E8-1

(1) $y' = 3y$

$y = Ae^{3x}$

(2)

右辺 = 0 とすると

$y = Ae^{-2x}$

A を x の関数として すると

$y' = A'(x)e^{-2x} - 2A(x)e^{-2x}$

与式に代入

$A'(x)e^{-2x} - 2A(x)e^{-2x} + 2Ae^{-2x} = x$

$\therefore A'(x)e^{-2x} = x$

$A'(x) = xe^{2x}$

$A(x) = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

$+x \quad \frac{1}{2}e^{2x}$

$-1 \quad \frac{1}{4}e^{2x}$

$\therefore y = Ce^{-2x} + \frac{1}{2}x - \frac{1}{4}$

S8-1

(1) $(1+x)y' + (1+y) = 0$

$\frac{y'}{(1+y)} = -\frac{1}{1+x}$

$\therefore \ln(1+y) = -\ln(1+x) + C$

$1+y = \frac{A}{1+x}$

$y = \frac{A}{1+x} - 1$

(2) $y' + 2xy = x$

ここで

$e^{\int 2x\,dx} = e^{x^2}$

これを両辺にかけると

$y'e^{x^2} + 2xe^{x^2}y = xe^{x^2}$

$\therefore (y \cdot e^{x^2})' = xe^{x^2}$

両辺 x で積分すると

$$y \cdot e^{x^2} = \int x e^{x^2} dx$$

$$= \frac{1}{2} e^{x^2} + C$$

$$\therefore y = Ce^{-x^2} + \frac{1}{2}$$

E8-2

(1)

$z = y^{-2}$ とおくと

$$z' = -2y^{-3} \cdot y' \quad \therefore y^{-3} \cdot y' = -\frac{z'}{2}$$

よって与式は

$$y' + y = xy^3$$

$$y^{-3} \cdot y' + y^{-2} = x$$

$$\therefore -\frac{z'}{2} + z = x$$

$$z' - 2z = -2x$$

(2)

両辺に $e^{-2x}$ をかけると

$$z'e^{-2x} - 2e^{-2x} z = -2xe^{-2x}$$

$$(z \cdot e^{-2x})' = -2xe^{-2x}$$

$$\therefore ze^{-2x} = \int -2xe^{-2x} dx$$

| | |
|---|---|
| $+\ x$ | $e^{-2x}$ |
| $-\ 1$ | $-\frac{1}{2}e^{-2x}$ |

$$= xe^{-2x} + \frac{1}{2} e^{-2x} + C$$

$$\therefore z = x + \frac{1}{2} + Ce^{2x}$$

$$\therefore y^{-2} = x + \frac{1}{2} + Ce^{2x}$$

$$\therefore \left(x + \frac{1}{2} + Ce^{2x}\right) y^2 = 1$$

S8-2

(1) $y' + (\sin x) y = (\sin x) y^2$

$z = y^{-1}$ とおく.

$$z' = -y^{-2} \cdot y'$$

与式の両辺を $y^2$ で割る.

$y^{-2}\cdot y' + (\sin x)\, y^{-1} = \sin x$

$\int -\sin x\, dx = \cos x$

$\therefore -z' + (\sin x)\, z = \sin x$

$z' - (\sin x)\, z = -\sin x$

両辺に $e^{\cos x}$ をかけて

$z' e^{\cos x} - e^{\cos x}\sin x \cdot z = -e^{\cos x}\sin x$

$(z\cdot e^{\cos x})' = -e^{\cos x}\sin x$

$\therefore z\cdot e^{\cos x} = \int -e^{\cos x}\sin x\, dx$

$= e^{\cos x} + C$

$\therefore z = 1 + Ce^{-\cos x}$

$z = y^{-1}$ とおいたので

$y^{-1} = 1 + Ce^{-\cos x}$

$\therefore (1 + Ce^{-\cos x})\, y = 1$

(2)

$z = y^{-1}$ とおく。

$z' = -y^{-2}\cdot y'$

与式の両辺を $y^2$ で割る。

$x y^{-2}\cdot y' + y^{-1} = \ln x$

$\int -\frac{1}{x}\, dx = -\ln x$

$\therefore -xz' + z = \ln x$

$z' - \frac{z}{x} = -\frac{\ln x}{x}$

$e^{-\ln x}$

$= x^{-1}$

両辺に $x^{-1}$ をかけて

$z' x^{-1} - \frac{z}{x^2} = -\frac{\ln x}{x^2}$

$\therefore (z\cdot x^{-1})' = -\frac{\ln x}{x^2}$

$\frac{1}{x^2}$

$\therefore z\cdot x^{-1} = \int -x^{-2}\ln x\, dx$

$= x^{-1}\ln x - \int x^{-1}\cdot\frac{1}{x}\, dx$

$= x^{-1}\ln x + x^{-1} + C$

$\therefore z = \ln x + 1 + Cx$

$z = y^{-1}$ 로 치환하면

$y^{-1} = \ln x + 1 + Cx$

$\therefore y = \dfrac{1}{\ln x + 1 + Cx}$

E8-3

(1) $\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 2, 1$

$\therefore y = Ae^{2x} + Be^{x}$

(2) $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda = 2$ (중근)

$\therefore y = (Ax + B)e^{2x}$

(3) $\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2}$

$= -1 \pm 2i$

$\therefore y = e^{-x}(A\cos 2x + B\sin 2x)$

S8-3

(1) $\lambda^2 - 2\lambda - 2 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4 - 4 \times 1 \times (-2)}}{2}$

$= 1 \pm \sqrt{3}$

$\therefore y = Ae^{(1+\sqrt{3})x} + Be^{(1-\sqrt{3})x}$

(2) $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$

$\therefore y = (Ax + B)e^{x}$

(3) $\lambda^2 + 3 = 0$

$\lambda = \pm\sqrt{3}\,i$

$\therefore\ y = A\cos\sqrt{3}x + B\sin\sqrt{3}x$

ES-4

(1)

右辺 = 0 とすると

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-2)(\lambda-1) = 0$

$\lambda = 2, 1$

$\therefore\ y = Ae^{2x} + Be^{x}$

与式の特殊解を

$y = ae^{-x}$ と予想

$\therefore\ y' = -ae^{-x},\ y'' = ae^{-x}$

これらを与式に代入

$ae^{-x} + 3ae^{-x} + 2ae^{-x} = e^{-x}$

$6ae^{-x} = e^{-x}$

係数比較して

$a = \frac{1}{6}$

よって、特殊解の1つは

$y = \frac{1}{6}e^{-x}$

同次の一般解と合わせると、与式の一般解は

$y = Ae^{2x} + Be^{x} + \frac{1}{6}e^{-x}$

(2) $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)^2 = 0$

$\lambda = 2$

$\therefore\ y = (Ax + B)e^{2x}$

特殊解を

$y = a\cos x + b\sin x$

と探す。

$y' = -a\sin x + b\cos x,\ y'' = -a\cos x - b\sin x$

与式に代入すると、

$$-a\cos x - b\sin x + 4a\sin x - 4b\cos x + 4a\cos x + 4b\sin x = \sin x$$

$\therefore\ (3a-4b)\cos x + (4a+3b)\sin x = \sin x$

$$\therefore \begin{cases} 3a-4b=0 \\ 4a+3b=1 \end{cases}$$

$4b = 3a$

$b = \frac{3}{4}a$

$$\begin{array}{rl} 9a-12b &= 0 \\ +)\ 16a+12b &= 4 \\ \hline 25a &= 4 \end{array}$$

$a = \frac{4}{25}$

$b = \frac{3}{25}$

特殊解の1つは $y = \frac{4}{25}\cos x + \frac{3}{25}\sin x$

同次の場合の一般解と合わせると、与式の一般解は

$$y = (Ax+B)e^{2x} + \frac{4}{25}\cos x + \frac{3}{25}\sin x$$

5.8-4

(1) $\lambda^2 - \lambda - 2 = 0$

$(\lambda-2)(\lambda+1) = 0$

$\lambda = 2, -1$

$y = Ae^{2x} + Be^{-x}$

$y = axe^{2x}$, $y' = ae^{2x} + 2axe^{2x}$, $y'' = 2ae^{2x} + 2ae^{2x} + 4axe^{2x}$

$= 4ae^{2x} + 4axe^{2x}$

$4ae^{2x} + 4axe^{2x} - ae^{2x} - 2axe^{2x} - 2axe^{2x} = 6e^{2x}$

$3ae^{2x} = 6e^{2x}$

$a = 2$

$\therefore\ y = Ae^{2x} + Be^{-x} + 2xe^{2x}$

(2) $\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$\therefore\ y = A\cos 2x + B\sin 2x$

$y = ax\cos 2x + bx\sin 2x$

$y' = a\cos 2x - 2ax\sin 2x + b\sin 2x + 2bx\cos 2x$

$y'' = -2a\sin 2x - 2a\sin 2x - 4ax\cos 2x + 2b\cos 2x$
$\qquad + 2b\cos 2x - 4bx\sin 2x$

$-2a\sin 2x - 2a\sin 2x - \cancel{4ax\cos 2x} + 2b\cos 2x + 2b\cos 2x - \cancel{4bx\sin 2x}$
$\qquad + \cancel{4ax\cos 2x} + \cancel{4bx\sin 2x} = 4\cos 2x$

$-4a\sin 2x + 4b\cos 2x = 4\cos 2x$

よって $a = 0,\ b = 1$

よって $y = A\cos 2x + B\sin 2x + x\sin 2x$

E8-5

(1)

$x = e^u$ より

$$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = \frac{dy}{dx}e^u = \frac{dy}{dx}x$$

$$\frac{d^2y}{du^2} = \frac{d^2y}{dx^2}\frac{dx}{du}\cdot\frac{dx}{du} + \frac{dy}{dx}\cdot\frac{d^2x}{du^2} = \frac{d^2y}{dx^2}e^u\cdot e^u + \frac{dy}{dx}\cdot e^u$$
$$= \frac{d^2y}{dx^2}x^2 + \frac{dy}{dx}x$$

$\therefore\ x\dfrac{dy}{dx} = \dfrac{dy}{du}$

$x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2} - \dfrac{dy}{du}$

よって与式は

$$\frac{d^2y}{du^2} - \frac{dy}{du} - 3\frac{dy}{du} - 12y = 0$$

$$\frac{d^2y}{du^2} - 4\frac{dy}{du} - 12y = 0$$

(2) $\lambda^2 - 4\lambda - 12 = 0$

$(\lambda - 6)(\lambda + 2) = 0$

$\lambda = 6,\ -2$

$\therefore\ y = Ae^{6u} + Be^{-2u}$

$\quad = Ax^6 + Bx^{-2}$

8 8 - 5

(1) $x = e^t$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = x\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\frac{dx}{dt}\cdot\frac{dx}{dt} + \frac{dy}{dx}\frac{d^2x}{dt^2}$$

$$= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$$

$$= x^2\frac{d^2y}{dx^2} + \frac{dy}{dt}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + y = 0$$

$$\frac{d^2y}{dt^2} + y = 0$$

$$y = A\cos t + B\sin t$$

$$= A\cos(\ln x) + B\sin(\ln x)$$

(2) $$\frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} + y = t$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda = 1$$

$$y = (At + B)e^t$$

$$y = at + b,\ y' = a,\ y'' = 0$$

$$-2a + at + b = t$$

$-2a + b = 0$

$a = 1$

$b = 2a$

$= 2$

$$y = (At + B)e^t + t + 2$$

$$= (A\ln x + B)x + \ln x + 2$$

$$= Ax\ln x + Bx + \ln x + 2$$

E8-6

(1) $x^3 y' + y^2 = 0$

$$\frac{y'}{y^2} = -\frac{1}{x^3}$$

$$\int \frac{1}{y^2} dy = \int -\frac{1}{x^3} dx$$

$$-\frac{1}{y} = \frac{1}{2}\frac{1}{x^2} + C$$

$$\frac{1}{y} = -\frac{1}{2x^2} + C$$

$$y = \frac{1}{-\frac{1}{2x^2} + C}$$

$$= \frac{2x^2}{2Cx^2 - 1}$$

$$= \frac{2x^2}{Ax^2 - 1}$$

(2) $y' = y^2 - y$

$$\frac{y'}{y^2 - y} = 1$$

$$\int \frac{dy}{y^2 - y} = \int dx$$

$$\int \frac{dy}{y(y-1)} = x + C$$

$$\int \left( \frac{-1}{y} + \frac{1}{y-1} \right) dy = x + C$$

$$-\ln|y| + \ln|y-1| = x + C$$

$$\ln\left|\frac{y-1}{y}\right| = x + C$$

$$\frac{y-1}{y} = Ae^x$$

$$-\frac{1}{y} = Ae^x - 1$$

$$y = \frac{1}{1 - Ae^x}$$

§8-6

(1) $y\sqrt{1+x^2}\,y' = -x\sqrt{1+y^2}$

$$\frac{y}{\sqrt{1+y^2}}y' = -\frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = \int -\frac{x}{\sqrt{1+x^2}}\,dx$$

$$\sqrt{1+y^2} = -\sqrt{1+x^2} + C$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = C$$

(2) $(1+x^2)y' + x\tan y = 0$

$$\frac{y'}{\tan y} = -\frac{x}{1+x^2}$$

$$\int \frac{dy}{\tan y} = \int -\frac{x}{1+x^2}\,dx$$

$$\ln|\sin y| = -\frac{1}{2}\ln(1+x^2) + C$$

$$\sin y = \frac{A}{\sqrt{1+x^2}}$$

$$\int \frac{\cos y}{\sin y}\,dy$$

$$= \ln|\sin y|$$

E8-7

(1) $y' = \dfrac{x-y}{x+y}$

$$= \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

$u = \dfrac{y}{x}$ とおくと

$$y = ux$$

$$y' = u'x + u$$

$$\therefore u'x + u = \frac{1-u}{1+u}$$

$$u'x = \frac{1-u}{1+u} - \frac{u(1+u)}{1+u}$$

$$= \frac{1-2u-u^2}{1+u}$$

$$\therefore \frac{1+u}{1-2u-u^2}\cdot u' = \frac{1}{x}$$

$$\frac{1+u}{u^2+2u-1}u' = -\frac{1}{x}$$

$$\int \frac{1+u}{u^2+2u-1}du = \int -\frac{1}{x}dx$$

$$\frac{1}{2}\ln|u^2+2u-1| = -\ln|x| + C$$

$$\ln x^2|u^2+2u-1| = 2C$$

$$x^2|u^2+2u-1| = A$$

$$x^2(u^2+2u-1) = A$$

$$\therefore x^2\left(\frac{y^2}{x^2}+2\frac{y}{x}-1\right) = A$$

$$y^2+2xy-x^2 = A$$

$y' = y$

$\ln|y| =$

(2) $y' = \dfrac{1}{(x+y)^2}$

$u = x+y$ とおくと

$$u' = 1+y'$$

$$\therefore u'-1 = \frac{1}{u^2}$$

$$u' = \frac{1}{u^2}+1$$

$$= \frac{1}{u^2}+\frac{u^2}{u^2}$$

$$= \frac{1+u^2}{u^2}$$

$$\int \frac{u^2}{1+u^2}du = \int dx$$

$$= x+C$$

$u = \tan\theta$ とおくと

$$\int \frac{u^2}{1+u^2}du = \int \frac{\tan^2\theta}{1+\tan^2\theta}\cdot\frac{1}{\cos^2\theta}d\theta$$

$$= \int \tan^2\theta\, d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos^2\theta}d\theta$$

$$= \int\left(\frac{1}{\cos^2\theta}-1\right)d\theta$$

$$= \tan\theta - \theta$$

$u = \tan\theta \quad du = \dfrac{1}{\cos^2\theta}d\theta$

$$\frac{\tan^2\theta}{1+\tan^2\theta}\cdot\frac{d\theta}{\cos^2\theta}$$

$$= \tan^2\theta\, d\theta$$

$$= \frac{\sin^2\theta}{\cos^2\theta}d\theta$$

$$= \frac{1-\cos^2\theta}{\cos^2\theta}d\theta$$

$$= u - \tan^{-1} u$$

$$= x + y - \tan^{-1}(x+y)$$

$$\therefore\ x + y - \tan^{-1}(x+y) = x + C$$

$$y - \tan^{-1}(x+y) = C$$

§8-17

(1) $(x^2 - y^2)y' = 2xy$

$$y' = \frac{2xy}{x^2 - y^2}$$

$$= \frac{2\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2}$$

$u = \frac{y}{x}$ とおくと

$$u'x + u = y'$$

$$\therefore\ u'x + u = \frac{2u}{1-u^2}$$

$$u'x = \frac{2u}{1-u^2} - \frac{u(1-u^2)}{1-u^2}$$

$$= \frac{u^3 + u}{1-u^2}$$

$$= \frac{u(u^2+1)}{1-u^2}$$

$$\frac{1-u^2}{u(u^2+1)} u' = \frac{1}{x}$$

$$\left(\frac{1}{u} - \frac{2u}{u^2+1}\right)u' = \frac{1}{x}$$

$$\int \left(\frac{1}{u} - \frac{2u}{u^2+1}\right) du = \int \frac{1}{x} dx$$

$$\ln|u| - \ln|u^2+1| = \ln|x| + C$$

$$\frac{u}{u^2+1} = Ax$$

$$\frac{\frac{y}{x}}{\frac{y^2}{x^2}+1} = Ax$$

$$\frac{xy}{x^2+y^2} = Ax$$

$$\frac{A}{u} + \frac{Bu+C}{u^2+1}$$

$$Au^2 + A + Bu^2 + Cu$$

$$(A+B)u^2 + Cu + A = 1 - u^2$$

$$A + B = -1$$

$$A = 1$$

$$B = -2$$

$$C = 0$$

(2)

$u = x + y$ とおく

$$u' = 1 + y'$$

$$u' - 1 = \tan u - 1$$

$$\frac{u'}{\tan u} = 1$$

$$\int \frac{du}{\tan u} = \int dx$$

$$\ln|\sin u| = x + C$$

$$\sin u = Ae^{x}$$

$$\therefore \sin(x + y) = Ae^{x}$$