Weissinger method

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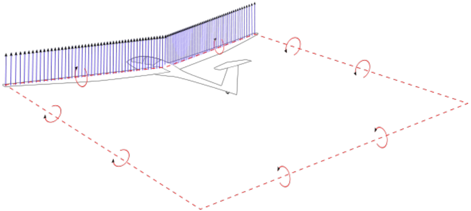
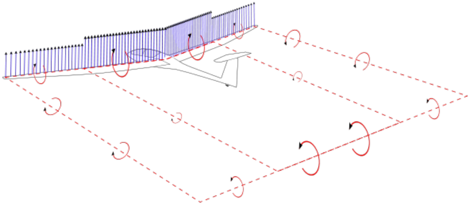
The Horseshoe Vortex



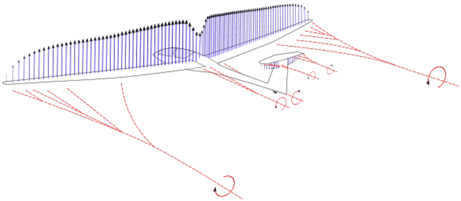
A finite wing can be (roughly) approximated by a Horseshoe Vortex, whose circulation is given by the Kutta–Joukowsky theorem.

The Horseshoe Vortex



A slightly less rough approximation is a series of spanwise horseshoe vortices, in order to represent the spanwise variation of the produced circulation.

The Horseshoe Vortex



The spanwise variation of the circulation generates a vorticity sheet downstream (the wake).

Weissinger Method: Hypothesis

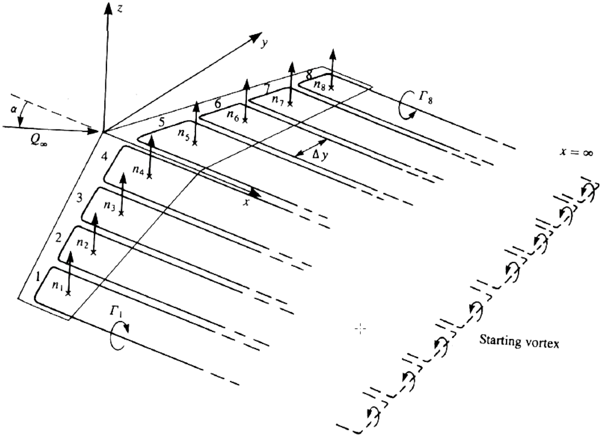
The flow field is:

- incompressible
- inviscid (viscous effects confined in a thin region such as the boundary layer and/or the thin wake)
- irrotational

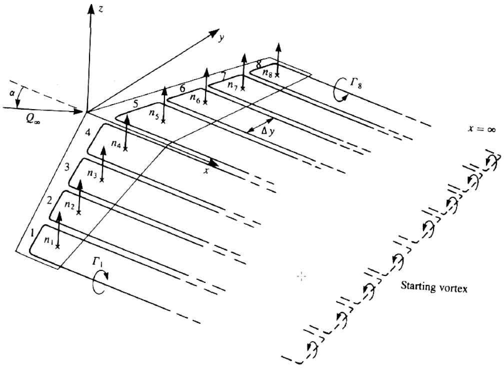
The surface is:

- flat
- thin
- with small angles

Spanwise and chordwise distribution of horse vortices



Weissinger Method: potentiality

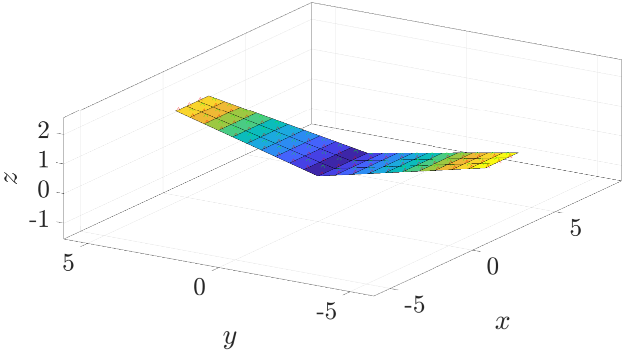


Versatile method, we can add the effect of:

- aspect ratio (AR)
- taper ratio
- sweep angle (Λ)
- dihedral angle (d)
- tail surface

Weissinger Method

On all the $2M \times N$ panels we put a horsoe vortex of constant intensity Γ_k



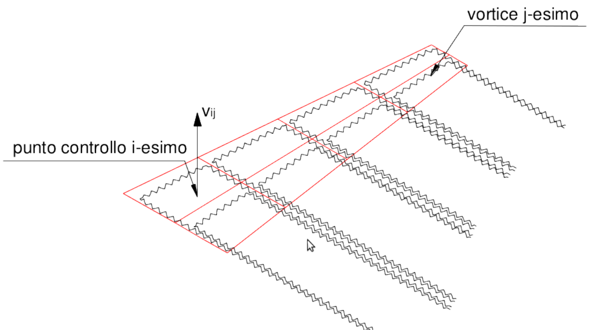
By knowing Γ_k we can evaluate the aerodynamic distributions and forces of the considered wing.

Weissinger Method

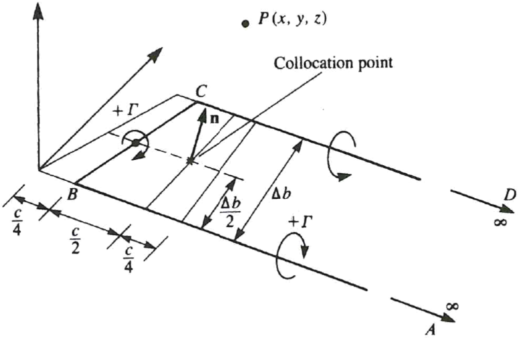
Γ_k are evaluated by imposing the no-penetration boundary condition on each panel:

$$\left(\sum_j \mathbf{v}_{jk} + \mathbf{U}_\infty\right) \cdot \hat{\mathbf{n}}_k = 0, \quad \forall k$$

$2M \times N$ unknowns and $2M \times N$ equations: $\rightarrow$ the problem is well posed.



Evaluation of v_{jk}

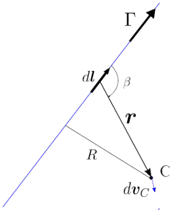


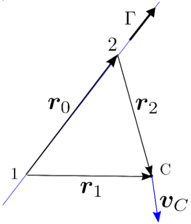
- Lifting vortexes are placed at 1/4 of the chord
- Control points are placed at the centre of the panel's 3/4 chord-line

This choice of the vortex and control points *approximately* satisfies the Kutta condition for each panel.

Evaluation of v_{jk}: Biot–Savart Law (1/2)

For a vortex filament:





$$d\mathbf{v}_C = \frac{\Gamma}{4\pi}\frac{d\mathbf{l}\times\mathbf{r}}{|\mathbf{r}|^3}, \quad \mathbf{v}_C = \frac{\Gamma}{4\pi}\int_1^2 \frac{d\mathbf{l}\times\mathbf{r}}{|\mathbf{r}|^3}$$

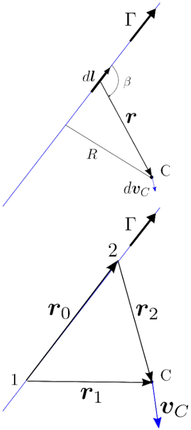
$$\mathbf{v}_C = \frac{\Gamma}{4\pi}\int_1^2 \frac{\sin\beta}{|\mathbf{r}|^2} dl\ \hat{\mathbf{e}}_C, \quad \hat{\mathbf{e}}_C = \frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1\times\mathbf{r}_2|}$$

Considering $dl = \frac{R}{\sin^2\beta}d\beta$, $|r| = \frac{R}{\sin\beta}$ and integrating:

$$\mathbf{v}_C = \frac{\Gamma}{4\pi R}(\cos\beta_1 - \cos\beta_2)\ \hat{\mathbf{e}}_C \tag{1}$$

Evaluation of v_{jk}: Biot–Savart Law (2/2)

For a vortex filament:



We define $\mathbf{r}_0 = \mathbf{x}_2 - \mathbf{x}_1$; hence:

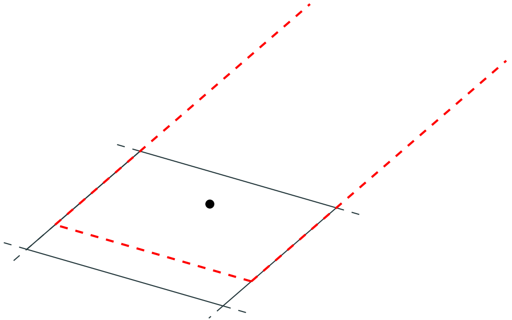
$$\cos\beta_i = \frac{\mathbf{r}_0 \cdot \mathbf{r}_i}{|\mathbf{r}_0||\mathbf{r}_i|}, \qquad R = \frac{|\mathbf{r}_1 \times \mathbf{r}_2|}{|\mathbf{r}_0|}$$

By substituting in equation (1):

$$\mathbf{v}_C = \frac{\Gamma}{4\pi}\mathbf{r}_0 \cdot \left(\frac{\mathbf{r}_1}{|\mathbf{r}_1|} - \frac{\mathbf{r}_2}{|\mathbf{r}_2|}\right) \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|^2} \tag{2}$$

Evaluation of v_{jk}

However, remember that on each panel we have put a horsoe vortex, formed by three segments..



Matrix assembly

$$\left(\sum_j \mathbf{v}_{jk} + \mathbf{U}_\infty\right) \cdot \hat{\mathbf{n}}_k = 0, \quad \forall k, \quad \mathbf{v}_{jk} = \mathbf{q}_{jk}\Gamma_j$$

where $\mathbf{q}_{jk}$ is the velocity induced by the $j-$th Horseshoe vortex with $\Gamma = 1$ on the control point of the $k - th$ panel. We obtain the Linear system:

$$\boldsymbol{A\Gamma} = \boldsymbol{b}$$

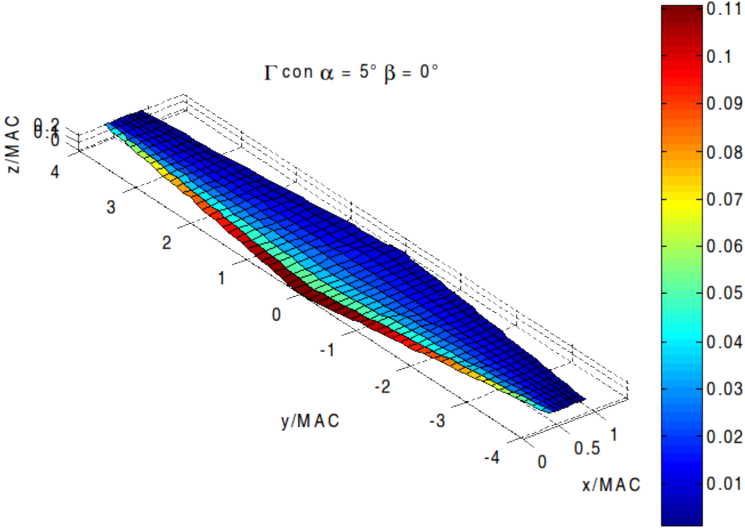
with:

$$A_{jk} = \mathbf{q}_{jk} \cdot \hat{\mathbf{n}}_k, \quad b_k = -\mathbf{U}_\infty \cdot \hat{\mathbf{n}}_k$$

A is a full matrix (every vortex induces on each panel).

Solving the linear system we obtain $\mathbf{\Gamma}$: the circulation associated to each panel

Circulation Distribution



Forces computation: lift

The Kutta-Joukowsky theorem is exploited to get the lift.

- The lift contribution of each section made is computed:

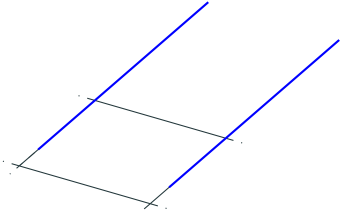
$$L_{2D,i} = \sum_{j=1}^{N} \rho U_\infty \Gamma_{ij} \cdot \cos(d) \quad \forall i = 1, 2M$$

- The total lift of the wing is computed considering all the sections' contributions:

$$L = \sum_{i=1}^{2M} \Delta b_i L_{2D,i}$$

Forces computation: induced drag

- For the computation of the induced drag the evaluation of the induced angle of incidence on each section is needed
- Only the wake contribution to the induced velocity must be considered



- In each section the induced velocity is evalauted in the quarter point of its chord-wise dimension
- The induced velocity is evaluated with the Biot-Savarts law

Forces computation: induced drag

- Determine the quarter point of each section
- Evaluate for each section the velocity induced by the wake vortex street on its quarter point, i.e. $\boldsymbol{v}_{ind,i}\ \forall i = 1..2M$
- Evaluate the induce angle of incidence of each section

$$\alpha_{ind,i} = \tan^{-1}\left(\frac{\boldsymbol{v}_{ind,i} \cdot \mathrm{e}_{z,i}}{U_\infty}\right)$$

- Evaluate the induced drag of each section:

$$D_{2D,i} = L_{2D,i}\sin(\alpha_{ind,i})$$

- Evaluate the total induced drag:

$$D = \sum_{i=1}^{2M} \Delta b_i D_{2D,i}$$

The program

We split the program into several sub-modules

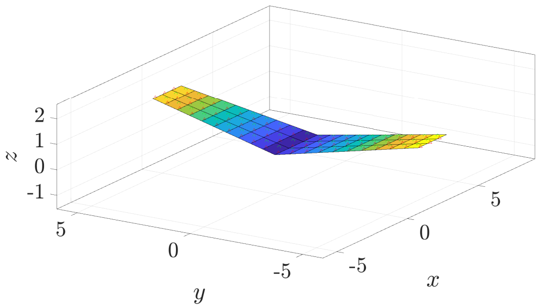
- Build the geometry
- Induced velocity calculation
- Assemble the influence matrix
- Load computations

The program: Build the geometry

Input $\longrightarrow$ Output

root chord, tip chord, taper ratio
sweep angle, dihedral angle, wing span
number of panels, etc.

wing geometry, panels' extrema
panels' quarter points, panels' control points
panels' normal vectors

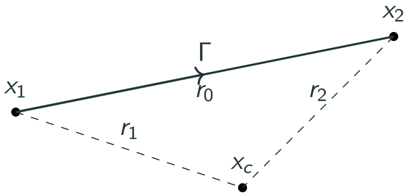


The program: Induced velocity calculation

It evaluates the induced velocity $\boldsymbol{v}_{c,i}$ of a vortex segment of extrema $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and circulation Γ into a point $\boldsymbol{x}_c$.

Input $\longrightarrow$ Output

$\boldsymbol{x_1}$, $\boldsymbol{x_2}$, $\boldsymbol{x_c}$, Γ $\qquad$ $v_{c,i}$



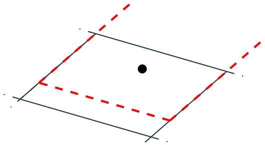
$$\boldsymbol{v_c} = \frac{\Gamma}{4\pi}\boldsymbol{r}_0 \cdot \left(\frac{\boldsymbol{r}_1}{|\boldsymbol{r}_1|} - \frac{\boldsymbol{r}_2}{|\boldsymbol{r}_2|}\right)\frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{|\boldsymbol{r}_1 \times \boldsymbol{r}_2|^2}$$

The program: Assemble the influence matrix

The linear system $\boldsymbol{A\Gamma} = \boldsymbol{b}$ is assembles

$$\begin{bmatrix} A_{11} & .. & A_{1,2M\times N} \\ .. & .. & .. \\ A_{2M\times N,1} & .. & A_{2M\times N,2M\times N} \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ .. \\ \Gamma_{2M\times N} \end{bmatrix} = \begin{bmatrix} b_1 \\ .. \\ b_{2M\times N} \end{bmatrix}$$

where



$$A_{ij} = \boldsymbol{q}_{ji} \cdot \boldsymbol{n}_i \quad b_i = -\boldsymbol{U}_\infty \cdot \boldsymbol{n}_i$$

$$\text{Input} \longrightarrow \text{Output}$$

panels' control points, panels' quarter points, panels'normal vector ,incoming velocity, etc.

A, b

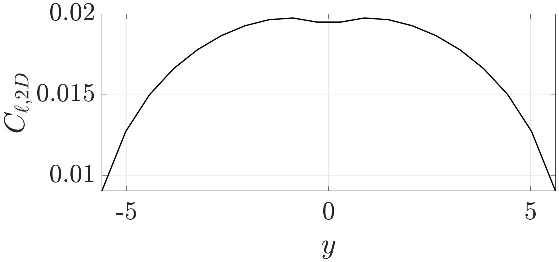
The Program: Lift Computation

Compute the lift ditribution over the wing span and the total lift

Input $\longrightarrow$ Output

Γ, geometry $\qquad$ L_{2D}, L

$$L_{2D,i} = \sum_{j=1}^{N} \rho U_\infty \Gamma_{ij} \quad \text{for } i = 1..M$$



The Program: α_{ind} and induced drag

Compute the induced angle of incidence on each section,

$$\text{Input} \longrightarrow \text{Output}$$

Γ, geometry $\qquad\qquad$ $\boldsymbol{\alpha}_{ind}$

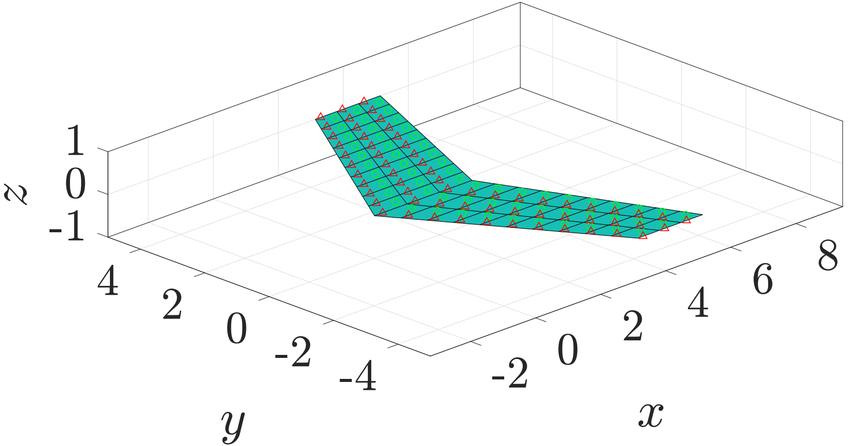
$$\alpha_{ind,i} = \tan^{-1}\left(\frac{\boldsymbol{v}_{ind,i} \cdot \mathrm{e}_{z,i}}{U_\infty}\right)$$

then

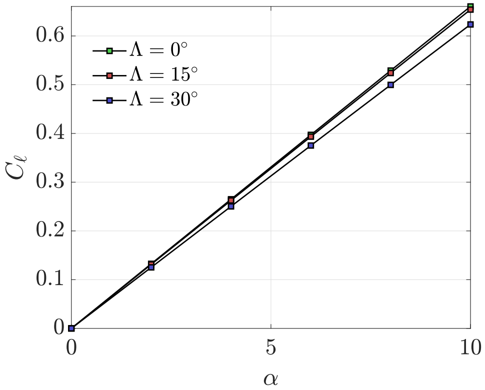
$$D_{2D,i} = L_{2D,i}\sin(\alpha_{ind,i}) \qquad D = \sum_{i=1}^{2M} \Delta b_i D_{2D,i}$$

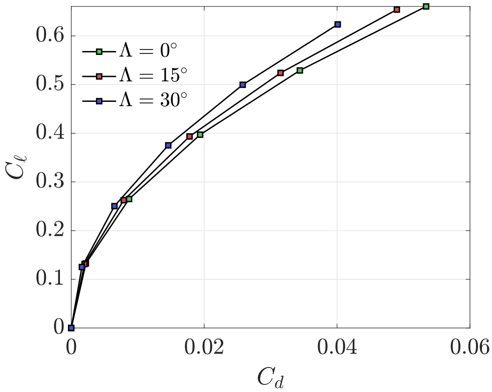
Example

$d = 0^\circ \quad \text{span} = 10 \quad \lambda = 1.5 \quad chord_{root} = 3 \quad \Lambda = 30^\circ$

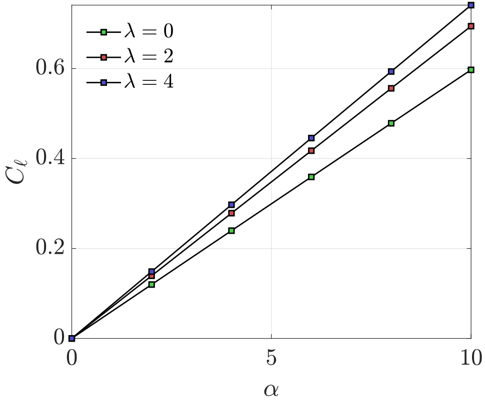


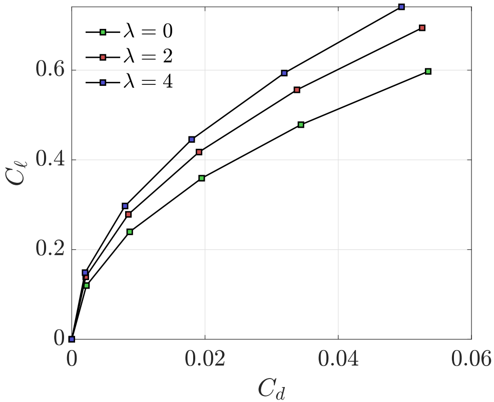
Example: Sweep angle effect





Example: taper ratio effect





Possible Outcomes

General studies

- Wing polar curves ($C_\ell - \alpha$, $C_d - \alpha$)
- Influence of the side angle β
- Influence of the taper ratio λ
- Influence of the sweep angle Λ
- Influence of the diheadral angle d
- Estimation of the stability derivatives ($C_{\ell,\beta}$, etc.)

Ground effect

- Influence of the ground on the aerodynamics performances
- Influence of different heights

Tail-plane interactions

- Influence of the presence of another wing. Study the effect of the aerodynamic interaction varying the tailplane height, size and distance.