[I]

(1) $g(-x) = \int_0^{-x} t \cdot f(-x-t)\,dt$

$-t = u$ とおくと

$dt = -du$

| $t$ | $0 \to -x$ |
|---|---|
| $u$ | $0 \to x$ |

$\therefore g(-x) = \int_0^{x} -u \cdot f(-x+u) \cdot (-du)$

$= \int_0^{x} u \cdot f(-(x-u))\,du$

(i) $f(x)$ が奇関数

$g(-x) = \int_0^{x} u \cdot (-f(x-u))\,du$

$= -\int_0^{x} u \cdot f(x-u)\,du$

$= -g(x)$

$\therefore g(x)$ は奇関数

(ii) $f(x)$ が奇関数

$g(-x) = \int_0^{x} u \cdot f(x-u)\,du$

$= g(x)$

$\therefore g(x)$ は偶関数

(2) $g(x) = \int_0^{x} t \cdot f(x-t)\,dt$

$x - t = s$ とおくと

$ds = -dt$

| $t$ | $0 \to x$ |
|---|---|
| $s$ | $x \to 0$ |

$$\therefore g(x) = \int_x^0 (x-s)\cdot f(s) \cdot (-ds)$$
$$= \int_0^x (x-s)\cdot f(s)\, ds$$
$$= x\int_0^x f(s)\,ds - \int_0^x s f(s)\,ds$$
$$= x\Big[F(s)\Big]_0^x - \left(\Big[sF(s)\Big]_0^x - \int_0^x F(s)\,ds\right)$$

(2)

$$g(x) = \int_0^x t\cdot f(x-t)\,dt$$
$$= \int_0^x t\cdot\cos(x-t)\,dt$$
$$= \Big[-t\cdot\sin(x-t)\Big]_0^x - \int_0^x -\sin(x-t)\,dt$$
$$= \Big[\cos(x-t)\Big]_0^x$$
$$= 1-\cos x$$

$$g'(x) = \sin x$$
$$g''(x) = \cos x$$

(3) $$g(x) = \int_0^x t\cdot f(x-t)\,dt$$

$x - t = s$ とおく

| $t$ | $0 \to x$ |
|---|---|
| $s$ | $x \to 0$ |

$dt = -ds$

$t = x - s$

$$g(x) = -\int_0^x (x-s)\cdot f(s)\cdot(-ds)$$
$$= x\int_0^x f(s)\,ds - \int_0^x s f(s)\,ds$$

$\to h(s)$

$H(s)$

$\to$

$H(x)-H(0)$

$$\therefore g'(x) = \int_0^x f(s)\,ds + x\cdot f(x) - x\cdot f(x)$$
$$= \int_0^x f(s)\,ds$$

$\therefore\ g'(0) = \int_0^0 f(s)\,ds = 0$

$g''(x) = f(x)$

$\therefore\ g''(0) = f(0) > 0$

したがって，$g(x)$ は $x>0$ で極小値をとる．

$\boxed{2}$

(1) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$= \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2}$$

$$= 2\,\frac{\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$= \frac{2u}{1+u^2}$$

$\cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right) >$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$$= \cos^2\frac{x}{2}\left(1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)$$

$$= \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$= \frac{1-u^2}{1+u^2}$$

$$\frac{du}{dx} = \frac{\frac{1}{2}}{\cos^2\frac{x}{2}} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) = \frac{1}{2}(1+u^2)$$

$$\therefore\ dx = \frac{2}{1+u^2}\,du$$

(2) $I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx$

| $x$ | $0 \to \frac{\pi}{2}$ |
|---|---|
| $u$ | $0 \to 1$ |

$$= \int_0^1 \frac{\frac{2u}{1+u^2}}{1+\frac{2u}{1+u^2}}\cdot\frac{2}{1+u^2}\,du$$

$$= \int_0^1 \frac{2u}{1+u^2+2u} \cdot \frac{2}{1+u^2}\, du$$

$$= \int_0^1 \frac{4u}{(u+1)^2(u^2+1)}\, du$$

ここで、

$$\frac{4u}{(u+1)^2(u^2+1)}$$

$$= \frac{Au+B}{(u+1)^2} + \frac{Cu+D}{u^2+1}$$

使えない

分母の因数の次数が1のとき限定

**ヘビサイドの展開定理**

$$\frac{-s+7}{(s+1)(s+5)} = \frac{A}{s+1} + \frac{B}{s+5}$$

両辺に $\times (s+1)$

$$\frac{-s+7}{s+5} = A + \cancel{\frac{B(s+1)}{s+5}}$$

恒等式において、$s$はどんな値でも成立する → $s=-1$を代入

$$\frac{-(-1)+7}{-1+5} = A \quad \therefore A = 2$$

$B$においても同様

$$\therefore (Au+B)(u^2+1) + (Cu+D)(u+1)^2$$

$$= Au^3 + Au + Bu^2 + B + (Cu+D)(u^2+2u+1)$$

$$= \cancel{Au^3} + \cancel{Au} + \cancel{Bu^2} + B + \cancel{Cu^3} + \cancel{2Cu^2} + \cancel{Cu} + \cancel{Du^2} + \cancel{2Du} + D$$

$$= (A+C)u^3 + (B+2C+D)u^2 + (A+C+2D)u + (B+D)$$

$$= 4u$$

$$\therefore \begin{cases} A + C = 0 \\ B + 2C + D = 0 \\ A + C + 2D = 4 \\ B + D = 0 \end{cases}$$

$$C = 0,\ A = 0,\ D = 2,\ B = -2$$

$$\therefore I_1 = \int_0^1 \left( \frac{-2}{(u+1)^2} + \frac{2}{(u^2+1)} \right) du$$

$$= 2\left[ \frac{1}{u+1} + \tan^{-1} u \right]_0^1$$

$$= 2\left( \frac{1}{2} + \frac{\pi}{4} - 1 \right)$$

$$= -1 + \frac{\pi}{2}$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x}\,dx$$

$$= \int_0^1 \frac{\frac{1-u^2}{1+u^2}}{1+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$

$$= \int_0^1 \frac{1-u^2}{2} \cdot \frac{2}{1+u^2}\,du$$

$$= \int_0^1 \frac{1-u^2}{1+u^2}\,du$$

$$= \int_0^1 \frac{2-(1+u^2)}{1+u^2}\,du$$

$$= \int_0^1 \left(\frac{2}{1+u^2} - 1\right) du$$

$$= \left[2\tan^{-1}u - u\right]_0^1$$

$$= 2 \cdot \frac{\pi}{4} - 1$$

$$= \frac{\pi}{2} - 1$$

$\boxed{3}$

(1)

$\underline{n=0}$

$$\int \frac{x^n}{x^2+1}\,dx = \tan^{-1}x + C$$

$\underline{n=1}$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C$$

$n=2$

$$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx$$

$$= \int 1 - \frac{1}{x^2+1}\,dx$$

$$= x - \tan^{-1}x + C$$

(2) $\displaystyle\int_1^\infty \frac{x^n}{x^2+1}dx = \lim_{\alpha\to\infty}\int_1^\alpha \frac{x^n}{x^2+1}dx$

$n=0$のとき

$$\int_1^\infty \frac{x^n}{x^2+1}dx = \lim_{\alpha\to\infty}\left[\tan^{-1}x\right]_1^\alpha = \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4} \rightarrow \text{収束}$$

$n=1$のとき

$$\int_1^\infty \frac{x^n}{x^2+1}dx = \lim_{\alpha\to\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_1^\alpha$$

$$= \lim_{\alpha\to\infty}\left(\frac{1}{2}\ln(\alpha^2+1)-\frac{1}{2}\ln 2\right)$$

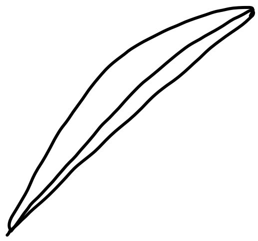
$$= \infty \rightarrow \text{発散}$$

$n>1$において

$$\int_1^\infty \frac{x^n}{x^2+1}dx > \int_1^\infty \frac{x}{x^2+1}dx$$

より、発散する。

したがって、$n=0$のとき$\frac{\pi}{4}$に収束、$n\geqq 1$のとき$\infty$に発散.



$\boxed{4}$

(1)

$0 \leq x \leq \frac{\pi}{2}$のとき

$$0 \leq \sin x \leq 1$$

したがって、

$$\sin^2 x \leq \sin x \leq 1$$

それぞれに$\sin^{2n-1}x$をかけて

$$\sin^{2n+1}x \leq \sin^{2n}x \leq \sin^{2n-1}x$$

(2)

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1}x\,dx$$

$$= \left[-\cos x\sin^{n-1}x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x\cdot(n-1)\sin^{n-2}x\cos x\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^{n-2} x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x \, dx$$

$$= (n-1)\left(\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \, dx\right)$$

$$= (n-1)\left(I_{n-2} - I_n\right)$$

$$\therefore \ nI_n = (n-1) I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

(3)

(2) より

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$

$$= \frac{2n-1}{2n} \cdot \frac{(2n-2)-1}{2n-2} I_{(2n-2)-2}$$

$$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} I_2$$

$$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} I_{2n-3}$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} I_1$$

(1) より

$$\sin^{2n+1} x \leq \sin^{2n} x \leq \sin^{2n-1} x$$

$$\therefore \ 0 < I_{2n+1} \leq I_{2n} \leq I_{2n-1}$$

$$\therefore \ \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} I_1 \leq \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} I_0 \leq \frac{2n-2}{2n-1} \cdots \frac{2}{3} I_1$$

ここで、

$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \quad , \quad I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = 1$$

$$\therefore \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot 1 \leq \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \leq \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot 1$$

$$\therefore \frac{(2n)!!}{(2n+1)!!} \leq \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \leq \frac{(2n-2)!!}{(2n-1)!!}$$

$$\therefore \frac{1}{2n+1} \leq \left( \frac{(2n-1)!!}{(2n)!!} \right)^2 \cdot \frac{\pi}{2} \leq \frac{1}{2n}$$

$$\therefore 2n \leq \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \cdot \frac{2}{\pi} \leq 2n+1$$

$$\therefore 2 \leq \frac{1}{n} \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \cdot \frac{2}{\pi} \leq 2 + \frac{1}{n}$$

$$\therefore \pi \leq \frac{1}{n} \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \leq \pi + \frac{\pi}{2n}$$

はさみうちの原理より

$$\lim_{n \to \infty} \frac{1}{n} \left( \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \right)^2 = \pi$$