Exercises on iterative methods for nonlinear systems
Advanced Numerical Analysis

2019–2020

1 Newton's method

Exercise 1 Write a MATLAB function that implements Newton's method to solve the nonlinear system $\mathbf{F}(\mathbf{x}) = 0$. The function can have the following syntax:

```
function [xstar,iter,resvec] = newton(x0,F,Jac,tol,itmax,lsol)
```

The vector function F and the Jacobian matrix (`Jac`) should be provided as function handles. The meaning of the additional input parameter `lsol` is as follows:

$$\texttt{lsol} = \begin{cases} 1 & \text{the inner linear system is solved by the a direct method (MATLAB "\textbackslash" or LU factorization).} \\ 0 & \text{the inner linear system is solved by an iterative method (the GMRES method, with suitable} \\ & \text{choice of } \texttt{restart, itmax, tol} \text{ parameters).} \end{cases}$$

Regarding the output parameters, `resvec` is a vector containing the norm of the residual at each nonlinear iteration. The iterative Newton procedure must be stopped whenever the following test is satisfied

$$\frac{\|\mathbf{F}(\mathbf{x}_k)\|}{\|\mathbf{F}(\mathbf{x}_0)\|} < \texttt{tol}.$$

Exercise 2 *Try your Newton implementation to solve the* 2×2 *nonlinear problem of Lecture 11, namely*

$$\begin{cases} x^2 + y^2 - 4 = 0 \\ xy - 1 = 0 \end{cases} \qquad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Use a tolerance $\texttt{tol} = 10^{-8}$ *and* $\texttt{lsol} = 1$ *for obvious reasons. Verify the concordance of your output with that of the slides.*

Exercise 3 Solve the following system of nonlinear equations

$$\mathbf{F}(\mathbf{x}) = 0, \quad \text{where} \quad \mathbf{F}(\mathbf{x}) = A\mathbf{x} - 0.1\sin(\mathbf{x}) - 5,$$

and A is the usual discretized Laplacian matrix provided by the `delsq` and `numgrid` functions with a number of internal nodes in each mesh dimension `nx = 48`.
Use an initial vector $\mathbf{x}_0$ such that $(x_0)_i = 1000 * \sin\left(\frac{i}{n}\right)$; $\texttt{tol} = 10^{-12}$ and $\texttt{itmax} = 100$.

1. Solve the problem twice using first $\texttt{lsol} = 1$ and then $\texttt{lsol} = 0$, namely the left-preconditioned GMRES method for the inner linear systems. In this second case suggested values for the parameters are:

   $\texttt{restart} = 50$; $\texttt{tol\_GMRES} = 10^{-8}$; $\texttt{itmax\_GMRES} = 20$; preconditioner: ILU(0).

2. Plot the nonlinear residual norm vs the Newton iteration number for both runs in the same picture. The two curves must roughly overlap.

Eigenvalue problem as a nonlinear system of equations

Exercise 4 Given an $n \times n$ matrix A:

1. Write an $(n+1) \times (n+1)$ system of nonlinear equations to find an eigenvalue/eigenvector pair (eigenpair) $(\mathbf{u}, \lambda)$. Use the definition of eigenvalue, where the eigenvector is subject to the constraint: $\|\mathbf{u}\| = 1$.
2. What is the Jacobian of this system?
3. Use Newton's method (with `lsol` $= 1$) to compute the smallest eigenvalue and corresponding eigenvector of the discretized Laplacian (with $nx = 50$) through the following steps:
   (a) Compute an initial approximation of $(\mathbf{u}, \lambda)$, namely $(\mathbf{u}_0, \lambda_0)$ by slightly perturbing the "exact" eigenpair computed by the Matlab function `eigs`:
   ```
   [u0,lambda0] = eigs(A,1,'sm');
   u0 = u0 + ones(n,1)*1e-2;
   lambda0 = lambda0 + 1e-2;
   ```
   (b) Solve $\mathbf{F}(\mathbf{x}) = 0$ using Newton's method with tolerance `tol` $= 10^{-12}$, `itmax` $= 20$, $\mathbf{x}_0 = \begin{bmatrix} \mathbf{u}_0 \\ \lambda_0 \end{bmatrix}$.
   (c) Plot the norm of the residuals vs the iteration number. Say how is the convergence of Newton's method in this case.
   (d) Is the computed eigenvalue accurate to machine precision?
4. **Difficult**: If $\mathbf{x} = (\mathbf{u}, \lambda) \in \mathbb{R}^{n+1}$ is an eigenpair, in which case is $F'(\mathbf{x})$ nonsingular?

2 Quasi-Newton method

Exercise 5 Write a MATLAB function that implements Broyden Quasi-Newton method to solve the nonlinear system $\mathbf{F}(\mathbf{x}) = 0$. The function can have the following syntax:

```
function [xstar,iter,resvec] = quasinewton(x0,F,B0,tol,itmax)
```

Try your function onto the 2×2 problem:

$$\begin{cases} x^2 + y^2 - 4 = 0 \\ xy - 1 = 0 \end{cases} \qquad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Use a tolerance `tol` $= 10^{-8}$. The method should converge in 8 iterations.

The vector function F (and possibly also the initial Jacobian approximation `B0`) should be provided as function handles. Solve the inner linear system using the LU factorization of `B0`.

3 Inexact Newton method

We consider the classical 2-dimensional Bratu problem [1] which is an elliptic nonlinear PDE with homogeneous Dirichlet boundary conditions. The problem is given by

$$\begin{aligned} \Delta u + \lambda \exp(u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{aligned} \tag{1}$$

[1] G. BRATU, *Sur les équations intégrales non linéaires*, Bulletin de la Société Mathématique de France, 42 (1914), pp. 113–142.

where $\Omega = [0,1]^2 \subset \mathbb{R}^2$ is an open set and $\partial\Omega$ defines its boundary; $\lambda \in [0, 6.8]$ is a real parameter. Discretization of (1) by the Finite Difference method yields the following system of nonlinear equations:

$$A\mathbf{u} + \lambda \exp(\mathbf{u}) = 0 \tag{2}$$

where $\exp(\mathbf{u}) \equiv \big(\exp(u_1), \ldots, \exp(u_n)\big)^T$, A is the Finite Difference discretization of the Laplacian.

Exercise 6 Write a MATLAB script to solve the nonlinear system (2). In detail your MATLAB script must

1. compute the (scaled) FD discretization of the Laplacian in the unit square with $h = 5 \times 10^{-3}$. **Note**: Recall that the delsq function returns the SPD matrix $B = -h^2 A$ and hence (2) reads
$$-B\mathbf{u} + h^2 \lambda \exp(\mathbf{u}) = 0. \tag{3}$$
2. set $\lambda = 6.5$ and solve (3) using $\mathbf{x}_0$ with all components equal to 0.1 and tolerance $\texttt{tol} = 10^{-13}$, with the following methods:
   (a) Newton's method with the solution of the linear systems by a direct method (LU factorization).
   (b) Broyden (Quasi-Newton) method with the solution of the linear systems by a direct method (LU factorization).
   (c) Inexact Newton method with the solution of the linear systems using the GMRES method with $p = 50$ (restart parameter), ILU preconditioner with `setup.type='ilutp'` and `droptol` $= 1e-2$, and the following four sequences of tolerance η_k.
   Set $\eta_{\max} = 0.1$, $\eta_0 = \eta_{\max}$ and, for all $k \geq 1$,
      i. $\eta_k = \eta_{\max}$
      ii. $\eta_k = \dfrac{\eta_{k-1}}{3}$
      iii. $\eta_k = \min\{\eta_{\max}, 0.95\|\mathbf{F}(\mathbf{x}_k)\|\}$
      iv. $\eta_k = \min\left\{\eta_{\max}, 0.95\dfrac{\|\mathbf{F}(\mathbf{x}_k)\|^2}{\|\mathbf{F}(\mathbf{x}_{k-1})\|^2}\right\}$
3. Display the results as a table in which every row represents a single Newton iteration showing the iteration number, the residual norm, and, in case of iterative solution of the Newton system, also the forcing term η_k and the number of GMRES iterations.

   Example of possible output:

```
 k        tol          ||F(x_k)||    LIN IT
...       ...             ....         ...
 4     1.25e-03      7.97266e-08        21
```

4. Among the four strategies suggested, which is the best choice for sequence η_k? Compare them with respect to the total number of **linear** iterations. Explain the fact that choice (iv) produces better results than choice (iii).
5. Produce two figures with the semilogarithmic convergence profile (residual norm vs number of nonlinear iteration) of the methods: the first picture should contain the Newton vs Quasi-Newton profiles; the second one should plot the convergence profiles of Newton's method with direct solution of the inner systems and Inexact Newton with the four choices of η_k.
6. Provide a table comparing execution times of all the methods in your computing environment.