$\boxed{1}$

(1) $\int \arcsin x \, dx$

$\arcsin x = t \qquad x = \sin t$

$\frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx}$

$dx = \sqrt{1-\sin^2 t} \, dt$

$= \cos t \, dt$

$$\int \arcsin x \, dx = \int t \cos t \, dt$$

$$= t \sin t - \int \sin t \, dt$$

$$= t \sin t + \cos t + C$$

$$= t \sin t + \sqrt{1-\sin^2 t} + C$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$

(2) $\int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$$= \int \frac{\sin x}{1-\cos^2 x} dx$$

$$= \int \left(\frac{\frac{1}{2}\sin x}{1+\cos x} + \frac{\frac{1}{2}\sin x}{1-\cos x} \right) dx$$

$$= \frac{-1}{2} \log(1+\cos x) + \frac{1}{2} \log(1-\cos x) + C$$

$$= \frac{1}{2} \log \frac{1-\cos x}{1+\cos x} + C$$

[2]

(1) $\int x \log x = \frac{x^2}{2} \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$= \frac{x^2}{2} \log x - \frac{x^2}{4} + C$

(2) $\int \frac{x^2 - x + 1}{(x-1)(x-2)^2}$

$\left(\frac{x^2 - x + 1}{x - 1} \right)'$

$= \frac{(2x-1)(x-1) - x^2 + x - 1}{(x-1)^2}$

$= \frac{2x^2 - 2x - x + 1 - x^2 + x - 1}{(x-1)^2}$

$= \frac{x^2 - 2x}{(x-1)^2} \Big|_{x=2}$

$= 0$

$= \int \left(\frac{1}{x-1} + \frac{3}{(x-2)^2} + \frac{0}{x-2} \right) dx$

$= \log|x-1| - \frac{3}{x-2} + C$

[3]

$\int \frac{x}{x^2 + 2x + 2} dx$

$= \int \left(\frac{\frac{1}{2}(2x+2)}{x^2 + 2x + 2} + \frac{-2}{x^2 + 2x + 2} \right) dx$

$= \frac{1}{2} \int \frac{2x+2}{x^2 + 2x + 2} dx - 2 \int \frac{1}{(x+1)^2 + 1} dx$

$= \frac{1}{2} \log|x^2 + 2x + 2| - 2 \tan^{-1}(x+1)^2 + C$

$4 - 4 \times 1 \times 2$
$= -4$

$\downarrow$

$D = 2^2 - 4 \times 1 \times 2 < 0$
かつ下凸なので常に正の値

$= \frac{1}{2} \log(x^2 + 2x + 2) - 2 \tan^{-1}(x+1)^2 + C$

[4]

$\int \frac{1}{x^4 - 1} dx$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 \\ \hline -1 & 1 & 1 & 1 & 1 & \boxed{0} \\ & & -1 & 0 & -1 & \\ \hline & 1 & 0 & 1 & \boxed{0} & \end{array}$$

$= \int \frac{1}{(x-1)(x+1)(x^2+1)} dx$

$$= \int \left(\frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{-\frac{1}{2}}{x^2+1} \right) dx$$

$$= \frac{1}{4} \log \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + C$$

$\frac{1}{4}(x+1)(x^2+1) - \frac{1}{4}(x-1)(x^2+1)$
$+ (Bx+C)(x^2-1)$

$\frac{1}{4}(x^2+1)(x+1-(x-1))$
$+ Bx^3 - Bx + Cx^2 - C$

$\frac{1}{2}x^2 + \frac{1}{2} + Bx^3 - Bx + Cx^2 - C = 1$

$C = -\frac{1}{2}$

$\boxed{5}$

$$\int \frac{6x^2+2x}{x^3+3x^2-x-3} dx$$

$$= \int \frac{6x^2+2x}{(x-1)(x+1)(x+3)} dx$$

$$= \int \left(\frac{1}{x-1} + \frac{-1}{x+1} + \frac{6}{x+3} \right) dx$$

$$= \log \left| \frac{(x-1)(x+3)^6}{x+1} \right| + C$$

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -1 & -3 \\ & & 1 & 4 & 3 \\ \hline & 1 & 4 & 3 & \boxed{0} \end{array}$$

$\frac{54-6}{8} = \frac{48}{8}$
$= 6$

$\boxed{6}$

(1) $L_n = x(\log x)^n - \int x \cdot n (\log x)^{n-1} \cdot \frac{1}{x} dx$

$$= x(\log x)^n - n \int (\log x)^{n-1} dx$$

$$= x(\log x)^n - n L_{n-1}$$

(2) (1)の両辺を$(-1)^n \cdot n!$でわると

$$\frac{L_n}{(-1)^n \cdot n!} = \frac{x(\log x)^n}{(-1)^n \cdot n!} + \frac{L_{n-1}}{(-1)^{n-1}(n-1)!}$$

$$\frac{L_n}{(-1)^n \cdot n!} - \frac{L_{n-1}}{(-1)^{n-1}(n-1)!} = \frac{x(\log x)^n}{(-1)^n \cdot n!}$$

これを$1, 2, \cdots, n$まで和をとると

$$\frac{L_n}{(-1)^n \cdot n!} - L_0 = \sum_{k=1}^{n} \frac{x(\log x)^k}{(-1)^k \cdot k!}$$

nのときの左辺第1項と1のときの左辺第2項のみ残る

$$\therefore\ L_n = (-1)^n \cdot n! \left(\underbrace{L_0}_{x+C} + \frac{x(\log x)^n}{(-1)^n \cdot n!} \right)$$

$$= (-1)^n n!\, x \left(1 + \sum_{k=1}^{n} \frac{(\log x)^k}{(-1)^k k!} \right) + C$$