NAPDE project - titolo provvisorio

Luca Ciarimboli, Lorenzo Citterio, Marianna Dell'Otto

July 21, 2023

Abstract

This document presents our mathematical model for predicting the evapotranspiration of the lettuce inside a simplified cubic cell.
This is only a sketch that, one day (hopefully soon), will become the basis of the report for this project.
The document is divided into three chapters, each one reporting the equations of our interest from one out of the three papers we have used to build the model.
Since this is only a sketch and not for diffusion, papers are not quoted yet, neither are the other papers quoted inside them; we refer to each paper using the name of the author: this is clear enough at the moment.
Of course we will provide to include a references list and to quote properly each source.

Contents

Chapter 1

Naranjani

1.1 Balance partial differential equations

In this section we write down the PDEs of our system, which are continuity equation, momentum balance and energy balance. These are taken from the Naranjani paper, converted in vectorial form for semplicity.
As assumpitons, we consider the air as an incompressible, steady-state, Newtonian fluid.

1.1.1 Navier-Stokes

$$\begin{cases} \nabla \cdot (\rho_a \vec{u}) = 0 \\ (\rho_a \vec{u} \cdot \nabla)\vec{u} + \vec{u}\nabla \cdot (\rho_a \vec{u}) = -\nabla P + \mu(\Delta \vec{u} + \nabla(\nabla \cdot \vec{u})) + \rho_a \vec{g} \\ +BCs \\ +(k - \epsilon) \end{cases} \tag{1.1}$$

Where the unkowns are $\vec{u}$ $[ms^{-1}]$, which is the air fluid velocity, and P $[W]$, which is the required power of the fans.
Moreover ρ_a $[kgm^{-2}]$, μ $[kgm^{1}s^{-1}]$ and $\vec{g}$ $[ms^{-2}]$ are respectively the air density, the dynamic viscosity and the gravitational acceleration.
From this system (**??**) we aim to compute the fluid velocity of the air ($\vec{u}$), in order to use it as known variable on the next equation.
Moreover, for the moment, we start for semplicity to consider the flow laminar, namely we are not integrating the $k - \epsilon$ turbulence model yet.

1.1.2 Energy Balance

The energy balance:

$$\begin{cases} \nabla \cdot (\rho_a \vec{u} C_p T_a) - \Delta(\lambda T_a) = 0 \\ +BCs \end{cases} \tag{1.2}$$

yields the unknown air temperature T_a $[K]$, which is of high interest for us, with C_p $[Jkg^{-1}K^{-1}]$ being the specific heat capacity and λ $[Wm^{-1}s^{-1}]$ being the thermal conductivity, both given values.
In accordance with the paper, we decided to put the source and sink term equal to 0; hence we will need to incorporate the presence of leds (heat sources in our system) inside the boundary conditions, since they are located on the upper surface of our cell.

1.2 Species Mass Fraction

We incorporate form Naranjani as well an equation which allows us to compute the mass fraction (γ_n) of the species composing the fluid (H_2O, CO_2, O_2 and N_2):

$$\nabla \cdot (\rho_a \vec{u} \gamma_n) = -\nabla \cdot (\vec{j_n}) + R_n + S_n \tag{1.3}$$

With n indicating the specie and $\vec{j_n}$ $[kgm^{-2}s^{-1}]$ being the turbolent mass diffusion flux.
In particular we compute the value of γ for H_2O and CO_2: we will need these values later on.
(indeed: γ_{H_2O} appears in (**??**), but may not be needed if the saturation assumption (**??**) is made; γ_{CO_2} is used

in (**??**))
About the sink and source terms here:

- R_n=0: term related with chemical reactions we don't consider;
- S_n is user-defined and related with the presence of lettuce, hence we consider it $\neq 0$ only on the exchange surface. This means that, changing the value of S_n, we can compute two different values for each specie: $\gamma_{n,sur}$ and $\gamma_{n,air}$.

Chapter 2

Van Henten

2.1 Dry weight model

In this section we use the model provided by the paper Van Henten, yielding the dry weigth DW $[g]$ of the crop. In particular the idea is to divide the dry weight between *non structural* weight X_{nsdw} (which consists of glucose, sucrose, starch, etc) and *structural* weight X_{sdw}.
Both X_{sdw}, X_{nsdw} are in $[gm^{-2}]$, meaning that:

$$DW(t) = \frac{X_{sdw}(t) + X_{nsdw}(t)}{n}$$

with n being the number of plants per squared meter.
The model is given by the following system of ODEs:

$$\begin{cases} \frac{d}{dt}X_{nsdw}(t) = c_\alpha f_{phot}(t) - r_{gr}(t)X_{sdw}(t) - f_{resp}(t) - \frac{1-c_\beta}{c_\beta} r_{gr}(t)X_{sdw}(t) \\ \frac{d}{dt}X_{sdw}(t) = r_{gr}(t)X_{sdw}(t) \end{cases} \tag{2.1}$$

Where the coefficients are known and given by the paper:

- $c_\alpha = 30/44 \approx 0.68$ is the ratio of molecular weigth of CH_2O (sugar) and CO_2;
- $c_\beta = 0.8$ indicates the respiratory and synthesis losses of non-structural material due to growht. The value 0.8 for lettuce was estimated on paper Sweeney.

Moreover, the other functions describe what follows:

- f_{phot} $[gm^{-2}s^{-1}]$ is the gross canopy photosyntesis;
- f_{resp} $[gm^{-2}s^{-1}]$ is the manteinance respiration;
- r_{gr} $[s^{-1}]$ is the specific growth rate.

The latters are computable through further relations we must add to our system of equation.

2.2 Further needed relations

First of all, remark that we already computed T_a (at (**??**)) and γ_{CO_2} (at (**??**)).
Moreover, for the canopy temperature T_c $[K]$, we introduce the following approximation suggested to us by *Agricola Moderna*:

$$T_c \approx T_a - 1 \tag{2.2}$$

2.2.1 Functions inside (??)

The equations yielding the missing functions of the model we are looking for are:

$$\begin{cases} f_{phot}(t) = (1 - exp(-c_K c_{lar}(1 - c_\tau)X_{sdw}(t)))f_{phot,max} \\ f_{resp}(t) = (c_{resp,sht}(1 - c_\tau)X_{sdw}(t) + c_{resp,rt}c_\tau X_{sdw}(t))c_{Q_{10},resp}^{\frac{T_c-25}{10}} \\ r_{gr}(t) = c_{gr,max}\frac{X_{nsdw}(t)}{X_{sdw}(t)+X_{nsdw}(t)}c_{Q_{10},gr}^{\frac{T_c-20}{10}} \end{cases} \tag{2.3}$$

where the given coefficients are:

- $c_K \approx o.9$ is the extinction coefficient;
- $c_{lar} \approx 75 \times 10^{-3}\ m^2 g^{-1}$ is the structural leaf area, measured by Lorenz & Wiebe (1980);
- $c_\tau = 0.15$ is the ratio of the root dry weigth and the total dry weigth, reported constant by assumption by Lorenz & Weibe and Sweeney;
- $c_{resp,sht} \approx 3.47 \times 10^{-7}\ s^{-1}$ is the shoot maintenance respiration coefficient;
- $c_{resp,rt} \approx 1.16 \times 10^{-7}\ s^{-1}$ is the root maintenance respiration coefficient;
- $c_{Q_{10},resp} = 2$ is the Q_{10} factor of the maintenance respiration;
- $c_{gr_max} \approx 5 \times 10^{-6}\ s^{-1}$ is the saturation growth rate estimated by Van Holsteijn (1981);
- $c_{Q_{10},gr} = 1.6$ is the Q_{10} factor for growth.

2.2.2 Functions inside (??)

For the parameters inside system (??) missing a given value, the paper reports the following relations:

$$\begin{cases} f_{phot,max} = \frac{\epsilon U_{par} g_{CO_2} c_\omega (\gamma_{CO_2} - \Gamma)}{\epsilon U_{par} + g_{CO_2} c_\omega (\gamma_{CO_2} - \Gamma)} \\ \Gamma = c_\Gamma c_{Q_{10},\Gamma}^{\frac{T_c - 20}{10}} \\ \epsilon = c_\epsilon \frac{\gamma_{CO_2} - \Gamma}{\gamma_{CO_2} + 2\Gamma} \\ \frac{1}{g_{CO_2}} = \frac{1}{g_{bnd}} + \frac{1}{g_{stm}} + \frac{1}{g_{car}} \\ g_{car} = c_{car,1} T_c^2 + c_{car,2} T_c + c_{car,3} \end{cases} \tag{2.4}$$

where

- ϵ $[gJ^{-1}]$ is the light use efficiency;
- Γ $[ppm]$ is the CO_2 compensation point accounting for photorespiration at high ligth levels;
- U_{par} $[Wm^{-2}]$ is the incident photosyntetically active radiation;
- g_{CO_2} $[ms^{-1}]$ is the canopy conductance to CO_2 diffusion;
- $c_\omega \approx 1.83 \times 10^{-3}\ gm^{-3}$ is the estimated CO_2 density at $15°$ temperature and ambient pressure;
- $c_\Gamma = 40\ ppm$ is the value of Γ at $20°$;
- $c_{Q_{10},\Gamma} = 2$ is the Q_{10} value which accounts for the effect of temperature on Γ;
- $c_\epsilon = 17.0 \times 10^{-6}\ gJ^{-1}$ is the value of ϵ at very high CO_2 concentrations (Gondurian et al., 1985);
- g_{bnd} $[ms^{-1}]$ is the boundary layer conductance;
- g_{stm} $[ms^{-1}]$ is the stomatal conductance;
- g_{car} $[ms^{-1}]$ is the carboxylation conductance, with $c_{car,1} = -1.32 \times 10^{-5}\ ms^{-1}[°C]^{-2}$, $c_{car,2} = 5.94 \times 10^{-4}\ ms^{-1}[°C]^{-2}$ and $c_{car,2} = -2.64 \times 10^{-3}\ ms^{-1}[°C]^{-2}$;

x

2.3 Leaf Area Index

After the huge system of equation discussed by far, one is now allowed to compute the dry weight as a function of the time. In particular, we aim to use X_{sdw} to compute a crucial parameter for us: the "Leaf Area Index" (LAI) $[m^2/g]$, that plays a role in the energy balance of the crop, that is the latter step of our model. Paper Van Henten gives the following relation:

$$LAI = \frac{(1 - c_\tau) c_{lar} X_{sdw}(t)}{n} \tag{2.5}$$

Notice that LAI was "hidden" inside the equation for f_{phot}, we did not call it LAI yet since we were not able to compute it without knowing X_{sdw}.
Notice moreover that paper Graamans (that treats the crop energy balance) indicates with LAI not the value

computed here, but another one we call $LAI_{effective}$. Indeed: LAI does not consider the fact that, when lettuce is grown enough, part of the leaves could overlap with the ones of adjoining plants.
The relation enabling us to update LAI into $LAI_{effective}$ is determined by the paper "Tei et al.".
For the moment we consider for semplicity the "old" estimate of LAI given by Van Henten good enough for us.

Chapter 3

Graamans

3.1 Crop energy balance

We extrapolated from paper Graamans the energy balance of the crop.
At the base of this treatment there is an important assumption introduced by Penman & Monteith (1965): the three dimentional crop canopy is reduced to a one-dimentional *big leaf* where net radiation is absorbed, heat is exchanged and water vapour escapes due to evapotranspiration.
We are allowed to make this assumption since in our cell crop is homogeneus, level, continuous and extensive enough.
At equilibrium, the energy arriving at the *big leaf* equals the amount leaving it:

$$R_{net} - H - \lambda E = 0 \tag{3.1}$$

Where:

- R_{net} $[Wm^{-2}]$ is the net radiation absorbed by the crop;
- H $[Wm^{-2}]$ is the sensible heat flux;
- λE $[Wm^{-2}]$ is the latent heat flux, thus the heat related with the evapotranspiration of the lettuce.

Clearly R_{net} is a data, since we can determine it after the light intensity is set, in the paper the following relation is presented:

$$R_{net} = (1 - \rho_r) I_{lighting} CAC \tag{3.2}$$

where the coefficients are known:

- $\rho_r \approx 0.05 - 0.08$ is the reflection coefficient;
- $CAC \approx 0.9$ is the ratio of projected leaf area to cultivation area, which is almost constant in lettuce throughout its development.

and $I_{lighting}$ $[Wm^{-2}]$ is a control variable.
On the other hand, for the heat fluxes, Gramaans presents the following relations:

$$\begin{cases} H = LAI \rho_a C_p \frac{T_c - T_a}{r_a} \\ \lambda E = LAI \lambda \frac{\gamma_{H_2O,sur} - \gamma_{H_2O,air}}{r_s + r_a} \end{cases} \tag{3.3}$$

Where:

- r_a $[sm^{-1}]$ is the aereodynamics resistence to heat;
- r_s $[sm^{-1}]$ is the stomatal resistence of vapour flow through the transpiring crop;
- λ $[Jg^{-1}]$is the latent heat of the evaporation of water and it is unknown.

Remark that we already computed the values of LAI (at (**??**)), $\gamma_{H_2O,sur}$ and $\gamma_{H_2O,air}$ (at (**??**)); moreover ρ_a and C_p are given data already used in (**??**) and (**??**).
Notice also that we made the assumption (**??**), thus the numerator of the fraction inside (**??**) in the H equation is 1.
About the difference $\gamma_{H_2O,sur} - \gamma_{H_2O,air}$ we can state that we are able to determine it since we already know from (**??**) each H_2O mass fraction. Anyway, Graamans presents a relation between the two, holding if assuming

that the transpiring surface is saturated at his temperature, which may save us some previous computations if adopted:

$$\gamma_{H_2O,sur} \approx \gamma_{H_2O,air} + \frac{\rho_a C_p}{\lambda} \epsilon (T_c - T_a) \tag{3.4}$$

In the end, the stomatal resistance is known and it depends on the photosyntetic photon flux density $PPFD$ $[\mu mol m^{-2} s^{-1}]$, which is a control variable. On the other hand, the aereodynamics resistance can be determined using a relation introduced by Fuchs (1993).
Explicitly:

$$\begin{cases} r_s = 60\frac{1500+PPFD}{200+PPFD} \\ r_a = 350(\frac{l}{u_\infty})^{0.5} LAI^{-1} \end{cases} \tag{3.5}$$

with l being the mean leaf diameter and u_∞ the uninhibited air speed, both known data.

3.2 Evapotranspiration

After having determined the flux of latent heat, we are able to compute in the end the value of our interest: ET $[g m^{-2} s^{-1}]$, which represents the amount of water vapour released by the crop due to the evapotranspiration process.