Problem 1 MLE for Geometric distribution [30 marks]

In this problem you will consider the maximum likelihood estimate (MLE) of the parameters of a *Geometric Distribution*.

Suppose we have a Bernoulli random variable with probability π of "success", and probability $1-\pi$ of "failure". The geometric distribution is the probability that we need x independent Bernoulli trials to see the first occurrence of "success". For example, if the Bernoulli r.v. is a coin and π is the probability of "Heads", then the geometric distribution models the number of times we need to flip the coin to see the first occurrence of "Heads".

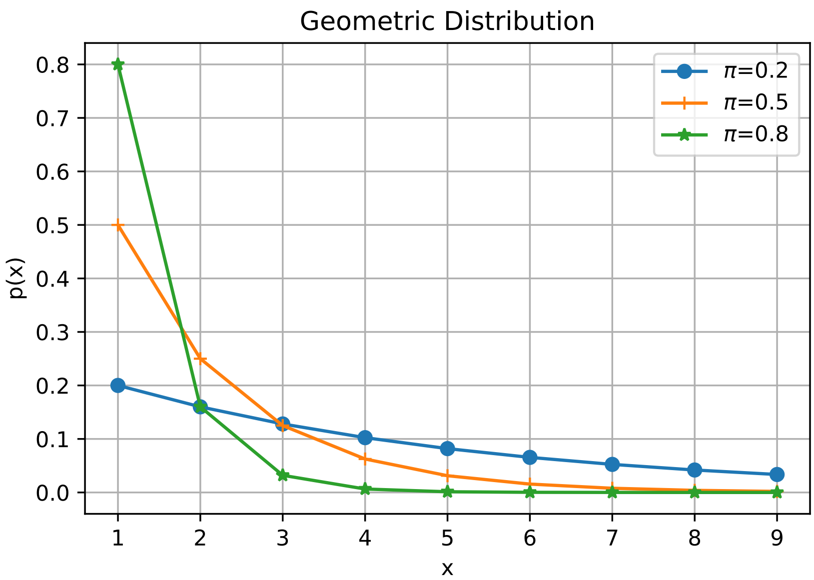
Formally, the geometric distribution is:

$$p(x|\pi) = (1-\pi)^{x-1}\pi \tag{1}$$

where $x \in \{1, 2, 3, \cdots\}$ is the count (the number of Bernoulli trials), and $0 < \pi \leq 1$ is the parameter (probability of "success"). The mean and variance of the geometric distribution are

$$E[x] = \frac{1}{\pi}, \qquad \text{var}(x) = \frac{1-\pi}{\pi^2}. \tag{2}$$

Here is a plot of the geometric distribution for different values of π:



Suppose we have a set of N samples of the number of trials, $\mathcal{D} = \{x_1, \cdots, x_N\}$, where $x_i \in \{1, 2, 3, \cdots\}$ is the number of trials for the i-th sample.

(a) [5 marks] Write down the log-likelihood of the data $\mathcal{D}$, i.e., $\log p(\mathcal{D}|\pi)$.

(b) [5 marks] Write down optimization problem for maximum-likelihood estimation of the parameter π.

(c) [15 marks] Derive the MLE for the parameter π.

(d) [5 marks] What is the intuitive interpretation of the derived MLE for π.

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