Dimensionless conservation equations

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

$$\mathrm{Ra} = \frac{gL^3\beta\Delta T}{\nu\kappa}$$

Navier-Stokes equation

$$\frac{1}{\mathrm{Pr}} \cdot \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \nabla^2 \vec{v} + \mathrm{Ra} \cdot T \vec{e}_y$$

Thermodynamic equation

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \nabla^2 T$$

Continuity equation

$$\nabla \cdot \vec{v} = 0$$

Pr: Prandtl number
Ra: Rayleigh number
$\vec{v}$: Velocity vector (u,v)
ρ: Density
p: Pressure
ν: Kinematic viscosity
g: Gravitational acceleration
β: Thermal expansion coefficient
T: Temperature
$\vec{e}_y$: Unit vector in y direction
κ: Diffusion coefficient

Navier-Stokes equation (individual components)

(1) $$\frac{1}{\mathrm{Pr}} \cdot \left(\frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u\right) = -\frac{\partial p}{\partial x} + \nabla^2 u$$

(2) $$\frac{1}{\mathrm{Pr}} \cdot \left(\frac{\partial v}{\partial t} + \vec{v} \cdot \nabla v\right) = -\frac{\partial p}{\partial y} + \nabla^2 v + \mathrm{Ra} \cdot T$$

Vector product

$$\frac{\partial(2)}{\partial x} - \frac{\partial(1)}{\partial y}$$

→ $$\frac{1}{\mathrm{Pr}} \cdot \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \nabla^2 \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \mathrm{Ra} \cdot \frac{\partial T}{\partial x}$$

Vorticity

$$\omega = \nabla \times \vec{v}$$

$$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

→ $$\frac{1}{\mathrm{Pr}} \cdot \left(\frac{\partial \omega}{\partial t} + \vec{v} \cdot \nabla \omega\right) = \nabla^2 \omega + \mathrm{Ra} \cdot \frac{\partial T}{\partial x}$$

New equations (with finite Prandtl number)

(1) $$\frac{1}{\mathrm{Pr}} \cdot \left(\frac{\partial \omega}{\partial t} + \vec{v} \cdot \nabla \omega \right) = \nabla^2 \omega + \mathrm{Ra} \cdot \frac{\partial T}{\partial x}$$

(2) $$\nabla^2 \psi = -\omega$$ ← only one Poisson equation

(3) $$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \nabla^2 T$$

ω now also depends on the initial conditions and must be integrated forward

Exercise 8

Extend your Fortran program from Exercise 7 such that it solves the conservation equations for a fluid with a finite Prandtl number (e.g., the atmosphere):

(1) $$\frac{1}{\mathrm{Pr}} \cdot \left(\frac{\partial \omega}{\partial t} + \vec{v} \cdot \nabla \omega \right) = \nabla^2 \omega + \mathrm{Ra} \cdot \frac{\partial T}{\partial x}$$

(2) $$\nabla^2 \psi = -\omega$$

(3) $$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \nabla^2 T$$

Exercise 8

1. Read in the input variables from a namelist:
   - Number of grid points `nx` and `ny`
   - Integration time `total_time`
   - Time step constants `a_adv` and `a_diff`
   - Convergence criterion for Poisson solver `max_err`
   - Rayleigh number `Ra`
   - Temperature initialization `T_ini_type`: `random` or `cosine`
   - NEW: Prandtl number `Pr`

2. Initialize the variables:
   - Temperature `T` with random numbers or a cosine function
   - Stream function and vorticity `S=W=0` (only at the beginning)
   - Grid spacing `h=1./(ny-1.)`

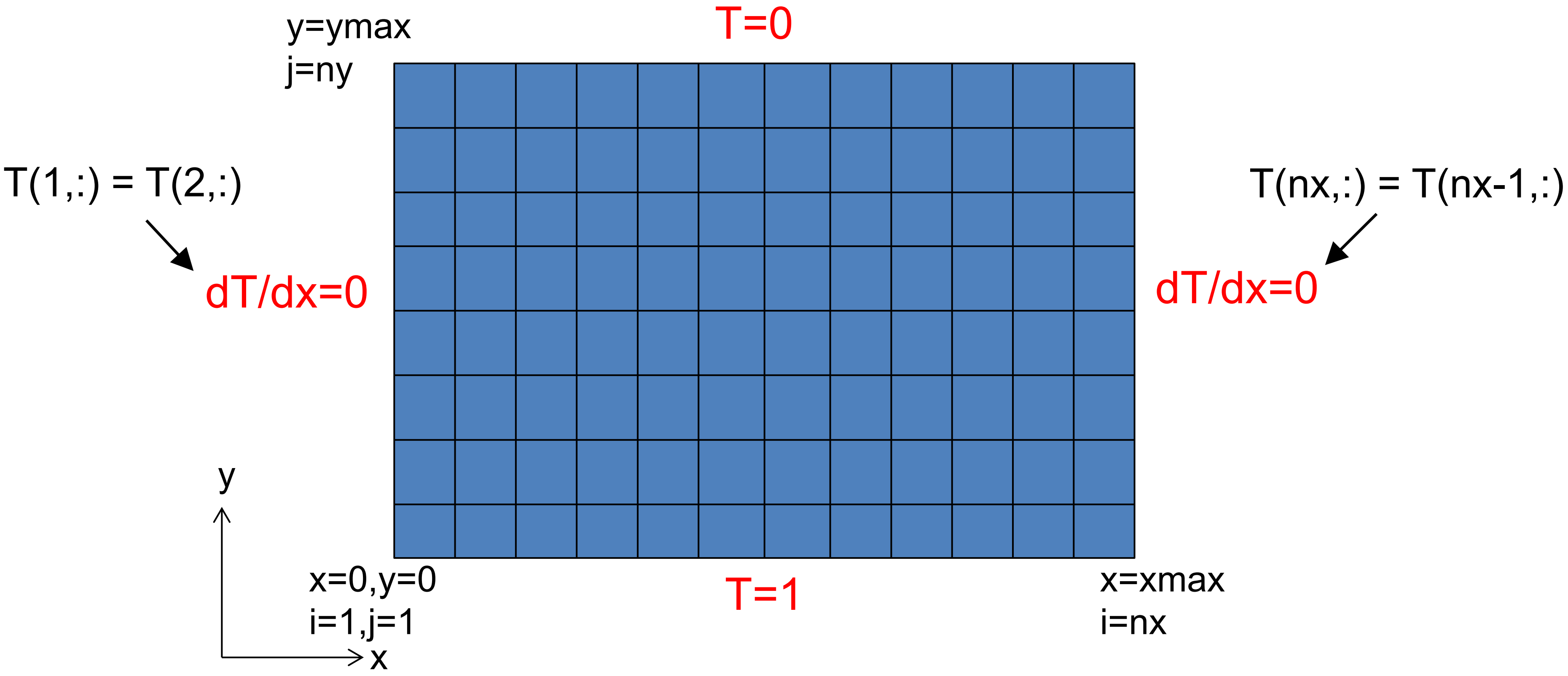
Exercise 8

3. Perform several time steps until the integration time is reached:
   - Determine ψ from ω using the Poisson solver (Equation 2)
   - Compute the wind speeds u and v from ψ
   - Compute the time step from a_adv, a_diff and the maximum wind speed in the model domain: $\Delta t = \min\left(a_{diff} \cdot \frac{h^2}{\max(1,\mathrm{Pr})}, a_{adv} \cdot \frac{h}{v_{max}}\right)$
   - Compute $\mathrm{Ra} \cdot \partial T/dx$
   - Compute $\nabla^2 T$ and $\vec{v} \cdot \nabla T$ as well as $\nabla^2 \omega$ and $\vec{v} \cdot \nabla\omega$ using the subroutines from Exercise 5.
   - Integrate forward in time:
     - $\omega_{i,j}^{n+1} = \omega_{i,j}^{n} +$ ... (Equation 1)
     - $T_{i,j}^{n+1} = T_{i,j}^{n} +$ ... (Equation 3)
     - $t = t + \Delta t$

Exercise 8

Boundary conditions for T:

ψ and ω: 0 at all boundaries



Exercise 8

4. Write a makefile and compile the program with `make`.

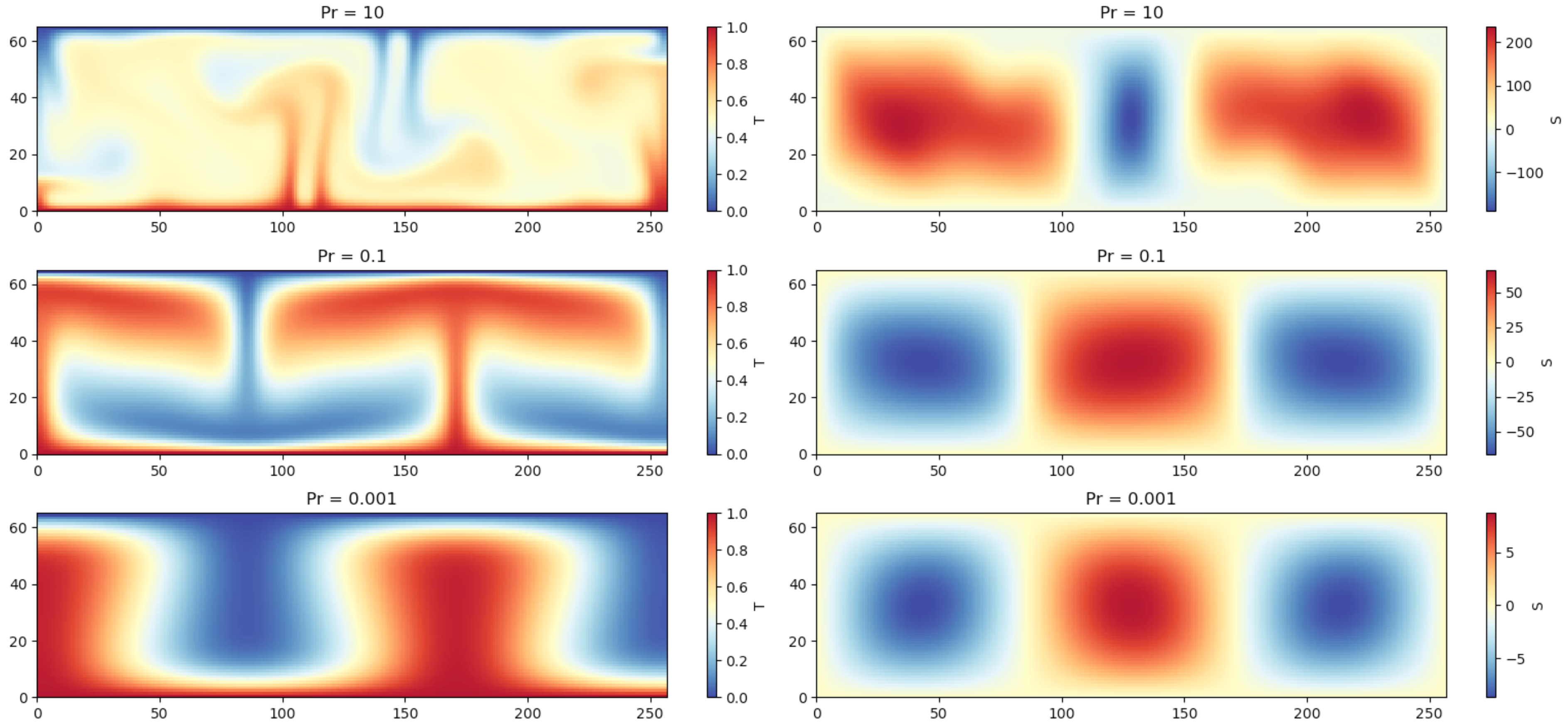
5. Test the program with different Prandtl numbers (between 0.001 and 10) and plot the resulting temperature fields.

Example namelist:

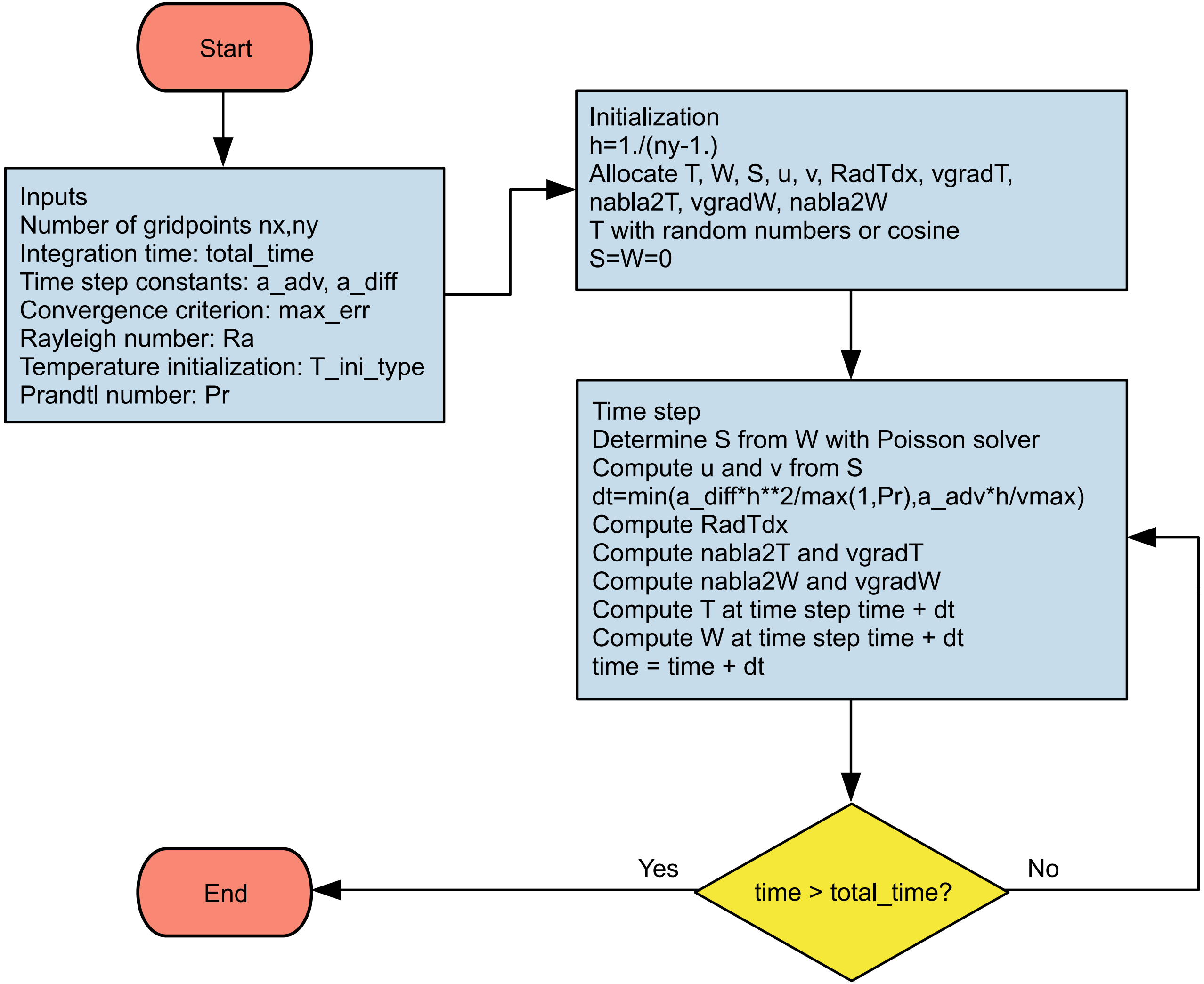
```
&INPUTS
  Pr=0.1
  nx=257
  ny=65
  a_diff=0.15
  a_adv=0.4
  total_time=0.1
  max_err=1.E-3
  Ra=1.E6
  T_ini_type='cosine'
/
```

Resulting temperature and flow fields

Time=0.01



Flowchart for Exercise 8



Exercise 8

Deadline:

- Please hand in your solutions (.f90 files and plots T) no later than Tuesday, **21 May 2024, 23:59**.

Questions?

- Email me (marina.duetsch@univie.ac.at)
  Or pass by my office (UZA II, 2G551).