Advanced Numerical Analysis Homework 1

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1 Stationary Iterative Methods

Jacobi, Guass Siedel and their respective relaxation methods for solving the linear system of equations $Ax = b$ have been studied in this assignment. An alternative formulation of the methods has been used for the codes. It involves the **vector form**[1] of the methods which is given below.
For Jacobi, a generalization of the Jacobi method has been used with a relaxation parameter ω. The vector form of the **Jacobi Over Relaxation method (JOR)** is:

$$x^{(k+1)} = x^{(k)} + \omega.D^{-1}.r \tag{1}$$

For the **Siedel Over Relaxation (SOR)**, the vector form is given as :

$$x^{(k+1)} = x^{(k)} + (\frac{1}{\omega}.D + L)^{-1}.r \tag{2}$$

where D is the diagonal part of A, L is the strictly lower triangular part of A, r is the residual vector given as

$$r = A.x^{(k)} - b \tag{3}$$

2 Codes

'HW.m' is the main code which solves the exercises 2 and 3. The files 'jor.m' and 'sor.m' implements the JOR & SOR methods respectively with $\omega = 1$ in JOR is Jacobi and the same in SOR is GS. The functions take as input the matrix A, right hand side vector b, intial guess vector x0, the tolerance tol, and the relaxation parameter omega and returns the solution x, number of iterations iter, and the error at each iteration err_J

3 Results

3.1 Exercise 2

The Poisson problem in (4) has been solved using the Jacobi and Guass Siedel methods and Finite Difference Approximation and Dirichlet Boundary Condition

$$-\nabla^2 u = f \tag{4}$$

Matlab codes were developed using the above vector form of the methods with stopping criteria given as

$$||x^{(k+1)} - x^{(k)}|| < \epsilon \tag{5}$$

where ϵ is user defined tolerance.
Both the methods are compared for 4 different grid sizes. The grid size (nx) vales are $[20, 40, 60, 80]$. The plot in Figure1 shows the results of the run. From the plot we can obtain the conclusions as following

- As the grid size (nx) increases the number of iterations increase for both the methods

Method	No. of Iterations	Estimated No. of Its	Error
Jacobi	2604	3581	9.9914×10^{-9}
GS	1371	1791	9.9048×10^{-9}
JOR ($\omega = 0.5$)	4944	–	9.9876×10^{-9}
SOR ($\omega = \omega_{opt}$)	105	91	8.6070×10^{-9}

Table 1: Comparison of the four methods for $nx = 30$

- The GS method requires roughly half the number of iterations of the Jacobi Method. The main reason is attributed to the fact that the FDM discretization of the Poisson problem with Dirichilet BC is a Tridiagonal Matrix.

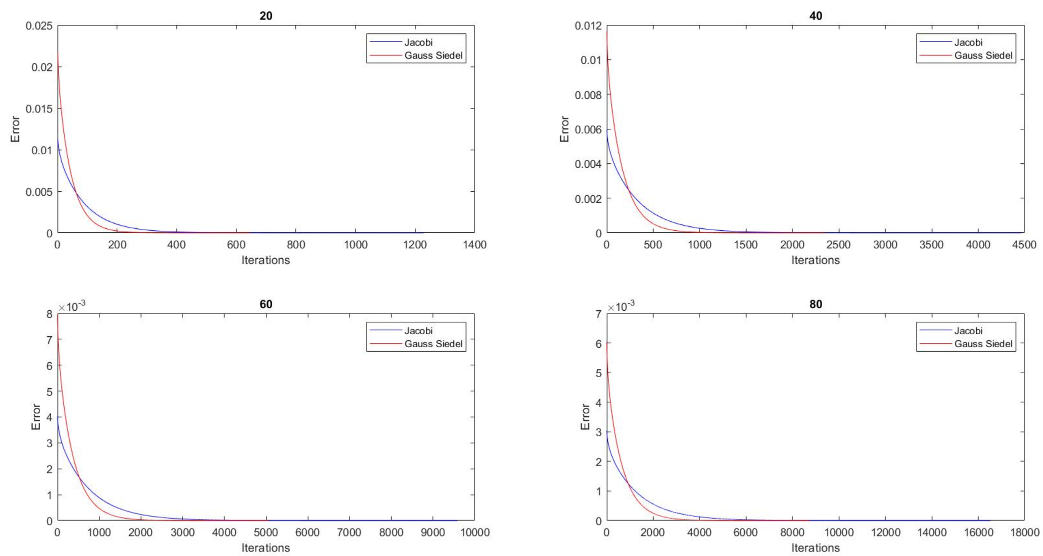

Figure 1: Comparison of Jacobi and Gauss Siedel methods for the Poisson problem for various grid sizes. Error as a function of number of iterations for each grid size.

3.2 Exercise 3

Here, we calcuate the optimum value of the relaxation parameter ω using the formula in the equation (6), where H_J is the Jacobi Iteration matrix and $\rho(H_J)$ is the spectral radius of the Jacobi Iteration Matrix. [1]

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2(H_J)}} \tag{6}$$

The value of ω_{opt} is found as *1.8163*.
We know that the SOR method converges for $0 < \omega < 2$ and the JOR method converges for $0 < \omega \leq 1$ if the Jacobi converges. Hence, for the purpose of study we choose a random ω for JOR as 0.5. The plot in the figure 2 and the table 1 shows the comparison of the four methods.

We find the SOR with the optimum value of ω takes the least number of iterations and it had reduced roughly by a factor more than 10 from that of the Gauss Seidel method.This is attributed to the fact that the matrix A enojys the A-property.
The GS method takes roughly half the iterations than the Jacobi method which is due to the fact that A is

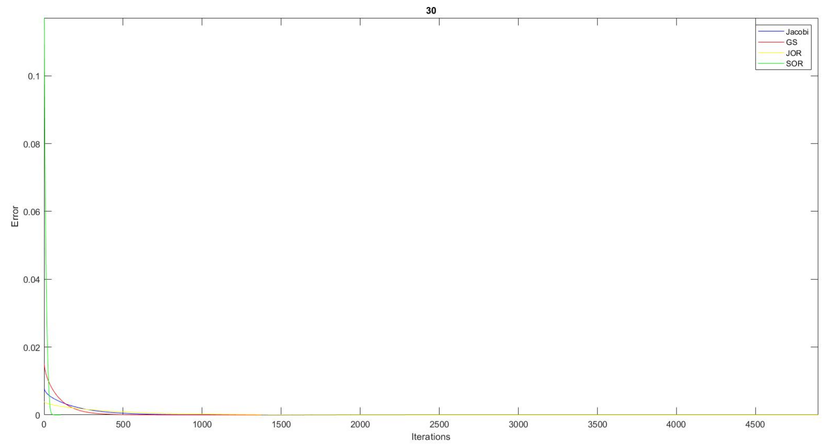

Figure 2: Comparison of Jacobi, GS, JOR, SOR methods for the Poisson problem with grid size nx = 30. $\omega = \omega_{opt}$ for SOR and $\omega = 0.5$ for JOR

tridiagonal.We know that if a matrix is tridiagonal, equation 7 follows.

$$\rho(H_{GS}) = \rho^2(H_J) \tag{7}$$

The estimated number of iterations(k) for an error reduction by a factor of 10^p is calculated per the equation 8

$$k = \frac{p}{-\log_{10}\rho(H)} \tag{8}$$

. Hence, subustituting (7) in (8), we obtain $k_J = 2 \times k_{GS}$

References

[1] Alfio Quarteroni, Riccardo Sacco, and Fausto Saleri. *Numerical Mathematics (Texts in Applied Mathematics)*. Berlin, Heidelberg: Springer-Verlag, 2006. ISBN: 3540346589.