E3-1

(1) $\int_0^1 x^2 \tan^{-1} x \, dx$

$= \left[\frac{x^3}{3} \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x^3}{3} \cdot \frac{1}{1+x^2} \, dx$

$$\begin{array}{r} x \\ 1+x^2 \overline{) x^3} \\ \underline{x^3 + x} \\ -x \end{array}$$

$= \frac{1}{3} \tan^{-1} 1 - \frac{1}{3} \int_0^1 \left(x - \frac{x}{1+x^2} \right) dx$

$= \frac{1}{3} \cdot \frac{\pi}{4} - \frac{1}{3} \left[\frac{x^2}{2} - \frac{1}{2} \log (1+x^2) \right]_0^1$

$= \frac{\pi}{12} - \frac{1}{3} \left(\frac{1}{2} - \frac{1}{2} \log 2 \right)$

$= \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \log 2$

(2) $\int_0^1 \log (1+\sqrt{x}) \, dx$

$1+\sqrt{x} = t$

$x = t^2 - 2t + 1$

$dx = (2t-2) dt$

x	0	1
t	1	2

$\int_1^2 (2t-2) \log t \, dt$

$= 2 \int_1^2 (t \log t - 2 \log t) \, dt$

$\log t \quad \frac{t^2}{2}$

$\frac{1}{t}$

$= 2 \left(\left[\frac{t^2}{2} \log t \right]_1^2 - \int_1^2 \frac{t^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{t}} \, dt \right) - 2 \left[t \log t - t \right]_1^2$

$= 2 \left(2 \log 2 - \left[\frac{t^2}{4} \right]_1^2 \right) - 4 \log 2 + 4 - 2$

$= \frac{1}{2}$

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{\pi}{4}$

§3-1

(1) $\displaystyle\int_0^1 (\sin^{-1} x)^2\, dx$

$\sin^{-1} x = t \qquad x = \sin t$

$\dfrac{1}{\sqrt{1-x^2}} = \dfrac{dt}{dx} \qquad dx = \sqrt{1-x^2}\, dt = \cos t\, dt$

$$\begin{array}{c|ccc} x & 0 & \to & 1 \\ t & 0 & \to & \frac{\pi}{2} \end{array}$$

$$\begin{array}{lll} + & t^2 & \sin t \\ - & 2t & -\cos t \\ + & 2 & -\sin t \end{array}$$

$\displaystyle\int_0^1 (\sin^{-1} x)^2\, dx$

$\displaystyle= \int_0^{\frac{\pi}{2}} t^2 \cos t\, dt$

$\displaystyle= \left[t^2 \sin t + 2t\cos t - 2\sin t \right]_0^{\frac{\pi}{2}}$

$\displaystyle= \frac{\pi^2}{4} - 2$

(2) $\displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{3+x^2}}\, dx$

$t = x + \sqrt{x^2+3}$

$\dfrac{dt}{dx} = 1 + \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}}$

$dt = \dfrac{x+\sqrt{x^2+3}}{\sqrt{x^2+3}}\, dx$

$\dfrac{dt}{t} = \dfrac{dx}{\sqrt{x^2+3}}$

$$\begin{array}{c|ccc} x & 0 & \to & \sqrt{3} \\ t & \sqrt{3} & \to & \sqrt{3}+\sqrt{6} \end{array}$$

$$\int_0^{\sqrt{3}} \frac{1}{\sqrt{3+x^2}}\,dx = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{1}{t}\,dt$$

$$= \Big[\log|t|\Big]_{\sqrt{3}}^{\sqrt{3}+\sqrt{6}}$$

$$= \log(\sqrt{3}+\sqrt{6}) - \log\sqrt{3}$$

$$= \log\frac{\sqrt{3}+\sqrt{6}}{\sqrt{3}}$$

$$= \log(\sqrt{2}+1)$$

(3) $\displaystyle\int_0^{\frac{\pi}{3}} \sin 3x \sin 2x\,dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (\cos x - \cos 5x)\,dx$$

$$= \frac{1}{2}\Big[\sin x - \frac{1}{5}\sin 5x\Big]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\frac{{}^{5}\sqrt{3}}{\cancel{2}_{\,10}} - \frac{1}{5}\cdot\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{1}{2}\cdot\frac{6\sqrt{3}}{10}$$

$$= \frac{3\sqrt{3}}{10}$$

E3-2

(1) $\displaystyle\int_1^{\infty} \frac{x^2}{(1+x^2)^2}\,dx$

$$= \lim_{\beta\to\infty}\int_1^{\beta} \frac{x^2}{(1+x^2)^2}\,dx$$

$$= \lim_{\beta \to \infty} \int_1^\beta x \cdot \frac{x}{(1+x^2)^2}\, dx$$

$$+ \quad x \qquad -\frac{1}{2}\frac{1}{1+x^2}$$

$$- \quad 1 \qquad -\frac{1}{2}\tan^{-1} x$$

$$= \lim_{\beta \to \infty} \left[-\frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1} x \right]_1^\beta$$

$$= \lim_{\beta \to \infty} \left(-\frac{\beta}{2+2\beta^2} + \frac{1}{2}\tan^{-1}\beta + \frac{1}{4} - \frac{\pi}{8} \right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{4} - \frac{\pi}{8}$$

$$= \frac{1}{4} + \frac{\pi}{8}$$

(2) $$\int_1^\infty \frac{1}{x(1+x^2)}\, dx$$

$$\frac{A}{x} + \frac{Bx+C}{1+x^2}$$

$$= A + Ax^2 + Bx^2 + Cx = 1$$

$A = 1$

$C = 0$

$B = -1$

$$= \lim_{\beta \to \infty} \int_1^\beta \left(\frac{1}{x} + \frac{-x}{1+x^2} \right) dx$$

$$= \lim_{\beta \to \infty} \left[\log|x| - \frac{1}{2}\log(1+x^2) \right]_1^\beta$$

$$= \lim_{\beta \to \infty} \left(\log\beta - \frac{1}{2}\log(1+\beta^2) + \frac{1}{2}\log 2 \right)$$

$$= \lim_{\beta \to \infty} \left(\frac{1}{2}\log\frac{\beta^2}{1+\beta^2} + \frac{1}{2}\log 2 \right)$$

$$= \frac{1}{2}\log 2$$

§3-2

(1) $$\int_1^\infty \frac{dx}{\sqrt{x}(1+x)}$$

$\sqrt{x} = t \qquad x = t^2$

$dx = 2t\, dt$

x	1	$\to \beta$
t	1	$\to \sqrt{\beta}$

$$\int_1^\infty \frac{dx}{\sqrt{x}(1+x)}$$

$$= \lim_{\beta\to\infty} \int_1^{\sqrt{\beta}} \frac{2\cancel{t}}{\cancel{t}(1+t^2)} dt$$

$$= \lim_{\beta\to\infty} \left[2\tan^{-1} t\right]_1^{\sqrt{\beta}}$$

$$= \lim_{\beta\to\infty} \left(2\tan^{-1}\sqrt{\beta} - 2\tan^{-1} 1\right)$$

$$= 2\cdot\frac{\pi}{2} - \cancel{2}\cdot\frac{\pi}{\cancel{4}_2}$$

$$= \frac{\pi}{2}$$

$$\tan^{-1}\frac{x}{a}$$

$$= \frac{\frac{1}{a}}{1+\left(\frac{x}{a}\right)^2}$$

$$= \frac{a}{a^2+x^2}$$

(2) $$\int_{-\infty}^{\infty} \frac{dx}{\pi + x^2}$$

$$= \lim_{\beta\to\infty} \int_{-\beta}^{\beta} \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\pi + x^2} dx$$

$$= \lim_{\beta\to\infty} \frac{1}{\sqrt{\pi}} \left[\tan^{-1}\frac{x}{\sqrt{\pi}}\right]_{-\beta}^{\beta}$$

$$= \lim_{\beta\to\infty} \frac{1}{\sqrt{\pi}} \left(\tan^{-1}\frac{\beta}{\sqrt{\pi}} + \tan\frac{\beta}{\sqrt{\pi}}\right)$$

$$= \sqrt{\pi}$$

E3-3

(1) $$\int_0^1 \log x \, dx$$

$$= \lim_{\alpha\to+0} \int_\alpha^1 \log x \, dx$$

$$= \lim_{\alpha\to+0} \left[x\log x - x\right]_\alpha^1$$

$= \lim_{\alpha \to +0} \left(-1 - \alpha \log \alpha + \alpha \right)$

$= -1 - \lim_{\alpha \to +0} \frac{\log \alpha}{\alpha^{-1}}$ $\qquad \frac{\frac{1}{\alpha}}{-\frac{1}{\alpha^2}} = -\alpha$

$= -1$

(2) $\int_1^3 \frac{1}{\sqrt{x^2-1}} dx$

$= \lim_{\alpha \to 1+0} \int_\alpha^3 \frac{1}{\sqrt{x^2-1}} dx$

$= \lim_{\alpha \to 1+0} \left[\log \left| x + \sqrt{x^2-1} \right| \right]_\alpha^3$

$= \lim_{\alpha \to 1+0} \left(\log (3 + 2\sqrt{2}) - \log (\alpha + \sqrt{\alpha^2 - 1}) \right)$

$= \log (3 + 2\sqrt{2})$

83-3

(1) $\int_0^1 \sqrt{\frac{x}{1-x}} dx$

$\sqrt{\frac{x}{1-x}} = t \qquad \frac{x}{1-x} = t^2 \qquad x = t^2 - t^2 x \qquad x = \frac{t^2}{1+t^2}$

$dx = \frac{2t + 2t^3 - 2t^3}{(1+t^2)^2} = \frac{2t}{(1+t^2)^2}$

$\begin{array}{c|ccc} x & 0 & \to & \beta \\ t & 0 & \to & \sqrt{\frac{\beta}{1-\beta}} \end{array}$

$\int \sqrt{\frac{x}{1-x}} dx$

$+ \, t \qquad - \frac{1}{1+t^2}$

$= \int t \cdot \frac{2t}{(1+t^2)^2} dt$

$- \, 1 \qquad - \tan^{-1} t$

$= t \cdot \frac{-1}{1+t^2} - \int \frac{-1}{1+t^2} dt$

$$= \frac{-t}{1+t^2} + \tan^{-1} t \, dt$$

$$= \frac{-\sqrt{\frac{x}{1-x}}}{1+\frac{x}{1-x}} + \tan^{-1}\sqrt{\frac{x}{1-x}}$$

$$= -\sqrt{x(1-x)} + \tan^{-1}\sqrt{\frac{x}{1-x}}$$

$$\therefore \int_0^1 \sqrt{\frac{x}{1-x}}\, dx$$

$$= \lim_{\beta \to 1-0} \left[-\sqrt{x(1-x)} + \tan^{-1}\sqrt{\frac{x}{1-x}} \right]_0^\beta$$

$$= \lim_{\beta \to 1-0} \left(-\sqrt{\beta - \beta^2} + \tan^{-1}\sqrt{\frac{\beta}{1-\beta}} \right)$$

$$= \frac{\pi}{2}$$

(2) $\displaystyle\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$

$$= \lim_{\substack{\alpha \to +0 \\ \beta \to 1-0}} \int_\alpha^\beta \frac{1}{\sqrt{x(1-x)}} dx$$

$$\int_\alpha^\beta \frac{1}{\sqrt{x(1-x)}} dx$$

$$= \int_\alpha^\beta \frac{1}{\sqrt{-(x^2-x)}} dx$$

$$= \int_\alpha^\beta \frac{1}{\sqrt{-\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}}} dx$$

$$= \left[\sin^{-1} 2\left(x - \frac{1}{2}\right) \right]_\alpha^\beta$$

$= \sin^{-1}(2\beta-1) - \sin^{-1}(2\alpha-1)$

$\therefore \lim_{\substack{\alpha\to+0\\ \beta\to 1-0}} \int_\alpha^\beta \frac{1}{\sqrt{x(1-x)}}dx$

$= \lim_{\substack{\alpha\to+0\\ \beta\to 1-0}} \left(\sin^{-1}(2\beta-1) - \sin^{-1}(2\alpha-1)\right)$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$= \pi$

E3-4

(1) $\int_0^\infty e^{-x^2}dx$

$\lim_{x\to\infty} x^2\cdot e^{-x^2} = \lim_{x\to\infty} \frac{2}{(-2x)^2\cdot e^{-x^2}} = 0$

$\therefore$ 十分大きいxについて $x^2\cdot e^{-x^2} < 1$

$\therefore e^{-x^2} < \frac{1}{x^2}$

$x > c$のとき $e^{-x^2} < \frac{1}{x^2}$ であるとすると

$\int_0^\infty e^{-x^2}dx = \int_0^c e^{-x^2}dx + \int_c^\infty e^{-x^2}dx$

第1項は明らかに収束. 第2項は

$\int_c^\infty e^{-x^2}dx \leq \int_c^\infty \frac{1}{x^2}dx$

$= \left[-\frac{1}{x}\right]_c^\infty$

$= -\frac{1}{c}$

よって，広義積分は収束する．

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\sin x}\,dx$

$0 < x < \dfrac{\pi}{2}$ のとき，$0 < \sin x < x$

$\therefore\ 0 < \dfrac{1}{x} < \dfrac{1}{\sin x}$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin x}\,dx \geq \int_0^{\frac{\pi}{2}} \frac{1}{x}\,dx$$

$$= \lim_{\alpha \to +0} \int_\alpha^{\frac{\pi}{2}} \frac{1}{x}\,dx$$

$$= \lim_{\alpha \to +0} \Big[\log x\Big]_\alpha^{\frac{\pi}{2}}$$

$$= \lim_{\alpha \to +0} \left(\log \frac{\pi}{2} - \log \alpha\right)$$

$$= +\infty$$

よって発散

§3-4

(1) $\displaystyle\int_1^{\infty} \frac{\tan^{-1} x}{x}\,dx$

$x > 1$ のとき $\tan^{-1} x > \dfrac{\pi}{4}$

$\therefore\ \dfrac{\tan^{-1} x}{x} > \dfrac{\pi}{4} \cdot \dfrac{1}{x}$

$$\therefore \int_1^{\infty} \frac{\tan^{-1} x}{x}\,dx > \int_1^{\infty} \frac{\pi}{4} \cdot \frac{1}{x}\,dx$$

$$= \lim_{\beta \to +\infty} \int_1^{\beta} \frac{\pi}{4} \cdot \frac{1}{x}\,dx$$

$$= \lim_{\beta \to +\infty} \frac{\pi}{4} \log \beta$$

$$= +\infty$$

∴ 発散

(2) $$\int_1^{\infty} \left(\frac{\pi}{2} \cdot \frac{1}{x} - \frac{\tan^{-1} x}{x} \right) dx$$

$$= \int_1^{\infty} \frac{1}{x} \left(\frac{\pi}{2} - \tan^{-1} x \right) dx$$

$f(x) = \frac{1}{x} - \left(\frac{\pi}{2} - \tan^{-1} x \right)$ とおく.

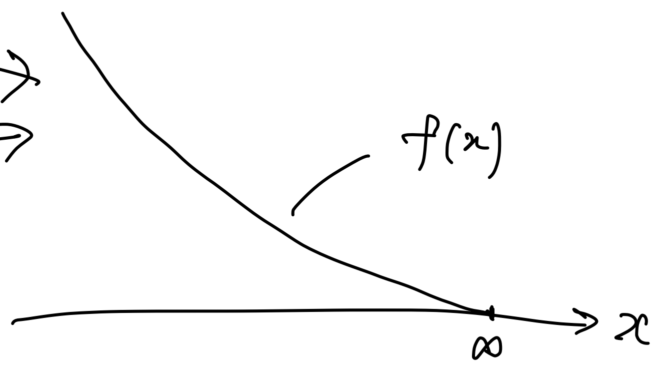
$$f'(x) = -\frac{1}{x^2} + \frac{1}{1+x^2}$$

$$= -\frac{1}{x^2(1+x^2)}$$

$$< 0$$

また,

$$\lim_{x \to \infty} f(x) = 0$$



よって, $f(x) > 0$.

$$\therefore \frac{\pi}{2} - \tan^{-1} x < \frac{1}{x}$$

$x > 0$ のとき $\frac{\pi}{2} - \tan^{-1} x < \frac{1}{x}$ より

$$\frac{1}{x} \left(\frac{\pi}{2} - \tan^{-1} x \right) < \frac{1}{x^2}$$

$$\therefore \int_1^{\infty} \left(\frac{\pi}{2} \cdot \frac{1}{x} - \frac{\tan^{-1} x}{x} \right) dx \leq \int_1^{\infty} \frac{1}{x^2} dx$$

ここで

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{\beta \to \infty} \int_1^{\beta} \frac{1}{x^2} dx$$

$$= \lim_{\beta \to \infty} \left[-\frac{1}{x} \right]_1^{\beta}$$

$$= -1 < \infty$$

∴ 収束する.

E3-5

$$(e^{-ax} \sin bx)' = -ae^{-ax} \sin bx + be^{-ax} \cos bx \quad -①$$

$$(e^{-ax} \cos bx)' = -be^{-ax} \sin bx - ae^{-ax} \cos bx \quad -②$$

∴ ① × b − ② × a より

$$\int e^{-ax} \cos bx \, dx = \frac{1}{a^2+b^2} e^{-ax} (b \sin bx - a \cos bx) + C$$

∴ ① × a + ② × b より

$$\int e^{-ax} \sin bx \, dx = -\frac{1}{a^2+b^2} e^{-ax} (a \sin bx + b \cos bx) + C$$

$$I = \lim_{\beta \to \infty} \int_0^{\beta} e^{-ax} \cos bx \, dx$$

$$= \lim_{\beta \to \infty} \left[\frac{1}{a^2+b^2} e^{-ax} (b \sin bx - a \cos bx) \right]_0^{\beta}$$

$$= \lim_{\beta \to \infty} \left(\frac{1}{a^2+b^2} \left(e^{-a\beta} (b \sin b\beta - a \cos b\beta) - (-a) \right) \right)$$

$$= \frac{a}{a^2+b^2}$$

$$J = \lim_{\beta \to \infty} \int_0^{\beta} e^{-ax} \sin bx \, dx$$

$$= \lim_{\beta \to \infty} \left[-\frac{1}{a^2+b^2} e^{-ax} (a \sin bx + b \cos bx) \right]_0^{\beta}$$

$$= \lim_{\beta \to \infty} \left(-\frac{1}{a^2+b^2} \left(e^{-a\beta} (a \sin b\beta + b \cos b\beta) - b \right) \right)$$

$$= \frac{b}{a^2+b^2}$$

S3-5

$$(e^{-ax}\sin bx)' = -ae^{-ax}\sin bx + be^{-ax}\cos bx$$

$$(e^{-ax}\cos bx)' = -be^{-ax}\sin bx - ae^{-ax}\cos bx$$

$$\therefore \int e^{-ax}\sin bx = -\frac{1}{a^2+b^2}e^{-ax}(a\sin bx + b\cos bx) + C$$

$$\int e^{-ax}\cos bx = \frac{1}{a^2+b^2}e^{-ax}(b\sin bx - a\cos bx) + C$$

(1) $$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C$$

(2) $$\int_0^\infty e^{-x}|\sin x|\,dx$$

$$= \sum_{k=0}^{\infty}(-1)^k\int_{\pi k}^{\pi(k+1)} e^{-x}\sin x\,dx$$

$$= \sum_{k=0}^{\infty}(-1)^k\left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_{\pi k}^{\pi(k+1)}$$

$$= \sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{1}{2}e^{-\pi(k+1)}\cdot(-1)^{k+1} - \frac{1}{2}e^{-\pi k}\cdot(-1)^k\right)$$

$$= \sum_{k=0}^{\infty}(-1)^{k+1}\cdot\frac{1}{2}e^{-\pi k}\cdot(-1)^{k+1}\cdot(e^{-\pi}+1)$$

$$= \sum_{k=0}^{\infty}\frac{1}{2}e^{-\pi k}(e^{-\pi}+1)$$

公比 $e^{-\pi}$ の無限等比級数 $\quad \dfrac{a_0(1-r^n)}{1-r}$

$$= \lim_{k\to\infty}\frac{\frac{1}{2}(e^{-\pi}+1)(1-e^{-\pi k})}{1-e^{-\pi}}$$

$$= \frac{1+e^{-\pi}}{2(1-e^{-\pi})}$$

$$= \frac{1}{2}\cdot\frac{1+e^{-\pi}}{1-e^{-\pi}}$$

E3-6

(1) $\lim_{x\to\infty} \frac{x^a}{e^x} := f(x)$

$$\ln \lim_{x\to\infty} f(x) = \lim_{x\to\infty} (a\ln x - x)$$
$$= \lim_{x\to\infty} x\left(\frac{a\ln x}{x} - 1\right)$$
$$= \lim_{x\to\infty} x \cdot \lim_{x\to\infty}\left(\frac{a\ln x}{x} - 1\right)$$
$$= -\infty$$

$$\therefore \lim_{x\to\infty} f(x) = e^{-\infty}$$
$$= 0$$

(2) $$\Gamma(s+1) = \int_0^\infty x^s e^{-x} dx$$
$$= \lim_{\beta\to\infty} \int_0^\beta x^s e^{-x} dx$$
$$= \lim_{\beta\to\infty} \left(\left[x^s(-e^{-x}) \right]_0^\beta - \int_0^\beta s x^{s-1}(-e^{-x}) dx \right)$$
$$= \lim_{\beta\to\infty} \left(\beta^s(-e^{-\beta}) + s\int_0^\beta x^{s-1} e^{-x} dx \right.$$
$$= \lim_{\beta\to\infty} \left(-\frac{\beta^s}{e^{-\beta}} \right) + s\Gamma(s)$$
$$= s\Gamma(s)$$

S3-6

(1) $$B_a(m, n+1) = \int_0^a x^{m-1}(a-x)^n dx$$
$$= \left[\frac{1}{m} x^m (a-x)^n \right]_0^a - \int_0^a \frac{1}{m} x^m \cdot (-n(a-x)^{n-1}) dx$$

$$= 0 + \frac{n}{m}\int_0^a x^m (a-x)^{n-1}\,dx$$

$$= \frac{n}{m} B_a(m+1,\, n)$$

(2) $B_a(m, n) = \frac{n-1}{\underbrace{m}_{1-1}} B_a(m+1,\, n-1)$ $\quad$ 1回

$$= \frac{n-1}{m}\cdot\frac{n-2}{\underbrace{m+1}_{2-1}} B_a(m+2,\, n-2) \quad 2回$$

$$= \frac{n-1}{m}\cdot\frac{n-2}{m+1}\cdot \cdots \cdot \frac{1}{\underbrace{m+n-2}_{(n-1)-1}} B_a(m+n-1,\, 1) \quad (n-1)回$$

$$B_a(m+n-1,\, 1) = \int_0^a x^{m+n-2}\,dx$$

$$= \left[\frac{1}{m+n-1} x^{m+n-1}\right]_0^a$$

$$= \frac{1}{m+n-1} a^{m+n-1}$$

$$\therefore\ B_a(m, n) = \frac{n-1}{m}\cdot\frac{n-2}{m+1}\cdot \cdots \cdot \frac{1}{m+n-2}\cdot\frac{1}{m+n-1} a^{m+n-1}$$

$$= \frac{(n-1)!\cdot(m-1)!}{(m+n-1)!} a^{m+n-1}$$

E3-7

(1) $x = \frac{\pi}{2} - t$

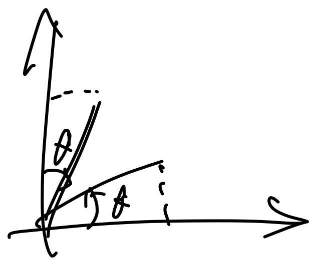
x	$0 \to \frac{\pi}{2}$
t	$\frac{\pi}{2} \to 0$

$dx = -dt$

$$\therefore\ I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$$

$$= \int_{\frac{\pi}{2}}^0 -\sin^n\left(\frac{\pi}{2} - t\right) dt$$

$$= \int_0^{\frac{\pi}{2}} \sin^n\left(\frac{\pi}{2} - t\right) dt$$



$$= \int_0^{\frac{\pi}{2}} \cos^n\left(\frac{\pi}{2}-t\right) dt$$

$$= J_n$$

(2) $$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x \cdot \sin^{n-1} x \, dx$$

$$= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x (n-1) \sin^{n-2} \cdot \cos x \, dx$$

$$= (n-1) \cdot \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cdot \cos^2 x \, dx$$

$$= (n-1) \cdot \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cdot (1-\sin^2 x) \, dx$$

$$= (n-1) \cdot \left(\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \, dx\right)$$

$$= (n-1) \cdot (I_{n-2} - I_n)$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2}$$

S3-7

(1) $$I_{n+1} = \int_0^1 \frac{1}{(x^2+1)^{n+1}} dx$$

$$= \int_0^1 1 \cdot \frac{1}{(x^2+1)^{n+1}} dx$$

$$= \left[x \cdot \frac{1}{(x^2+1)^{n+1}}\right]_0^1 - \int_0^1 -(n+1) \cdot x \cdot \frac{2x}{(x^2+1)^{n+2}} dx$$

$$= \frac{1}{2^{n+1}} - 0 + 2(n+1) \int_0^1 \frac{x^2}{(x^2+1)^{n+2}} dx$$

$$= \frac{1}{2^{n+1}} + 2(n+1) \int_0^1 \left(\frac{x^2+1}{(x^2+1)^{n+2}} - \frac{1}{(x^2+1)^{n+2}}\right) dx$$

$$= \frac{1}{2^{n+1}} + 2(n+1)\int_0^1 \left(\frac{1}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+2}}\right) dx$$

$n+1 \to n$ とおきかえ

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \left(\frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}\right) dx$$

$$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$

$$\therefore 2nI_{n+1} = \frac{1}{2^n} + (2n-1)I_n$$

$$I_{n+1} = \frac{1}{2n\cdot 2^n} + \frac{2n-1}{2n}I_n$$

$$= \frac{1}{n\cdot 2^{n+1}} + \frac{2n-1}{2n}I_n$$

(2) $$I_1 = \int_0^1 \frac{1}{x^2+1}dx$$

$$= \left[\tan^{-1}x\right]_0^1$$

$$= \frac{\pi}{4}$$

$$I_2 = \frac{1}{1\cdot 2^{1+1}} + \frac{2\cdot 1-1}{2\cdot 1}I_1$$

$$= \frac{1}{4} + \frac{\pi}{8}$$

$$I_3 = \frac{1}{2\cdot 2^{2+1}} + \frac{2\cdot 2-1}{2\cdot 2}\left(\frac{1}{4}+\frac{\pi}{8}\right)$$

$$= \frac{1}{16} + \frac{3}{16} + \frac{3}{32}\pi$$

$$= \frac{1}{4} + \frac{3}{32}\pi$$