Workbook on mapping simplexes affinely

Preprint · May 2019

DOI: 10.13140/RG.2.2.16346.18881

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August 10, 2019

Abstract

This workbook is intended to demonstrate the utility of the unusual method to define affine transformations we have presented in [1]. We will perform a walkthrough different problems starting with a straightforward application of the formula to recover the transformation from its action on a handful of points, different interpolation problems, and miscellaneous examples that occasionally came to our mind. Essential excerpts from theory are presented in the very first section, but Reader is always welcome to check [1] for more detailed discussion.

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Theory

Affine transformation from n-dimensional to m-dimensional vector space can be uniquely defined by defining its action on vertices $\vec{x}^{(1)}, \ldots, \vec{x}^{(n+1)}$ of any simplex. The explicit form of the transformation is given by the following formula (hereafter referred as SAM for Simplex Affine Map)

$$\vec{X}(\vec{x}) = (-1)\frac{\det\begin{pmatrix} 0 & \vec{X}^{(1)} & \vec{X}^{(2)} & \ldots & \vec{X}^{(n+1)} \\ x_1 & x_1^{(1)} & x_1^{(2)} & & x_1^{(n+1)} \\ x_2 & x_2^{(1)} & x_2^{(2)} & \ldots & x_2^{(n+1)} \\ \ldots & \ldots & \ldots & & \ldots \\ x_n & x_n^{(1)} & x_n^{(2)} & & x_n^{(n+1)} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}}{\det\begin{pmatrix} x_1^{(1)} & x_1^{(2)} & & x_1^{(n+1)} \\ x_2^{(1)} & x_2^{(2)} & \ldots & x_2^{(n+1)} \\ \ldots & \ldots & & \ldots \\ x_n^{(1)} & x_n^{(2)} & & x_n^{(n+1)} \\ 1 & 1 & \ldots & 1 \end{pmatrix}},$$

where coordinates $x_1, \ldots, x_n$ of vector-argument are color-coded with blue, the vertices of the simplex in the codomain $\vec{X}^{(1)}, \ldots, \vec{X}^{(n+1)}$ are shown in red, and the i-th coordinate of the j-th vertex of the domain simplex $x_i^{(j)}$ remains black. The last row of both matrices is filled with ones and the intersection of the first row and first column of the matrix in the numerator always contains zero. We will often write $\vec{X}^{(i)}$-s in form of vector-columns

$$\vec{X}^{(1)} = \begin{pmatrix} X_1^{(1)} \\ \ldots \\ X_m^{(1)} \end{pmatrix}; \quad \ldots; \quad \vec{X}^{(n)} = \begin{pmatrix} X_1^{(n)} \\ \ldots \\ X_m^{(n)} \end{pmatrix};$$

so don't be confused with braces inside the matrix. Please note as well that dimensions of domain and codomain may be different.

Any affine transformation can be presented this way, as well as any transformation defined by SAM can be rewritten in canonical form. If points $\vec{x}^{(1)}, \ldots, \vec{x}^{(n+1)}$ do not constitute a simplex, the determinant in the denominator will be equal to zero alerting us that there is not enough information to recover affine transformation.

SAM is inherently connected to barycentric coordinates — performing Laplace expansion along the first row of the matrix in the numerator we can rewrite SAM as linear combination of $\vec{X}^{(1)}, \ldots, \vec{X}^{(n+1)}$ with coefficients equal to barycentric coordinates of the point $\vec{x}$ with respect to simplex $\vec{x}^{(1)}, \ldots, \vec{x}^{(n+1)}$.

For derivation and in-depth discussion of SAM please refer to the paper [1]. Now let's solve some problems with it.

Affine transformations of vector spaces

This section contains problems on the most straightforward application of SAM — finding affine transformation given how points of a simplex are transformed. Such problems appear in computer graphics when you want to recover the affine transformation that was applied to a certain model, but the only information available are its initial and final positions, or when you want to perform texturing by mapping triangle in the "texture space" to the triangle in the "geometrical space."

1D transformations

General 1D transformation

Suppose affine transformation maps points $x^{(1)}$ and $x^{(2)}$ into $X^{(1)}$ and $X^{(2)}$ respectively. Find the transformation first using common sense only then using SAM. Compare the results.

► The result is easy to guess: we can shift the real line by $-x^{(1)}$, scale it by $(X^{(2)}-X^{(1)})/(x^{(2)}-x^{(1)})$ to make distance between the two points appropriate, and then again shift by $X^{(1)}$. All in all we have an affine map

$$X(x) = X^{(1)} + (x - x^{(1)})\frac{X^{(2)} - X^{(1)}}{x^{(2)} - x^{(1)}}.$$

Verify that

$$X(x^{(1)}) = X^{(1)},$$
$$X(x^{(2)}) = X^{(2)}.$$

If we use SAM we get

$$X(x) = -\frac{\det\begin{pmatrix} 0 & X^{(1)} & X^{(2)} \\ x & x^{(1)} & x^{(2)} \\ 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} x^{(1)} & x^{(2)} \\ 1 & 1 \end{pmatrix}} = -\frac{-x(X^{(1)} - X^{(2)}) + X^{(1)}x^{(2)} - X^{(2)}x^{(1)}}{x^{(1)} - x^{(2)}} =$$

$$= \frac{x(X^{(1)} - X^{(2)}) - X^{(1)}x^{(2)} + X^{(2)}x^{(1)} - X^{(1)}x^{(1)} + X^{(1)}x^{(1)}}{x^{(1)} - x^{(2)}} =$$

$$= \frac{x(X^{(1)} - X^{(2)}) - x^{(1)}(X^{(1)} - X^{(2)}) + X^{(1)}(x^{(1)} - x^{(2)})}{x^{(1)} - x^{(2)}} =$$

$$= (x - x^{(1)})\frac{X^{(2)} - X^{(1)}}{x^{(2)} - x^{(1)}} + X^{(1)}\cancel{\frac{x^{(2)} - x^{(1)}}{x^{(2)} - x^{(1)}}} = X^{(1)} + (x - x^{(1)})\frac{X^{(2)} - X^{(1)}}{x^{(2)} - x^{(1)}}.$$

That matches the formula we derived earlier. ◄

2D transformations

Identity mapping (sanity check)

Using SAM, find affine mapping of triangle with vertices

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

into itself ☺.

▶ Since our initial and final points coincide, we plug them in twice — elementwise into the body of matrix and as vector-columns into the first row. SAM looks as follows

$$\vec{X}(x_1;x_2) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}2\\3\end{pmatrix} & \begin{pmatrix}3\\2\end{pmatrix} \\ x_1 & 1 & 2 & 3 \\ x_2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 1 & 1\end{pmatrix}} =$$

$$= \frac{1}{3}\left[\begin{pmatrix}1\\1\end{pmatrix}(5 - x_1 - x_2) + \begin{pmatrix}2\\3\end{pmatrix}(2x_2 - x_1 - 1) + \begin{pmatrix}3\\2\end{pmatrix}(2x_1 - x_2 - 1)\right] =$$

$$= x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

The result is an identity map (as expected). ◀

Pure scale

Vertices of the triangle

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix},$$

are affinely mapped into

$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\9\end{pmatrix}, \begin{pmatrix}6\\6\end{pmatrix},$$

respectively. Using SAM, show that the mapping is pure scale: by factor 2 along $x-$coordinate and by factor 3 along $y-$coordinate.

▶

$$\vec{X}(x;y) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}2\\3\end{pmatrix} & \begin{pmatrix}4\\9\end{pmatrix} & \begin{pmatrix}6\\6\end{pmatrix} \\ x & 1 & 2 & 3 \\ y & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 1 & 1\end{pmatrix}} =$$

$$= \frac{1}{3}\left[\begin{pmatrix}2\\3\end{pmatrix}(5 - x - y) + \begin{pmatrix}4\\9\end{pmatrix}(2y - x - 1) + \begin{pmatrix}6\\6\end{pmatrix}(2x - y - 1)\right] =$$

$$= x\begin{pmatrix}2\\0\end{pmatrix} + y\begin{pmatrix}0\\3\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

As we can see, SAM correctly recovered scale factor for $x-$ and $y-$components. ◀

Pure translation

Vertices of the triangle

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix},$$

are affinely mapped into

$$\begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix},$$

respectively. Using SAM, show that the mapping is translation by vector $(-1, 2)^{\mathsf{T}}$.

▶

$$\vec{X}(x_1;x_2) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}0\\3\end{pmatrix} & \begin{pmatrix}1\\5\end{pmatrix} & \begin{pmatrix}2\\4\end{pmatrix} \\ x_1 & 1 & 2 & 3 \\ x_2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\\ 1 & 1 & 1\end{pmatrix}} =$$

$$= \frac{1}{3}\left[\begin{pmatrix}0\\3\end{pmatrix}(5 - x_1 - x_2) + \begin{pmatrix}1\\5\end{pmatrix}(2x_2 - x_1 - 1) + \begin{pmatrix}2\\4\end{pmatrix}(2x_1 - x_2 - 1)\right] =$$

$$= x_1\begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}-1\\2\end{pmatrix}.$$ ◀

Pure rotation

Consider counterclockwise rotation through an angle α. Find where triangle's vertices

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}$$

are mapped into. Show that SAM correctly recovers this transformation, when initial and rotated triangles are given.

▶ It is known, counterclockwise rotation through an angle α is performed by rotation matrix

$$\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}.$$

First we multiply rotation matrix by the vertices of the triangle to get points

$$\begin{pmatrix}\cos\alpha - \sin\alpha\\ \sin\alpha + \cos\alpha\end{pmatrix}, \begin{pmatrix}2\cos\alpha - 3\sin\alpha\\ 2\sin\alpha + 3\cos\alpha\end{pmatrix}, \begin{pmatrix}3\cos\alpha - 2\sin\alpha\\ 3\sin\alpha + 2\cos\alpha\end{pmatrix}.$$

Now we can use SAM

$$\vec{X}(x_1;x_2) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}\cos\alpha - \sin\alpha\\ \sin\alpha + \cos\alpha\end{pmatrix} & \begin{pmatrix}2\cos\alpha - 3\sin\alpha\\ 2\sin\alpha + 3\cos\alpha\end{pmatrix} & \begin{pmatrix}3\cos\alpha - 2\sin\alpha\\ 3\sin\alpha + 2\cos\alpha\end{pmatrix} \\ x_1 & 1 & 2 & 3 \\ x_2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\\ 1 & 1 & 1\end{pmatrix}} =$$

$$= \frac{1}{3}\left[\begin{pmatrix}\cos\alpha - \sin\alpha\\ \sin\alpha + \cos\alpha\end{pmatrix}(5 - x_1 - x_2) + \begin{pmatrix}2\cos\alpha - 3\sin\alpha\\ 2\sin\alpha + 3\cos\alpha\end{pmatrix}(2x_2 - x_1 - 1) + \right.$$

$$\left. + \begin{pmatrix}3\cos\alpha - 2\sin\alpha\\ 3\sin\alpha + 2\cos\alpha\end{pmatrix}(2x_1 - x_2 - 1)\right] = x_1\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} + x_2\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

As we can see, rotation matrix was successfully retained. ◀

90 degrees rotation around arbitrary point

Using SAM, find transformation that rotates plane 90° counterclockwise around arbitrary point $(a;b)$.

▶ To use SAM we need 3 points and their images. One point is really easy to obtain — rotation center — affine transformation should map $(a;b)$ to itself, i.e. $(a;b) \mapsto (a;b)$. To get two more points let's consider the following: if I do a step right from $(a;b)$ I get at $(a+1;b)$. After rotation around $(a;b)$ this point should become $(a;b+1)$, thus $(a+1;b) \mapsto (a;b+1)$. The same reasoning yields $(a;b-1) \mapsto (a+1;b)$.

Now we can use SAM

$$\vec{X}(\vec{x}) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix} a \\ b \end{pmatrix} & \begin{pmatrix} a \\ b+1 \end{pmatrix} & \begin{pmatrix} a+1 \\ b \end{pmatrix} \\ x_1 & a & a+1 & a \\ x_2 & b & b & b-1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} a & a+1 & a \\ b & b & b-1 \\ 1 & 1 & 1 \end{pmatrix}} = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix} a \\ b \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ x_1-a & 0 & 1 & 0 \\ x_2-b & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{pmatrix}}{\det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}} =$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(x_1-a) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(b-x_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a+b \\ b-a \end{pmatrix}.$$

◀

Arbitrary transformation

Affine transformation maps triangle with vertices

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

into triangle with vertices

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Find the transformation.

► Using SAM we can immediately write the transformation

$$\vec{X}(x_1;x_2) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix} 3 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 \\ 5 \end{pmatrix} & \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ x_1 & 1 & 2 & 3 \\ x_2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}} =$$

$$= \frac{1}{3}\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}(5-x_1-x_2) + \begin{pmatrix} 1 \\ 5 \end{pmatrix}(2x_2-x_1-1) + \begin{pmatrix} -2 \\ 1 \end{pmatrix}(2x_1-x_2-1)\right] =$$

$$= x_1\begin{pmatrix} -8/3 \\ -5/3 \end{pmatrix} + x_2\begin{pmatrix} 1/3 \\ 7/3 \end{pmatrix} + \begin{pmatrix} 16/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} -8/3 & 1/3 \\ -5/3 & 7/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 16/3 \\ 4/3 \end{pmatrix}.$$

It's easy to verify that input points are correctly mapped into output points. ◀

Degenerate output triangle (degenerate mapping)

Affine transformation maps vertices of the triangle

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

into points

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

respectively. Find the transformation.

Two vertices are mapped into the same point. Show that the obtained affine transformation is degenerate.

▶ Using SAM we can write the affine transformation

$$\vec{X}(x_1;x_2) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}4\\5\end{pmatrix} & \begin{pmatrix}4\\5\end{pmatrix} & \begin{pmatrix}-2\\1\end{pmatrix} \\ x_1 & 1 & 2 & 3 \\ x_2 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 1 & 1\end{pmatrix}} =$$

$$= \frac{1}{3}\left[\begin{pmatrix}4\\5\end{pmatrix}(5 - x_1 - x_2) + \begin{pmatrix}4\\5\end{pmatrix}(2x_2 - x_1 - 1) + \begin{pmatrix}-2\\1\end{pmatrix}(2x_1 - x_2 - 1)\right] =$$

$$= x_1\begin{pmatrix}-4\\-8/3\end{pmatrix} + x_2\begin{pmatrix}2\\4/3\end{pmatrix} + \begin{pmatrix}6\\19/3\end{pmatrix} = \begin{pmatrix}-4 & 2\\-8/3 & 4/3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}16/3\\4/3\end{pmatrix}.$$

To show that transformation is degenerate (i.e. includes projection), we consider rank of he matrix $\begin{pmatrix}-4 & 2\\-8/3 & 4/3\end{pmatrix}$ — that is equal to 1, that means that the plane is transformed into the line. If the rank had been equal to 0, the plane would have been transformed into a single point. ◀

Degenerate input triangle (no mapping)

Show that there is no affine transformation that maps points

$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}$$

into points

$$\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}.$$

Consider using SAM.

▶ The fast answer is "point $\begin{pmatrix}2\\3\end{pmatrix}$ is mapped into two different points that no function is capable of."

But there is the other way to think about this problem. We know that SAM "works" if and only if there exists a unique affine transformation that maps initial points into final, otherwise its denominator is equal to zero and no meaningful result can be obtained. Matrix in the denominator with coordinates of the points plugged in looks as follows

$$\det\begin{pmatrix}2 & 2 & 3\\3 & 3 & 2\\1 & 1 & 1\end{pmatrix} = 0.$$

It's equal to zero, since two columns are equal, thus no such affine mapping exists.

Please note, if the rows had been not simply equal but proportional (or some more complicated form of linear dependence had been spotted), SAM would still have correctly alert us that the given information is not enough to recover the affine mapping unambiguously, while "fast answer" as above would have been not possible. ◀

3D transformations

Mapping a point

Certain affine transformation maps simplex with vertices

$$\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}, \begin{pmatrix}-2\\2\\3\end{pmatrix}$$

into simplex with vertices

$$\begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}-2\\-1\\6\end{pmatrix}, \begin{pmatrix}4\\1\\-3\end{pmatrix}.$$

Find the image of the point $(2,2,2)^{\mathsf{T}}$ under this transformation.

► Using SAM we can immediately write

$$\vec{X}(2;2;2) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}0\\2\\1\end{pmatrix} & \begin{pmatrix}1\\2\\2\end{pmatrix} & \begin{pmatrix}-2\\-1\\6\end{pmatrix} & \begin{pmatrix}4\\1\\-3\end{pmatrix} \\ 2 & 1 & 2 & 3 & -2 \\ 2 & 1 & 3 & 2 & 2 \\ 2 & 2 & 0 & -2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 & -2 \\ 1 & 3 & 2 & 2 \\ 2 & 0 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}} =$$

$$= \frac{1}{15}\left[15\begin{pmatrix}0\\2\\1\end{pmatrix} + 15\begin{pmatrix}1\\2\\2\end{pmatrix} - 9\begin{pmatrix}-2\\-1\\6\end{pmatrix} - 6\begin{pmatrix}4\\1\\3\end{pmatrix}\right] = \begin{pmatrix}-3/5\\-21/5\\-3/5\end{pmatrix}.$$

Worth noting, SAM maps rational points rational points. If one prefers integers, homogeneous coordinates can be used, e.g. we can write obtained vector as $(3, 21, 3, -5)^{\mathsf{T}}$. ◄

Mapping a line

Certain affine transformation maps simplex with vertices

$$\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}, \begin{pmatrix}-2\\2\\3\end{pmatrix}$$

into simplex with vertices

$$\begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}-2\\-1\\6\end{pmatrix}, \begin{pmatrix}4\\1\\-3\end{pmatrix}.$$

Find the image of the line $(t, 0, 0)^{\mathsf{T}}$, where $t \in \mathbb{R}$, under this transformation.

► If line is described as $\vec{\mathcal{L}}(t) = \vec{v}t + \vec{u}$, where $t \in \mathbb{R}$, we can directly plug this expression into SAM (as elements of the first column). In this example we have $\vec{u} = \vec{0}$ and $\vec{v} = (1, 0, 0)^{\mathsf{T}}$, thus SAM should be written as follows

$$\vec{\mathcal{L}}(t) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}0\\2\\1\end{pmatrix} & \begin{pmatrix}1\\2\\2\end{pmatrix} & \begin{pmatrix}-2\\-1\\6\end{pmatrix} & \begin{pmatrix}4\\1\\-3\end{pmatrix} \\ t & 1 & 2 & 3 & -2 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 2 & 0 & -2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 & -2 \\ 1 & 3 & 2 & 2 \\ 2 & 0 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}} =$$

$$= \frac{1}{15}\left[\begin{pmatrix}0\\2\\1\end{pmatrix}(5+5t) + \begin{pmatrix}1\\2\\2\end{pmatrix}(5t-25) + \begin{pmatrix}-2\\-1\\6\end{pmatrix}(23-4t) + \begin{pmatrix}4\\1\\-3\end{pmatrix}(12-6t)\right] =$$

$$= t\begin{pmatrix}11/15\\-18/15\\-9/15\end{pmatrix} + \begin{pmatrix}23/15\\51/15\\-57/15\end{pmatrix}.$$

Worth noting, one could perform a mapping of all sorts of parametrically defined curves, surfaces or hypersurfaces — SAM does not care, how do we set the input vector. ◄

Texturing

Overlay triangular texture on another triangle.

▶ With the help of GLM library for calculation of determinants in SAM one could obtain the result shown in figure 1. The image was not just reshaped in graphical editor — the texture coordinates were actually mapped into rotated and deformed triangles using SAM. ◀

(a) Initial triangle with texture (b) Rotated triangle with texture (c) Deformed triangle with texture

Figure 1: Texturing with SAM

Augmented reality

You develop an augmented reality application that should position a 3D model on top of certain marker as in figure 2. Recognition engine detects a *square* marker printed on a piece of paper and returns 4 coordinates of its corner points in 3D

$$\vec{P}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \quad \vec{P}^{(2)} = \begin{pmatrix} 1+\sqrt{2} \\ 2+2\sqrt{2} \\ 3+2\sqrt{2} \end{pmatrix}; \quad \vec{P}^{(3)} = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix}; \quad \vec{P}^{(4)} = \begin{pmatrix} 1+\sqrt{2} \\ 2\sqrt{2}-1 \\ 6+2\sqrt{2} \end{pmatrix};$$

Assume that the vertices of a loaded 3D model are such that if the marker had had coordinates

$$\vec{p}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \vec{p}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \vec{p}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \vec{p}^{(4)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

there would have been no need to transform the model (identity transformation). Find affine transformation to place the augmented model appropriately. Note that only homogeneous scales are allowed.

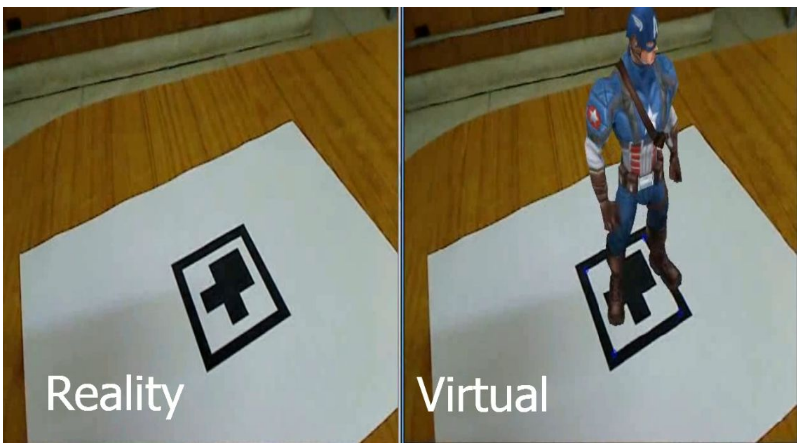


Figure 2: Augmented reality application. Image credits to [2]

▶ The problem is more complicated than previous — we do not have predefined simplexes (all 4 given points lie in a plane), which means we cannot recover affine transformation. On the other hand, we don't need any transformation — rotations, translations, and homogeneous scales would suffice. This leads to the idea of "inventing" the missing point of simplex so that it conformed to our restrictions. A good choice may be to make a cube from the square given (easily defined with the help of cross product) and take 4 of its verices as vertices of the simplex needed. Let's do this with the values given.

First we take 3 corners of the square, say $\vec{x}^{(1)} = \vec{p}^{(1)}$, $\vec{x}^{(2)} = \vec{p}^{(2)}$, and $\vec{x}^{(3)} = \vec{p}^{(3)}$, as is. The fourth point can be "artificially created" as

$$\vec{x}^{(4)} = \vec{p}^{(1)} + \frac{\left[\left(\vec{p}^{(2)} - \vec{p}^{(1)}\right) \times \left(\vec{p}^{(3)} - \vec{p}^{(1)}\right)\right]}{\left\|\left[\left(\vec{p}^{(2)} - \vec{p}^{(1)}\right) \times \left(\vec{p}^{(3)} - \vec{p}^{(1)}\right)\right]\right\|} \left\|\vec{p}^{(2)} - \vec{p}^{(1)}\right\|$$

where $\times$ is the vector product. Note that we suppose $\left\|\vec{p}^{(2)} - \vec{p}^{(1)}\right\| = \left\|\vec{p}^{(3)} - \vec{p}^{(1)}\right\|$, thus define the distance to the fourth point to be the same (think about 3 perpendicular edges of the cube). Thus we can write

$$\vec{x}^{(1)} = \begin{pmatrix}0\\0\\0\end{pmatrix}; \quad \vec{x}^{(2)} = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \vec{x}^{(3)} = \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad \vec{x}^{(4)} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Now we consider codomain. Obviously $\vec{X}^{(1)} = \vec{P}^{(1)}$, $\vec{X}^{(2)} = \vec{P}^{(2)}$, and $\vec{X}^{(3)} = \vec{P}^{(3)}$. The fourth point should be defined the same way as we did for domain

$$\vec{X}^{(4)} = \vec{P}^{(1)} + \frac{\left[\left(\vec{P}^{(2)} - \vec{P}^{(1)}\right) \times \left(\vec{P}^{(3)} - \vec{P}^{(1)}\right)\right]}{\left\|\left[\left(\vec{P}^{(2)} - \vec{P}^{(1)}\right) \times \left(\vec{P}^{(3)} - \vec{P}^{(1)}\right)\right]\right\|} \left\|\vec{P}^{(2)} - \vec{P}^{(1)}\right\|.$$

Here we suppose $\left\|\vec{P}^{(2)} - \vec{P}^{(1)}\right\| = \left\|\vec{P}^{(3)} - \vec{P}^{(1)}\right\|$ and define the distance to the fourth point to be the same. Note that such definition restricts possible scalings.

Now we have in codomain

$$\vec{X}^{(1)} = \begin{pmatrix}1\\2\\3\end{pmatrix}; \quad \vec{X}^{(2)} = \begin{pmatrix}1+\sqrt{2}\\2+2\sqrt{2}\\3+2\sqrt{2}\end{pmatrix}; \quad \vec{X}^{(3)} = \begin{pmatrix}1\\-1\\6\end{pmatrix}; \quad \vec{X}^{(4)} = \begin{pmatrix}5\\1\\2\end{pmatrix};$$

thus application of SAM is straightforward

$$\vec{X}(x_1;x_2;x_3) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}1\\2\\3\end{pmatrix} & \begin{pmatrix}1+\sqrt{2}\\2+2\sqrt{2}\\3+2\sqrt{2}\end{pmatrix} & \begin{pmatrix}1\\-1\\6\end{pmatrix} & \begin{pmatrix}5\\1\\2\end{pmatrix} \\ x_1 & 0 & 1 & 0 & 0 \\ x_2 & 0 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&1&1&1\end{pmatrix}} =$$

$$= (1 - x_1 - x_2 - x_3)\begin{pmatrix}1\\2\\3\end{pmatrix} + x_1\begin{pmatrix}1+\sqrt{2}\\2+2\sqrt{2}\\3+2\sqrt{2}\end{pmatrix} + x_2\begin{pmatrix}1\\-1\\6\end{pmatrix} + x_3\begin{pmatrix}5\\1\\2\end{pmatrix}. \qquad \blacktriangleleft$$

Linear transformations of vector spaces

Linear transformations are very important subset of affine transformations. In this section we will consider them more meticulously, find out that SAM can be simplified a little bit, and use obtained expression to solve systems of linear equations.

General linear transformation

Using SAM, find linear transformation that matches n given linearly independent vectors $\vec{x}^{(i)}$ into another n given vectors $\vec{X}^{(i)}$

$$\vec{x}^{(1)} \mapsto \vec{X}^{(1)}, \ \vec{x}^{(2)} \mapsto \vec{X}^{(2)}, \ \ldots, \ \vec{x}^{(n)} \mapsto \vec{X}^{(n)}.$$

Consider the fact that linear transformation is an affine transformation that maps $\vec{0}$ to $\vec{0}$.

▶ Since all vectors $\vec{x}^{(i)}$ are linearly independent, we can append $\vec{0}$ and treat obtained collection of points as vertices of certain simplex. Images of all $\vec{x}^{(i)}$ are given, while any linear transformation maps $\vec{0}$ to $\vec{0}$, thus we have all information needed to apply SAM

$$\vec{X}(\vec{x}) = (-1)\frac{\det\begin{pmatrix} 0 & \vec{X}^{(1)} & \vec{X}^{(2)} & \dots & \vec{X}^{(n)} & \vec{0} \\ x_1 & x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(2)} & 0 \\ x_2 & x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(2)} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n & x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(2)} & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n)} & 0 \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(n)} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(n)} & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}}.$$

Determinants could be expanded along the last columns.

$$\vec{X}(\vec{x}) = (-1)\frac{\det\begin{pmatrix} 0 & \vec{X}^{(1)} & \vec{X}^{(2)} & \dots & \vec{X}^{(n)} \\ x_1 & x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n)} \\ x_2 & x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(n)} \\ \dots & \dots & \dots & \dots & \dots \\ x_n & x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(n)} \end{pmatrix}}{\det\begin{pmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(n)} \\ \dots & \dots & \dots & \dots \\ x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(n)} \end{pmatrix}}. \tag{1}$$

This formula may also be used to find new coordinates of the point $\vec{x}$ when the basis is changed. ◀

Linear transformation by points

Using result of "General linear transformation," find linear transformation that maps given vectors as follows

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix} \mapsto \begin{pmatrix}3\\1\\1\end{pmatrix}, \begin{pmatrix}5\\-1\\2\end{pmatrix} \mapsto \begin{pmatrix}3\\-2\\-2\end{pmatrix}.$$

▶ Using formula (1) from the previous section one gets

$$\vec{X}(x_1; x_2; x_3) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}1\\2\\0\end{pmatrix} & \begin{pmatrix}3\\1\\1\end{pmatrix} & \begin{pmatrix}3\\-2\\-2\end{pmatrix} \\ x_1 & 1 & 2 & 5 \\ x_2 & 1 & 1 & -1 \\ x_3 & 1 & 2 & 2 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 5 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix}} =$$

$$= \frac{4x_1 + 6x_2 - 7x_3}{3}\begin{pmatrix}1\\2\\0\end{pmatrix} + \frac{6x_3 - 3x_1 - 3x_2}{3}\begin{pmatrix}3\\1\\1\end{pmatrix} + \frac{x_1 - x_3}{3}\begin{pmatrix}3\\-2\\-2\end{pmatrix} =$$

$$= \begin{pmatrix} -2/3 & -1 & 8/3 \\ 1 & 3 & -2 \\ -5/3 & -1 & 8/3 \end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

◀

Linear transformation: matrix (canonical form) to SAM

Using result of "General linear transformation," rewrite linear transformation given as matrix (canonical form)

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

in a form of SAM (determinants ratio). Note that many solutions are possible, try finding the simplest one.

Hint: consider action of $\hat{A}$ on coordinate vectors.

▶ The transformation $\hat{A}$ acts on coordinate vectors yielding its own columns

$$\hat{A} : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \ \hat{A} : \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \ \hat{A} : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix},$$

thus we immediately get all the necessary information to use SAM for linear transformations (1)

$$\hat{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-1) \det \begin{pmatrix} 0 & \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} & \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}.$$

Please note, we have no denominator. This is due to the fact, it had to be determinant of the identity matrix — simple one. ◀

Inverse linear transformation

Using result of "General linear transformation," rewrite inverse of the linear transformation given as matrix (canonical form)

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

in a form of SAM (determinants ratio). Note that many solutions are possible, try finding the simplest one.

Hint: consider action of $\hat{A}$ on coordinate vectors.

▶ The transformation $\hat{A}$ acts on coordinate vectors yielding its own columns

$$\hat{A} : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \ \hat{A} : \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \ \hat{A} : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix},$$

thus inverse transformation does exactly the opposite

$$\hat{A}^{-1} : \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \hat{A}^{-1} : \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \hat{A}^{-1} : \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

thus we immediately get all the necessary information to use SAM for linear transformations (1)

$$\hat{A}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-1) \frac{\det \begin{pmatrix} 0 & \vec{i} & \vec{j} & \vec{k} \\ x & a_{11} & a_{12} & a_{13} \\ y & a_{21} & a_{22} & a_{23} \\ z & a_{31} & a_{32} & a_{33} \end{pmatrix}}{\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}, \tag{2}$$

where $\vec{i} = (1;0;0)^{\mathsf{T}}$, $\vec{j} = (0;1;0)^{\mathsf{T}}$, and $\vec{k} = (0;0;1)^{\mathsf{T}}$. ◀

Inverse matrix

Using results of "Inverse linear transformation," find inverse of

$$\hat{A} = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{pmatrix}.$$

Hint: consider action of $\hat{A}^{-1}$ on arbitrary vector $\vec{x} = (x; y; z)^\mathsf{T}$.

► We can use (2) and immediately write the result of action of $\hat{A}^{-1}$ on arbitrary vector $\vec{x}$

$$\hat{A}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-1)\frac{\det\begin{pmatrix} 0 & \vec{i} & \vec{j} & \vec{k} \\ x & 1 & 0 & 5 \\ y & 2 & 1 & 6 \\ z & 3 & 4 & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{pmatrix}} = \left(-24\vec{i} + 18\vec{j} + 5\vec{k}\right)x + \left(20\vec{i} - 15\vec{j} - 4\vec{k}\right)y +$$

$$+\left(-5\vec{i} + 4\vec{j} + \vec{k}\right)z = \begin{pmatrix} -24 \\ 18 \\ 5 \end{pmatrix} x + \begin{pmatrix} 20 \\ -15 \\ -4 \end{pmatrix} y + \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} z = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\vec{i} = (1; 0; 0)^\mathsf{T}$, $\vec{j} = (0; 1; 0)^\mathsf{T}$, and $\vec{k} = (0; 0; 1)^\mathsf{T}$. Thus inverse matrix $\hat{A}^{-1}$ can be written as

$$\hat{A}^{-1} = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & 4 & 1 \end{pmatrix}.$$

◄

System of linear equations

Cramer's rule

Using results of section "Inverse linear transformation," solve the following linear equation

$$\hat{A}\vec{x} = \vec{b}$$

in terms of SAM. Consider 3D case for simplicity

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

► Solution of linear equation

$$\hat{A}\vec{x} = \vec{b}$$

(if one exists and is unique) can be written as

$$\vec{x} = \hat{A}^{-1}\vec{b}.$$

To write it in terms of SAM we can use equation (2) (section "Inverse linear transformation") and plug in $\vec{b}$

$$\vec{x} = A^{-1}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (-1)\frac{\det\begin{pmatrix} 0 & \vec{i} & \vec{j} & \vec{k} \\ b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{21} & a_{22} & a_{23} \\ b_3 & a_{31} & a_{32} & a_{33} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}, \tag{3}$$

where $\vec{i} = (1; 0; 0)^\mathsf{T}$, $\vec{j} = (0; 1; 0)^\mathsf{T}$, and $\vec{k} = (0; 0; 1)^\mathsf{T}$. The latter is Cramer's rule written in a more compact form. ◄

System of linear equations

Using results of section “Cramer’s rule”, find solution to the following system of linear equations

$$\begin{cases} -2x + 3y + z = 2, \\ -x - y + 4z = 1, \\ x + y + z = 1. \end{cases}$$

► Consider the equation in matrix form

$$\begin{pmatrix} -2 & 3 & 1 \\ -1 & -1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Using formula (3) from section “Cramer’s rule,” we can write its solution as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-1)\frac{\det \begin{pmatrix} 0 & \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 3 & 1 \\ 1 & -1 & -1 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det \begin{pmatrix} -2 & 3 & 1 \\ -1 & -1 & 4 \\ 1 & 1 & 1 \end{pmatrix}} = \frac{1}{25}\vec{i} + \frac{14}{25}\vec{j} + \frac{2}{5}\vec{k} = \begin{pmatrix} 1/25 \\ 14/25 \\ 2/5 \end{pmatrix},$$

thus $x = 1/25$, $y = 14/25$, and $z = 2/5$. ◄

Initial conditions of linear homogeneous ordinary differential equation

Initial conditions of linear homogeneous ordinary differential equation: general case

Suppose that $\{y_1, y_2 \dots y_n\}$ is a system of linearly independent functions that are solutions of linear homogeneous ordinary differential equation. Find a function y that is a linear combination of these functions that match initial conditions

$$\begin{aligned} y(x_0) &= y_0, \\ y'(x_0) &= y_0', \\ &\dots \\ y^{(n-1)}(x_0) &= y_0^{(n-1)}. \end{aligned}$$

Hint: consider $y(x)$ not as function of x, rather as linear function acting on $y_1(x), \dots, y_n(x)$; use SAM.

► In essence everything boils down to obtaining such coefficients α that

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x).$$

Let’s treat y as the following linear transformation

$$y(y_1(x); y_2(x); \dots; y_n(x)) = \alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x),$$

that acts from n-dimensional space to $\mathbb{R}$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} \mapsto \alpha_1 \xi_1 + \alpha_2 \xi_2 + \dots + \alpha_n \xi_n.$$

Moreover, we know its action on handful of points

$$\begin{pmatrix} y_1(x_0) \\ y_2(x_0) \\ \dots \\ y_n(x_0) \end{pmatrix} \mapsto y_0, \quad \begin{pmatrix} y_1'(x_0) \\ y_2'(x_0) \\ \dots \\ y_n'(x_0) \end{pmatrix} \mapsto y_0', \quad \dots, \quad \begin{pmatrix} y_1^{(n-1)}(x_0) \\ y_2^{(n-1)}(x_0) \\ \dots \\ y_n^{(n-1)}(x_0) \end{pmatrix} \mapsto y_0^{(n-1)},$$

that allows us immediately obtain the solution with SAM

$$y(x) = (-1)\frac{\det\begin{pmatrix} 0 & y_0 & y_0' & \dots & y_0^{(n-1)} \\ y_1(x) & y_1(x_0) & y_1'(x_0) & \dots & y_1^{(n-1)}(x_0) \\ y_2(x) & y_2(x_0) & y_2'(x_0) & \dots & y_2^{(n-1)}(x_0) \\ \dots & \dots & \dots & \dots & \dots \\ y_n(x) & y_n(x_0) & y_n'(x_0) & \dots & y_n^{(n-1)}(x_0) \end{pmatrix}}{\det\begin{pmatrix} y_1(x_0) & y_1'(x_0) & \dots & y_1^{(n-1)}(x_0) \\ y_2(x_0) & y_2'(x_0) & \dots & y_2^{(n-1)}(x_0) \\ \dots & \dots & \dots & \dots \\ y_n(x_0) & y_n'(x_0) & \dots & y_n^{(n-1)}(x_0) \end{pmatrix}}. \tag{4}$$

The determinant in the denominator up to the sign is equal to the Wronskian determinant. ◀

Solution of linear homogeneous ordinary differential equation

Solve differential equation

$$y''(x) = y(x),$$

use results of "Initial conditions of linear homogeneous ordinary differential equation: general case" section to satisfy initial conditions

$$y(0) = 0,$$
$$y'(0) = 1.$$

▶ One can easily integrate the equation and find its fundamental system of solutions — $\{e^x, e^{-x}\}$. Using (4) from "Initial conditions of linear homogeneous ordinary differential equation: general case" section it is easy to obtain

$$y(x) = (-1)\frac{\det\begin{pmatrix} 0 & 0 & 1 \\ e^x & 1 & 1 \\ e^{-x} & 1 & -1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = \frac{e^x - e^{-x}}{2} = \sinh(x).$$

◀

Change of basis

New basis by two known vectors

Consider three vectors

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

After the basis was changed, coordinates of $\vec{u}$ and $\vec{v}$ become

$$\vec{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}', \vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}',$$

where primes designate that coordinates are given in a new basis. Find coordinates of vector $\vec{x}$ in the new basis.

▶ Changing basis is actually a linear transformation. Thus we should find the linear transformation that makes from coordinates of $\vec{u}$ and $\vec{v}$ in old basis their coordinates in the new basis. Using SAM we can write

$$\vec{x} = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix} 3 \\ 0 \end{pmatrix}' & \begin{pmatrix} 4 \\ 3 \end{pmatrix}' \\ 2 & 1 & -2 \\ 3 & 2 & 3 \end{pmatrix}}{\det\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}} = \begin{pmatrix} 32/7 \\ -3/7 \end{pmatrix}'.$$

◀

New basis by basis vectors transformation

Suppose basis vectors are changed so that new basis in old coordinates have following components

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}.$$

Find coordinates of vector $(3, 4)^{\mathsf{T}}$ in a new basis.

▶ In a new coordinate system vectors

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}$$

will become orths with "new" coordinates

$$\begin{pmatrix}1\\0\end{pmatrix}', \begin{pmatrix}0\\1\end{pmatrix}'.$$

As change of coordinates is a linear transformation we can use SAM.

$$\vec{x} = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}1\\0\end{pmatrix}' & \begin{pmatrix}0\\1\end{pmatrix}' \\ 3 & 1 & 2 \\ 4 & 1 & 0 \end{pmatrix}}{\det\begin{pmatrix}1 & 2\\1 & 0\end{pmatrix}} = \begin{pmatrix}4\\-1/2\end{pmatrix}'.$$

◀

Interpolation problems

Looking on SAM from a different point of view, we may notice it can perform multilinear interpolation of vectors. In some sense it means we map from "geometric space" to some very different linear space, but the only thing we need is to obtain a vector or number out of there. Such problems often appear in computer graphics in form of colors-normals-whatever interpolation.

Interpolating color across the triangle

Consider triangle with vertices

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}.$$

A color is assigned to each vertex: yellow, cyan, and magenta respectively. Perform multilinear interpolation of color within the triangle; RGB coordinates of the colors mentioned are as follows

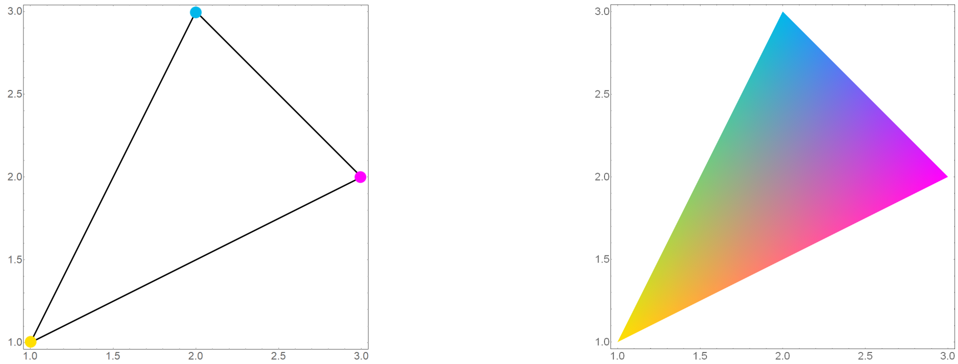
$${\color{yellow}\bullet} = \begin{pmatrix}255\\223\\0\end{pmatrix}, {\color{cyan}\bullet} = \begin{pmatrix}0\\183\\235\end{pmatrix}, {\color{magenta}\bullet} = \begin{pmatrix}255\\0\\255\end{pmatrix}.$$

▶ In essence we are looking for an affine transformation between the two triangles, but this time one triangle "lives" in color space, while the other is an ordinary "geometric" triangle. SAM can be used to obtain the explicit form of the transformation

$$\vec{C}(x; y) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix}255\\223\\0\end{pmatrix} & \begin{pmatrix}0\\183\\235\end{pmatrix} & \begin{pmatrix}255\\0\\255\end{pmatrix} \\ x & 1 & 2 & 3 \\ y & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\\1 & 1 & 1\end{pmatrix}} = \left[\begin{pmatrix}255\\223\\0\end{pmatrix}(5 - x - y) + \right.$$

$$+\begin{pmatrix}0\\183\\235\end{pmatrix}(2y-x-1)+\begin{pmatrix}255\\0\\255\end{pmatrix}(2x-y-1)\Bigg] = x\begin{pmatrix}255/3\\-406/3\\275/3\end{pmatrix}+y\begin{pmatrix}-510/3\\143/3\\215/3\end{pmatrix}+\begin{pmatrix}1020/3\\932/3\\-490/3\end{pmatrix}.$$

Vector $\vec{\mathcal{C}}(x;y)$ contains three components— RGB color triplet of the point (x,y). Please note, how interpolation is hidden under the hood of the mapping spaces. If one performs visualisation, it should look as in figure 3.



(a) Initial triangle with colors assigned to vertices

(b) Color interpolated across the triangle

Figure 3: Interpolation of color across the triangle

◀

Interpolating normals across the triangle

Consider triangle with vertices

$$\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}.$$

A unit vector is assigned to each vertex

$$\vec{\mathcal{N}}_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix},\ \vec{\mathcal{N}}_2=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\\0\end{pmatrix},\ \vec{\mathcal{N}}_3=\begin{pmatrix}0\\0\\-1\end{pmatrix}.$$

Perform multilinear interpolation of these vectors within the triangle.

⋆ **Technical note.** *The problem of linear interpolation of vectors arises in computer graphics in the context of Phong shading (a.k.a. normal-vector interpolation shading). Specifically, this shading algorithm linearly interpolates surface normals across rasterized polygons (often triangles), normalizes the result, and computes pixels' colors based on the reflection model and interpolated normals. Application of Phong shading looks as in figure 4.*

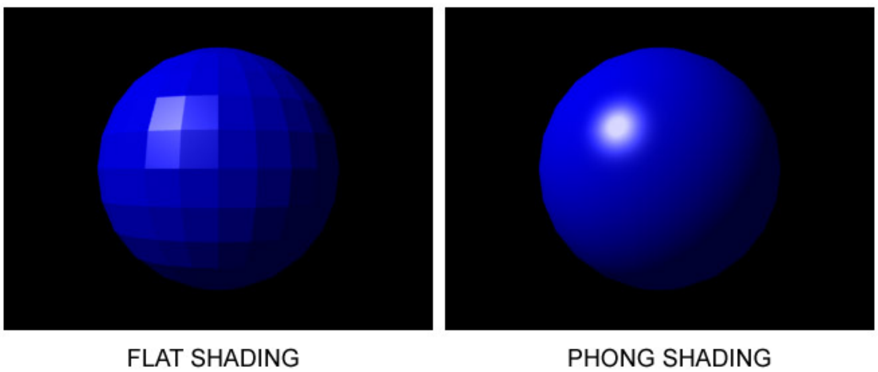


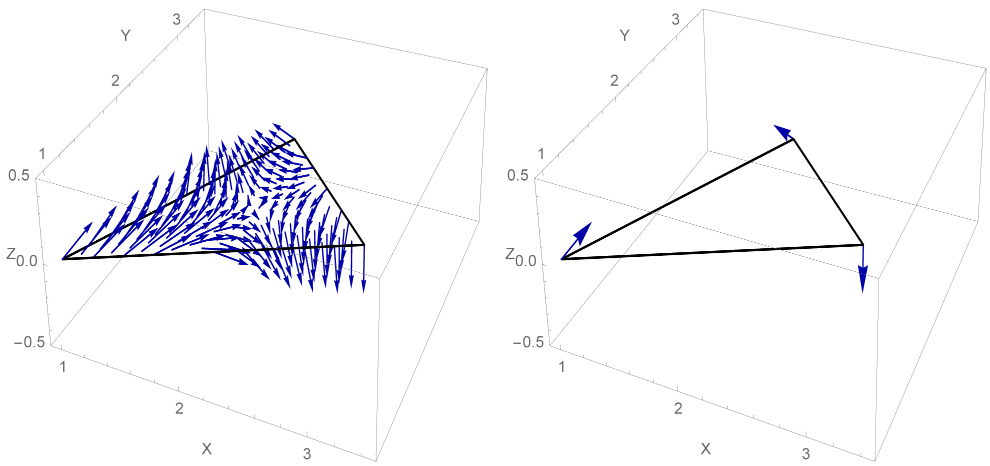
Figure 4: Phong shading. Image credits to [3]

▶ As in the previous example we map two triangles — one is "geometric", while the other "lives" in the space of "arrows". We use SAM as usual

$$\vec{n}(x;y) = (-1)\frac{\det\begin{pmatrix} 0 & \vec{\mathcal{N}}_1 & \vec{\mathcal{N}}_2 & \vec{\mathcal{N}}_3 \\ x & 1 & 2 & 3 \\ y & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}} = \frac{1}{3}\left[\vec{\mathcal{N}}_1(5-x-y) + \vec{\mathcal{N}}_2(2y-x-1) + \vec{\mathcal{N}}_3(2x-y-1)\right] \approx$$

$$\approx x\begin{pmatrix} 0.106 \\ -0.342 \\ -0.859 \end{pmatrix} + y\begin{pmatrix} -0.789 \\ 0.106 \\ 0.141 \end{pmatrix} + \begin{pmatrix} 1.26 \\ 0.813 \\ 1.30 \end{pmatrix}.$$

The result should look like in the figure 5.



(a) Triangle with normals in vertices (b) Normals interpolated across the triangle

Figure 5: Interpolation of normals (scaled for better visuals) across the triangle

Please note that vector $\vec{n}(x;y)$ may be non-unitary. To use it for Phong interpolation, normalization is required

$$\vec{\mathcal{N}}(x;y) = \frac{\vec{n}(x;y)}{\|\vec{n}(x;y)\|}.$$ ◀

Lagrange interpolation

Lagrange polynomial formula

Consider a polynomial

$$P(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0$$

as a linear transformation that maps vector $(x^n; x^{n-1}; \ldots; 1)^T$ to $\mathbb{R}$. Given that $P(x)$ passes through points $(a_0; b_0)$, $(a_1; b_1)$, …, $(a_n; b_n)$, find expression for $P(x)$. Use results of "General linear transformation" section.

⋆ **Technical note.** *As a result we will get the Lagrange interpolation of the given points. The proof that obtained polynomial is indeed the Lagrange interpolation can be found in [1].*

▶ Let's treat P not as function of x but as linear transformation

$$P(x^n; x^{n-1}; \ldots; 1) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0,$$

that acts from $n+1$-dimensional space to $\mathbb{R}$

$$\begin{pmatrix} \xi_n \\ \xi_{n-1} \\ \cdots \\ \xi_0 \end{pmatrix} \mapsto \alpha_n\xi_n + \alpha_{n-1}\xi_{n-1} + \cdots + \alpha_1\xi_1 + \alpha_0\xi_0.$$

Moreover, we know its action on handful of points

$$\begin{pmatrix} a_0^n \\ a_0^{n-1} \\ \cdots \\ 1 \end{pmatrix} \mapsto b_0, \quad \begin{pmatrix} a_1^n \\ a_1^{n-1} \\ \cdots \\ 1 \end{pmatrix} \mapsto b_1, \cdots, \begin{pmatrix} a_n^n \\ a_n^{n-1} \\ \cdots \\ 1 \end{pmatrix} \mapsto b_n,$$

that allows us immediately obtain the solution with (1) (from section "General linear transformation")

$$P(x) = (-1)\frac{\det\begin{pmatrix} 0 & b_0 & b_1 & \cdots & b_n \\ x^n & a_0^n & a_1^n & \cdots & a_n^n \\ x^{n-1} & a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}.$$

The matrix in the numerator can be expanded along the first row to get the classical form of the Lagrange polynomial. ◀

Second-degree Lagrange interpolation

Find second-degree polynomial $P(x)$ that interpolates following points: $P(-1) = 2$, $P(3) = 4$, and $P(2) = 7$. Use the result from the "Lagrange polynomial formula" section.

▶ Using result from the "Lagrange polynomial formula" section, we can immediately write

$$P(x) = (-1)\frac{\det\begin{pmatrix} 0 & 2 & 4 & 7 \\ x^2 & (-1)^2 & 3^2 & 2^2 \\ x & -1 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} (-1)^2 & 3^2 & 2^2 \\ -1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}} = -\frac{7}{6}x^2 + \frac{17}{6}x + 6.$$

For visuals see figure 6. ◀

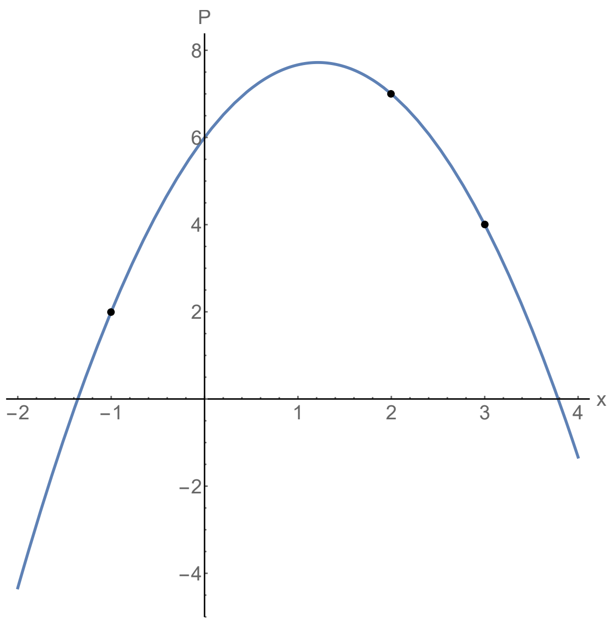


Figure 6: Polynomial interpolation.

Coefficients of Lagrange polynomials

Consider result of the "Lagrange polynomial formula" section as an expression for Lagrange polynomial

$$P(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0$$

that interpolates $(a_0; b_0)$, ..., $(a_n; b_n)$. Find explicit expression for its coefficients.

► Due to "Lagrange polynomial formula" section, Lagrange polynomial that interpolates $(a_0; b_0)$, ..., $(a_n; b_n)$ can be written as

$$P(x) = (-1)\frac{\det\begin{pmatrix} 0 & b_0 & b_1 & \cdots & b_n \\ x^n & a_0^n & a_1^n & \cdots & a_n^n \\ x^{n-1} & a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}.$$

Now we can perform Laplace expansion along the first column and see that

$$\alpha_i = (-1)^{n-i}\frac{\det\begin{pmatrix} b_0 & b_1 & \cdots & b_n \\ a_0^n & a_1^n & \cdots & a_n^n \\ \cdots & \cdots & \cdots & \cdots \\ a_0^{i+1} & a_1^{i+1} & \cdots & a_n^{i+1} \\ a_0^{i-1} & a_1^{i-1} & \cdots & a_n^{i-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}},$$

where α_i is the coefficient at the x^i in the polynomial. ◄

Sum of the Lagrange basis polynomials

Consider Lagrange interpolation polynomial that passes through points $(a_0; b_0)$, ..., $(a_n; b_n)$ as a linear combination of Lagrange basis polynomials $L_i(x)$

$$P(x) = \sum_{i=0}^{n} b_i L_i(x).$$

Using results of the "Lagrange polynomial formula" section, show that sum of Lagrange basis polynomials

$$\sum_{i=0}^{n} L_i(x) = 1.$$

► To get the sum of interest, let us set set all $b_i = 1$ in the general expression for the Lagrange polynomial (result from the section "Lagrange polynomial formula")

$$P(x) = \sum_{i=0}^{n} \underbrace{b_i}_{=1} L_i(x) = \sum_{i=0}^{n} L_i(x) = (-1)\frac{\det\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ x^n & a_0^n & a_1^n & \cdots & a_n^n \\ x^{n-1} & a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}},$$

than perform simplification through Laplace expansion along the first row

$$P(x) = (-1)\frac{\det\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ x^n & a_0^n & a_1^n & \cdots & a_n^n \\ x^{n-1} & a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}} = \frac{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}} = 1$$

that completes the proof. ◄

Weighted sum of basis Lagrange polynomials

Consider Lagrange interpolation polynomial that passes through points $(a_0; b_0)$, …, $(a_n; b_n)$ as a linear combination of Lagrange basis polynomials $L_i(x)$

$$P(x) = \sum_{i=0}^{n} b_i L_i(x).$$

Using results of the “Lagrange polynomial formula” section, show that the weighted sum of the Lagrange basis polynomials at $x = 0$

$$\sum_{i=0}^{n} L_i(0) a_i^{n+1} = (-1)^n a_0 \cdot \dots \cdot a_n.$$

► To get the sum of interest, let us set set all $b_i = a_i^{n+1}$ in the general expression for the Lagrange polynomial (result from the section “Lagrange polynomial formula”)

$$P(0) = (-1)\frac{\det\begin{pmatrix} 0 & a_0^{n+1} & a_1^{n+1} & \cdots & a_n^{n+1} \\ 0 & a_0^n & a_1^n & \cdots & a_n^n \\ 0 & a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}} = (-1)^n \frac{\det\begin{pmatrix} a_0^{n+1} & a_1^{n+1} & \cdots & a_n^{n+1} \\ a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_0 & a_1 & \cdots & a_n \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}$$

and use properties of determinant

$$P(0) = (-1)^n a_0 a_1 \dots a_n \frac{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} a_0^n & a_1^n & \cdots & a_n^n \\ a_0^{n-1} & a_1^{n-1} & \cdots & a_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}} = (-1)^n a_0 a_1 \dots a_n.$$

The latter finishes the proof. ◄

Lagrange nodal basis

Consider n-th degree polynomial $q(x)$ and its values at $n+1$ distinct points a_i. Using results of the “Lagrange polynomial formula” section, show that it is equal to the Lagrange polynomial interpolating points $(a_0; q(a_0))$, …, $(a_n; q(a_n))$.

Note: q is the only polynomial of a degree less than or equal to n that passes through $(a_0; q(a_0))$, …, $(a_n; q(a_n))$. This fact can be proven by contradiction: consider polynomial q' of a degree $\leq n$ that passes through the same points. Difference $q(x) - q'(x)$ should be polynomial of a degree $\le n$ that has zeros at a_0, …, a_n, but it can't have $n+1$ zeros thus contradiction.

▶ Using results of the "Lagrange polynomial formula" section, we can write expression for the Lagrange polynomial that passes through $(a_0; q(a_0))$, …, $(a_n; q(a_n))$

$$P(x) = (-1)\frac{\det\begin{pmatrix} 0 & q(x_0) & q(x_1) & \cdots & q(x_n) \\ x^n & x_0^n & x_1^n & \cdots & x_n^n \\ x^{n-1} & x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} x_0^n & x_1^n & \cdots & x_n^n \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}.$$

Now I can subtract from the very first row a linear combination of all the rest rows, sot that its entries (except the left-most) become zero. One such combination certainly exists — I can multiply rows by coefficients of q. Moreover, this is the solely linear combination with such a property — no other polynomial of a degree $\le n$ that has values $q(a_i)$ at a_i. The latter means, we will recover $-q$ at the left-most entry

$$P(x) = (-1)\frac{\det\begin{pmatrix} -q(x) & 0 & 0 & \cdots & 0 \\ x^n & x_0^n & x_1^n & \cdots & x_n^n \\ x^{n-1} & x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} x_0^n & x_1^n & \cdots & x_n^n \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}} = q(x)\frac{\det\begin{pmatrix} x_0^n & x_1^n & \cdots & x_n^n \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} x_0^n & x_1^n & \cdots & x_n^n \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}} = q(x).$$

The latter finishes the proof. ◀

Trigonometric interpolation

Consider function

$$F(x) = \alpha_1 \sin(x) + \alpha_2 \cos(x) + \alpha_3 \sin(2x) + \alpha_4 \cos(2x) + \alpha_5$$

and find such coefficients α_1, …, α_5 that it passes through following points

$$F(0) = 1,\ F\left(\frac{\pi}{4}\right) = 5,\ F\left(\frac{\pi}{2}\right) = 3,\ F(\pi) = 3,\ F\left(\frac{3\pi}{2}\right) = 1.$$

Hint: consider F not as a function of x, rather as a linear function acting on 1, $\sin(x)$, $\cos(x)$, etc; use results of "General linear transformation" section.

▶ Let's treat P not as function of x but as linear transformation

$$F(\sin(x); \cos(x); \sin(2x); \cos(2x); 1) = \alpha_1 \sin(x) + \alpha_2 \cos(x) + \alpha_3 \sin(2x) + \alpha_4 \cos(2x) + \alpha_5,$$

that acts from 5-dimensional space to $\mathbb{R}$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix} \mapsto \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 + \alpha_4\xi_4 + \alpha_5\xi_5.$$

Moreover, we know its action on handful of points

$$\begin{pmatrix} \sin(0) \\ \cos(0) \\ \sin(0) \\ \cos(0) \\ 1 \end{pmatrix} \mapsto 1;\quad \begin{pmatrix} \sin(\pi/4) \\ \cos(\pi/4) \\ \sin(\pi/2) \\ \cos(\pi/2) \\ 1 \end{pmatrix} \mapsto 5;\quad \begin{pmatrix} \sin(\pi/2) \\ \cos(\pi/2) \\ \sin(\pi) \\ \cos(\pi) \\ 1 \end{pmatrix} \mapsto 3;\quad \begin{pmatrix} \sin(\pi) \\ \cos(\pi) \\ \sin(2\pi) \\ \cos(2\pi) \\ 1 \end{pmatrix} \mapsto 3;\quad \begin{pmatrix} \sin(3\pi/2) \\ \cos(3\pi/2) \\ \sin(3\pi) \\ \cos(3\pi) \\ 1 \end{pmatrix} \mapsto 1,$$

thus we can evaluate all these sines-cosines and imply (1) (from section "General linear transformation")

$$F(x) = (-1)\frac{\det\begin{pmatrix} 0 & 1 & 5 & 3 & 3 & 1 \\ \sin(x) & 0 & 1/\sqrt{2} & 1 & 0 & -1 \\ \cos(x) & 1 & 1/\sqrt{2} & 0 & -1 & 0 \\ \sin(2x) & 0 & 1 & 0 & 0 & 0 \\ \cos(2x) & 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 0 & 1/\sqrt{2} & 1 & 0 & -1 \\ 1 & 1/\sqrt{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}} = \sin(x) + 3\sin(2x) - \cos(x) + 2.$$

For visuals see figure 7. Worth noting, presented approach can be used for DFT. ◄

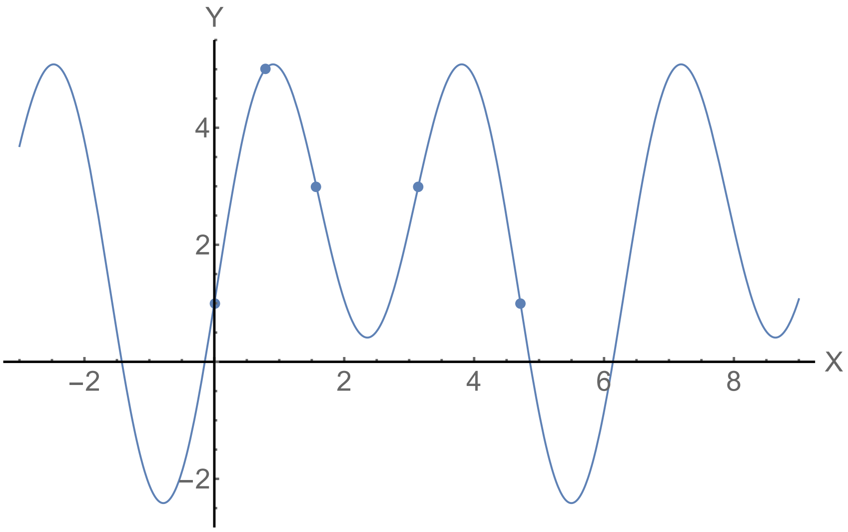


Figure 7: Trigonometric interpolation.

Barycentric coordinates

Barycentric coordinates of a point

Find barycentric coordinates of a point $(2, 2)^{\intercal}$ with respect to triangle with vertices

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

► As we know, plugging any orthonormal vectors (at least formally) as the codomain' points, SAM produces vector, whose entries are barycentric coordinates of the point with respect to domain simplex (please note how the first row is formed)

$$\vec{X}(2;2) = (-1)\frac{\det\begin{pmatrix} 0 & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ 2 & 1 & 4 & 2 \\ 2 & 1 & 3 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 4 & 2 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{pmatrix}} = \frac{1}{10}\left[6\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 3/5 \\ 3/10 \\ 1/10 \end{pmatrix}.$$

As a result barycentric coordinates of the point $(2, 2)^{\intercal}$ are 0.6, 0.3, and 0.1. ◄

Point inside a simplex

Verify that the point $(2,2)^\mathsf{T}$ is contained within the triangle with vertices

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}.$$

▶ For some point to be inside a simplex, its barycentric coordinates with respect to that simplex should all be positive and less than 1. Thus the problem is reduced to finding barycentric coordinates of the point $(2,2)^\mathsf{T}$ with respect to the triangle given and verification that they satisfy restrictions.

Using results of the "Barycentric coordinates of a point" problem, we state that barycentric coordinates, we are interested in, are equal to 0.6, 0.3, and 0.1. Since $0 < 0.6 < 1$, $0 < 0.3 < 1$, and $0 < 0.1 < 1$ we conclude that the point $(2,2)^\mathsf{T}$ is inside the triangle. ◀

View direction

Consider simplex with vertices $\vec{x}^{(1)}, \ldots, \vec{x}^{(n+1)}$. An ant sits at the vertex $\vec{x}^{(1)}$ and looks along direction $\vec{v}$. Does it look inside the simplex? (The problem is easy for ants, but we should do some calculations.)

▶ The question can be answered if we know whether the ray of view punches the opposite face of the simplex. We can gain this information by considering appropriate line in barycentric coordinates and solving for first coordinate to be zero (it corresponds to the simplex' face on the other side of $\vec{x}^{(1)}$).

As we know, plugging any orthonormal vectors (at least formally) as the codomain points, SAM produces vector, whose entries are barycentric coordinates of the point with respect to domain simplex. Thus equation of the "line of view" can be written as

$$\vec{L}(t) = (-1)\frac{\det\begin{pmatrix} 0 & \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_{n+1} \\ x_1^{(1)} + v_1 t & x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n+1)} \\ x_2^{(1)} + v_2 t & x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n^{(1)} + v_n t & x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n+1)} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} x_1^{(1)} & x_1^{(2)} & & x_1^{(n+1)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(n+1)} \\ \cdots & \cdots & & \cdots \\ x_n^{(1)} & x_n^{(2)} & & x_n^{(n+1)} \\ 1 & 1 & \cdots & 1 \end{pmatrix}},$$

where $\vec{x}^{(1)} + \vec{v}t$ is supposed to be line of view.

We perform Laplace expansion along first row and select summand with $\vec{e}_1$ — the one that corresponds to the face of interest. We demand line of view to intersect plane containing this face, thus set $L_1(t) = 0$ and solve for t. As a result we get

$$t = (-1)\frac{\det\begin{pmatrix} x_1^{(1)} & x_1^{(2)} & & x_1^{(n+1)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(n+1)} \\ \cdots & \cdots & & \cdots \\ x_n^{(1)} & x_n^{(2)} & & x_n^{(n+1)} \\ 1 & 1 & \cdots & 1 \end{pmatrix}}{\det\begin{pmatrix} v_1 & x_1^{(2)} & & x_1^{(n+1)} \\ v_2 & x_2^{(2)} & \cdots & x_2^{(n+1)} \\ \cdots & \cdots & & \cdots \\ v_n & x_n^{(2)} & & x_n^{(n+1)} \\ 0 & 1 & \cdots & 1 \end{pmatrix}}.$$

If the determinant in denominator equals zero, line of view newer crosses plane containing the face of interest, thus the answer is "NO" immediately. If $t \leq 0$ intersection exists, but the ant looks exactly opposite direction and can't see the face as well — answer "NO" once again. If $t > 0$ intersection exists at the point

$$\vec{x} = t\vec{v} + \vec{x}^{(1)},$$

but this point can be either on the face of interest, or somewhere else on the plain containing this face. We can check this by considering barycentric coordinates of $\vec{x}$

$$\vec{\Lambda} = (-1)\frac{\det\begin{pmatrix} 0 & \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_{n+1} \\ x_1 & x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n+1)} \\ x_2 & x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(n+1)} \\ \dots & \dots & \dots & \dots & \dots \\ x_n & x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(n+1)} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}}{\det\begin{pmatrix} x_1^{(1)} & x_1^{(2)} & & x_1^{(n+1)} \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(n+1)} \\ \dots & \dots & & \dots \\ x_n^{(1)} & x_n^{(2)} & & x_n^{(n+1)} \\ 1 & 1 & \dots & 1 \end{pmatrix}}.$$

We expect $\Lambda_1 = 0$ and $0 < \Lambda_{i>1} < 1$ for the answer to be “YES”, otherwise “NO.” ◀

View direction: example

Consider triangle with vertices $(0;0)^\mathsf{T}$, $(0;1)^\mathsf{T}$, and $(1;0)^\mathsf{T}$. If one stands at the vertex $(0;0)^\mathsf{T}$ and looks toward $(1;1)^\mathsf{T}$, is he looking inside the triangle? Use results of “View direction” to find the answer, verify the result with picture.

▶ We use algorithm presented in section “View direction.” First, we calculate parameter t

$$t = (-1)\frac{\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}} = \frac{1}{2}.$$

Since $t > 0$ there is a chance one looks inside, but we cannot tell definitely. Let’s find the only possible point of intersection with the plane that contains face of interest

$$\vec{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

and check its barycentric coordinates

$$\vec{\Lambda} = (-1)\frac{\det\begin{pmatrix} 0 & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} = 0\vec{e}_1 + \frac{1}{2}\vec{e}_2 + \frac{1}{2}\vec{e}_3.$$

The latter means, one looks inside the simplex. Sketching a picture is left to the Reader as an exercise. ◀

Miscellaneous geometrical problems

A line by two points

Find any parametric equation of the line that passes through two points $\vec{a}$ and $\vec{b}$, try to use SAM.

▶ One of the possible solutions is to take any easy-to-define parametrization of line and affinely map it so that it passes through the points of interest. Line remains line under affine transformation, thus we will get a valid result.

To find the affine mapping from parameter space $\mathbb{R}$ to $\mathbb{R}^2$ we define its action on two points, say $0 \mapsto \vec{a}$, $1 \mapsto \vec{b}$, while SAM is the Johnny-on-the-spot to do the rest

$$\vec{\mathcal{L}}(t) = (-1)\frac{\det\begin{pmatrix}0 & \vec{a} & \vec{b}\\ t & 0 & 1\\ 1 & 1 & 1\end{pmatrix}}{\det\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}} = (1-t)\vec{a} + t\vec{b} = \vec{a} + t(\vec{b} - \vec{a}).$$

As we can see, $\vec{b} - \vec{a}$ defines the direction of the line. If we consider different from 0 and 1 values for initial points, we will get different parametric representations of the same line. ◀

Plane by three points I

Three points in 3D space are given

$$\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}.$$

Find parametric equation of the plane that passes through these points in form $z = z(x; y)$.

▶ We can reformulate problem as follows: find affine mapping $(x; y)^\mathsf{T} \to (x; y; z(x; y))^\mathsf{T}$ such that the resulting plane passes through given points. We need affine mapping for the $\mathbb{R}^2$ plane to still remain plane, thus the problem will be solved.

We have 3 codomain points but since the transformation should be $(x; y)^\mathsf{T} \to (x; y; z(x; y))^\mathsf{T}$, the domain points are easily recovered. Thus we need affine transformation that maps

$$\begin{pmatrix}1\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix} \mapsto \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix} \mapsto \begin{pmatrix}3\\2\\-2\end{pmatrix}.$$

Now we can use SAM

$$\vec{\mathcal{P}}(x; y) = (-1)\frac{\det\begin{pmatrix}0 & \begin{pmatrix}1\\1\\2\end{pmatrix} & \begin{pmatrix}2\\3\\1\end{pmatrix} & \begin{pmatrix}3\\2\\-2\end{pmatrix}\\ x & 1 & 2 & 3\\ y & 1 & 3 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}}{\det\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\\ 1 & 1 & 1\end{pmatrix}} =$$

$$= \frac{1}{3}\left[\begin{pmatrix}1\\1\\2\end{pmatrix}(-x - y + 5) + \begin{pmatrix}2\\3\\1\end{pmatrix}(-x + 2y - 1) + \begin{pmatrix}3\\2\\-2\end{pmatrix}(2x - y - 1)\right] = \begin{pmatrix}x\\ y\\ -\frac{7}{3}x + \frac{2}{3}y + \frac{11}{3}\end{pmatrix}.$$

From the formula above we read off the expression for $z(x, y)$

$$z(x, y) = -\frac{7}{3}x + \frac{2}{3}y + \frac{11}{3}.$$

The result is visualized in figure 8. Please note that codomain points are orthogonal projections of domain points.

Plane by three points II

Plane passes through three points $\vec{a}$, $\vec{b}$, and $\vec{c}$. By using SAM show that the plane can be defined as $\vec{\mathcal{P}}(s; t) = \vec{a} + s\vec{u} + t\vec{v}$, where $\vec{u} = \vec{b} - \vec{a}$ and $\vec{v} = \vec{c} - \vec{a}$.

Hint: consider mapping $(0, 0)^\mathsf{T} \mapsto \vec{a}$, $(1, 0)^\mathsf{T} \mapsto \vec{b}$, and $(0, 1)^\mathsf{T} \mapsto \vec{c}$.

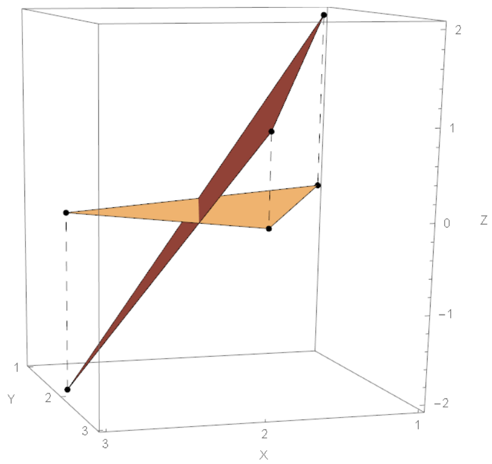


Figure 8: Orthogonal projection of a triangle on XOY-plane

► Since SAM yields affine mapping, plane $\mathbb{R}^2$ will still remain plane, thus passing through the points of interest is the only concern. Now suppose $(0,0)^\mathsf{T} \mapsto \vec{a}$, $(1,0)^\mathsf{T} \mapsto \vec{b}$, and $(0,1)^\mathsf{T} \mapsto \vec{c}$ and SAM immediately yields the transformation

$$\vec{\mathcal{P}}(s;t) = (-1)\frac{\det\begin{pmatrix} 0 & \vec{a} & \vec{b} & \vec{c} \\ s & 0 & 1 & 0 \\ t & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}} = \vec{a}(1-t-s) + \vec{b}s + \vec{c}t.$$

The latter can be rewritten as

$$\vec{\mathcal{P}}(s;t) = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a}) = \vec{a} + s\vec{u} + t\vec{v},$$

where $\vec{u} = \vec{b} - \vec{a}$ and $\vec{v} = \vec{c} - \vec{a}$ that completes the proof. ◄

References

[1] https://www.researchgate.net/publication/332410209_Beginner%27s_guide_to_mapping_simplexes_affinely

[2] By Kkraoj - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=26296104

[3] By en:User:T-tus, Image taken from en.wiki (en:Phong-shading-sample.jpg). - enWP, Public Domain, https://commons.wikimedia.org/w/index.php?curid=1556366