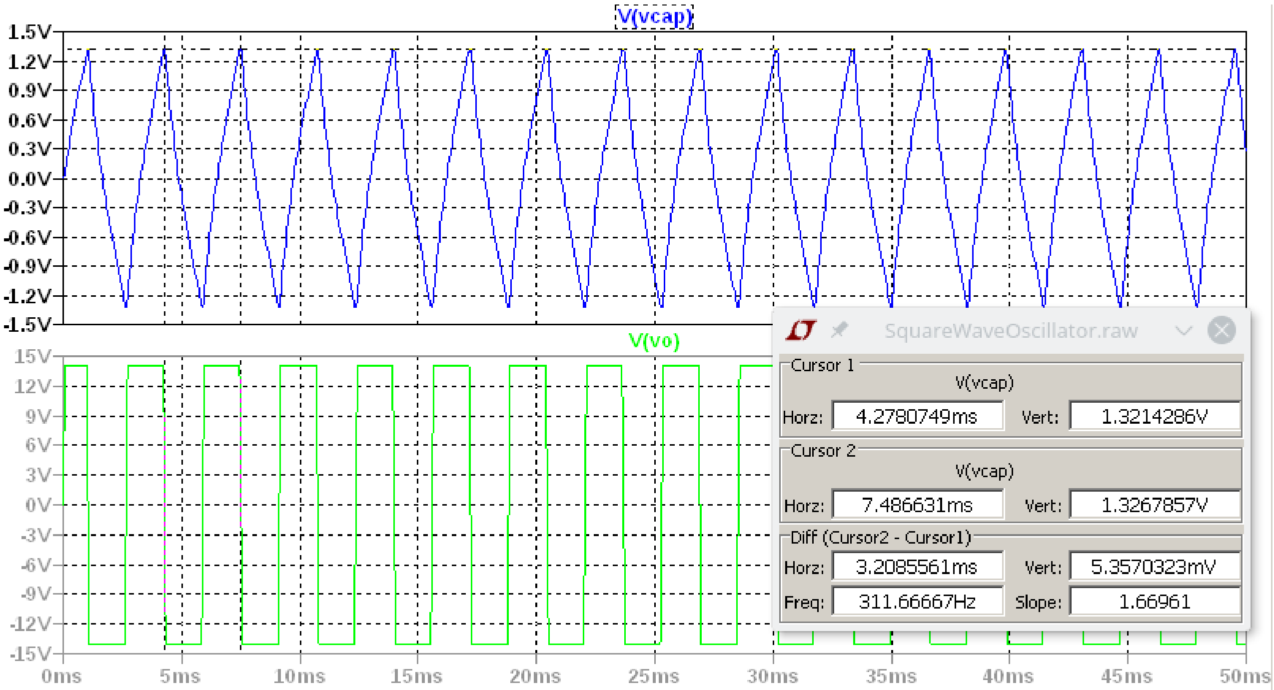


Figure 50 - Square wave oscillator.

Appendix

In this section we find the general unit-step response form, Figure 51, for first-order systems. Given a system with input x(t), output y(t) and transfer function H(s) [UnitStep].

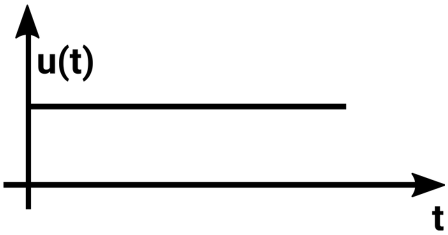


Figure 51 - Unit step.

$$H(s)=\frac{Y(s)}{X(s)} \quad \text{(Eq.89)}$$

The output with zero initial conditions (that is, the zero-state output) is simply given by:

$$Y(s)=X(s)H(s) \quad \text{(Eq.90)}$$

Then the unit step response, $Y_\gamma(s)$, is given by:

$$Y_\gamma(s)=\frac{1}{s}H(s) \quad \text{(Eq.91)}$$

We determine two characteristics of the unit step response, the initial and final values, of the step response invoking the initial value theorems, Eq.92, and final Eq.93.

$$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s) \quad \text{(Eq.92)}$$

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s) \quad \text{(Eq.93)}$$

Therfore

$$\lim_{t\to 0} y_\gamma(t)=\lim_{s\to\infty} sY_\gamma(s)=\lim_{s\to\infty} s\frac{1}{s}H(s)=\lim_{s\to\infty} H(s) \quad \text{(Eq.94)}$$

$$\lim_{t\to\infty} y_\gamma(t)=\lim_{s\to 0} sY_\gamma(s)=\lim_{s\to 0} s\frac{1}{s}H(s)=\lim_{s\to 0} H(s) \quad \text{(Eq.95)}$$

Developing

$$y_\gamma(0)=H(\infty) \quad \text{(Eq.96)}$$

$$y_\gamma(\infty)=H(0) \quad \text{(Eq.97)}$$

Applying these concepts into a first-order system:

$$H(s)=\frac{bs+c}{s+a} \quad \text{(Eq.98)}$$

In this the variables a, b and c are arbitrary real numbers and b or c (but not both) can be zero. To find the unit-step answer, we multiply H(s) by 1/s:

$$Y_\gamma(s)=\frac{1}{s}H(s)=\frac{1}{s}\frac{bs+c}{s+a} \quad \text{(Eq.99)}$$

Separating into partial fractions and developing

$$Y_\gamma(s)=\frac{A}{s}+\frac{B}{s+a} \quad \text{(Eq.100)}$$

$$Y_\gamma(s)=\frac{As+Aa+Bs}{s(s+a)} \quad \text{(Eq.101)}$$

$$Aa=c \quad \text{(Eq.102)}$$

$$s(A+B)=sb \quad \text{(Eq.103)}$$

.

$$A=\frac{c}{a} \quad \text{(Eq.104)}$$

$$B=b-A=b-\frac{c}{a} \quad \text{(Eq.105)}$$

$$Y_\gamma(s)=\frac{c}{a}\frac{1}{s}+\left(b-\frac{c}{a}\right)\frac{1}{s+a} \quad \text{(Eq.106)}$$

$$y_\gamma(t)=\frac{c}{a}+\left(b-\frac{c}{a}\right)e^{at},t>0 \quad \text{(Eq.107)}$$

Observing some characteristics of this equation, we arrive at:

$$y_\gamma(0)=H(\infty)=b \quad \text{(Eq.108)}$$

$$y_\gamma(\infty)=H(0)=\frac{c}{a} \quad \text{(Eq.109)}$$

$$\tau=\frac{1}{a} \quad \text{(Eq.110)}$$

Thus, we can write the general form of the unit step response as:

$$y_\gamma(t)=y_\gamma(\infty)+\left(y_\gamma(0)-y_\gamma(\infty)\right)e^{-t/\tau} \quad \text{(Eq.111)}$$

$$y_\gamma(t)=H(0)+\left(H(\infty)-H(0)\right)e^{-t/\tau} \quad \text{(Eq.112)}$$

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