$\boxed{\text{I}}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & \\ \vdots & & \ddots & \\ b_{n1} & & & b_{nn} \end{pmatrix}$$

とする。

$$AB = \begin{pmatrix} \sum_{i=1}^{n} a_{1i} b_{i1} & \sum_{i=1}^{n} a_{1i} b_{i2} & \cdots & \sum_{i=1}^{n} a_{1i} b_{in} \\ \sum_{i=1}^{n} a_{2i} b_{i1} & & \ddots & \\ \vdots & & & \\ \sum_{i=1}^{n} a_{ni} b_{i1} & & & \sum_{i=1}^{n} a_{ni} b_{in} \end{pmatrix}$$

$$\therefore tr(AB) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} b_{ij}$$

$$BA = \begin{pmatrix} \sum_{i=1}^{n} b_{1i} a_{i1} & \sum_{i=1}^{n} b_{1i} a_{i2} & \cdots & \sum_{i=1}^{n} b_{1i} a_{in} \\ \sum_{i=1}^{n} b_{2i} a_{i1} & & \ddots & \\ \vdots & & & \\ \sum_{i=1}^{n} b_{ni} a_{i1} & & & \sum_{i=1}^{n} b_{ni} a_{in} \end{pmatrix}$$

$$\therefore tr(BA) = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} b_{ij}$$

$$\therefore tr(AB) = tr(BA)$$

$\boxed{2}$

拡大係数行列は

$$\begin{pmatrix} 1 & 2 & 1 & a \\ 2 & 3 & -1 & -1 \\ 3 & a & -12 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & a \\ 0 & -1 & -3 & -1-2a \\ 0 & a-6 & -15 & -12-3a \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & a \\ 0 & 1 & 3 & 1+2a \\ 0 & 0 & -15-3(a-6) & -12-3a-(1+2a)(a-6) \end{pmatrix}$$

$-15-3a+18$
$=3-3a$

$-12-3a-(a-6+2a^2-12a)$
$=-12-3a+11a+6-2a^2$
$=-2a^2+8a-6$
$=-2(a^2-4a+3)=-2(a-3)(a-1)$

$$\to \begin{pmatrix} 1 & 2 & 1 & a \\ 0 & 1 & 3 & 1+2a \\ 0 & 0 & 3-3a & -2a^2+8a-6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & a \\ 0 & 1 & 3 & 1+2a \\ 0 & 0 & -3(a-1) & -2(a-3)(a-1) \end{pmatrix}$$

$a=1$ のとき

$x+2y+z=1$

$y+3z=3$

$x+2(-3t+3)+t=1$

$x=6t-6-t+1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}$$

$a \neq 1$ のとき

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & a & -12 \end{vmatrix} = -36 - 6 + 2a - (9 - 48 - a)$$

$$= -42 + 2a + 39 + a$$

$$= 3a - 3$$

$$= 3(a-1)$$

$$\begin{vmatrix} a & 2 & 1 \\ -1 & 3 & -1 \\ -12 & a & -12 \end{vmatrix} = -36a + 24 - a - (-36 - a^2 + 24)$$

$$= a^2 - 37a + 36$$

$$= (a-36)(a-1)$$

$$\begin{vmatrix} 1 & a & 1 \\ 2 & -1 & -1 \\ 3 & -12 & -12 \end{vmatrix} = 12 - 24 - 3a - (-3 - 24a + 12)$$

$$= -3a - 24 + 24a + 3$$

$$= 21a - 21$$

$$= 21(a-1)$$

$$\begin{vmatrix} 1 & 2 & a \\ 2 & 3 & -1 \\ 3 & a & -12 \end{vmatrix} = -36 - 6 + 2a^2 - (9a - 48 - a)$$

$$= -42 + 2a^2 - 8a + 48$$

$$= 2(a^2 - 4a + 3)$$

$$= 2(a-1)(a-3)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{a-36}{3} \\ 7 \\ \frac{2(a-3)}{3} \end{pmatrix}$$

$\boxed{3}$

(1) $\begin{pmatrix} 1 & 0 & a & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 2 & a-3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & a & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & a-1 \end{pmatrix}$

$\mathrm{rank}(A) = 3$

(2) (1)において, $\mathrm{rank}(A) = 2$となるとき解をもつ.

よって,

$a = 1$

このときの解は

$\begin{cases} x + \quad z = 3 \\ \quad y + z = -1 \end{cases}$

$z = t$とおいて

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

$\boxed{4}$

(1) (a) 最大次数

(b) 1次独立

(c) 1次独立

(d) 次元

(2) $\begin{pmatrix} 1 & 0 & 4 & 2 \\ 3 & 1 & 2 & 6 \\ 1 & -2 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -10 & 0 \\ 0 & -2 & -5 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & -25 & 0 \end{pmatrix}$ $\quad \therefore \mathrm{rank}(A) = 3$

(3) $\begin{pmatrix} 1 & 0 & 4 & 2 & 2 \\ 3 & 1 & 2 & 6 & 3 \\ 1 & -2 & -1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 4 & 2 & 2 \\ 0 & 1 & -10 & 0 & -3 \\ 0 & -2 & -6 & 0 & -3 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 4 & 2 & 2 \\ 0 & 1 & -10 & 0 & -3 \\ 0 & 0 & +25 & 0 & +9 \end{pmatrix}$

$x_4 = t$ (tは任意定数) とおけば

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{14}{25} \\ \frac{3}{5} \\ \frac{9}{25} \\ 0 \end{pmatrix}$$

$x_2 = 10x_3 - 3$

$= 2 \cdot \frac{9}{5} - \frac{15}{5}$

$x_1 = -4x_3 - 2x_4 + 2$

$= \frac{-36}{25} - 2t + \frac{50}{25}$