Disjoint, sliding blocks and runs estimators for heavy tailed time series

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September 1, 2022

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3. **Estimation of cluster indices**
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Motivation : What is an extreme event?

- Are rare by definition;
- High impact event:
  - Tornado outbreaks; large wildfires;
  - El Nino : a climate pattern that describes the unusual warming of surface waters (brings rains and extreme floods which destroys homes, hospitals, businesses, ...);

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Hence, extremes remain a subject of active research and widely used in many other disciplines.

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Record heat under the dome : Lytton (northeast of Vancouver) set a record temperature of 50 °C on June 29, 2021, nearly 24 °C higher than normal. The next day, 90% of the small town of Lytton burned to the ground.

Figure: **Source: Environment and Climate Change Canada (600 people died in Vancouver, 650 000 farm animals perished).**

Motivation

Consider a regularly varying sequence of i.i.d. nonnegative random variables $\{X_j^\dagger, j \in \mathbb{Z}\}$ with tail distribution $\overline{F}$. In particular:

- $\lim_{n\to\infty} \overline{F}(tx)/\overline{F}(x) = t^{-\alpha}$ for some $\alpha > 0$. (e.g. Pareto, Student).
- There exists a sequence $a_n \to \infty$ s.t.

$$\lim_{n\to\infty} \mathbb{P}(a_n^{-1} \max_{j=1,\dots,n} \{X_j^\dagger\} \le x) = \exp(-x^{-\alpha}) , \quad x > 0 .$$

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Let $\{X_j, j \in \mathbb{Z}\}$ be stationary, regularly varying with the same marginal tail df $\overline{F}$. Then

$$\lim_{n\to\infty} \mathbb{P}(a_n^{-1} \max_{j=1,\ldots,n} \{X_j\} \leq x) = \exp(-\theta x^{-\alpha}) \ , \quad x > 0 \ ,$$

where $\theta \in (0, 1]$ is called the *extremal index* (whenever exists).

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$$\lim_{x\to\infty} \mathbb{E}[H(X_j/x,\ j\in\mathbb{Z})]$$

for some $H\ :\ \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}\ :\quad H(\boldsymbol{x}) = \mathbb{1}\left\{\max_{j\in\mathbb{Z}} x_j > 1\right\}.$

Questions:

- Can we consider different functionals $H\ :\ \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}$?

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- Can we consider different functionals $H : \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}$?
- Yes, for specific choices of H we will define **H-cluster indices**.
- How to estimate H-cluster indices? disjoint blocks, **sliding blocks** and **runs estimators**.

Tail process

Consider a stationary, regularly varying nonnegative time series $\boldsymbol{X} = \{X_j, j \in \mathbb{Z}\}$ with marginal distribution function F with tail index $\alpha > 0$.

[1] Basrak and Segers (2009)

Tail process

Consider a stationary, regularly varying nonnegative time series $\boldsymbol{X} = \{X_j, j \in \mathbb{Z}\}$ with marginal distribution function F with tail index $\alpha > 0$. Then, there exists $\boldsymbol{Y} = \{Y_j, j \in \mathbb{Z}\}$, called **tail process**[1], such that

$$\lim_{x\to\infty} \mathbb{P}(x^{-1}(X_i, \ldots, X_j) \in \cdot \mid X_0 > x) = \mathbb{P}((Y_i, \ldots, Y_j) \in \cdot).$$

The process $\boldsymbol{Y}$ is not stationary. Explicit formulas do exist for some time series models.

[1] Basrak and Segers (2009)

Clusters of extremes, cluster functionals

Cluster functionals H

For $\boldsymbol{X} = \{X_j, j \in \mathbb{Z}\} \in (\mathbb{R})^{\mathbb{Z}}$. We denote $\boldsymbol{X}_{i,j} = (X_i, \ldots, X_j) \in (\mathbb{R})^{(j-i+1)}$ with $i \leq j \in \mathbb{Z}$. Then, we identify $H(\boldsymbol{X}_{i,j})$ with $H((\boldsymbol{0}, \boldsymbol{X}_{i,j}, \boldsymbol{0}))$, where $\boldsymbol{0} \in (\mathbb{R})^{\mathbb{Z}}$ is the zero sequence.

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Given H on $(\mathbb{R})^{\mathbb{Z}}$, we want to consider the limiting quantity (cluster index)

$$\nu^*(H) = \lim_{n\to\infty} \nu^*_{n,r_n}(H) = \lim_{n\to\infty} \frac{\mathbb{E}\Big[H(\boldsymbol{X}_{1,r_n}/u_n)\Big]}{r_n \mathbb{P}\Big(X_0 > u_n\Big)} ,$$

with $r_n,\ u_n \to \infty$.

Question :

What are the conditions for the existence of such limit?

Assumptions

Assumptions on $r_n,\ u_n$ and the functional H are needed.

- $\lim_{n\to\infty} n\mathbb{P}(X_0 > u_n) = \infty$ and $\lim_{n\to\infty} r_n\mathbb{P}(X_0 > u_n) = 0$.

[2] Davis and Hsing (1995)

[3] Kulik, Soulier and Wintenberger (2019)

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- $\lim_{n\to\infty} n\mathbb{P}(X_0 > u_n) = \infty$ and $\lim_{n\to\infty} r_n\mathbb{P}(X_0 > u_n) = 0$.
- **Anticlustering condition** $\mathcal{AC}(r_n, u_n)$ Condition (extremes cannot persist for a infinite horizon time) holds if for all $x, y > 0$,[2]

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\max_{k\le|j|\le r_n} X_j > u_n x \;\Big|\; X_0 > u_n y\right) = 0\ .$$

It's valid e.g. geometrically ergodic Markov chains, short-memory linear or max-stable processes.[3]

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  It's valid e.g. geometrically ergodic Markov chains, short-memory linear or max-stable processes.[3]
- However, H cannot be arbitrary. For e.g: of $H = 1$, then $\boldsymbol{\nu}^*(H) = \infty$, and if $H(\boldsymbol{x}) = \sum_{j\in\mathbb{Z}} \mathbb{1}\left\{x_j > 1\right\}$, then $\boldsymbol{\nu}^*(H) = 1$.

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Example of H-cluster indices

Some cluster indices of interest are, among others:

- the **extremal index** obtained with $H_1(\boldsymbol{x}) = \mathbb{1}\left\{\sup_{j\in\mathbb{Z}} x_j > 1\right\}$.
- the **cluster size** distribution obtained with

$$H_{2,m}(\boldsymbol{x}) = \mathbb{1}\left\{\sum_{j\in\mathbb{Z}} \mathbb{1}\left\{\boldsymbol{x}_j > 1\right\} = m\right\}, \quad m \in \mathbb{N}\ ;$$

- the **large deviation index** of a univariate time series obtained with[4]

$$H_3(\boldsymbol{x}) = \mathbb{1}\{K(\boldsymbol{x}) > 1\}\ , \quad K(\boldsymbol{x}) = \left(\sum_{j\in\mathbb{Z}} x_j\right)_+ \ .$$

[4]Mikosh and Wintenberger (2013, 2014)

Existence and representation

Theorem (1)

Let condition $\mathcal{AC}(r_n, u_n)$ hold. The sequence of measures converges vaguely $\boldsymbol{\nu}^*_{n,r_n} \to \boldsymbol{\nu}^*$, that is,

$$\lim_{n\to\infty} \boldsymbol{\nu}^*_{n,r_n}(H) = \lim_{n\to\infty} \frac{\mathbb{E}\left[H(u_n^{-1}\boldsymbol{X}_{1,r_n})\right]}{r_n\mathbb{P}\left(X_0 > u_n\right)} = \boldsymbol{\nu}^*(H) \; .$$

for all bounded, continuous and shift invariant functions H with support separated from $\mathbf{0}$.

It has the following representation [5].

$$\boldsymbol{\nu}^*(H) = \mathbb{E}\left[H(\boldsymbol{Y})\mathbb{1}\left\{\boldsymbol{Y}^*_{-\infty,-1} \le 1\right\}\right] = \mathbb{E}\left[\sup_{j\le -1} |Y_j| < 1\right] \; .$$

[5] Kulik and Soulier (2020), Chapter VI

Disjoint blocks estimator

Consider the disjoint blocks statistics

$$\widetilde{\boldsymbol{D}B}_n(H) := \frac{1}{n\mathbb{P}(X_0 > u_n)} \sum_{i=1}^{m_n} H\left(\boldsymbol{X}_{(i-1)r_n+1, ir_n}/u_n\right) ,$$

where $m_n = [n/r_n]$. Note that

$$\nu^*(H) = \lim_{n\to\infty} \mathbb{E}[\widetilde{\boldsymbol{D}B}_n(H)].$$

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where $m_n = [n/r_n]$. Note that

$$\boldsymbol{\nu}^*(H) = \lim_{n\to\infty} \mathbb{E}[\widetilde{\boldsymbol{DB}}_n(H)].$$

For sequence of integers $k \to \infty$ such that $k/n \to \infty$, define $u_n = F^{\leftarrow}(1-k/n)$. Define the disjoint blocks estimator

$$\widehat{\boldsymbol{DB}}_n(H) = \frac{1}{k} \sum_{i=1}^{m_n} H\left(\boldsymbol{X}_{(i-1)r_n+1,ir_n}\Big/X_{(n:n-k)}\right) ,$$

where $X_{(n:1)} \leq \cdots \leq X_{(n:n)}$.

Sliding blocks estimator

Consider the sliding blocks statistics

$$\widetilde{SB}_n(H) := \frac{1}{q_n r_n \mathbb{P}(X_0 > u_n)} \sum_{i=0}^{q_n-1} H\left(\boldsymbol{X}_{i+1,i+r_n}/u_n\right) ,$$

where $q_n = n - r_n + 1$,

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where $q_n = n - r_n + 1$,
and

$$\widehat{SB}_n(H) = \frac{1}{kr_n} \sum_{i=0}^{q_n-1} H\left(\boldsymbol{X}_{i+1,i+r_n}\Big/X_{(n:n-k)}\right) .$$

Runs estimator

Consider the runs statistics

$$\widetilde{\boldsymbol{R}}_{n,r_n}(H^C) = \frac{1}{n\mathbb{P}(\boldsymbol{X}_0 > u_n)} \sum_{i=1}^{n-r_n} H^C\left(\boldsymbol{X}_{i-r_n,i+r_n}/u_n\right) \,,$$

where H^C is defined as

$$H^C(\boldsymbol{x}) = H(\boldsymbol{x})\mathbb{1}\{C(\boldsymbol{x}) = 0\}\mathbb{1}\{\boldsymbol{x}_0 > 1\} \,,$$

with C being an anchoring map, e.g. $C(\boldsymbol{x}) = \inf\{j : |x_j| > 1\}$.

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$$\widehat{\boldsymbol{R}}_{n,r_n}(H^C) = \frac{1}{k} \sum_{i=1}^{n-r_n} H^C\left(\boldsymbol{X}_{i-r_n,i+r_n}/X_{(n:n-k)}\right) .$$

Sliding blocks estimator-CLT

Theorem (Cissokho and Kulik (2021), *Electronic Journal of Statistics*)

Let $\{X_j, j \in \mathbb{Z}\}$ *be a stationary, regularly varying* $\mathbb{R}$*-valued and* β*-mixing time series and* $s > 0$*. Under the "appropriate" conditions*

$$\sqrt{k}\left\{\widehat{SB}_n(H) - \nu^*(H)\right\} \xrightarrow{\text{d}} \mathbb{G}^*(H) \ ,$$

where $\mathbb{G}$ *is a centered Gaussian process with covariance* $\nu^*(H\widetilde{H})$ *and* $\mathbb{G}^*(H) = \mathbb{G}(H - \nu^*(H)\mathcal{E})$, $\ \mathcal{E}(\boldsymbol{x}) = \sum_{j\in\mathbb{Z}} \mathbb{1}\left\{x_j > 1\right\}$.

Sliding blocks estimator-CLT

Theorem (Cissokho and Kulik (2021), *Electronic Journal of Statistics*)

Let $\{X_j, j \in \mathbb{Z}\}$ be a stationary, regularly varying $\mathbb{R}$-valued and β-mixing time series and $s > 0$. Under the "appropriate" conditions

$$\sqrt{k}\left\{\widehat{SB}_n(H) - \boldsymbol{\nu}^*(H)\right\} \xrightarrow{\text{d}} \mathbb{G}^*(H) \ ,$$

where $\mathbb{G}$ is a centered Gaussian process with covariance $\boldsymbol{\nu}^*(H\widetilde{H})$ and $\mathbb{G}^*(H) = \mathbb{G}(H - \boldsymbol{\nu}^*(H)\mathcal{E}), \ \ \mathcal{E}(\boldsymbol{x}) = \sum_{j\in\mathbb{Z}} \mathbb{1}\left\{x_j > 1\right\}$.

The same asymptotics holds for disjoint blocks estimator as well.

Runs estimator-CLT

Theorem (Cissokho and Kulik (2022), *Electronic Journal of Statistics*)

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$$\sqrt{k}\left\{\widehat{\boldsymbol{R}}_n(H) - \boldsymbol{\nu}^*(H)\right\} \xrightarrow{\mathrm{d}} \mathbb{G}^*(H) ,$$

where $\mathbb{G}$ is a centered Gaussian process with covariance $\boldsymbol{\nu}^*(H\widetilde{H})$ and
$\mathbb{G}^*(H) = \mathbb{G}(H - \boldsymbol{\nu}^*(H)\mathcal{E})$, $\mathcal{E}(\boldsymbol{x}) = \sum_{j\in\mathbb{Z}} \mathbb{1}\left\{x_j > 1\right\}$.

Simulations-Stationary AR process

We start with a simple AR(1) process. For this process we have explicit formulas for all cluster indices. Samples of size $n = 1000$ are generated from AR(1) with $\alpha = 4$ and $\rho = 0.5, 0.9$.

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Extremal index.

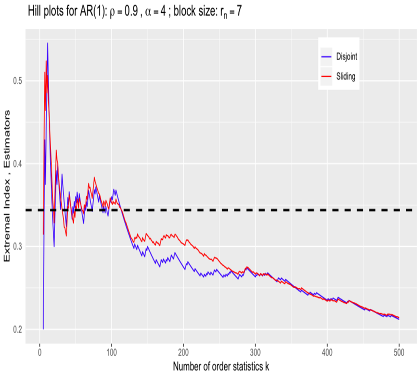
For AR(1) with $\rho > 0$ the extremal index is $\theta = 1 - \rho^{\alpha}$; (Kulik and Soulier (2020)).

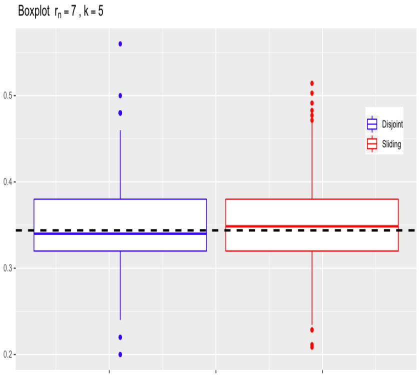
Simulations-Extremal index

	$\rho = 0.9$, Extremal Index=0.34				$\rho = 0.5$, Extremal Index= 0.94			
(k %)	$k = 5$		$k = 10$		$k = 5$		$k = 10$	
$r_n = 7$								
Disjoint bl	**0.34**	(0.05)	**0.31**	(0.03)	**0.68**	(0.05)	**0.58**	(0.03)
Sliding bl	**0.35**	(0.04)	**0.31**	(0.03)	**0.68**	(0.04)	**0.58**	(0.03)
$r_n = 8$								
Disjoint bl	**0.32**	(0.05)	**0.29**	(0.03)	**0.67**	(0.05)	**0.56**	(0.03)
Sliding bl	**0.33**	(0.04)	**0.29**	(0.03)	**0.67**	(0.04)	**0.56**	(0.03)
$r_n = 9$								
Disjoint bl	**0.32**	(0.05)	**0.28**	(0.03)	**0.66**	(0.05)	**0.53**	(0.03)
Sliding bl	**0.32**	(0.04)	**0.28**	(0.03)	**0.65**	(0.05)	**0.53**	(0.03)
$r_n = 10$								
Disjoint bl	**0.30**	(0.05)	**0.26**	(0.03)	**0.64**	(0.05)	**0.52**	(0.03)
Sliding bl	**0.30**	(0.04)	**0.26**	(0.03)	**0.63**	(0.05)	**0.52**	(0.03)

Figure: The median and the variance (in brackets) of disjoint and sliding blocks estimators for the extremal index. Data are simulated from AR(1) with $\alpha = 4$, $\rho = 0.5$ (thus, $\theta = 0.94$), and $\rho = 0.9$ (thus $\theta = 0.34$). Block size $r_n = 7, 8, 9, 10$. The number of order statistics is $k = 5\%$ and 10% for a sample $n = 1000$ based on $N = 1000$ Monte Carlo simulations.

Simulations-Extremal index





PoT vs. Block maxima

PoT method

- Drees and Neblung (2020) studied asymptotic normality of the sliding blocks and runs estimators in general setting. they showed that it's limiting variance **does not exceed that of the disjoint blocks estimators**.
- For the extremal index, they **showed that the variances are equal**.

Note: we worked under PoT method.

Block maxima

- Robert, Segers, Ferro (2009) and Bücher and Segers (2018a, 2918b), Zou, Volgusher and Bücher (2021): Sliding blocks estimators have smaller variance that the disjoint blocks.

Our contribution

To the best of our knowledge, this thesis makes the following contribution:

- Central limit theorem for the data-based sliding blocks and runs estimators under easy to verify assumptions.
- We give an explicit formula for the asymptotic variance. As such, we can conclude that **the sliding, disjoint blocks and runs estimators yield the same asymptotics**.
- This solves **the longstanding problem in the context of cluster functionals**.

Open questions

- Extend CLT for sliding blocks estimators (Theorem 1) to piecewise stationary processes. This line of research was proposed recently by Axel Bücher and his student. Piecewise stationary processes may be used in climate modeling.
- Obtain the results of (Theorem 1) under minimal conditions (that is, without relying on β-mixing and linear ordering of function classes). Do these results are valid under long range dependence?
- Can we extend the asymptotic results presented here to Gumbel domain of attraction? note that the probabilistic methods have to be completely different.
- Since the disjoint and sliding blocks statistics have the same asymptotic behaviour, is it possible to obtain an asymptotic expansion for the difference between these two statistics?
- Can we compare results between Peak-over-Threshold and Block Maxima methods?

Thank you and questions please. . .