E9-1

(1)
$$\begin{pmatrix} -1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 1 \\ 3 & 9 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\begin{pmatrix} 3 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 2 & -4 \\ -3 & -1 & 2 \\ 12 & 4 & -8 \end{pmatrix}$$

(3)
$$\begin{pmatrix} 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = -3$$

S9-1

(1)
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 13 & 16 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 12 \\ 1 & 2 & 3 \\ 8 & 10 & 12 \end{pmatrix}$$

(3)
$$\begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 4 \\ 8 & -24 & 32 \\ 3 & -9 & 12 \end{pmatrix}$$

E9-2

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & -1 & -3 & -1 \\ -2 & 1 & 4 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 \\ 0 & 5 & 10 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\operatorname{rank} A = 3$

S9-2

(1)
$$A = \begin{pmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -2 & -5 & -1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & -6 & -2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{rank} A = 2$

(2)
$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -2 \\ 0 & -3 & 6 \\ 0 & 3 & -6 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{rank} A = 2$

E9-3

拡大係数行列

$$(A \ \ b) = \begin{pmatrix} 1 & -1 & 6 & 5 \\ 5 & -2 & 12 & 4 \\ 3 & -1 & 6 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 6 & 5 \\ 0 & \cancel{3}^{1} & \cancel{-18}^{-6} & \cancel{-21}^{-7} \\ 0 & \cancel{2}^{1} & \cancel{-12}^{-6} & \cancel{-14}^{-7} \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -1 & 6 & 5 \\ 0 & 1 & -6 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -6 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{cases} x = -2 \\ y - 6z = -7 \end{cases}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 6t-7 \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -7 \\ 0 \end{pmatrix}$$

S9-3

(1)

拡大係数行列は

$$\begin{pmatrix} 1 & 2 & -1 & 3 & -3 \\ 2 & 3 & 0 & 4 & -2 \\ 0 & 1 & -2 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 3 & -3 \\ 0 & -1 & 2 & -2 & 4 \\ 0 & 1 & -2 & 2 & -4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & -1 & 3 & -3 \\ 0 & -1 & 2 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 3 & -1 & 5 \\ 0 & -1 & 2 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{cases} x + 3z - w = 5 \\ -y + 2z - 2w = 4 \end{cases} \qquad \begin{aligned} x &= -3z + w + 5 \\ y &= 2z - 2w - 4 \end{aligned}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = a\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -4 \\ 0 \\ 0 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 1 & 3 & 5 & -8 \\ 2 & -1 & -4 & 5 \\ 3 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & -8 \\ 0 & \overset{1}{\cancel{-7}} & \overset{2}{\cancel{-14}} & \overset{-3}{\cancel{21}} \\ 0 & -8 & -16 & 25 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 3 & 5 & -8 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore$ 解なし

E9-4

$$\begin{pmatrix} 2 & -1 & 9 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x + 6z = 0 \\ y + 3z = 0 \end{cases} \qquad \begin{aligned} x &= -6z \\ y &= -3z \end{aligned}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix}$$

S9-4

(1) $$\begin{pmatrix} 3 & 1 & 4 & 2 \\ 2 & -1 & 1 & -2 \\ 8 & 1 & 9 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & -2 \\ 8 & 1 & 9 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -5 & -10 \\ 0 & -15 & -15 & -30 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{aligned} x + z &= 0 \\ y + z + 2w &= 0 \end{aligned} \qquad \begin{aligned} x &= -z \\ y &= -z - 2w \end{aligned}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = a \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 2 & -1 & 3 \\ 3 & -2 & 4 \\ 1 & -4 & 10 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & -4 & 10 \end{pmatrix}$$
$$\to \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -3 & 9 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

E9-5
$$\begin{pmatrix} 1 & 1 & 3 & k+7 \\ 3 & 4 & 11 & k \\ 2 & 5 & 12 & k+1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & k+7 \\ 1 & -1 & -1 & -1 \\ 2 & 5 & 12 & k+1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & 3 & k+7 \\ 1 & -1 & -1 & -1 \\ 1 & 4 & 9 & -6 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 4 & 9 & -6 \\ 1 & 1 & 3 & k+7 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & \overset{1}{\cancel{5}} & \overset{2}{\cancel{10}} & \overset{-1}{\cancel{-5}} \\ 0 & 2 & 4 & k+8 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & k+10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & k+10 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ k+10 \end{pmatrix}$$

解をもつためには，係数行列のランク＝拡大係数行列のランク
となる必要がある．よって，

$$k = -10$$

このとき

$$x + z = -2$$
$$y + 2z = -1$$

aを任意実数とすると

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

S9-5

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 3 & k & 2 \\ 2 & k & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & k-1 & 1 \\ 0 & k+2 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & \frac{k-1}{4} & \frac{1}{4} \\ 0 & k+2 & 1 & 1 \end{pmatrix}$$

$1-\frac{1}{4}(k^2+k-2)$

$-\frac{1}{4}(k^2+k-2-4)$

$$\to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & \frac{k-1}{4} & \frac{1}{4} \\ 0 & 0 & 1-\frac{1}{4}(k+2)(k-1) & 1-\frac{1}{4}(k+2) \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & \frac{k-1}{4} & \frac{1}{4} \\ 0 & 0 & -\frac{1}{4}(k+3)(k-2) & -\frac{1}{4}(k-2) \end{pmatrix}$$

よって，連立方程式が解をもたない条件は $k=-3$

E9-6

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 7 & 4 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 \\ 0 & 0 & 1 & -10 & 2 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 9 & -2 & -2 \\ 0 & 1 & 0 & 6 & -1 & -2 \\ 0 & 0 & 1 & -10 & 2 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 6 & -1 & -2 \\ 0 & 0 & 1 & -10 & 2 & 3 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 6 & -1 & -2 \\ -10 & 2 & 3 \end{pmatrix}$$

S9-6

(1) $$\left(\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & {}^{0}\cancel{4} & {}^{1}\cancel{-3} & 1 & 2 & 3 & {}^{4}\cancel{0} \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & {}^{0}\cancel{-2} & {}^{0}\cancel{1} & {}^{1}\cancel{0} & {}^{2}\cancel{0} & {}^{2}\cancel{-1} & {}^{2}\cancel{0} \\ 0 & 0 & 1 & {}^{0}\cancel{-1} & {}^{1}\cancel{0} & {}^{2}\cancel{0} & {}^{3}\cancel{0} & {}^{3}\cancel{-1} \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & -1 & -3 & -3 & -3 \\ 0 & 1 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 \end{array}\right)$$

$\therefore$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

(2)

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 9 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

-10 8
4 -4
-10 8
25 -16
-10 8
-16 8

$$|A| = \overbrace{25+16+16}^{57} - \overbrace{(16+20+20)}^{56}$$

$$= 1$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 9 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

E9-7

$$\begin{pmatrix} A^{-1} & O \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}\begin{pmatrix} A & O \\ C & D \end{pmatrix} = \begin{pmatrix} A^{-1}A + O & O + O \\ -D^{-1}CA^{-1}A + D^{-1}C & O + D^{-1}D \end{pmatrix}$$

$$= \begin{pmatrix} E & O \\ -D^{-1}C + D^{-1}C & E \end{pmatrix}$$

$$= \begin{pmatrix} E & O \\ O & E \end{pmatrix}$$

$$\therefore \begin{pmatrix} A & O \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

S9-7

$$\begin{pmatrix} A & B \\ O & D \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} AP + BR & AQ + BS \\ O + DR & O + DS \end{pmatrix}$$

これが $= E$ となるように $P \sim S$ を定めていく.

$P = A^{-1}$, $S = D^{-1}$, $R = O$, $Q = -A^{-1}BD^{-1}$ とする.

$$\begin{pmatrix} AP + BR & AQ + BS \\ DR & DS \end{pmatrix} = \begin{pmatrix} E & -BD^{-1} + BD^{-1} \\ O & E \end{pmatrix}$$

$$= \begin{pmatrix} E & O \\ O & E \end{pmatrix}$$

$$= E$$

よって、

$$\begin{pmatrix} A & B \\ O & D \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{pmatrix} = E$$

したがって、Xは正則であり、

$$X^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{pmatrix}$$