Model evaluation under Non-constant Class Imbalance: Supplementary Material

Jan Brabec[1,2], Tomáš Komárek[1,2], Vojtěch Franc[2], and Lukáš Machlica[3]

[1] Cisco Systems, Inc., Karlovo Namesti 10 Street, Prague, Czech Republic
{janbrabe,tomkomar}@cisco.com

[2] Czech Technical University in Prague, Faculty of Electrical Engineering, Czech Rep.
xfrancv@cmp.felk.cvut.cz

[3] Bulletproof.ai, Prague, Czech Republic
lukas.machlica@bulletproof.ai

Theorem 1. *Let* $TPR \in (\widehat{TPR} - \sigma_{TPR}, \widehat{TPR} + \sigma_{TPR})$ *and* $FPR \in (\widehat{FPR} - \sigma_{FPR}, \widehat{FPR} + \sigma_{FPR})$. *Let further* $\widehat{TPR} > \sigma_{TPR}$ *and* $\widehat{FPR} > \sigma_{FPR}$. *Then*

$$\varDelta \leq \max\{CV_{TPR}, CV_{FPR}\}$$

and the equality is attain iff $CV_{TPR} = CV_{FPR}$.

PROOF: The value of $\varDelta$ is defined as the maximal width of the interval $(\mathrm{LB}(\eta), \mathrm{UB}(\eta))$ w.r.t. η, that is,

$$\varDelta = \max_{\eta \in (0,1)} \left(\mathrm{UB}(\eta) - \mathrm{LB}(\eta) \right),$$

where

$$\mathrm{LB}(\eta) = \min_{\substack{\mathrm{TPR} \in \mathcal{I}_{\mathrm{TPR}} \\ \mathrm{FPR} \in \mathcal{I}_{\mathrm{FPR}}}} \mathrm{Prec}(\eta, \mathrm{TPR}, \mathrm{FPR}),$$
$$\mathrm{UB}(\eta) = \max_{\substack{\mathrm{TPR} \in \mathcal{I}_{\mathrm{TPR}} \\ \mathrm{FPR} \in \mathcal{I}_{\mathrm{FPR}}}} \mathrm{Prec}(\eta, \mathrm{TPR}, \mathrm{FPR}),$$

and

$$\mathrm{Prec}(\eta, \mathrm{TPR}, \mathrm{FPR}) = \frac{\eta \cdot \mathrm{TPR}}{\eta \cdot \mathrm{TPR} + (1-\eta) \cdot \mathrm{FPR}} = \frac{1}{1 + \frac{(1-\eta)}{\eta} \frac{\mathrm{FPR}}{\mathrm{TPR}}}. \tag{1}$$

The equation (1) shows that $\mathrm{Prec}(\eta, \mathrm{TPR}, \mathrm{FPR})$ is a monotonically decreasing function of the ratio $\frac{\mathrm{FPR}}{\mathrm{TPR}}$ which together with $\mathrm{FPR} \in (\widehat{\mathrm{FPR}} - \sigma_{\mathrm{FPR}}, \widehat{\mathrm{FPR}} + \sigma_{\mathrm{FPR}})$ and $\mathrm{TPR} \in (\widehat{\mathrm{TPR}} - \sigma_{\mathrm{TPR}}, \widehat{\mathrm{TPR}} + \sigma_{\mathrm{TPR}})$ implies that

$$\mathrm{UB}(\eta) = \frac{1}{1 + \frac{(1-\eta)}{\eta} \frac{\widehat{\mathrm{FPR}} - \sigma_{\mathrm{FPR}}}{\widehat{\mathrm{TPR}} + \sigma_{\mathrm{TPR}}}} \quad \text{and} \quad \mathrm{LB}(\eta) = \frac{1}{1 + \frac{(1-\eta)}{\eta} \frac{\widehat{\mathrm{FPR}} + \sigma_{\mathrm{FPR}}}{\widehat{\mathrm{TPR}} - \sigma_{\mathrm{TPR}}}}.$$

Using $x = \frac{1-\nu}{\nu} \in (0, \infty)$, we can re-parametrize $\mathrm{UB}(\eta)$ and $\mathrm{LB}(\eta)$ as

$$\mathrm{UB}(x) = \frac{1}{1 + x \cdot r_1} \quad \text{and} \quad \mathrm{LB}(x) = \frac{1}{1 + x \cdot r_2}$$

where $r_1 = \frac{\widehat{\mathrm{FPR}} - \sigma_{\mathrm{FPR}}}{\widehat{\mathrm{TPR}} + \sigma_{\mathrm{TPR}}}$ and $r_2 = \frac{\widehat{\mathrm{FPR}} + \sigma_{\mathrm{FPR}}}{\widehat{\mathrm{TPR}} - \sigma_{\mathrm{TPR}}}$. The value of Δ can be then equivalently defined as

$$\Delta = \max_{x \in (0,\infty)} f(x) \quad \text{and} \quad f(x) = \mathrm{UB}(x) - \mathrm{LB}(x) = \frac{1}{1 + x \cdot r_1} - \frac{1}{1 + x \cdot r_2} \,.$$

The maximum of $f(x)$ can be found analytically by solving $f'(x) = 0$ for x which yields

$$x^* = \frac{1}{\sqrt{r_1 \cdot r_2}} \,,$$

and hence

$$\Delta = f(x^*) = \frac{1}{1 + \frac{r_1}{\sqrt{r_1 \cdot r_2}}} - \frac{1}{1 + \frac{r_2}{\sqrt{r_1 \cdot r_2}}} = \frac{1 - \sqrt{\frac{r_1}{r_2}}}{1 + \sqrt{\frac{r_1}{r_2}}} \,.$$

It is seen that Δ is monotonically decreasing with the ratio

$$\frac{r_1}{r_2} = \frac{\widehat{\mathrm{FPR}} - \sigma_{\mathrm{FPR}}}{\widehat{\mathrm{TPR}} + \sigma_{\mathrm{TPR}}} \cdot \frac{\widehat{\mathrm{TPR}} - \sigma_{\mathrm{TPR}}}{\widehat{\mathrm{FPR}} + \sigma_{\mathrm{FPR}}} = \frac{1 - \mathrm{CV}_{\mathrm{FPR}}}{1 + \mathrm{CV}_{\mathrm{FPR}}} \cdot \frac{1 - \mathrm{CV}_{\mathrm{TPR}}}{1 + \mathrm{CV}_{\mathrm{TPR}}} \,.$$

If $\mathrm{CV}_{\mathrm{TPR}} = \mathrm{CV}_{\mathrm{FPR}} = \mathrm{C}$ then

$$\Delta = \frac{1 - \sqrt{\frac{(1-C)^2}{(1+C)^2}}}{1 + \sqrt{\frac{(1-C)^2}{(1+C)^2}}} = C = \max\{\mathrm{CV}_{\mathrm{TPR}}, \mathrm{CV}_{\mathrm{FPR}}\} \,.$$

Decreasing either $\mathrm{CV}_{\mathrm{TPR}}$ or $\mathrm{CV}_{\mathrm{FPR}}$ increases the ratio $\frac{r_1}{r_2}$ resulting in the decrease of Δ. Hence $\Delta \leq \max\{\mathrm{CV}_{\mathrm{TPR}}, \mathrm{CV}_{\mathrm{FPR}}\}$ and $\Delta = \max\{\mathrm{CV}_{\mathrm{TPR}}, \mathrm{CV}_{\mathrm{FPR}}\}$ iff $\mathrm{CV}_{\mathrm{TPR}} = \mathrm{CV}_{\mathrm{FPR}}$.

$\blacksquare$