E1-1

(1) $f^{(1)}(x) = 2\cos 2x = 2\sin\left(2x+\frac{\pi}{2}\right)$

$f^{(2)}(x) = -4\sin 2x = 4\sin\left(2x+\pi\right)$

$f^{(3)}(x) = -8\cos 2x = 8\sin\left(2x+\frac{3}{2}\pi\right)$

$\therefore f^{(n)}(x) = 2^n\sin\left(2x+\frac{\pi}{2}n\right)$

(2) $f(x) = \frac{1}{(x-3)(x-2)} = \frac{1}{x-3} + \frac{-1}{x-2}$

$f'(x) = \frac{-1}{(x-3)^2} + \frac{1}{(x-2)^2}$

$f''(x) = \frac{2}{(x-3)^3} + \frac{-2}{(x-2)^3}$

$f'''(x) = \frac{-6}{(x-3)^4} + \frac{6}{(x-2)^4}$

$\therefore f(x) = \frac{(-1)^n \cdot n!}{(x-3)^{n+1}} + \frac{(-1)^{n+1} \cdot n!}{(x-2)^{n+1}}$

$= (-1)^n \cdot n!\left(\frac{1}{(x-3)^{n+1}} + \frac{-1}{(x-2)^{n+1}}\right)$

S1-1

(1) $f'(x) = 2\cos\left(2x+\frac{\pi}{2}\right)$

$f''(x) = 4\cos\left(2x+\pi\right)$

$\therefore f(x) = 2^n\cos\left(2x+\frac{\pi}{2}n\right)$

(2) $f(x) = \frac{1}{(1+x)(1-x)} = \frac{1}{2}\frac{1}{1+x} + \frac{1}{2}\frac{1}{1-x}$

$f'(x) = \frac{1}{2}\frac{-1}{(1+x)^2} + \frac{1}{2}\frac{1}{(1-x)^2}$

$f''(x) = \frac{1}{2}\frac{2}{(1+x)^3} + \frac{1}{2}\frac{2}{(1-x)^3}$

$$f'''(x) = \frac{1}{2}\frac{-6}{(1+x)^4} + \frac{1}{2}\frac{6}{(1-x)^4}$$

$$\therefore f^{(n)}(x) = \frac{1}{2}\frac{(-1)^n \cdot n!}{(1+x)^{n+1}} + \frac{1}{2}\frac{n!}{(1-x)^4}$$

E1-2

(1) $$(fg)^{(n)} = \sum_{k=0}^{n} {}_nC_k\, f^{(n-k)} g^{(k)}$$

$$f := x^2,\ g := e^{3x}$$

$$f' = 2x,\ f'' = 2,\ f''' = 0$$

$$g' = 3e^{3x},\ g'' = 9e^{3x},\ g''' = 27e^{3x} \quad \therefore g^{(n)} = 3^n e^{3x}$$

$$\therefore f^{(n)}(x) = {}_nC_0 \cdot x^2 \cdot 3^n e^{3x} + {}_nC_1 \cdot 2x \cdot 3^{n-1} e^{3x} + {}_nC_2 \cdot 2 \cdot 3^{n-2} e^{3x}$$

$$= (9x^2 + 6nx + n(n-1))\,3^{n-2} e^{3x}$$

(2) $$f(x) = \underbrace{e^x}_{:=f}\underbrace{(1+x)^{-1}}_{:=g}$$

$$f^{(n)} = e^x$$

$$g^{(n)} = \frac{(-1)^n \cdot n!}{(1+x)^{n+1}}$$

$$\therefore f^{(n)}(x) = \sum_{k=0}^{n} {}_nC_k\, (e^x)^{(n-k)} \cdot ((1+x)^{-1})^{(k)}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)! \cdot k!} \cdot e^x \cdot \frac{(-1)^k \cdot k!}{(1+x)^{k+1}}$$

$$= n!\, e^x \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! \cdot (1+x)^{k+1}}$$

S1-2

(1) $$f := x,\ g := \sin 2x$$

$$f' = 1,\ f'' = 0$$

$$g' = 2\sin\left(2x+\frac{\pi}{2}\right),\quad g'' = 4\sin\left(2x+\pi\right)$$

$$\therefore f^{(n)}(x) = \sum_{k=0}^{n} {}_nC_k f^{(n-k)} g^{(k)}$$

$$= {}_nC_0\, x\, 2^n \sin\left(2x+\frac{\pi}{2}n\right) + {}_nC_1\, 2^{n-1} \sin\left(2x+\frac{\pi}{2}(n-1)\right)$$

$$= x\cdot 2^n \sin\left(2x+\frac{\pi}{2}n\right) + n\, 2^{n-1} \sin\left(2x+\frac{\pi}{2}(n-1)\right)$$

(2) $f := x^2,\ g := \cos x$

$$f' = 2x,\ f'' = 2$$

$$g^{(n)} = \cos\left(x+\frac{\pi}{2}n\right)$$

$$\therefore f(x) = {}_nC_0\cdot x^2\cdot\cos\left(x+\frac{\pi}{2}n\right) + {}_nC_1\cdot 2x\cdot\cos\left(x+\frac{\pi}{2}(n-1)\right) + {}_nC_2\cdot 2\cdot\cos\left(x+\frac{\pi}{2}(n-2)\right)$$

$$= x^2\cos\left(x+\frac{\pi}{2}n\right) + 2nx\cos\left(x+\frac{\pi}{2}(n-1)\right) + n(n-1)\cos\left(x+\frac{\pi}{2}(n-2)\right)$$

E1-3

テイラーの定理

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}(a)\cdot\frac{(x-a)^k}{k!} + \underbrace{f^{(n)}(c)\cdot\frac{(x-a)^n}{n!}}_{\text{じょーよこー}}$$

$$(a\le c\le x) \xrightarrow[\text{マクローリンの定理}]{a=0\text{なら}} (0\le c\le x) \xrightarrow{c=\theta x\text{とく}} (0\le\theta\le 1)$$

(1) $f(x) = x - \frac{1}{3!}x^3 + \frac{\sin(\theta x)}{4!}x^4 \quad (0\le\theta\le 1)$

(2) $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{e^{\theta x}}{4!}x^4 \quad (0\le\theta\le 1)$

S1-3

(1) $f(x) = 1 - \frac{x^2}{2!} + \frac{\cos(\theta x)}{4!} x^4$

(2) $(1+x)^p = \binom{p}{0} x^0 + \binom{p}{1} x^1 + \binom{p}{2} x^2 + \cdots\cdots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

$\therefore\ f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{(-3)\cdot(-2)\cdot(-1)}{(1+\theta x)^4} \cdot \frac{x^4}{4!}$

$= x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{4} \frac{-1}{(1+\theta x)^4} x^4$

E1-4

(1) $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{1 - \cos x}$

$\frac{2e^{2x} - 2}{\sin x} = \frac{4e^{2x}}{\cos x}$

$\to 4 \quad (x \to 0)$

(2) $\lim_{x \to +0} \frac{e^{\sqrt{x}} - 1 - \sqrt{x}}{x}$

$\frac{e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x}}}{1} = \frac{e^{\sqrt{x}} - 1}{2\sqrt{x}}$

$= \frac{e^{\sqrt{x}} \frac{1}{2\sqrt{x}}}{\frac{1}{\sqrt{x}}}$

$= \frac{e^{\sqrt{x}}}{2}$

$\to \frac{1}{2} \quad (x \to +0)$

(3) $\lim_{x\to+0} x^2 \log x = \lim_{x\to+0} \frac{\log x}{x^{-2}}$

$$\frac{\log x}{x^{-2}} = \frac{\frac{1}{x}}{\frac{-2}{x^{3}}}$$

$$= \frac{x^2}{-2}$$

$$\to 0 \quad (x\to+0)$$

S1-4

(1) $\lim_{x\to\infty} \frac{x^n}{e^x}$

$$\frac{x^n}{e^x} = \frac{nx^{n-1}}{e^x} = \frac{n(n-1)x^{n-2}}{e^x} = \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{e^x}$$

$$= \frac{n!}{e^x}$$

$$\to 0 \quad (x\to\infty)$$

(2) $\lim_{x\to+0} x\log(\sin x) = \lim_{x\to+0} \frac{\log(\sin x)}{\frac{1}{x}}$

$$\frac{\log(\sin x)}{\frac{1}{x}} = \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}}$$

$$= \frac{-x^2}{\tan x}$$

$$= \frac{-2x}{\frac{1}{\cos^2 x}}$$

$$\to 0 \quad (x\to+0)$$

E1-5

$\lim_{x\to+0} \left(\frac{1}{x}\right)^{\sin x} \Rightarrow \lim_{x\to+0} \ln\left(\frac{1}{x}\right)^{\sin x} = \lim_{x\to+0} \sin x \ln\frac{1}{x}$

$$= \lim_{x\to+0} -\sin x \ln x$$

$$-\sin x \ln x = -\frac{\ln x}{(\sin x)^{-1}}$$

$$= -\frac{\frac{1}{x}}{-(\sin x)^{-2}\cos x}$$

$$= \frac{\sin^2 x}{x\cos x}$$

$$= \tan x \frac{\sin x}{x}$$

$$\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

$$\lim_{x\to +0}\ln\left(\frac{1}{x}\right)^{\sin x} = 0\times 1$$

$$= 0$$

$$\therefore \lim_{x\to +0}\left(\frac{1}{x}\right)^{\sin x} = e^0$$

$$= 1$$

51-5

(1) $\lim_{x\to 1} x^{\frac{1}{1-x}} \Rightarrow \lim_{x\to 1}\ln x^{\frac{1}{1-x}}$

$$\frac{1}{1-x}\ln x = \frac{\frac{1}{x}}{-1} = -1$$

$$\therefore \lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1}$$

$$= \frac{1}{e}$$

(2) $\lim_{x\to\infty}(1+x)^{\frac{1}{x}} \Rightarrow \lim_{x\to\infty}\ln(1+x)^{\frac{1}{x}}$

$$\frac{1}{x}\ln(1+x) = \frac{1}{1+x} \to 0 \quad (x\to\infty)$$

$$\therefore \lim_{x\to\infty}(1+x)^{\frac{1}{x}} = e^0$$

$$= 1$$

E1-6

(1) $(\sin^{-1}2x)' = \dfrac{2}{\sqrt{1-(2x)^2}}$

(2) $(\cos^{-1}\frac{1}{x})' = \dfrac{-\left(-\frac{1}{x^2}\right)}{\sqrt{1-\left(\frac{1}{x}\right)^2}} = \dfrac{1}{x\sqrt{x^2-1}}$

(3) $((\tan^{-1}3x)^2)' = 2\tan^{-1}3x \cdot \dfrac{3}{1+(3x)^2}$

$$= \frac{6\tan^{-1}3x}{1+9x^2}$$

E1-6

(1) $(\sin^{-1}\sqrt{x})' = \dfrac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}} = \dfrac{1}{2\sqrt{x-x^2}}$

(2) $((\cos^{-1}2x)^3)' = 3(\cos^{-1}2x)^2 \cdot \dfrac{-2}{\sqrt{1-4x^2}}$

$$= \frac{-6(\cos^{-1}2x)^2}{\sqrt{1-4x^2}}$$

(3) $(\tan^{-1}\frac{1}{x})' = \dfrac{1}{1+\frac{1}{x^2}} \cdot \dfrac{-1}{x^2}$

$$= \frac{-1}{x^2+1}$$

E1-7

$f(x) = x - \sin x$

$f(0) = 0$

$f'(x) = 1 - \cos x \geq 0$

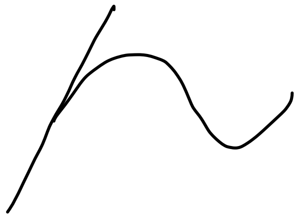
$\therefore$ $f(x)$ は単調増加

∴ $x>0$ のとき $f(x)>0$ なので、$\sin x < x$

$$g(x) = \sin x - \left(x - \frac{x^3}{6}\right)$$

$$g'(x) = \cos x - 1 + \frac{x^2}{2}$$

$$g''(x) = -\sin x + x > 0$$



$g'(0) = 0$ より $x>0$ で $g'(x) > 0$

よって $g(x)$ は単調に増加

また、$g(0) = 0$

∴ $x - \frac{x^3}{6} < \sin x \quad (x>0)$

したがって

$$x - \frac{x^3}{6} < \sin x < x \quad (x>0)$$

8(-7

$$f(x) = x - \tan^{-1} x$$

$$f'(x) = 1 - \frac{1}{1+x^2} > 0$$

$$f(0) = 0$$

∴ $\tan^{-1} x < x$

$$g(x) = \tan^{-1} x - \frac{x}{1+x^2}$$

$$g'(x) = \frac{1}{1+x^2} - \frac{1+x^2-2x}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2} > 0 \quad (x>0)$$

$$g(0) = 0$$

∴ $\frac{x}{1+x^2} < \tan^{-1} x$

$\therefore \dfrac{x}{1+x^2} < \tan^{-1}x < x$

1章末

[1]

(1) $y' = \cos(a\sin^{-1}x)\cdot a\dfrac{1}{\sqrt{1-x^2}} = a\dfrac{\cos(a\sin^{-1}x)}{\sqrt{1-x^2}}$

$$y'' = \frac{-a\sin(a\sin^{-1}x)\cdot\dfrac{a}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2} - a\cos(a\sin^{-1}x)\dfrac{-2x}{2\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{-a^2\sin(a\sin^{-1}x) + a\cos(a\sin^{-1}x)\dfrac{x}{\sqrt{1-x^2}}}{1-x^2}$$

$\therefore$ (左辺) $= -a^2\sin(a\sin^{-1}x) + a\cos(a\sin^{-1}x)\dfrac{x}{\sqrt{1-x^2}}$

$\qquad - a\cos(a\sin^{-1}x)\dfrac{x}{\sqrt{1-x^2}} + a^2\sin(a\sin^{-1}x)$

$\qquad = 0 =$ (右辺)

(2) 与式の両辺をn階微分

$((1-x^2)\cdot y'')^{(n)} - (xy')^n + a^2(y)^{(n)} = 0$

第1項について、ライプニッツの公式より

$((1-x^2)\cdot y'')^{(n)}$ $\qquad\qquad -2x \quad -2$

$= {}_nC_0(1-x^2)y^{(n+2)} + {}_nC_1(-2x)y^{(n+1)} + {}_nC_2(-2)y^{(n)}$

$= (1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - n(n-1)y^{(n)}$

第2項についても同様に

$-(xy')^n$

$= -xy^{(n+1)} - ny^{(n)}$

$\therefore (1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - n(n-1)y^{(n)} - xy^{(n+1)} - ny^{(n)} + a^2y^{(n)}$

$= (1-x^2)y^{(n+2)} - x(2n+1)y^{(n+1)} + (a^2-n^2)y^{(n)} = 0$

$x=0$ を代入

$y^{(n+2)}(0) + (a^2-n^2)y^{(n)}(0) = 0$

$\boxed{2}$

(1) $R_n = e^{\theta x} \cdot \dfrac{x^n}{n!}$

(2) $\displaystyle \lim_{n\to\infty} R_n = e^{\theta x} \lim_{n\to\infty} \frac{x^n}{n!}$

?

x^3

$3x^2$

$3 \cdot 2x^1$

$3 \cdot 2 \cdot 1$

(3) $e = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{(n-1)!} + \cdots$ より明らかに $2 < e$.

また、

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \cdots$$

$$< 1 + 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-2}} + \cdots$$

$$= 1 + \frac{1}{1 - \frac{1}{2}}$$

$$= 3$$

よって、$e < 3$.

したがって、$2 < e < 3$.

[3]

$F(t)=f(t)-g(t)$ とすると、平均値の定理より

$$\frac{F(1)-F(0)}{1-0}=F'(t_0)$$

を満たす $t_0\ (0<t_0<1)$ が存在する.

$$F(0)=f(0)-g(0)=0$$

$$F(1)=f(1)-g(1)=0$$

また、

$$F'(t)=f'(t)-g'(t)$$

よって、

$$\frac{0-0}{1-0}=f'(t_0)-g'(t_0)$$

だから、$f'(t_0)=g'(t_0)$ を満たす $t_0\ (0<t_0<1)$ が存在する.

$$\frac{dx(t_0)}{dt}=(f'(t_0),\ g'(t_0))$$

$$=(k,\ k)$$

$$=k\overrightarrow{OA}$$

ただし $f'(t_0)=g'(t_0)=k$

[4]

(1) $f'(x)=\dfrac{\sqrt{x^2+1}-\dfrac{2x^2}{2\sqrt{x^2+1}}}{x^2+1}$

$$=\frac{1}{(x^2+1)\sqrt{x^2+1}}$$

$$>0$$

$\therefore f(x)$ は単調増加

$$g'(x) = \frac{1}{\sqrt{1 - \frac{x^2}{x^2+1}}} \cdot \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

$$= \frac{1}{x^2+1}$$

$$> 0$$

∴ $g(x)$ は単調増加

(2) $$h'(x) = -\sin\left(\sin^{-1}\frac{x}{\sqrt{x^2+1}}\right) \cdot \frac{1}{x^2+1}$$

$$= -\frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{x^2+1}$$

(3) $$F(0) = 1 - 1$$

$$= 0$$

$$F'(x) = h'(x) - \left(\frac{1}{\sqrt{x^2+1}}\right)'$$

$$= -\frac{x}{(x^2+1)\sqrt{x^2+1}} - \frac{-\frac{2x}{2\sqrt{x^2+1}}}{x^2+1}$$

$$= 0$$

したがって，$F(x)$ は恒等的に0である.