E2-1

(1) $\int x \sin 2x \, dx$

$$\begin{array}{lll} + & x & -\frac{1}{2}\cos 2x \\ - & 1 & -\frac{1}{4}\sin 2x \end{array}$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

(2) $\int \log x \, dx = x\log x - \int x \cdot \frac{1}{x} dx$

$$= x\log x - x + C$$

(3) $\int \frac{x^2}{(1+x^2)^2} dx = \int x \cdot \frac{x}{(1+x^2)^2}$

$$\begin{array}{lll} + & x & -\frac{1}{2}\frac{1}{1+x^2} \\ - & 1 & -\frac{1}{2}\tan^{-1}x \end{array}$$

$$= \frac{1}{2}\left(-\frac{x}{1+x^2} + \tan^{-1}x\right) + C$$

S2-1

(1) $\int x^2 \cos x \, dx$

$$\begin{array}{lll} + & x^2 & \sin x \\ - & 2x & -\cos x \\ + & 2 & -\sin x \end{array}$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

(2) $\int x^2 \log x \, dx$

$$= \frac{x^3}{3}\log x - \int \frac{x^{\cancel{3}\,2}}{3}\cdot\frac{1}{\cancel{x}} dx$$

$$= \frac{1}{3}x^3\left(\log x - \frac{1}{3}\right) + C$$

(3) $\int \tan^{-1}x \, dx$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2} dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$$

E 2-2

(1) $\int \frac{1}{1+\sqrt{x}} dx$

$\sqrt{x} = t$ とおくと $x = t^2 \quad dx = 2t\,dt$

$\therefore \int \frac{1}{1+\sqrt{x}} dx = \int \frac{2t}{1+t} dt$

$$\begin{array}{r} 2 \\ 1+t\,\overline{)\,2t\qquad} \\ \underline{2t+2} \\ -2 \end{array}$$

$= \int \left(2 - \frac{2}{1+t}\right) dt$

$= 2t - 2\log|1+t| + C$

$= 2\sqrt{x} - 2\log(1+\sqrt{x}) + C$

(2) $\int \frac{1}{e^x+1} dx$

$e^x = t$ とおくと $e^x = \frac{dt}{dx} \quad \therefore dx = \frac{1}{e^x} dt = \frac{1}{t} dt$

$\int \frac{1}{e^x+1} dx = \int \frac{1}{t+1} \frac{1}{t} dt$

$= \int \left(\frac{-1}{t+1} + \frac{1}{t}\right) dt$

$= -\log|t+1| + \log|t|$

$= \log\left(\frac{e^x}{e^x+1}\right) + C$

S 2-1

(1) $\int \frac{1}{x\sqrt{x+1}} dx$

$\sqrt{x+1} = t \qquad x+1 = t^2 \quad x = t^2-1 \quad dx = 2t\,dt$

$\therefore \int \frac{1}{x\sqrt{x+1}} dx = \int \frac{2\cancel{t}}{\cancel{t}(t^2-1)} dt$

$= \int \frac{2}{(t+1)(t-1)} dt$

$$= \int \left(\frac{-1}{t+1} + \frac{1}{t-1} \right) dt$$

$$= -\log|t+1| + \log|t-1| + C$$

$$= \log \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C$$

(2) $\displaystyle\int \frac{e^x}{\sqrt{e^x-1}}\,dx$

$\sqrt{e^x-1} = t \quad e^x-1 = t^2 \quad e^x = t^2+1 \quad e^x\frac{dx}{dt} = 2t$

$e^x dx = 2t\,dt$

$$\therefore \int \frac{e^x}{\sqrt{e^x-1}}\,dx = \int \frac{2t}{t}\,dt$$

$$= 2t + C$$

$$= 2\sqrt{e^x-1} + C$$

E2-3

(1) $\displaystyle\int \frac{1}{4x^2-1}\,dx = \int \frac{1}{(2x+1)(2x-1)}\,dx$

$$= \int \left(\frac{-\frac{1}{2}}{2x+1} + \frac{\frac{1}{2}}{2x-1} \right) dx$$

$$= -\frac{1}{4}\log|2x+1| + \frac{1}{4}\log|2x-1| + C$$

$$= \frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right| + C$$

(2) $\displaystyle\int \frac{1}{x^3-x}\,dx = \int \frac{1}{x(x+1)(x-1)}\,dx$

$$= \int \left(\frac{-1}{x} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} \right) dx$$

$$= -\log|x| + \frac{1}{2}\log|x+1| + \frac{1}{2}\log|x-1| + C$$

$$= \frac{1}{2}\left(\log|x+1| + \log|x-1| - 2\log|x|\right) + C$$

$$= \frac{1}{2}\log\left|\frac{(x+1)(x-1)}{x^2}\right| + C$$

S 2-3

(1) $\displaystyle\int \frac{1}{x^3+3x}\,dx = \int \frac{1}{x(x^2+3)}\,dx$

$$= \int\left(\frac{\frac{1}{3}}{x} + \frac{-\frac{1}{3}x}{x^2+3}\right)dx$$

$$= \frac{1}{3}\log|x| - \frac{1}{6}\log(x^2+3) + C$$

$$= \frac{1}{6}\left(\log x^2 - \log(x^2+3)\right) + C$$

$$= \frac{1}{6}\log\frac{x^2}{x^2+3} + C$$

$\dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$

$Ax^2 + 3A + Bx^2 + Cx = 1$

$A + B = 0$

$C = 0$

$3A = 1$

$\therefore A = \frac{1}{3},\ B = -\frac{1}{3},$

$C = 0$

(2) $\displaystyle\int \frac{1}{x^3+8}\,dx$

$$= \int \frac{1}{(x+2)(x^2-2x+4)}\,dx$$

$$= \int\left(\frac{\frac{1}{12}}{x+2} + \frac{-\frac{1}{12}x+\frac{1}{3}}{x^2-2x+4}\right)dx$$

$$= \int\left(\frac{1}{12(x+2)} - \frac{x-4}{12(x^2-2x+4)}\right)dx$$

$$= \int\left(\frac{1}{12(x+2)} - \frac{2x-2-6}{24(x^2-2x+4)}\right)dx$$

$$= \int\left(\frac{1}{12(x+2)} - \frac{2x-2}{24(x^2-2x+4)}\right.$$

$$\left. + \frac{1}{4(x^2-2x+4)}\right)dx$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

$Ax^2 + 2Ax + 4A + Bx^2 + Cx - 2Bx + 2C = 1$

$$\begin{cases} A + B = 0 \\ 2A - 2B + C = 0 \\ 4A + 2C = 1 \end{cases}$$

$4A - 4B + 2C = 0$

$-)\ 4A \qquad + 2C = 1$

$-4B \qquad = 1$

$+)\ 4A + 4B \qquad = 0$

$12A \qquad = 1$

$A = \frac{1}{12}$

$B = -\frac{1}{12}$

$2C = 4 \times \frac{1}{12} - 1$

$= \frac{1}{3} - 1 = -\frac{2}{3} \quad \therefore C = \frac{1}{3}$

$$= \int \left(\frac{1}{12(x+2)} - \frac{2x-2}{24(x^2-2x+4)} + \frac{1}{4} \frac{1}{(x-1)^2+3} \right) dx$$

$$= \int \frac{1}{12(x+2)} - \frac{2x-2}{24(x^2-2x+4)} + \frac{1}{4\sqrt{3}} \frac{\sqrt{3}}{(x-1)^2+3} \Bigg) dx$$

$$= \frac{1}{12} \log|x+2| - \frac{1}{24} \log\left(x^2-2x+4\right)$$ 下に凸でx軸と交わらない→正

$$\left(\tan^{-1}\frac{x}{a}\right)' = \frac{1}{a} \frac{1}{1+\left(\frac{x}{a}\right)^2} = \frac{a}{a^2+x^2}$$

$$+ \frac{1}{4\sqrt{3}} \tan^{-1} \frac{x-1}{\sqrt{3}} + C$$

$$= \frac{1}{24} \left(\log \frac{(x+2)^2}{x^2-2x+4} + 2\sqrt{3} \tan^{-1} \frac{x-1}{\sqrt{3}} \right) + C$$

$$\frac{6}{\sqrt{3}} = \frac{\cancel{6}^{2}\sqrt{3}}{\cancel{3}}$$

E 2-4

$$\int \frac{1}{\cos x + 2} dx$$

$\tan \frac{x}{2} = t$ とおくと

$$\cos \frac{x}{2} = \frac{\sin \frac{x}{2}}{t}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$$

$$= 2\cos^2 \frac{x}{2} - 1$$

$$= \frac{2}{\frac{1}{\cos^2 \frac{x}{2}}} - 1$$

$$= \frac{2}{1 + \tan^2 \frac{x}{2}} - 1$$

$$= \frac{2}{1+t^2} - 1$$

$$\frac{\frac{1}{2}}{\cos^2 \frac{x}{2}} = \frac{dt}{dx}$$

$$dx = 2\cos^2\frac{x}{2}\,dt$$

$$= \frac{2}{1+\tan^2\frac{x}{2}}\,dt$$

$$= \frac{2}{1+t^2}\,dt$$

$$\int \frac{1}{\cos x + 2}\,dx = \int \frac{1}{\frac{2}{1+t^2} - 1 + 2}\,\frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{2+1+t^2}\,dt$$

$$= \int \frac{2}{t^2+3}\,dt$$

$$= \int \frac{2}{\sqrt{3}}\,\frac{\sqrt{3}}{t^2+3}\,dt$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{3}} + C$$

S2-4

(1) $\int \frac{\sin x}{1+\sin x}\,dx$

$\tan\frac{x}{2} = t$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\tan\frac{x}{2}\cos^2\frac{x}{2}$$

$$= 2\tan\frac{x}{2}\cdot\frac{1}{1+\tan^2\frac{x}{2}}$$

$$= \frac{2t}{1+t^2}$$

$$\frac{1}{2}\frac{1}{\cos^2\frac{x}{2}} = \frac{dt}{dx} \qquad dx = 2\cos^2\frac{x}{2}\,dt = \frac{2}{1+t^2}\,dt$$

$$\int \frac{\sin x}{1+\sin x}\,dx = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int \frac{4t}{1+t^2+2t} \cdot \frac{1}{1+t^2}\,dt$$

$$= \int \frac{4t}{(1+t)^2(1+t^2)}\,dt$$

$$A + At^2 + B + Bt^2 + 2Bt = 4t$$

$$A+B=0$$

$$2B = 4 \qquad B=2$$

$$A=-2$$

$$= \int \left(\frac{-2}{(1+t)^2} + \frac{2}{1+t^2} \right) dt$$

$$= 2\frac{1}{1+t} + 2\tan^{-1}t + C$$

$$= 2\left(\frac{1}{1+\tan^{-1}\frac{x}{2}} + \frac{x}{2} \right) + C$$

$$= \frac{2}{1+\tan^{-1}\frac{x}{2}} + x + C$$

(2) $\tan\frac{x}{2} = t$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\tan\frac{x}{2}\cos^2\frac{x}{2}$$

$$= \frac{2t}{1+t^2}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= \frac{2}{1+t^2} - 1$$

$$= \frac{1-t^2}{1+t^2}$$

$\tan\frac{x}{2} = t$

$\frac{1}{2\cos^2\frac{x}{2}} = \frac{dt}{dx}$

$dx = 2\cos^2\frac{x}{2}\,dt$

$= \frac{2}{1+t^2}\,dt$

$$\int \frac{1}{3\sin x + 4\cos x}\,dx$$

$$= \int \frac{1}{3\frac{2t}{1+t^2} + 4\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{6t + 4 - 4t^2}\,dt$$

$$= \int \frac{-1}{2t^2 - 3t - 2}\,dt$$

$$\begin{array}{cccc} 2 & & 1 & 1 \\ 1 & \times & -2 & -4 \\ \hline 2 & & -2 & -3 \end{array}$$

$$= \int \frac{-1}{(2t+1)(t-2)}\,dt$$

$$= \int \left(\frac{\frac{2}{5}}{2t+1} + \frac{-\frac{1}{5}}{t-2} \right) dt$$

$$\frac{-1}{-\frac{1}{2}-2} = \frac{2}{5}$$

$$= \frac{1}{5}\left(\log|2t+1| - \log|t-2| \right) + C$$

$$= \frac{1}{5}\log\left|\frac{2t+1}{t-2}\right| + C$$

$$= \frac{1}{5}\log\left|\frac{2\tan\frac{x}{2}+1}{\tan\frac{x}{2}-2}\right| + C$$

E2-5

(1) $\int \sin^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(2) $\displaystyle\int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx$

$$= \int \frac{\cos x}{1-\sin^2 x}\,dx$$

$A - A\sin x + B + B\sin x$
$= 1$

$$= \int \frac{\cos x}{(1+\sin x)(1-\sin x)}\,dx$$

$A+B=1$
$A-B=0$
$A=\frac{1}{2},\ B=\frac{1}{2}$

$$= \int \frac{1}{2}\cos x\left(\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right)dx$$

$$= \frac{1}{2}\int\left(\frac{\cos x}{1+\sin x}+\frac{\cos x}{1-\sin x}\right)dx$$

$$= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}+C$$

(3) $\displaystyle\int \cos 3x \sin 2x\,dx$

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$

$$= \int \frac{1}{2}\left(\sin 5x + \sin(-x)\right)dx$$

$$= \frac{1}{2}\int(\sin 5x - \sin x)\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x\right)+C$$

S 2-5

(1) $\displaystyle\int \cos^2 2x\,dx = \int \frac{1+\cos 4x}{2}\,dx$

$$= \frac{1}{2}x + \frac{1}{8}\sin 4x + C$$

(2) $\displaystyle\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)\,dx$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

(3) $\displaystyle\int \cos 4x \cos 3x \, dx = \int \frac{1}{2}(\cos 7x + \cos x)\, dx$

$$= \frac{1}{14}\sin 7x + \frac{1}{2}\sin x + C$$

E2-6

$$\int \frac{1}{x\sqrt{x^2 - x + 2}}\, dx$$

2乗すると左辺のx^2と相殺

$\sqrt{x^2 - x + 2} = t - x$ とおく

$$x^2 - x + 2 = t^2 - 2tx + x^2$$

$$(2t - 1)x = t^2 - 2$$

$$x = \frac{t^2 - 2}{2t - 1}$$

$$dx = \frac{2t(2t-1) - 2(t^2 - 2)}{(2t-1)^2} = \frac{2(t^2 - t + 2)}{(2t-1)^2}\, dt$$

$$\int \frac{1}{x\sqrt{x^2 - x + 2}}\, dx$$

$$= \int \frac{1}{\frac{t^2-2}{2t-1} \cdot \left(t - \frac{t^2-2}{2t-1}\right)} \cdot \frac{2(t^2 - t + 2)}{(2t-1)^2}\, dt$$

$$\frac{2t^2 - t - t^2 + 2}{2t - 1} = \frac{t^2 - t + 2}{2t - 1}$$

$$= \int \frac{1}{\frac{(t^2 - 2)(t^2 - t + 2)}{(2t-1)^2}} \cdot \frac{2(t^2 - t + 2)}{(2t-1)^2}\, dt$$

$$= \int \frac{2}{t^2 - 2}\, dt$$

$$= \int \left(\frac{-\frac{1}{\sqrt{2}}}{t + \sqrt{2}} + \frac{\frac{1}{\sqrt{2}}}{t - \sqrt{2}} \right) dt$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{t-\sqrt{2}}{t+\sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{x^2-x+2}+x-\sqrt{2}}{\sqrt{x^2-x+2}+x+\sqrt{2}} \right| + C$$

S 2-6

(1) $\sqrt{x^2+1} = t - x \longrightarrow t = \sqrt{x^2+1} + x$

$$x^2+1 = t^2 - 2tx + x^2$$

$$2tx = t^2 - 1$$

$$x = \frac{t^2-1}{2t}$$

$$dx = \frac{4t^2 - 2t^2 + 2}{4t^2}\,dt = \frac{\cancel{2}(t^2+1)}{{}_2\cancel{4}t^2}\,dt$$

$$\int \frac{1}{1+\sqrt{x^2+1}}\,dx$$

$$= \int \frac{1}{\underbrace{1 + t - \frac{t^2-1}{2t}}} \cdot \frac{t^2+1}{2t^2}\,dt$$

$$\frac{2t + 2t^2 - t^2 + 1}{2t} = \frac{t^2+2t+1}{2t} = \frac{(t+1)^2}{2t}$$

$$= \int \frac{1}{\frac{(t+1)^2}{\cancel{2t}}} \cdot \frac{t^2+1}{\cancel{2t^2}}\,dt$$

$$= \int \frac{t^2+1}{t(t+1)^2}\,dt$$

$$= \int \left(\frac{1}{t} + \frac{-2}{(t+1)^2} \right) dt$$

$$= \log|t| + \frac{2}{t+1} + C$$

$$= \log\left(\sqrt{x^2+1}+x\right) + \frac{2}{\sqrt{x^2+1}+x+1} + C$$

(2) $\sqrt{x^2+1} = t$

$x^2+1 = t^2 \qquad x^2 = t^2-1$

$2x\,dx = 2t\,dt \quad \therefore x\,dx = t\,dt$

$$\int \frac{1}{x\sqrt{x^2+1}}\,dx$$

$$= \int \frac{1}{x^2\sqrt{x^2+1}}\,x\,dx$$

$$= \int \frac{\cancel{t}}{(t^2-1)\cdot\cancel{t}}\,dt$$

$$= \int \left(\frac{-\frac{1}{2}}{t+1} + \frac{\frac{1}{2}}{t-1}\right)dt$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + C$$

E 2-7

(1) $\displaystyle\int \frac{1}{\sqrt{2-x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{-\left(x+\frac{1}{2}\right)^2+\frac{9}{4}}}\,dx$$

$$\left(\sin^{-1}\frac{x}{a}\right)' = \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{1}{\sqrt{a^2-x^2}}$$

$$= \sin^{-1}\frac{x+\frac{1}{2}}{\frac{3}{2}} + C$$

$$= \sin^{-1}\frac{2x+1}{3} + C$$

(2) $\int \sqrt{4-x^2}\,dx$

$x = 2\sin\theta \qquad \theta = \sin^{-1}\frac{x}{2}$

$dx = 2\cos\theta\,d\theta$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4(1-\sin^2\theta)} \cdot 2\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta$$

$$= 4\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 2\theta + \sin 2\theta + C$$

$$= 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\theta + 2\sin\theta\sqrt{1-\sin^2\theta} + C$$

$$= 2\sin^{-1}\frac{x}{2} + x\cdot\sqrt{1-\left(\frac{x}{2}\right)^2} + C$$

$$= 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$$

S 2-7

(1) $\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{-(x+1)^2+4}}\,dx$$

$$= \sin^{-1}\frac{x+1}{2} + C$$

(2) $\int \sqrt{2x - x^2}\, dx$

$= \int \sqrt{-(x-1)^2 + 1}$

$x - 1 = \sin\theta \qquad \theta = \sin^{-1}(x-1)$

$dx = -\cos\theta\, d\theta$

$\therefore$

$= \int \sqrt{1 - \sin^2\theta} \cdot (-\cos\theta)\, d\theta$

$= \int -\cos^2\theta\, d\theta$

$= \int -\frac{1 + \cos 2\theta}{2}\, d\theta$

$= -\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

$= -\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C$

$= -\frac{1}{2}\theta - \frac{1}{2}\sin\theta\sqrt{1 - \sin^2\theta} + C$

$= -\frac{1}{2}\sin^{-1}(x-1) - \frac{1}{2}(x-1)\sqrt{1 - x^2 + 2x - 1} + C$

$= -\frac{1}{2}\sin^{-1}(x-1) - \frac{1}{2}(x-1)\sqrt{2x - x^2} + C$