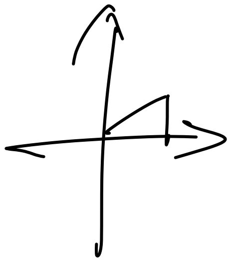
[1]

$$\iint_D \frac{x\,dx\,dy}{\sqrt{4x^2-y^2}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{x}^{2x\sin x} \frac{x}{\sqrt{4x^2-y^2}}\,dy\,dx$$



$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[x \sin^{-1}\frac{y}{2x} \right]_{x}^{2x\sin x} dx$$

$$\sin^{-1}\frac{x}{a}$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2-x^2}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(x\cdot x - x\cdot\frac{\pi}{6} \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{\pi}{12}x^2 \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{1}{3}\cdot\frac{\pi^3}{8} - \frac{\pi}{12}\cdot\frac{\pi^2}{4} - \frac{1}{3}\cdot\frac{\pi^3}{6^3} + \frac{\pi}{12}\cdot\frac{\pi^2}{6^2}$$

$$= \frac{\pi^3}{24} - \frac{\pi^3}{48} - \frac{\pi^3}{648} + \frac{\pi^3}{432}$$

$$= \frac{54}{1296}\pi^3 - \frac{27}{1296}\pi^3 - \frac{2}{1296}\pi^3 + \frac{3}{1296}\pi^2$$

$$= \frac{28}{1296}\pi^3$$

$$= \frac{7}{324}\pi^3$$

$36 \times 12 = 72 + 36 = 432$

$36 \times 6 = 216$

$216 \times 3 = 648$

$1296 \div 48 = 27$

$1296 \div 432 = 3$

$\cancel{324 \div 7}$

$1296 \div 4 = 324$

$1296 \div 24 = 54$

[2]

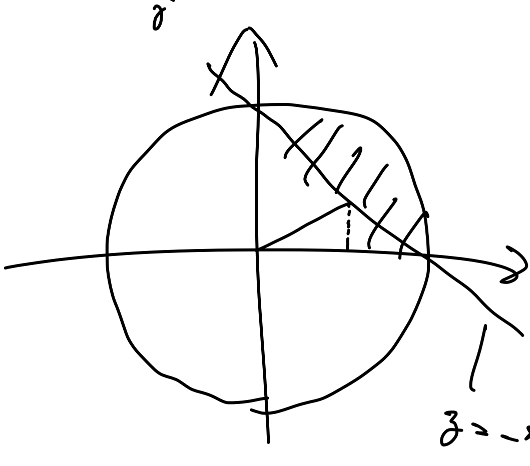
(1) $$\iint_D x^2\,dx\,dy$$

$$= \int_0^1 \int_{-x+1}^{\sqrt{1-x^2}} x^2\,dy\,dx$$

$y = \pm\sqrt{1-x^2}$

$$= \int_0^1 \left[x^2 y \right]_{-x+1}^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(x^2\sqrt{1-x^2} + x^3 - x^2 \right) dx$$

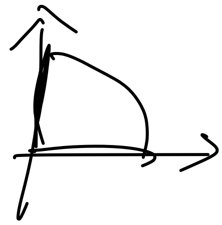


$y = -x + 1$

$= -\cos\theta + 1$

$= \int_0^1 x^2\sqrt{1-x^2}\,dx + \left[\frac{x^4}{4}-\frac{x^3}{3}\right]_0^1$

$= \frac{1}{4}-\frac{1}{3}+\int_0^1 x^2\sqrt{1-x^2}\,dx$



$x=\sin\theta$ とおくと、　　　$dx=\cos\theta\,d\theta$

x	$0 \to 1$
θ	$0 \to \frac{\pi}{2}$

$\int_0^1 x^2\sqrt{1-x^2}\,dx$

$= \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} (\sin^2\theta-\sin^4\theta)\,d\theta$

$= \frac{1}{2}\cdot\frac{\pi}{2}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$= \frac{\pi}{4}-\frac{3}{16}\pi$

$\therefore \iint_D x^2\,dx\,dy$

$= \frac{\cancel{1}^{3}}{\cancel{4}_{12}}-\frac{\cancel{1}^{4}}{\cancel{3}_{12}}+\frac{4\pi}{\cancel{4}_{16}}-\frac{3}{16}\pi$

$= -\frac{1}{12}+\frac{\pi}{16}$

(2)

$u=xy,\ v=\frac{y}{x}$ とおくと

$x=\pm\sqrt{\frac{u}{v}},\ y=\pm\sqrt{uv}$

$u = x\cdot xv$

$x^2=\frac{u}{v}$　　$x=\pm\sqrt{\frac{u}{v}}$

$y = xv$

$= \frac{u}{y}\cdot v$　　$y^2=uv$　　$y=\pm\sqrt{uv}$

$\therefore x_u = \pm\frac{1}{v}\cdot\frac{1}{2\sqrt{\frac{u}{v}}} = \pm\frac{1}{2\sqrt{uv}}$

$x_v = \pm\sqrt{u}\cdot\left(-\frac{1}{2}\right)\cdot v^{-\frac{3}{2}} = \mp\frac{1}{2}\cdot\frac{\sqrt{u}}{v\sqrt{v}}$

$y_u = \pm\frac{v}{2\sqrt{uv}} = \pm\frac{1}{2}\sqrt{\frac{v}{u}}$

$y_v = \pm\frac{u}{2\sqrt{uv}} = \pm\frac{1}{2}\sqrt{\frac{u}{v}}$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \frac{1}{4v} + \frac{1}{4v}$$

$$= \frac{1}{2v}$$

$1 \le u \le 3$

$1 \le \frac{y}{x} \le 4$

$\Downarrow$

$1 \le v \le 4$

$$\therefore \iint_D x^2 e^{-x^2y^2}\,dx\,dy$$

$$= \int_1^4 \int_1^3 \frac{u}{v} e^{-u^2} \cdot \frac{1}{2v}\,du\,dv$$

$$= \int_1^4 \left[-\frac{1}{4v^2} e^{-u^2}\right]_1^3 dv$$

$$= \int_1^4 \left(-\frac{1}{4v^2} e^{-9} + \frac{1}{4v^2} e^{-1}\right) dv$$

$$= \left[\frac{1}{4v} e^{-9} - \frac{1}{4v} e^{-1}\right]_1^4$$

$$= \frac{1}{16}\left(e^{-9} - e^{-1}\right) + \frac{1}{4}\left(e^{-1} - e^{-9}\right)$$

$$= \frac{1}{16e^9}\left(1 - e^8 + 4e^8 - 4\right)$$

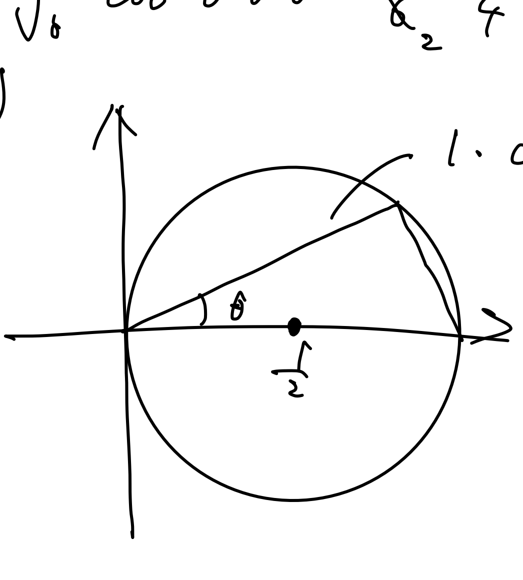
$$= \frac{3}{16} \cdot \frac{e^8 - 1}{e^9}$$

$\boxed{3}$

$\cancel{24 \times 4 = 96}$

(1) $\int_0^{\frac{\pi}{2}} \cos^6\theta\,d\theta = \frac{5}{\cancel{6}_2} \cdot \frac{\cancel{3}^1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5}{32}\pi$

(2)



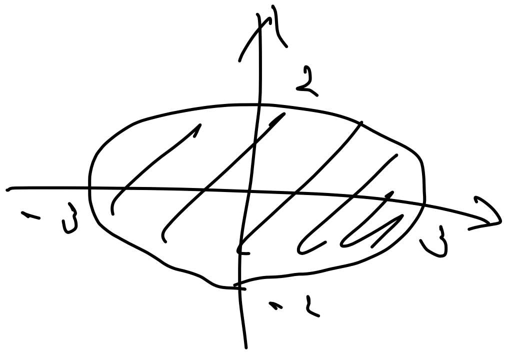
$x^2 + y^2 - x \le 0$

$(x - \frac{1}{2})^2 + y^2 \le \frac{1}{4}$

$$\iint_D x^2\,dx\,dy$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} r^3 \cos^2\theta\,dr\,d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{\cos\theta} \cos^2\theta\,d\theta$$
$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\theta\,d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^6\theta\,d\theta$$
$$= \frac{5}{64}\pi$$

$\boxed{4}$

$$D : \frac{x^2}{9} + \frac{y^2}{4} \leq 1$$



$x = 3r\cos\theta,\ y = 2r\sin\theta$ とおく。

$$x_r = 3\cos\theta,\ x_\theta = -3r\sin\theta$$
$$y_r = 2\sin\theta,\ y_\theta = 2r\cos\theta$$
$$\frac{\partial(x,y)}{\partial(r,\theta)} = 6r\cos^2\theta + 6r\sin^2\theta$$
$$= 6r$$

$$\iint_D (xy+1)\,dx\,dy$$

36

9

$$= \int_0^{2\pi}\int_0^1 \left(6r^2\cos\theta\sin\theta + 1\right) 6r\,dr\,d\theta$$
$$= \int_0^{2\pi} \left[9r^4\cos\theta\sin\theta + 3r^2\right]_0^1 d\theta$$

$2\sin\theta\cos\theta = \sin 2\theta$

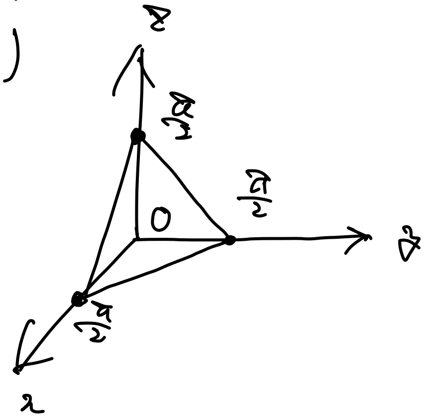
$$= \int_0^{2\pi} \left(9\cos\theta\sin\theta + 3\right) d\theta$$
$$= \frac{9}{2}\int_0^{2\pi} \sin 2\theta\,d\theta + \left[3\theta\right]_0^{2\pi}$$
$$= \frac{9}{2}\left[-\frac{1}{2}\cos 2\theta\right]_0^{2\pi} + 6\pi$$

$= \frac{9}{2}\left(-\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1\right) + 6\pi$

$= 6\pi$

$\boxed{5}$

(1)



(2) $\int x \sin(a+x)\,dx$

$+ \quad x \quad -\cos(a+x)$

$- \quad 1 \quad -\sin(a+x)$

$= -x\cos(a+x) + \sin(a+x) + C$

(3) $\iiint_D z\sin(x+y+z)\,dx\,dy\,dz$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-z} \int_0^{\frac{\pi}{2}-y-z} z\sin(x+y+z)\,dx\,dy\,dz$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-z} \Big[-z\cos(x+y+z)\Big]_0^{\frac{\pi}{2}-y-z} dy\,dz$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-z} \left(-z\cos\frac{\pi}{2} + z\cos(y+z)\right) dy\,dz$

$= \int_0^{\frac{\pi}{2}} \Big[z\sin(y+z)\Big]_0^{\frac{\pi}{2}-z} dz$

$+ \quad z \quad -\cos z$

$- \quad 1 \quad -\sin z$

$= \int_0^{\frac{\pi}{2}} \left(z\sin\frac{\pi}{2} - z\sin z\right) dz$

$= \left[\frac{z^2}{2}\right]_0^{\frac{\pi}{2}} - \Big[-z\cos z + \sin z\Big]_0^{\frac{\pi}{2}}$

$= \frac{\pi^2}{8} - 1$