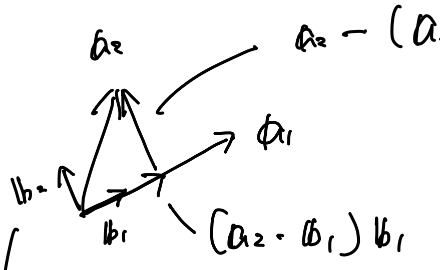
S13-1



$a_2 - (a_2 \cdot b_1) b_1 := u_2$

$b_2 = \frac{u_2}{|u_2|}$

$$u_2 = a_2 - (a_2 \cdot b_1) b_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$|u_2| = \frac{1}{2} \cdot \sqrt{1+1+4}$$

$$b_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$b_2 = \frac{\frac{1}{2}}{\frac{\sqrt{6}}{2}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$b_3 = \frac{\frac{2}{3}}{\frac{2\sqrt{3}}{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_3 = a_3 - \underbrace{(a_3 \cdot b_2)}_{= \frac{1}{\sqrt{6}}(0-1+2)} b_2 - \underbrace{(a_3 \cdot b_1)}_{= \frac{1}{\sqrt{2}}(0+1+0)} b_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$|u_3| = \frac{2}{3} \sqrt{1+1+1} = \frac{2\sqrt{3}}{3}$$

S13-2

$x + x^2 + x^2 + x^3$

$\{a_1, a_2, a_3\} = \{1+x,\ x+x^2,\ 1\}$ とする.

正規直交基底を $\{b_1, b_2, b_3\}$ とすると

$$|a_1|^2 = \int_{-1}^{1} (1+x)^2 \, dx = \int_{-1}^{1} (x^2+2x+1) \, dx = 2\left[\frac{x^3}{3} + x\right]_0^1 = \frac{8}{3} \qquad |a_1| = \frac{2\sqrt{2}}{\sqrt{3}}$$

$$b_1 = \frac{1}{|a_1|} a_1 = \frac{\sqrt{6}}{4}(1+x)$$

$$a_2 \cdot b_1 = \int_{-1}^{1} (x+x^2) \frac{\sqrt{6}}{4} (1+x) \, dx = \frac{\sqrt{6}}{4} \int_{-1}^{1} (x^3 + 2x^2 + x) \, dx$$

$$u_2 = a_2 - (a_2 \cdot b_1) b_1$$

$$= x + x^2 - \underbrace{\frac{\sqrt{6}}{3}\cdot\frac{\sqrt{6}}{4}}_{\frac{1}{2}}(1+x) \qquad = \frac{\sqrt{6}}{2}\left[\frac{2}{3}x^3\right]_0^1$$

$$= x^2 + \frac{1}{2}x - \frac{1}{2} \qquad = \frac{\sqrt{6}}{3}$$

$$\left(x^2 + \frac{1}{2}x - \frac{1}{2}\right)^2$$

$$= x^4 + \frac{1}{2}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{2}x^2 - \frac{1}{4}x + \frac{1}{4}$$

$$= x^4 + x^3 - \frac{3}{4}x^2 - \frac{x}{2} + \frac{1}{4}$$

$$\|h_2\|^2 = \int_{-1}^{1}\left(x^2 + \frac{1}{2}x - \frac{1}{2}\right)^2 dx$$

$$= 2\left[\frac{x^5}{5} - \frac{x^3}{4} + \frac{x}{4}\right]_0^1$$

$$= 2\left(\frac{1}{5} - \frac{1}{4} + \frac{1}{4}\right)$$

$$= \frac{2}{5} \qquad \therefore \|h_2\| = \sqrt{\frac{2}{5}}$$

$$\therefore b_2 = \sqrt{\frac{5}{2}}\left(x^2 + \frac{1}{2}x - \frac{1}{2}\right)$$

⋮ ばんじこちん

§13-3

(1)

$$|A - \lambda E| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 3 \\ 1 & 3 & -1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 3-\lambda & -1-\lambda & 3 \\ 3-\lambda & 3 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\lambda & 3 \\ 1 & 3 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2-\lambda & 2 \\ 0 & 2 & -2-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ -\lambda & -2-\lambda \end{vmatrix}$$

$$= -\lambda(3-\lambda)\begin{vmatrix} 1 & 2 \\ 0 & -4-\lambda \end{vmatrix}$$

$$= \lambda(3-\lambda)(4+\lambda)$$

$\lambda = -4, 0, 3$

$\lambda = -4$ のとき

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore\ u_1 = a\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$\lambda = 0$ のとき

$$\begin{pmatrix}1&1&1\\1&-1&3\\1&3&-5\end{pmatrix}\to\begin{pmatrix}1&1&1\\1&-1&3\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&-2&2\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&2\\0&-1&1\\0&0&0\end{pmatrix}$$

$\therefore\ u_2 = b\begin{pmatrix}-2\\1\\1\end{pmatrix}$

$\lambda = 3$ のとき

$$\begin{pmatrix}-2&1&1\\1&-4&3\\1&3&-4\end{pmatrix}\to\begin{pmatrix}1&3&-4\\0&-7&7\\0&7&-7\end{pmatrix}\to\begin{pmatrix}1&3&-4\\0&-1&1\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&-1\\0&-1&1\\0&0&0\end{pmatrix}$$

$\therefore\ u_3 = c\begin{pmatrix}1\\1\\1\end{pmatrix}$

$a_1 = \begin{pmatrix}0\\-1\\1\end{pmatrix},\ a_2 = \begin{pmatrix}-2\\1\\1\end{pmatrix},\ a_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$　　$\frac{-2}{\sqrt{6}} = \frac{-2\sqrt{2}}{2\sqrt{3}}$

$\{a_1, a_2, a_3\}$ を正規直交化する。異なる固有値に対する固有ベクトルは直交しているので、正規直交系 $\{b_1, b_2, b_3\}$ は

$$\{b_1, b_2, b_3\} = \left\{\begin{pmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}, \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\right\}$$

よって直交行列 P は

$$P = \begin{pmatrix}0 & -2/\sqrt{6} & 1/\sqrt{3}\\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\end{pmatrix}$$

また、

$$P^{-1}AP = \begin{pmatrix}-4 & & 0\\ & 0 & \\ 0 & & 3\end{pmatrix}$$

S13-4

$$f(x, y) = ax^2 + bxy + cy^2$$

$$= \underbrace{(x \ \ y)}_{:= {}^t\boldsymbol{x}} \underbrace{\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}}_{:= A} \begin{pmatrix} x \\ y \end{pmatrix}$$

← 対称行列

$$P^{-1}AP = {}^tPAP = D$$

$$\therefore A = PDP^{-1} = PD\,{}^tP$$

$$= {}^t\boldsymbol{x} PD\,{}^tP \boldsymbol{x}$$

$$= {}^t(\underbrace{{}^tP\boldsymbol{x}}_{:= \boldsymbol{y} = \begin{pmatrix} X \\ Y \end{pmatrix}}) D ({}^tP\boldsymbol{x})$$

$$= (X \ \ Y) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= (\lambda_1 X \ \ \lambda_2 Y) \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= \lambda_1 X^2 + \lambda_2 Y^2$$

$$Q(x, y, z) = \underbrace{(x \ \ y \ \ z)}_{:= {}^t\boldsymbol{x}} \underbrace{\begin{pmatrix} 5 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}}_{:= A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{:= \boldsymbol{x}}$$

$$= {}^t\boldsymbol{x} A \boldsymbol{x}$$

$$= P\,{}^t\boldsymbol{x} P^{-1}APP^{-1}\boldsymbol{x}$$

$$|A - \lambda E| = \begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 1-\lambda & 3 \\ 1 & 3 & 1-\lambda \end{vmatrix}$$

$$= (5-\lambda)(1 - 2\lambda + \lambda^2) + 3 + 3 - 1 + \lambda - 1 + \lambda - 45 + 9\lambda$$

$$= 5 - 10\lambda + 5\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 4 + 11\lambda - 45$$

$$= -\lambda^3 + 7\lambda^2 - 36$$

$$= -(\lambda + 2)(\lambda^2 - 9\lambda + 18)$$

$$= -(\lambda + 2)(\lambda - 3)(\lambda - 6)$$

$$\begin{array}{c|cccc} & & & 8 & 28 \quad 36 \\ -2 & -1 & 7 & 0 & -36 \\ & & 2 & -18 & 36 \\ \hline & -1 & 9 & -18 & \boxed{0} \end{array}$$

$\therefore \lambda = -2, 3, 6$

$\lambda = -2$ のとき

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore u_1 = C_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = C_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad (C_1 \neq 0)$$

$\lambda = 3$ のとき

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore u_2 = C_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = C_2 \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad (C_2 \neq 0)$$

$\lambda = 6$ のとき

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -5 & 3 \\ 1 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} 0 & 4 & -4 \\ 0 & -8 & 8 \\ 1 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$\therefore u_3 = C_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = C_3 \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

P は直交行列なので、

$$\begin{aligned} Q(x, y, z) &= P\,{}^t x\, P^{-1} A P P^{-1} x \\ &= P\,{}^t x\, P^{-1} A P\,{}^t P x \\ &= {}^t({}^t P x)\, P^{-1} A P\, ({}^t P x) \end{aligned}$$

${}^t P x = {}^t(X \;\; Y \;\; Z)$ とおくと

$$\begin{aligned} Q(x, y, z) &= (X \;\; Y \;\; Z) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \\ &= -2X^2 + 3Y^2 + 6Z^2 \end{aligned}$$

S13-5

$Q(x,y) = x^2 + 2\sqrt{3}xy - y^2 = 2$ とする.

$$Q(x,y) = \underbrace{\begin{pmatrix} x & y \end{pmatrix}}_{:= {}^t x} \underbrace{\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}}_{:= A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{:= x}$$

Aは実対称行列で対角化可能.

$$|A - \lambda E| = \begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = -\begin{vmatrix} 1-\lambda & \sqrt{3} \\ -\sqrt{3} & 1+\lambda \end{vmatrix}$$

$$= -((1-\lambda^2) + 3)$$

$$= \lambda^2 - 4$$

$$= (\lambda+2)(\lambda-2)$$

$$= 0$$

$\therefore \lambda = -2, 2$

$\lambda = -2$ のとき

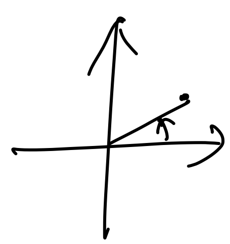
$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \to \begin{pmatrix} 3 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix}$$

$$\therefore u_1 = c_1 \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} = c_1 \frac{\sqrt{3}}{2} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$\lambda = 2$ のとき

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{3} \\ 0 & 0 \end{pmatrix} \qquad (c_1 \neq 0) \qquad \frac{1}{3} + \frac{3}{3} = \frac{4}{3}$$

$$\therefore u_2 = c_2 \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = c_2 \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$



固有ベクトルとして

$$\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

がとれる.

$$\therefore P = \begin{pmatrix} u_2 & u_1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}$$

u_2 と u_1 は正規直交基底なので $P^{-1} = {}^tP$

ここで，

$$Q(x, y) = {}^t\boldsymbol{x} A \boldsymbol{x}$$
$$= P\,{}^t\boldsymbol{x}\,{}^tPAP\,{}^tP\boldsymbol{x}$$
$$= {}^t({}^tP\boldsymbol{x})\,{}^tPAP\,({}^tP\boldsymbol{x})$$

P に対応した対角行列 ${}^tPAP$ は

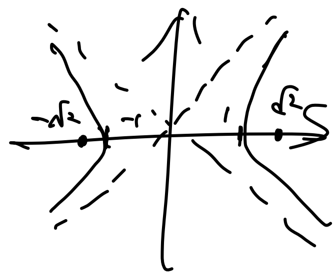
$${}^tPAP = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

また，${}^tP\boldsymbol{x} = \begin{pmatrix} X \\ Y \end{pmatrix}$ とおくと

$$Q(x, y) = (x\ y)\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= (X\ Y)\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$
$$= 2X^2 - 2Y^2 = 2$$

よって与式は

$$X^2 - Y^2 = 1$$



${}^tP\boldsymbol{x} = \begin{pmatrix} X \\ Y \end{pmatrix}$ とおいたので，$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} X \\ Y \end{pmatrix}$.

したがって，与えられた曲線は，$x^2 - y^2 = 1$ を原点まわりに $\frac{\pi}{6}$ 回転したもの

よって以下のようになる.