Assignment 7: Dynamic Decision

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Simulate data

Suppose that there is a firm and it makes decisions for $t = 1, \cdots, \infty$. We solve the model under the infinite-horizon assumption, but generate data only for $t = 1, \cdots, T$. There are $L = 5$ state $s \in \{1, 2, 3, 4, 5\}$ states for the player. The firm can choose $K + 1 = 2$ actions $a \in \{0, 1\}$.

The mean period payoff to the firm is:

$$\pi(a, s) := \alpha \ln s - \beta a,$$

where $\alpha, \beta > 0$. The period payoff is:

$$\pi(a, s) + \epsilon(a),$$

and $\epsilon(a)$ is an i.i.d. type-I extreme random variable that is independent of all the other variables.

At the beginning of each period, the state s and choice-specific shocks $\epsilon(a), a = 0, 1$ are realized, and the the firm chooses her action. Then, the game moves to the next period.

Suppose that $s > 1$ and $s < L$. If $a = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves up by 1 with probability γ, moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa - \gamma$.

Suppose that $s = 1$. If $a = 0$, the state stays at the same state with probability 1. If $a = 1$, the state moves up by 1 with probability γ and stays at the same with probability $1 - \gamma$.

Suppose that $s = L$. If $a = 0$, the state stays at the same state with probability $1 - \kappa$ and moves down by 1 with probability κ. If $a = 1$, the state moves down by 1 with probability κ, and stays at the same with probability $1 - \kappa$.

The mean period profit is summarized in Π as:

$$\Pi := \begin{pmatrix} \pi(0, 1) \\ \vdots \\ \pi(K, 1) \\ \vdots \\ \pi(0, L) \\ \vdots \\ \pi(K, L) \end{pmatrix}$$

The transition law is summarized in G as:

$$g(a, s, s') := \mathbb{P}\{s_{t+1} = s' | s_t = s, a_t = a\},$$

$$
G := \begin{pmatrix} g(0,1,1) & \cdots & g(0,1,L) \\ \vdots & & \vdots \\ g(K,1,1) & \cdots & g(K,1,L) \\ & \vdots & \\ g(0,L,1) & \cdots & g(0,L,L) \\ \vdots & & \vdots \\ g(K,L,1) & \cdots & g(K,L,L) \end{pmatrix}.
$$

The discount factor is denoted by δ. We simulate data for N firms for T periods each.

1. Set constants and parameters as follows:

```
# set seed
set.seed(1)
# set constants
L <- 5
K <- 1
T <- 100
N <- 1000
lambda <- 1e-10
# set parameters
alpha <- 0.5
beta <- 3
kappa <- 0.1
gamma <- 0.6
delta <- 0.95
```

2. Write function `compute_pi(alpha, beta, L, K)` that computes Π given parameters and compute the true Π under the true parameters. Don't use methods in `dplyr` and deal with matrix operations.

```
PI <-
  compute_PI(
    alpha = alpha,
    beta = beta,
    L = L,
    K = K
    );
PI
```

```
##              [,1]
## k0_l1  0.0000000
## k1_l1 -3.0000000
## k0_l2  0.3465736
## k1_l2 -2.6534264
## k0_l3  0.5493061
## k1_l3 -2.4506939
## k0_l4  0.6931472
## k1_l4 -2.3068528
## k0_l5  0.8047190
## k1_l5 -2.1952810
```

3. Write function `compute_G(kappa, gamma, L, K)` that computes G given parameters and compute the true G under the true parameters. Don't use methods in `dplyr` and deal with matrix operations.

```r
G <-
  compute_G(
    kappa = kappa,
    gamma = gamma,
    L = L,
    K = K
    );
G
```

```
##        l1  l2  l3  l4  l5
## k0_l1 1.0 0.0 0.0 0.0 0.0
## k1_l1 0.4 0.6 0.0 0.0 0.0
## k0_l2 0.1 0.9 0.0 0.0 0.0
## k1_l2 0.1 0.3 0.6 0.0 0.0
## k0_l3 0.0 0.1 0.9 0.0 0.0
## k1_l3 0.0 0.1 0.3 0.6 0.0
## k0_l4 0.0 0.0 0.1 0.9 0.0
## k1_l4 0.0 0.0 0.1 0.3 0.6
## k0_l5 0.0 0.0 0.0 0.1 0.9
## k1_l5 0.0 0.0 0.0 0.1 0.9
```

The exante-value function is written as a function of a conditional choice probability as follows:

$$\varphi^{(\theta_1,\theta_2)}(p) := [I - \delta\Sigma(p)G]^{-1}\Sigma(p)[\Pi + E(p)],$$

where $\theta_1 = (\alpha, \beta)$ and $\theta_2 = (\kappa, \gamma)$ and:

$$\Sigma(p) = \begin{pmatrix} p(1)' & & \\ & \ddots & \\ & & p(L)' \end{pmatrix}$$

and:

$$E(p) = \gamma - \ln p.$$

Note that γ in the formula of $E(p)$ refers to the Euler constant, not the parameter defined before.

4. Write a function `compute_exante_value(p, PI, G, L, K, delta)` that returns the exante value function given a conditional choice probability. Don't use methods in `dplyr` and deal with matrix operations. When a choice probability is zero at some element, the corresponding element of $E(p)$ can be set at zero, because anyway we multiply the zero probability to the element and the corresponding element in $E(p)$ does not affect the result.

```r
p <-
  matrix(
    rep(0.5, L * (K + 1)),
    ncol = 1
    );
p
```

```
##       [,1]
##  [1,]  0.5
##  [2,]  0.5
##  [3,]  0.5
##  [4,]  0.5
##  [5,]  0.5
##  [6,]  0.5
##  [7,]  0.5
```

```
##  [8,]  0.5
##  [9,]  0.5
## [10,]  0.5
```

```r
V <-
  compute_exante_value(
    p = p,
    PI = PI,
    G = G,
    L = L,
    K = K,
    delta = delta
    );
V
```

```
##          [,1]
## l1  5.777876
## l2  7.597282
## l3  9.126304
## l4 10.115439
## l5 10.593438
```

The optimal conditional choice probability is written as a function of an exante value function as follows:

$$\Lambda^{(\theta_1,\theta_2)}(V)(a,s) := \frac{\exp[\pi(a,s)+\delta\sum_{s'}V(s')g(a,s,s')]}{\sum_{a'}\exp[\pi(a',s)+\delta\sum_{s'}V(s')g(a',s,s')]},$$

where V is an exante value function.

5. Write a function `compute_ccp(V, PI, G, L, K, delta)` that returns the optimal conditional choice probability given an exante value function. Don't use methods in `dplyr` and deal with matrix operations. To do so, write a function `compute_choice_value(V, PI, G, delta)` that returns the choice-specific value function. Use this for debugging by checking if the results are intuitive.

```r
value <-
  compute_choice_value(
    V = V,
    PI = PI,
    G = G,
    delta = delta
    );
value
```

```
##             [,1]
## k0_l1  5.488982
## k1_l1  3.526044
## k0_l2  7.391148
## k1_l2  5.262691
## k0_l3  9.074038
## k1_l3  6.637845
## k0_l4 10.208846
## k1_l4  7.481306
## k0_l5 10.823075
## k1_l5  7.823075
```

```r
p <-
  compute_ccp(
    V = V,
```

```
    PI = PI,
    G = G,
    L = L,
    K = K,
    delta = delta
    );
p
```

```
##              [,1]
## k0_l1 0.87685057
## k1_l1 0.12314943
## k0_l2 0.89363847
## k1_l2 0.10636153
## k0_l3 0.91954591
## k1_l3 0.08045409
## k0_l4 0.93863232
## k1_l4 0.06136768
## k0_l5 0.95257413
## k1_l5 0.04742587
```

6. Write a function that find the equilibrium conditional choice probability and ex-ante value function by iterating the update of an exante value function and an optimal conditional choice probability. The iteration should stop when $\max_s |V^{(r+1)}(s) - V^{(r)}(s)| < \lambda$ with $\lambda = 10^{-10}$.

```
output <-
  solve_dynamic_decision(
    PI = PI,
    G = G,
    L = L,
    K = K,
    delta = delta,
    lambda = lambda
    );
output
```

```
## $p
##              [,1]
## k0_l1 0.82218962
## k1_l1 0.17781038
## k0_l2 0.80024354
## k1_l2 0.19975646
## k0_l3 0.83074516
## k1_l3 0.16925484
## k0_l4 0.87691534
## k1_l4 0.12308466
## k0_l5 0.95257413
## k1_l5 0.04742587
##
## $V
##         [,1]
## l1 15.46000
## l2 18.03675
## l3 20.86514
## l4 23.33721
## l5 25.15557
```

```r
p <- output$p
V <- output$V
value <-
  compute_choice_value(
    V = V,
    PI = PI,
    G = G,
    delta = delta
    );
value
```

```
##            [,1]
## k0_l1 14.68700
## k1_l1 13.15574
## k0_l2 17.23669
## k1_l2 15.84887
## k0_l3 20.10249
## k1_l3 18.51157
## k0_l4 22.62865
## k1_l4 20.66511
## k0_l5 24.52976
## k1_l5 21.52976
```

7. Write a function `simulate_dynamic_decision(p, s, PI, G, L, K, T, delta, seed)` that simulate the data for a single firm starting from an initial state for T periods. The function should accept a value of seed and set the seed at the beginning of the procedure inside the function, because the process is stochastic. To match the generated random numbers, for each period, generate action using `rmultinom` and then state using `rmultinom`.

```r
# set initial value
s <- 1
# draw simulation for a firm
seed <- 1
df <-
  simulate_dynamic_decision(
    p = p,
    s = s,
    G = G,
    L = L,
    K = K,
    T = T,
    delta = delta,
    seed = seed
    );
df
```

```
## # A tibble: 100 x 3
##       t     s     a
##   <int> <dbl> <dbl>
## 1     1     1     0
## 2     2     1     0
## 3     3     1     0
## 4     4     1     1
## 5     5     2     1
## 6     6     1     0
```

```
##  7     7     1     0
##  8     8     1     0
##  9     9     1     0
## 10    10     1     0
## # i 90 more rows
```

8. Write a function `simulate_dynamic_decision_across_firms(p, s, PI, G, L, K, T, N, delta)` that returns simulation data for N firm. For firm i, set the seed at i

```
df <-
  simulate_dynamic_decision_across_firms(
    p = p,
    s = s,
    G = G,
    L = L,
    K = K,
    T = T,
    N = N,
    delta = delta
    )
saveRDS(
  df,
  file = "lecture/data/a7/df.rds" %>% here::here()
)
```

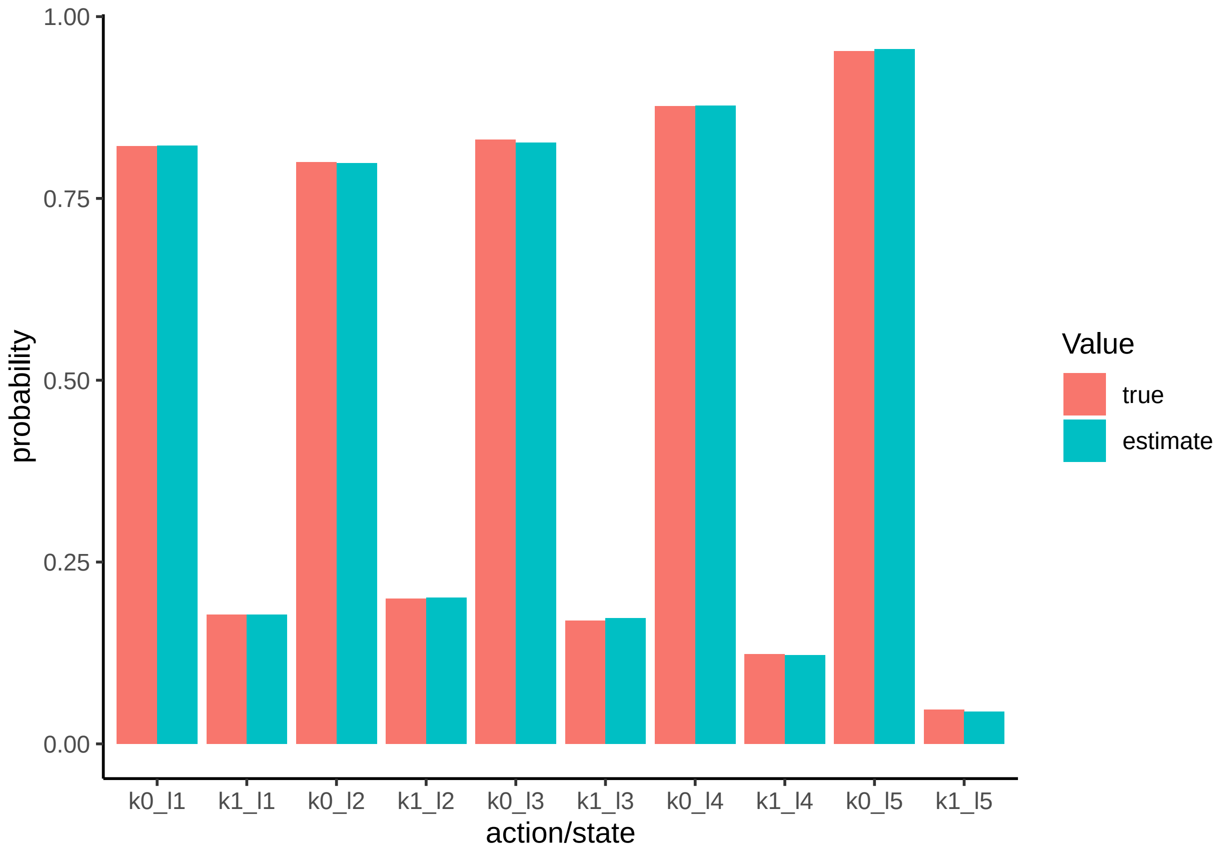
```
df <-
  readRDS(
    file = "lecture/data/a7/df.rds" %>% here::here()
  )
df
```

```
## # A tibble: 100,000 x 4
##        i     t     s     a
##    <int> <int> <dbl> <dbl>
##  1     1     1     1     0
##  2     1     2     1     0
##  3     1     3     1     0
##  4     1     4     1     1
##  5     1     5     2     1
##  6     1     6     1     0
##  7     1     7     1     0
##  8     1     8     1     0
##  9     1     9     1     0
## 10     1    10     1     0
## # i 99,990 more rows
```

9. Write a function `estimate_ccp(df)` that returns a non-parametric estimate of the conditional choice probability in the data. Compare the estimated conditional choice probability and the true conditional choice probability by a bar plot.

```
p_est <- estimate_ccp(df = df)
check_ccp <-
  cbind(
    p,
    p_est
    )
colnames(check_ccp) <-
```

```r
  c(
    "true",
    "estimate"
    )
check_ccp <-
  check_ccp %>%
  reshape2::melt()
ggplot(
  data = check_ccp,
  aes(
    x = Var1,
    y = value,
    fill = Var2
    )
  ) +
  geom_bar(
    stat = "identity",
    position = "dodge"
    ) +
  labs(fill = "Value") +
  xlab("action/state") +
  ylab("probability")  +
  theme_classic()
```



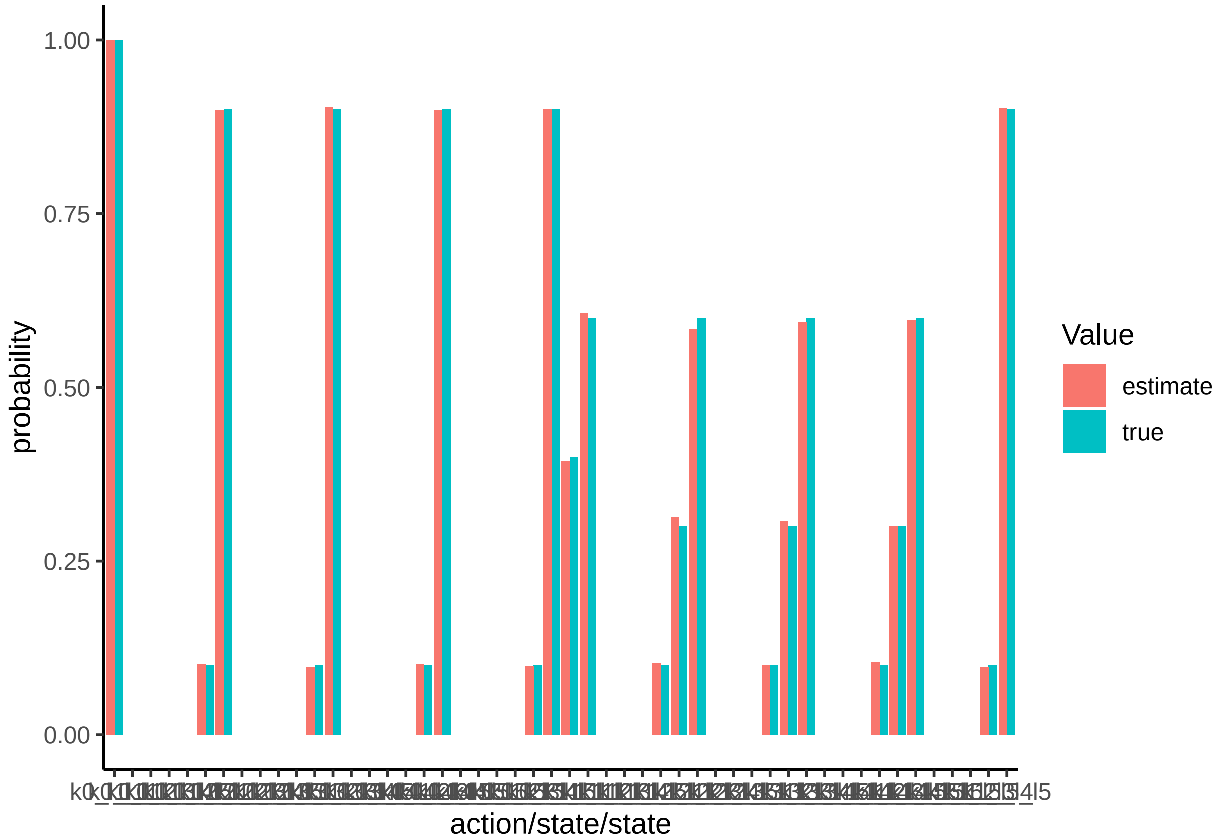
10. Write a function `estimate_G(df)` that returns a non-parametric estiamte of the transition matrix in the data. Compare the estimated transition matrix and the true transition matrix by a bar plot.

```r
G_est <-
  estimate_G(
    df = df
  );
G_est
```

```
##              l1         l2        l3         l4        l5
## k0_l1 1.0000000 0.00000000 0.0000000 0.00000000 0.0000000
## k1_l1 0.3930818 0.60691824 0.0000000 0.00000000 0.0000000
## k0_l2 0.1012162 0.89878384 0.0000000 0.00000000 0.0000000
## k1_l2 0.1031410 0.31276454 0.5840945 0.00000000 0.0000000
## k0_l3 0.0000000 0.09660837 0.9033916 0.00000000 0.0000000
## k1_l3 0.0000000 0.09974569 0.3071489 0.59310540 0.0000000
## k0_l4 0.0000000 0.00000000 0.1012564 0.89874358 0.0000000
## k1_l4 0.0000000 0.00000000 0.1039339 0.29966003 0.5964060
## k0_l5 0.0000000 0.00000000 0.0000000 0.09891400 0.9010860
## k1_l5 0.0000000 0.00000000 0.0000000 0.09751037 0.9024896
```

```r
check_G <-
  data.frame(
    type = "true",
    reshape2::melt(G)
    )
check_G_est <-
  data.frame(
    type = "estimate",
    reshape2::melt(G_est)
    )
check_G <-
  rbind(
    check_G,
    check_G_est
    )
check_G$variable <-
  paste(
    check_G$Var1,
    check_G$Var2,
    sep = "_"
    )
ggplot(
  data = check_G,
  aes(
    x = variable,
    y = value,
    fill = type
    )
  ) +
    geom_bar(
      stat = "identity",
      position = "dodge"
      ) +
  labs(fill = "Value") +
  xlab("action/state/state") +
  ylab("probability") +
```

```r
  theme(axis.text.x = element_blank())  +
  theme_classic()
```



Estimate parameters

1. Vectorize the parameters as follows:

```r
theta_1 <-
  c(
    alpha,
    beta
    )
theta_2 <-
  c(
    kappa,
    gamma
    )
theta <-
  c(
    theta_1,
    theta_2
    )
```

First, we estimate the parameters by a nested fixed-point algorithm. The loglikelihood for $\{a_{it}, s_{it}\}_{i=1,\cdots,N,t=1,\cdots,T}$ is:

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\log \mathbb{P}\{a_{it}|s_{it}\} + \log \mathbb{P}\{s_{i,t+1}|a_{it}, s_{it}\}],$$

with $\mathbb{P}\{s_{i,T+1}|a_{iT}, s_{iT}\} = 1$ for all i as $s_{i,T+1}$ is not observed.

2. Write a function `compute_loglikelihood_NFP(theta, df, delta, L, K)` that compute the loglikelihood.

```
loglikelihood <-
  compute_loglikelihood_NFP(
    theta = theta,
    df = df,
    delta = delta,
    L = L,
    K = K
    );
loglikelihood
```

```
## [1] -0.7474961
```

3. Check the value of the objective function around the true parameter.

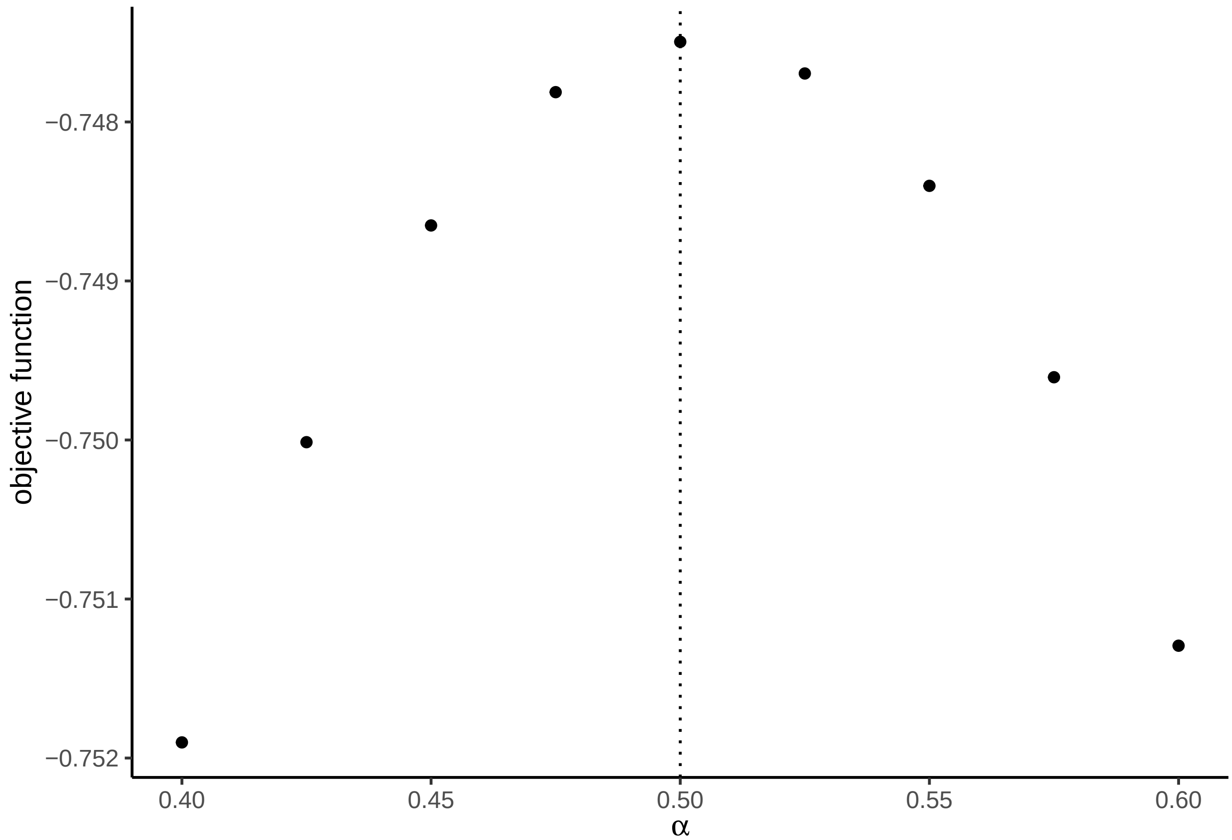
```
# label
label <-
  c(
    "\\alpha",
    "\\beta",
    "\\kappa",
    "\\gamma"
    )
label <-
  paste(
    "$",
    label,
    "$",
    sep = ""
    )
# compute the graph
graph <-
  foreach (
    i = 1:length(theta)
    ) %do% {
  theta_i <- theta[i]
  theta_i_list <-
    theta_i * seq(
    0.8,
    1.2,
    by = 0.05
    )
  objective_i <-
    foreach (
      j = 1:length(theta_i_list),
      .combine = "rbind"
      ) %do% {
               theta_ij <- theta_i_list[j]
               theta_j <- theta
               theta_j[i] <- theta_ij
               objective_ij <-
                 compute_loglikelihood_NFP(
```

```r
                   theta_j,
                   df,
                   delta,
                   L,
                   K
                   );
               loglikelihood

               return(objective_ij)
             }
  df_graph <-
    data.frame(
      x = theta_i_list,
      y = objective_i
      )
  g <-
    ggplot(
      data = df_graph,
      aes(
        x = x,
        y = y
        )
      ) +
    geom_point() +
    geom_vline(
      xintercept = theta_i,
      linetype = "dotted"
      ) +
    ylab("objective function") +
    xlab(TeX(label[i]))  +
    theme_classic()
  return(g)
}
saveRDS(
  graph,
  file = "lecture/data/a7/NFP_graph.rds" %>% here::here()
)
```

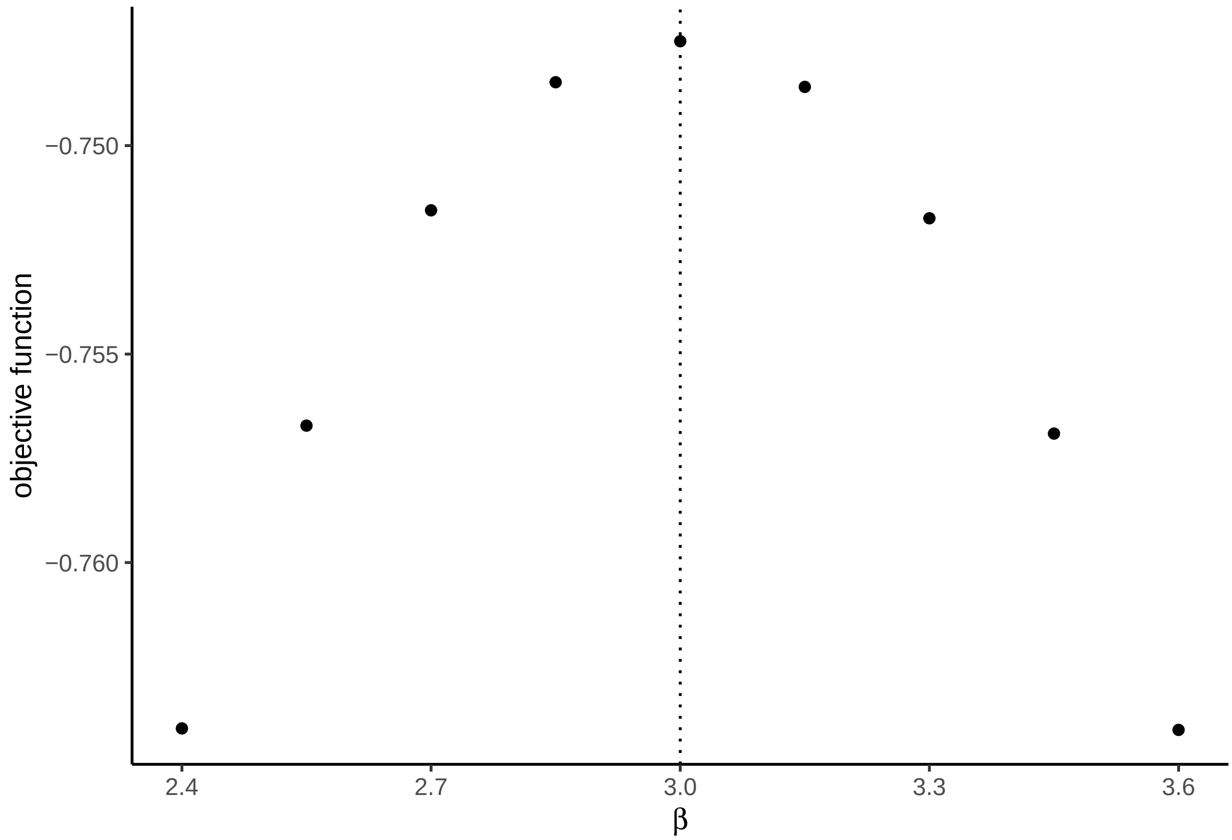
```r
graph <-
  readRDS(
    file = "lecture/data/a7/NFP_graph.rds" %>% here::here()
  )
graph
```

```
## [[1]]
```

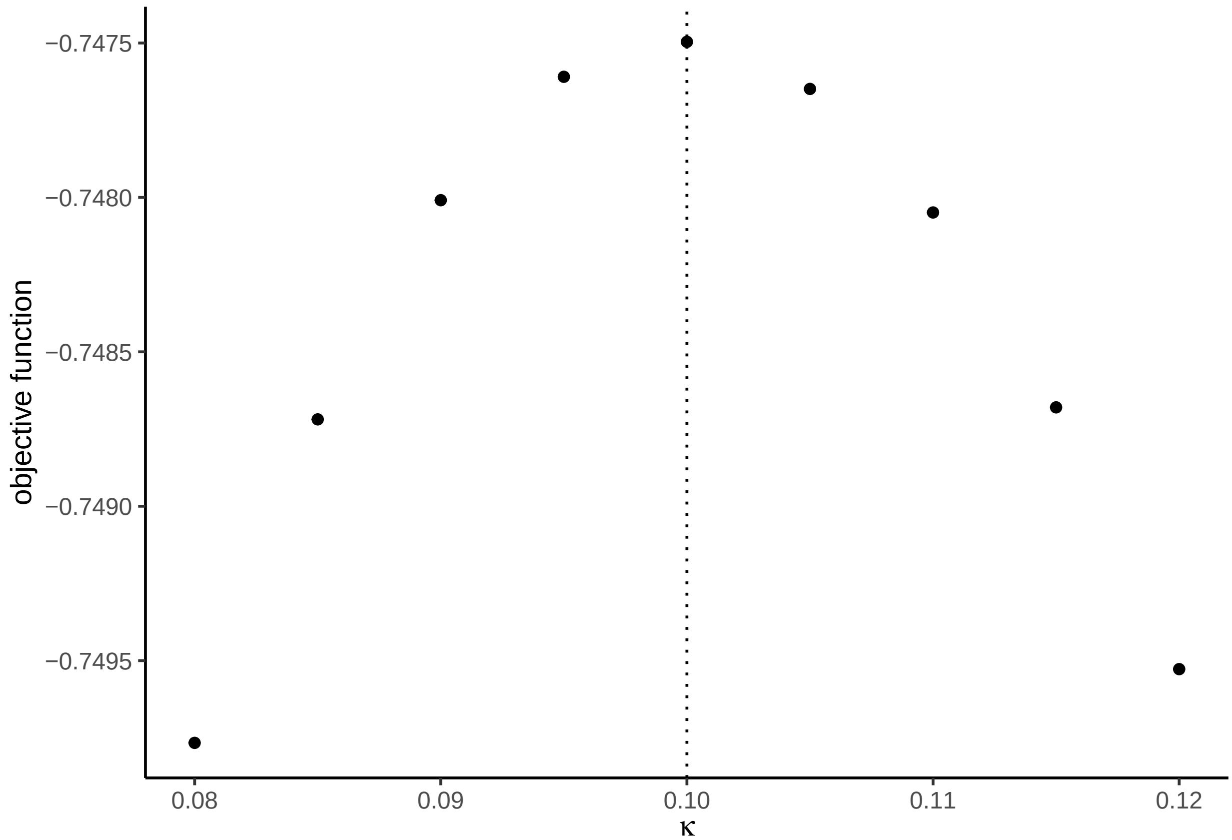


objective function
−0.748
−0.749
−0.750
−0.751
−0.752
0.40
0.45
0.50
0.55
0.60
α


```
##
## [[2]]
```

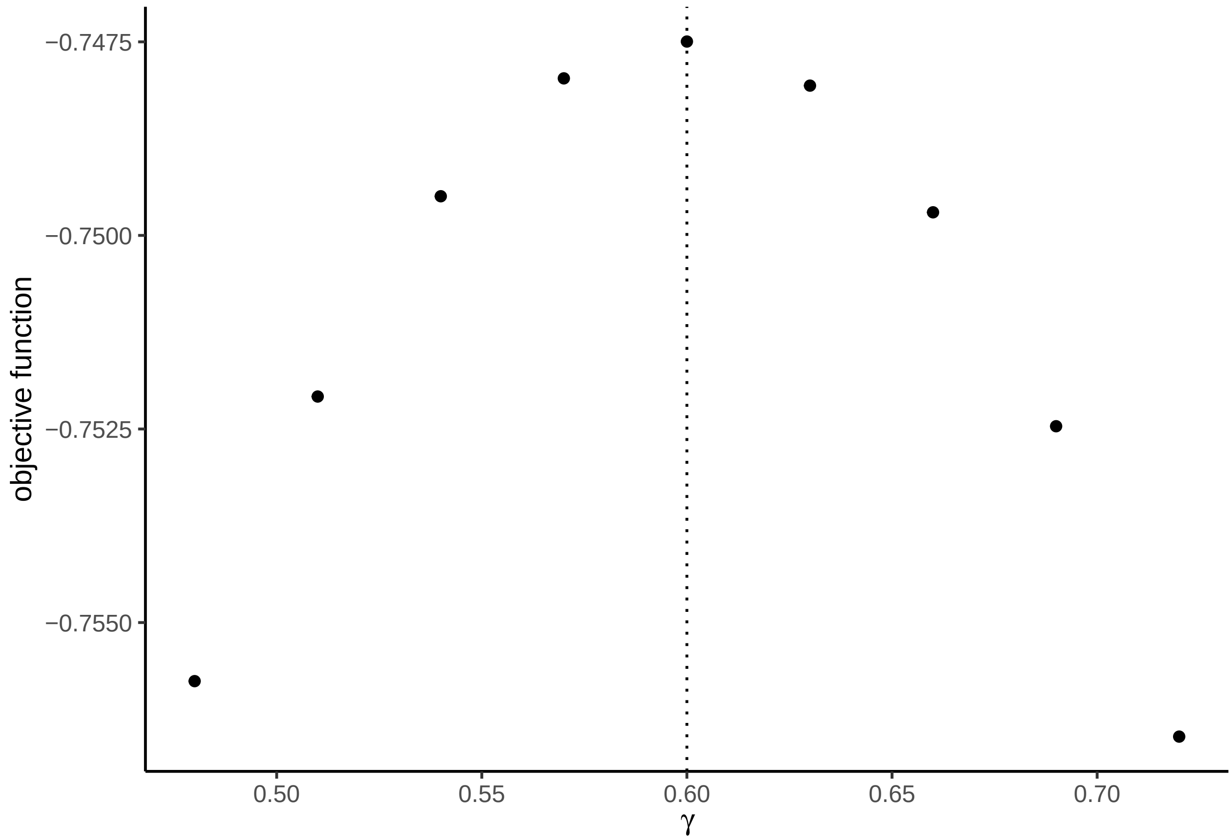



```
##
## [[3]]
```



−0.7475
−0.7480
−0.7485
−0.7490
−0.7495
objective function
0.08
0.09
0.10
0.11
0.12
κ


```
##
## [[4]]
```



4. Estiamte the parameters by maximizing the loglikelihood. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1], \beta \in [0, 5], \kappa \in [0, 0.2], \gamma \in [0, 0.7]$.

```
lower <- rep(0, length(theta))
upper <- c(1, 5, 0.2, 0.7)
NFP_result <-
  optim(
    par = theta,
    fn = compute_loglikelihood_NFP,
    method = "L-BFGS-B",
    lower = lower,
    upper = upper,
    control = list(fnscale = -1),
    df = df,
    delta = delta,
    L = L,
    K = K
  )
saveRDS(
  NFP_result,
  file = "lecture/data/a7/NFP_result.rds" %>% here::here()
)
```

```
NFP_result <-
  readRDS(
    file = "lecture/data/a7/NFP_result.rds" %>% here::here()
  )
NFP_result
```

```
## $par
## [1] 0.5273235 3.0652558 0.1000122 0.5955431
##
## $value
## [1] -0.7474743
##
## $counts
## function gradient
##       21       21
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```
compare <-
  data.frame(
    true = theta,
    estimate = NFP_result$par
  );
compare
```

```
##   true  estimate
## 1  0.5 0.5273235
## 2  3.0 3.0652558
## 3  0.1 0.1000122
## 4  0.6 0.5955431
```

Next, we estimate the parameters by CCP approach.

5. Write a function `estimate_theta_2(df)` that returns the estimates of κ and γ directly from data by counting relevant events.

```
theta_2_est <-
  estimate_theta_2(
    df = df
  );
theta_2_est
```

```
## [1] 0.09988488 0.59551895
```

The objective function of the minimum distance estimator based on the conditional choice probability approach is:

$$\frac{1}{KL}\sum_{s=1}^{L}\sum_{a=1}^{K}\{\hat{p}(a,s)-p^{(\theta_1,\theta_2)}(a,s)\}^2,$$

where $\hat{p}$ is the non-parametric estimate of the conditional choice probability and $p^{(\theta_1,\theta_2)}$ is the optimal conditional choice probability under parameters θ_1 and θ_2.

6. Write a function `compute_CCP_objective(theta_1, theta_2, p_est, L, K, delta)` that returns the objective function of the above minimum distance estimator given a non-parametric estimate of the conditional choice probability and θ_1 and θ_2.

```
compute_CCP_objective(
  theta_1 = theta_1,
  theta_2 = theta_2,
  p_est = p_est,
```

```r
  L = L,
  K = K,
  delta = delta
  )
```

```
## [1] 5.000511e-06
```

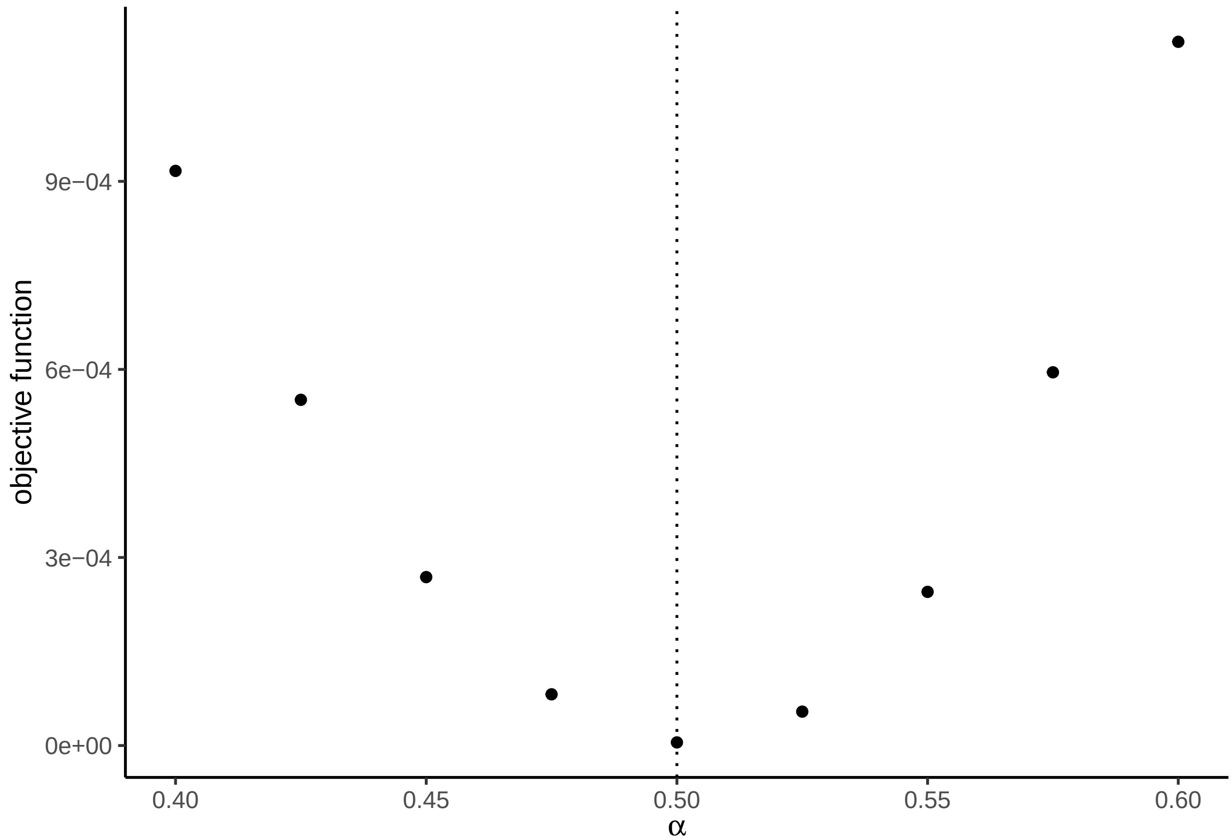
3. Check the value of the objective function around the true parameter.

```r
# label
label <-
  c(
    "\\alpha",
    "\\beta"
    )
label <-
  paste(
    "$",
    label,
    "$",
    sep = ""
    )
# compute the graph
graph <-
  foreach (
    i = 1:length(theta_1)
    ) %do% {
  theta_i <- theta_1[i]
  theta_i_list <-
    theta_i * seq(
      0.8,
      1.2,
      by = 0.05
      )
  objective_i <-
    foreach (
      j = 1:length(theta_i_list),
      .combine = "rbind"
      ) %do% {
              theta_ij <- theta_i_list[j]
              theta_j <- theta_1
              theta_j[i] <- theta_ij
              objective_ij <-
                compute_CCP_objective(
                  theta_j,
                  theta_2,
                  p_est,
                  L,
                  K,
                  delta
                  )
              return(objective_ij)
            }
  df_graph <-
    data.frame(
```

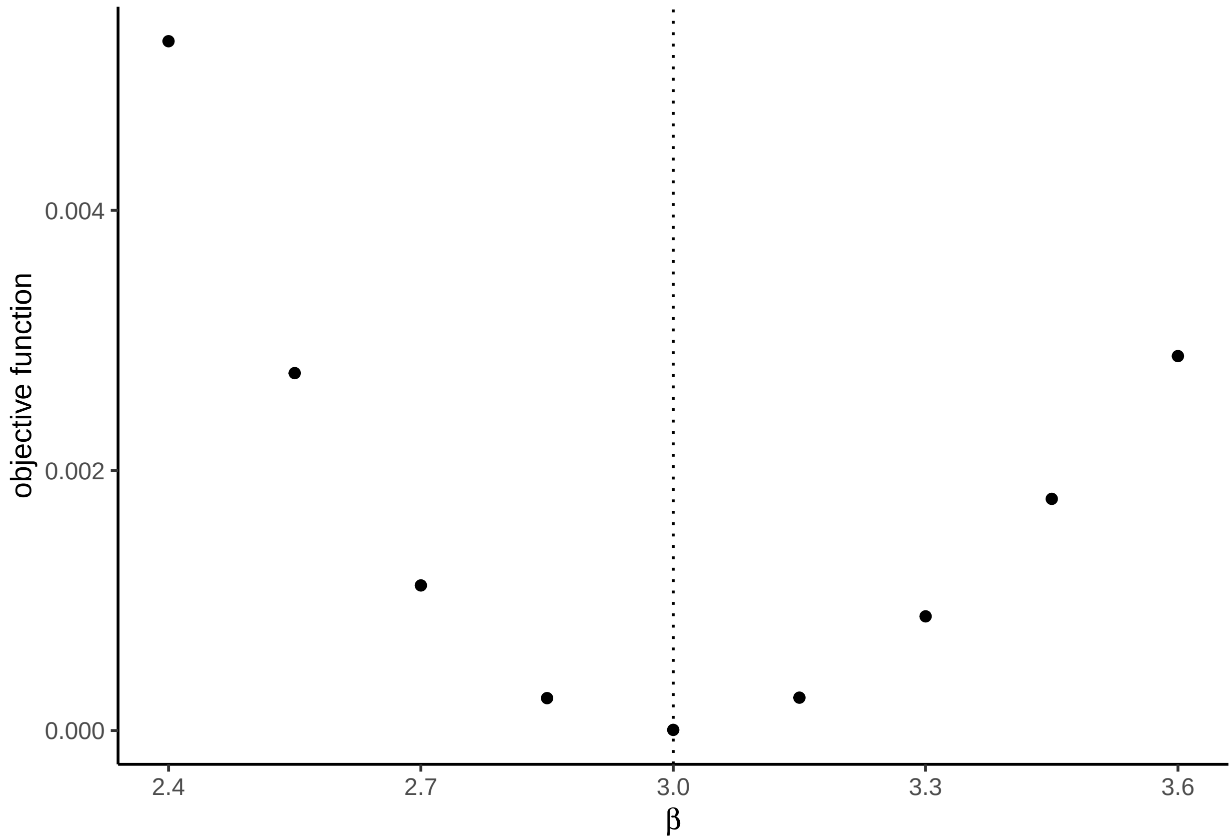
```r
      x = theta_i_list,
      y = objective_i
      )
  g <-
    ggplot(
      data = df_graph,
      aes(
        x = x,
        y = y)
      ) +
    geom_point() +
    geom_vline(
      xintercept = theta_i,
      linetype = "dotted"
      ) +
    ylab("objective function") +
    xlab(TeX(label[i]))  +
    theme_classic()
  return(g)
}
saveRDS(
  graph,
  file = "lecture/data/a7/CCP_graph.rds" %>% here::here()
)
```

```r
graph <-
  readRDS(
    file = "lecture/data/a7/CCP_graph.rds" %>% here::here()
  )
graph
```

```
## [[1]]
```




```
##
## [[2]]
```



4. Estiamte the parameters by minimizing the objective function. To keep the model to be well-defined, impose an ad hoc lower and upper bounds such that $\alpha \in [0, 1], \beta \in [0, 5]$.

```
lower <- rep(0, length(theta_1))
upper <- c(1, 5)
CCP_result <-
  optim(
        par = theta_1,
        fn = compute_CCP_objective,
        method = "L-BFGS-B",
        lower = lower,
        upper = upper,
        theta_2 = theta_2_est,
        p_est = p_est,
        L = L,
        K = K,
        delta = delta
        )
saveRDS(
  CCP_result,
  file = "lecture/data/a7/CCP_result.rds" %>% here::here()
)
```

```
CCP_result <-
  readRDS(
    file = "lecture/data/a7/CCP_result.rds" %>% here::here()
  )
CCP_result
```

```
## $par
## [1] 0.5271684 3.0644600
##
## $value
## [1] 1.790528e-06
##
## $counts
## function gradient
##       11       11
##
## $convergence
## [1] 0
##
## $message
## [1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```
compare <-

  data.frame(
    true = theta_1,
    estimate = CCP_result$par
  );
compare
```

```
##   true  estimate
## 1  0.5 0.5271684
## 2  3.0 3.0644600
```