S10-2

(1)
$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & 1 & -4 \\ 1 & 0 & 1 & 2 \\ -4 & 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & -7 \\ 1 & -2 & 1 & 1 \\ -4 & 9 & 3 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 & -7 \\ -2 & 1 & 1 \\ 9 & 3 & 8 \end{vmatrix}$$

$$= \cancel{-16} + 9 + 42 - (-63 - 6 \cancel{-16})$$

$$= 63 + 9 + 42 + 6$$

$$= 120$$

105 ~~185~~1

114

(2)
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix}$$

$$\begin{array}{r} 49 \\ \times\ 4 \\ \hline \end{array}$$

$$= -2 \begin{vmatrix} 1 & 2 & 7 \\ 1 & \cancel{2}\,4 & \cancel{8}\,5\ \cancel{10} \\ 7 & 10 & 13 \end{vmatrix}$$

$$= -2 \left(\cancel{52} + \cancel{70} + \cancel{70} - (196 + \cancel{26 + 50}) \right)$$

$$= -2 (2 - 56 - 26)$$

$$= 160$$

E10-1

(1)
$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix} = \begin{vmatrix} a+b+c & a^2 & b+c \\ a+b+c & b^2 & c+a \\ a+b+c & c^2 & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & a^2 & b+c \\ 1 & b^2 & c+a \\ 1 & c^2 & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & a^2 & b+c \\ 0 & b^2-a^2 & a-b \\ 0 & c^2-a^2 & a-c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} a^2-b^2 & a-b \\ a^2-c^2 & a-c \end{vmatrix} \times (-1)$$

$$= (a+b+c)(a-b)(a-c) \begin{vmatrix} a+b & 1 \\ a+c & 1 \end{vmatrix} \times (-1)$$

$$= (a+b+c)(a-b)(b-c)(c-a)$$

(2)
$$\begin{vmatrix} b+c+2a & b & c \\ a & c+a+2b & c \\ a & b & a+b+2c \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c+a+2b & c \\ 1 & b & a+b+2c \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)^3$$

§10-3

(1)
$$\begin{vmatrix} x & 1 & a & b \\ y^2 & y & 1 & c \\ yz^2 & z^2 & z & 1 \\ yzt & zt & t & 1 \end{vmatrix} = \begin{vmatrix} x-y & 1 & a & b \\ 0 & y & 1 & c \\ 0 & z^2 & z & 1 \\ 0 & zt & t & 1 \end{vmatrix}$$

$$= (x-y)\begin{vmatrix} y & 1 & c \\ z^2 & z & 1 \\ zt & t & 1 \end{vmatrix}$$

$$= (x-y)\begin{vmatrix} y-z & 1 & c \\ 0 & z & 1 \\ 0 & t & 1 \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} z & 1 \\ t & 1 \end{vmatrix}$$

$$= (x-y)(y-z)(z-t)$$

(2)
$$\begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix} = \begin{vmatrix} a+b & b & \cdots & b \\ a+b & a & \cdots & b \\ b+b & \vdots & \ddots & \vdots \\ \vdots & & & \\ b+b & b & \cdots & \end{vmatrix} \leftarrow n 列$$

$$= \begin{vmatrix} a+(n-1)b & b & \cdots & b \\ a+(n-1)b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ a+(n-1)b & b & \cdots & a \end{vmatrix} \leftarrow n 列$$

$$= (a+(n-1)b)\begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b & \cdots & a \end{vmatrix} \leftarrow n 列$$

$$= (a+(n-1)b)\begin{vmatrix} 1 & b & \cdots & & 0 \\ 0 & a-b & 0 & \cdots & 0 \\ \vdots & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a-b \end{vmatrix} \leftarrow n \geq 2$$

$$= (a+(n-1)b)\begin{vmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a-b \end{vmatrix} \leftarrow n-1 \geq 2$$

$$= (a+(n-1)b)(a-b)^{n-1}$$

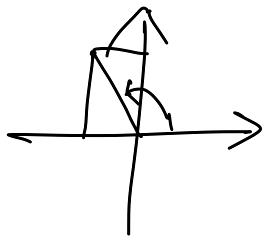
ECO-4

$$\begin{pmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{pmatrix} = \begin{pmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{pmatrix}\begin{pmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{pmatrix}$$

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = -2abc \cdot (-2abc) = 4a^2b^2c^2$$

SCO-4

$$\omega = \frac{-1+\sqrt{3}i}{2} = e^{i\frac{2}{3}\pi} \longrightarrow \omega^3 = 1$$



$$|AB| = \begin{vmatrix} a+c+b & a+b\omega+c\omega^2 & a+b\omega^2+c\omega \\ c+a+b & c+a\omega+b\omega^2 & c+a\omega^2+b\omega \\ b+c+a & b+c\omega+a\omega^2 & b+c\omega^2+a\omega \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b\omega+c\omega^2 & a+b\omega^2+c\omega \\ 1 & c+a\omega+b\omega^2 & c+a\omega^2+b\omega \\ 1 & b+c\omega+a\omega^2 & b+c\omega^2+a\omega \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b\omega+c\omega^2 & a+b\omega^2+c\omega \\ 1 & \omega(a+b\omega+c\omega^2) & \omega^2(a+b\omega^2+c\omega) \\ 1 & \omega^2(a+b\omega+c\omega^2) & \omega(a+b\omega^2+c\omega \end{vmatrix}$$

$$= (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)\,|B|$$

$$|B| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= \omega^2+\omega^2+\omega^2-\omega-\omega-\omega^4$$

$$= 3\omega^2-3\omega$$

$$= 3\omega(\omega-3)$$

$$\neq 0$$

$$\therefore |A| = \frac{|AB|}{|B|} = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

810-5

(2) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} a-c & b-c \\ a^2-c^2 & b^2-c^2 \end{vmatrix} = (a-c)(b-c)\begin{vmatrix} 1 & 1 \\ a+c & b+c \end{vmatrix}$

$= (a-c)(b-c)(b-a)$

$$\begin{vmatrix} 1 & 1 & 1 \\ d & b & c \\ d^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} d-c & b-c \\ d^2-c^2 & b^2-c^2 \end{vmatrix} = (d-c)(b-c)(b-d)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & d & c \\ a^2 & d^2 & c^2 \end{vmatrix} = \begin{vmatrix} a-c & d-c \\ a^2-c^2 & d^2-c^2 \end{vmatrix} = (a-c)(d-c)(d-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ a^2 & b^2 & d^2 \end{vmatrix} = \begin{vmatrix} a-d & b-d \\ a^2-d^2 & b^2-d^2 \end{vmatrix} = (a-d)(b-d)(b-a)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{(d-c)(b-d)}{(a-c)(b-a)} \\ \frac{(d-c)(d-a)}{(b-c)(b-a)} \\ \frac{(a-d)(b-d)}{(a-c)(b-c)} \end{pmatrix}$$

S10-6

$$|A| = 4+5+4-10$$
$$= 3$$

$$\begin{matrix} -3 & 1 & 5 \\ 9 & -2 & -10 \\ 6 & -1 & -8 \end{matrix}$$

$$\tilde{A} = \begin{pmatrix} -3 & 9 & 6 \\ 1 & -2 & -1 \\ 5 & -10 & -8 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} -3 & 9 & 6 \\ 1 & -2 & -1 \\ 5 & -10 & -8 \end{pmatrix}$$

E10-7

$$D_1 = 1+x^2$$

$$D_2 = \begin{vmatrix} 1+x^2 & x \\ x & 1+x^2 \end{vmatrix} = (1+x^2)^2 - x^2 = 1+x^2+x^4$$

$$D_n = (1+x^2)A_{11} + xA_{12}$$

$$= (1+x^2)\begin{vmatrix} 1+x^2 & x & \cdots & 0 \\ x & 1+x^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+x^2 \end{vmatrix} - x\begin{vmatrix} x & x & \cdots & 0 \\ 0 & 1+x^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+x^2 \end{vmatrix}$$

$$= (1+x^2)D_{n-1} - x^2 D_{n-2}$$

$$\therefore D_n - D_{n-1} = x^2(D_{n-1} - D_{n-2})$$

$$\therefore D_{n+1} - D_n = x^2(D_n - D_{n-1})$$

次数下げ 1回 $\Rightarrow$ $n-1$回繰り返すと $\underbrace{D_2 - D_1}_{=x^4}$ になる

$$\therefore \quad = x^{2(n-1)} \cdot x^4$$

$$\therefore D_n - D_{n-1} = x^{2(n-2)} \cdot x^4 = x^{2n} \cdot x^{-4} \cdot x^4 = x^{2n}$$

$n=4$なら

$$D_{4+1} - D_4 = x^2(D_4 - D_{4-1}) = x^2 \cdot x^2 (D_3 - D_2) = \underbrace{x^2 \cdot x^2 \cdot x^2}_{3つ}(D_2 - D_1)$$

$$D_n = D_1 + (D_2 - D_1) + (D_3 - D_2) + \cdots + (D_n - D_{n-1}) = 1 + x^2 + x^4 + x^6 + \cdots + x^{2n}$$

510-7

nに対する数学的帰納法で証明.

$n=1$のとき

$$\begin{vmatrix} a_0 & -1 \\ a_1 & x \end{vmatrix} = a_0 x + a_1$$

よって$n=1$のとき成立.

$n=k-1$で成り立つと仮定.

$n=k$のとき

$k+1$次　　k次

$$\begin{vmatrix} a_0 & -1 & 0 & \cdots & 0 \\ a_1 & x & -1 & \cdots & 0 \\ a_2 & 0 & x & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_k & 0 & 0 & \cdots & x \end{vmatrix} = a_0 \begin{vmatrix} x & -1 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{vmatrix} - 1 \cdot (-1)^3 \begin{vmatrix} a_1 & -1 & \cdots & 0 \\ a_2 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_k & 0 & \cdots & x \end{vmatrix}$$

$$= a_0 x^k + \begin{vmatrix} a_1 & -1 & \cdots & 0 \\ a_2 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_k & 0 & \cdots & x \end{vmatrix}$$

$$= a_0 x^k + a_1 x^{k-1} + \cdots + a_k$$

よって, $n=k-1$で成り立てば, $n=k$のときでも成り立つ.
したがって, すべての自然数nに対して等式が成立する.

E10-8

$$\begin{vmatrix} x & x+1 & x^2 \\ 1 & x^2+1 & 1 \\ x & x^2+x & x^2 \end{vmatrix} = \begin{vmatrix} x & 1 & x^2 \\ 1 & x^2 & 1 \\ x & x^2 & x^2 \end{vmatrix}$$

$$= x \begin{vmatrix} x & 1 & x^2 \\ 1 & x^2 & 1 \\ 1 & x & x \end{vmatrix}$$

$$= x(\cancel{x^4} + 1 + x^3 - \cancel{x^4} - x - x^2)$$

$$= x(x^3 - x^2 - x + 1)$$

$$= x(x-1)(x^2-1)$$

$$= x(x-1)^2(x+1)$$

1	1	-1	-1	1
		1	0	-1
	1	0	-1	0

(i) $x \neq 0, 1, -1$のとき ランクは3

(ii) $x = 0$のとき, ランクは2

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(iii) $x = 1$のとき, ランクは1

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(iv) $x = -1$のとき, ランクは2

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

S10-8

$$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a-b & b & b & b \\ b-a & a & b & b \\ 0 & b & a & b \\ 0 & b & b & a \end{vmatrix}$$

$$= -(a-b)\begin{vmatrix} 1 & b & b & b \\ 1 & -a & -b & -b \\ 0 & b & a & b \\ 0 & b & b & a \end{vmatrix}$$

$$= -(a-b)\begin{vmatrix} 0 & a+b & 2b & 2b \\ 1 & -a & -b & -b \\ 0 & b & a & b \\ 0 & b & b & a \end{vmatrix}$$

$$= (a-b)\begin{vmatrix} a+b & 2b & 2b \\ b & a & b \\ b & b & a \end{vmatrix}$$

$$= (a-b)\begin{vmatrix} a+3b & a+3b & a+3b \\ b & a & b \\ b & b & a \end{vmatrix}$$

$$= (a-b)(a+3b)\begin{vmatrix} 1 & 1 & 1 \\ b & a & b \\ b & b & a \end{vmatrix}$$

$$= (a-b)(a+3b)\begin{vmatrix} 1 & 0 & 0 \\ b & a-b & 0 \\ b & 0 & a-b \end{vmatrix}$$

$$= (a-b)^3(a+3b)$$

(i) $a \neq b$ かつ $a \neq -3b$ のとき ランク4

(ii) $a \neq b$ かつ $a = -3b$ のとき, ランク3

$$\begin{pmatrix} -3b & b & b & b \\ b & -3b & b & b \\ b & b & -3b & b \\ b & b & b & -3b \end{pmatrix} \to \begin{pmatrix} -3b & b & b & b \\ b & -3b & b & b \\ b & b & -3b & b \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(iii) $a=b$, $a \neq -3b$ のとき, ランク1

$$\begin{pmatrix} b & b & b & b \\ b & b & b & b \\ b & b & b & b \\ b & b & b & b \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(iv) $a=b$ かつ $a=-3b$ のとき, $a=b=0$.

よって, ランク0

E10-9

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ A+B & A \end{vmatrix}$$

$$= \begin{vmatrix} A+B & B \\ 0 & A-B \end{vmatrix}$$

$$= |A+B||A-B|$$

S10-9

(1) $A = \begin{pmatrix} 10 & 12 \\ 4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 13 & 17 \\ 7 & 11 \end{pmatrix}$, $C = \begin{pmatrix} -5 & 7 \\ 2 & -3 \end{pmatrix}$ とおき,

$$(\text{与式}) = \begin{vmatrix} A & B \\ 0 & C \end{vmatrix}$$

$$= |A||C|$$

$$= (50-48)(15-14)$$

$$= 2$$

(2) $$\begin{vmatrix} A & -A \\ B & B \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & 2B \end{vmatrix}$$

$$= |A||2B|$$

ここで $|2B|$ は

$$|2B| = \begin{vmatrix} 2b_{11} & 2b_{12} & \cdots & 2b_{1n} \\ 2b_{21} & 2b_{22} & \cdots & 2b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2b_{n1} & 2b_{n2} & \cdots & 2b_{nn} \end{vmatrix}$$

$$= 2^n \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$\therefore \begin{vmatrix} A & -A \\ B & B \end{vmatrix} = 2^n |A||B|$$