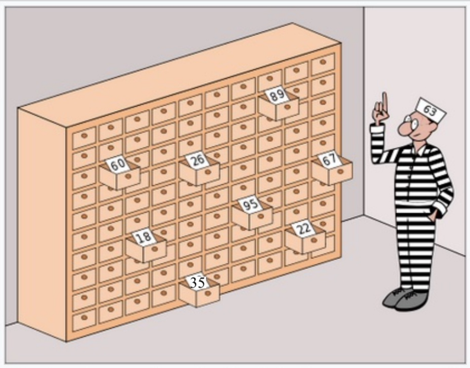
Bonus Group Project 2 - 100 prisoners problem

Stat 2122 - Intro. to Probability & Statistics
Section 001, Spring 2023 **Name(s):** ____________________________ **Points:** _____
Due: **Monday, 03/20/2023**

A maximum of 2 students are allowed in a group! Do NOT copy between groups!

The **100 prisoners problem**, originally proposed by Danish computer scientist Peter Bro Miltersen in 2003, is a mathematical problem in probability theory and combinatorics. In this problem, 100 numbered prisoners must find their own numbers in one of 100 drawers in order to survive. The rules state that each prisoner may open only 50 drawers and cannot communicate with other prisoners. If, during this search, **even one** prisoner does not find their number, all prisoners are executed. At first glance, the situation appears hopeless for the prisoners, but a clever strategy offers them a realistic chance of survival that is more than 30%!



Each prisoner has to find their own number in one of 100 drawers, but may open only 50 of the drawers.

1. **Task I:** First read the Wikipedia description of the problem available at

   https://en.wikipedia.org/wiki/100_prisoners_problem

   which includes the detailed description of the problem, the **best strategy**, and the analysis of the strategy and its probability of success.

   Then complete the following table for a general version of the problem: "The director of a prison offers n death row prisoners, who are numbered from 1 to n, a last chance. A room contains a cupboard with n drawers. The director randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into k drawers in any order $(n/2 \le k < n)$. The drawers are closed again afterwards. If, during this search, every prisoner finds their number in one of the drawers, all prisoners are pardoned. If even one prisoner does not find their number, all prisoners are executed. Before the first prisoner enters the room, the prisoners may discuss strategy – but may not communicate once the first prisoner enters to look in the drawers."

   If all prisoners select drawers completely at random, then $P(n,k)$, the probability that ALL prisoners find their number by looking into k drawers and consequently they are all pardoned, clearly is

   $$P(n,k) = \left(\frac{k}{n}\right)^n$$

   If all prisoners select drawers based on the best strategy, then what is $P(n,k)$?

   $$P(n,k) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

   (Note: $P(100, 50)$ can be found from the Wikipedia website.)

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n	k	$P(n,k)$ Probability when using the best strategy	$P(n,k)$ Probability when selecting completely at random
2	1		$\left(\frac{1}{2}\right)^2 = 0.25$
4	2		$\left(\frac{2}{4}\right)^4 = 0.0625$
4	3		$\left(\frac{3}{4}\right)^4 \approx 0.31641$
10	5		$\left(\frac{5}{10}\right)^{10} \approx 0.00098$
10	9		$\left(\frac{9}{10}\right)^{10} \approx 0.34868$
100	50	0.31183	$\left(\frac{50}{100}\right)^{100} \approx 7.89 \times 10^{-31}$
100	90		$\left(\frac{90}{100}\right)^{100} \approx 2.66 \times 10^{-5}$
100	99		$\left(\frac{99}{100}\right)^{100} \approx 0.36603$
$n \approx \infty$	$n/2$	$1 - \ln 2 \approx 0.30685$	0
$n \approx \infty$	$n-1$	1	$\frac{1}{e} \approx 0.36788$

2. **Task II:** If using the best strategy, once you find the correct formula for computing $P(n,k)$ in Task I, you should be able to see that $\lim_{n\to\infty} P(n, n-1) = 1$ immediately.

   On the other hand, as stated in Task I, if all prisoners select drawers completely at random, then

$$P(n,k) = \left(\frac{k}{n}\right)^n$$

   So we have

$$P(n, n-1) = \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n$$

   Prove that $\lim_{n\to\infty} P(n, n-1) \equiv \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$. (Recall: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$)

3. **Task III: Implement the best strategy in Python.** Run your code with the required number of trials, record the number (frequency) of successes, and calculate the corresponding relative frequency of the successes which should in general approach to the exact probability as the number of trials increases. Report your simulation results in the following tables. Here, in one trial, all n prisoners try to find their numbers. The outcome of the trial is a success if ALL n prisoners find their number after looking into (at most) k drawers (so all prisoners are pardoned).

   Run 1: $n = 100, k = 50$: the exact probability is 0.31183.

Number of trials	1000	10,000	100,000
Number of successes			
Relative frequency of successes			

   Run 2: $n = 100, k = 99$: the exact probability is 0.99.

Number of trials	1000	10,000	100,000
Number of successes			
Relative frequency of successes			

   Run 3: $n = 500, k = 250$: the exact probability is 0.30735.

Number of trials	1000	10,000	100,000
Number of successes			
Relative frequency of successes			

4. **Bonus points to the final average: 2.5 points**

   Task I: 0.5 point
   Task II: 0.5 point
   Task III: 1.5 point

5. **What to submit for the bonus points**
   (a) This assignment with completed tables (No point for Task III without the accompanying code)
   (b) Your Python code

6. **How to submit**
   (a) Online through Canvas
   (b) Each group submits only once.