$\boxed{1}$

$$\frac{A}{n(n+1)} + \frac{B}{(n+1)(n+3)}$$
$$A(n^2+4n+3) + B(n^2+n) = 1$$
$$A+B=0$$
$$4A+B=0$$
$$3A=1$$
$$A=\frac{1}{3},\ B=-\frac{1}{3}$$

(1) $$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)(n+3)}$$
$$= \frac{1}{3}\sum_{n=1}^{N}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+3)}\right)$$
$$= \frac{1}{3}\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{\frac{1}{2}}{n+1} + \frac{-\frac{1}{2}}{n+3}\right)$$
$$= \frac{1}{3}\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$
$$= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} + \frac{1}{N} - \frac{1}{N+1}\right)$$
$$-\frac{1}{6}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots \right.$$
$$\left. + \frac{1}{N} - \frac{1}{N+2} + \frac{1}{N+1} - \frac{1}{N+3}\right)$$
$$= \frac{1}{3}\left(1 - \frac{1}{N+1}\right) - \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{N+2} - \frac{1}{N+3}\right)$$
$$S = \frac{1}{3} - \frac{1}{6}\left(\frac{5}{6}\right)$$
$$= \frac{12}{36} - \frac{5}{6} = \frac{7}{36}$$

(2) $$S_N = 3r + 4r^2 + \cdots + (N+2)r^N$$
$$rS_N = 3r^2 + 4r^3 + \cdots + (N+2)r^{N+1}$$
$$\underbrace{S_N - rS_N}_{(1-r)S_N} = 3r + r^2 + \cdots + r^N - (N+2)r^{N+1}$$
$$\therefore S_N = \frac{2r + r\frac{1-r^N}{1-r} - (N+2)r^{N+1}}{1-r}$$
$$= \frac{2r(1-r) + r(1-r^N) - (1-r)(N+2)r^{N+1}}{(1-r)^2}$$
$$= \frac{3r - 2r^2 - r^{N+1} - r^{N+1}(N+2-r(N-2r))}{(1-r)^2}$$
$$= \frac{r(3-2r) - (N+3)r^{N+1} + (N+2)r^{N+2}}{(1-r)^2}$$
$$S = \frac{r(3-2r)}{(1-r)^2}$$

$\boxed{2}$

$$u_n = \left| \frac{(-1)^n x^n}{2^n \sqrt{n}} \right| = \frac{|x|^n}{2^n \sqrt{n}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{|x|^{n+1}}{2^{n+1}\sqrt{n+1}} \cdot \frac{2^n\sqrt{n}}{|x|^n}$$

$$= \frac{1}{2} \cdot \sqrt{\frac{n}{n+1}} \cdot |x|$$

$$= \frac{1}{2\sqrt{1+\frac{1}{n}}} |x|$$

$$= \frac{|x|}{2}$$

∴収束半径は2

(i) $x = 2$のとき

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (-1)^n$$

∴ライプニッツの定理より収束.

(ii) $x = -2$のとき

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-2)^n}{2^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

∴ゼータ級数より発散.

したがって, 収束域は $-2 < x \leq 2$

$\boxed{3}$

(1) $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$$f^{(n)}(x) = (-1)^{n+1} \cdot \frac{1}{2^n} \cdot (2(n-1)-1)! \cdot (1+x)^{-\frac{2(n-1)+1}{2}}$$

$$= (-1)^{n+1} \frac{(2n-3)!}{2^n} (1+x)^{-\frac{2n-1}{2}}$$

(2) $f'(0) = \frac{1}{2}$

$f''(0) = -\frac{1}{4}$

$f^{(n)}(0) = (-1)^{n+1}\frac{(2n-3)!}{2^n}$

$\therefore f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots + (-1)^{n+1}\frac{(2n-3)!}{2^n n!}x^n + \cdots$

(3) $\sqrt{101} = 10\sqrt{\frac{101}{100}}$

$\frac{10}{16} = \frac{5}{8} = 0.625$

$= 10\sqrt{1+\frac{1}{100}}$

$= 10\left(1 + \frac{1}{2}\cdot\frac{1}{100} - \frac{1}{8}\cdot\frac{1}{100^2} + \frac{1}{16}\cdot\frac{1}{100^3} + \cdots\right)$

$= 10 + 0.05 - 0.000125 + 0.000000625 + \cdots$

≈ 10.04987

$$\begin{array}{r} 0.05 \\ -\,0.000125 \\ \hline 0.049875 \\ +\,0.000000625 \\ \hline \approx 0.04987 \end{array}$$

$\boxed{4}$

(1) $\cos x = 1 - \frac{1}{2!}x^2 + \cdots$

$\therefore f(x) = x\cos x = x - \frac{1}{2!}x^3 + \cdots$

(2) $\frac{1}{1+x} = 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \cdots$

$= 1 - x + x^2$

$\therefore g(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

(3) $x\sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots$

(1), (2) より

$\frac{x\cos x - \ln(1+x)}{x\sin x}$

$= \frac{x - \frac{1}{2!}x^3 + \cdots - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots}$

$= \frac{-\frac{1}{2!}x + \cdots + \frac{1}{2} + \frac{1}{3}x + \cdots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots}$

$$\therefore \lim_{x\to 0}\frac{x\cos x-\ln(1+x)}{x\sin x}=\frac{1}{2}$$

手在，

$$e^{x^2}=1+x^2+\frac{1}{2}x^4+\cdots$$

$$\therefore \frac{x\ln(1+x)}{e^{x^2}-1}=\frac{x^2-\frac{x^3}{2}+\frac{x^4}{3}+\cdots}{1+x^2+\frac{1}{2}x^4-1}$$

$$=\frac{1-\frac{1}{2}x+\cdots}{1+\frac{1}{2}x^2}$$

$$\therefore \lim_{x\to 0}\frac{x\ln(1+x)}{e^{x^2}-1}=1$$