Exercises on stationary iterative methods for sparse linear systems
Advanced Numerical Analysis (2019–2020).

Stationary iterative methods

If A is nonsingular and $a_{ii} \neq 0,\ i = 1, \ldots, n$, let us split $A = M - N$, with M nonsingular. The linear system $A\mathbf{x} = \mathbf{b}$ can be rewritten as

$$(M - N)\mathbf{x} = \mathbf{b} \quad \Longrightarrow \quad M\mathbf{x} = N\mathbf{x} + \mathbf{b}$$

From this equation we can write a class of iterative methods like

$$M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \mathbf{b}, \qquad k \geq 0. \tag{1}$$

Equation (1) is suitable for implementation.
Recall that matrix A can be written as

$$A = L + D + U,$$

where L is the strictly lower triangular part, D is the diagonal and U is the strictly upper triangular part of A. Different definitions of M, N give raise to different methods:

1. **Jacobi**. $M = D$; $N = -(L + U)$.
2. **Gauss-Seidel**. $M = L + D$; $N = -U$.
3. **SOR**. The splitting is applied to the system $\omega A\mathbf{x} = \omega\mathbf{b}$,
   $M = \omega L + D$; $N = (1 - \omega)D - \omega U$, with $\omega \in (0, 2)$, and the iteration is $M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \omega\mathbf{b}$ (recall: for $\omega = 1$ SOR is Gauss-Seidel).

Exercises

1. Implement in Matlab the methods of Jacobi, Gauss-Seidel and SOR (note that Gauss-Seidel implementation can be avoided since it coincides with SOR with $\omega = 1$). The Matlab functions should have the following syntax:

```
function [x,iter,vdiff]=jacobi(A,b,x0,itmax,tol)
function [x,iter,vdiff]=SOR(A,b,x0,itmax,tol,omega)
```

   where A is the coefficient matrix, $\mathbf{b}$ the right-hand-side, `x0` the initial vector, `tol` the tolerance, `maxit` the maximum number of iterations (`omega` the SOR relaxation parameter). On output `x` is the approximate solution, `iter` the number of iterations employed, `vdiff` the vector of the norms of the differences $\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$. Stop the iterative process if the following test is satisfied:

$$\|\mathbf{s}^{(k+1)}\| = \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| < \texttt{tol}.$$

2. Use the Jacobi and Gauss-Seidel methods to solve the linear system $A\mathbf{x} = \mathbf{b}$, where A is the matrix arising from the discretization of the Poisson equation in the unit square

$$-\Delta u = f \ ,$$

yielded by the command

```
A = delsq(numgrid('S',nx+2))
```

and

$$\mathbf{b} = \frac{1}{(nx+1)^2} \cdot \texttt{ones(n,1)}.$$

Variable `nx` indicates the number of internal nodes on each dimension of the mesh. The total number of nodes on each mesh dimension will be then `nx+2` counting the boundary nodes. Using the values of $nx \in \{20, 40, 60, 80\}$, $\texttt{tol} = 10^{-8}$, and the vector with all zeros as the initial guess:

- Compare the number of iterations required for both methods
- Produce a picture where the number of iterations is graphed against the size of the systems for both methods (use the `figure` command to produce a different graphic window for each value of nx).

3. The largest eigenvalue of the Poisson discretized matrix A is

$$\lambda_1 = 8\cos^2\left(\frac{h\pi}{2}\right)$$

Since the diagonal of A is $D = 4I$ it is straightforward to compute also the spectral radius of the Jacobi iteration matrix $\rho(H_J)$ as

$$H_J = I - \frac{1}{4}A.$$

Therefore

$$\rho(H_J) = 2\cos^2\left(\frac{h\pi}{2}\right) - 1.$$

Knowing that A satisfies the hypotheses of the Young-Varga Theorem, also the optimal value of $\omega(\omega_{opt})$, can be analytically computed. Use $nx = 30$, compute ω_{opt} and solve the linear system with Jacobi, Gauss-Seidel and SOR(ω_{opt}) methods, use $\texttt{tol} = 10^{-8}$.

Compare the convergence profiles of the three methods.

Why Gauss-Seidel iterations are roughly half than those of Jacobi method?

The number of iterations to reduce the initial error by a factor 10^p for stationary iterative method is roughly

$$k \approx \frac{p}{R}$$

where $R = -\log_{10}\rho(H)$. Give an estimate of the number of iterations required by the 3 methods and compare it with the actual number of iterations employed.