E7-1

(1) $\iint_D \sin(x+y)\,dx\,dy$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-x} \sin(x+y)\,dy\,dx$

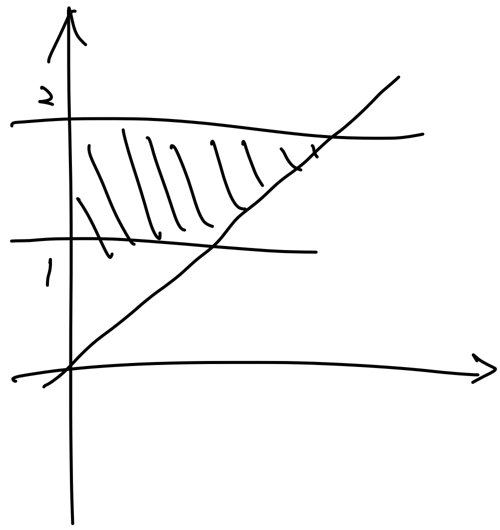
$= \int_0^{\frac{\pi}{2}} \Big[-\cos(x+y)\Big]_0^{\frac{\pi}{2}-x} dx$

$= \int_0^{\frac{\pi}{2}} \left(-\cos\frac{\pi}{2} + \cos x\right) dx$

$= \Big[\sin x\Big]_0^{\frac{\pi}{2}}$

$= 1$

(2) $\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$



$= \int_1^2 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$

$= \int_1^2 \Big[\sqrt{x^2+y^2}\Big]_0^y dy$

$= \int_1^2 (\sqrt{2}-1)\,y\,dy$

$= \left[\frac{\sqrt{2}-1}{2} y^2\right]_1^2$

$= \frac{4(\sqrt{2}-1)}{2} - \frac{\sqrt{2}-1}{2}$

$= \frac{3}{2}(\sqrt{2}-1)$

S7-1

(1) $\iint_D \frac{x}{1+y}\,dx\,dy$

$= \int_0^1 \int_0^{x^2} \frac{x}{1+y}\,dy\,dx$

$$= \int_0^1 \left[x \ln(1+y) \right]_0^{x^2} dx$$

$$= \int_0^1 x \ln(1+x^2)\, dx$$

$$= \left[\frac{x^2}{2} \ln(1+x) \right]_0^1 - \int_0^1 \frac{x^2}{2} \frac{2x}{1+x^2}\, dx$$

$$= \left[\frac{x^2}{2} \ln(1+x) \right]_0^1 - \int_0^1 \left(x - \frac{x}{1+x^2} \right) dx$$

$$= \frac{1}{2} \ln 2 - \left[\frac{x^2}{2} - \frac{1}{2} \ln(1+x^2) \right]_0^1$$

$$= \frac{1}{2} \ln 2 - \left(\frac{1}{2} - \frac{1}{2} \ln 2 \right)$$

$$= \ln 2 - \frac{1}{2}$$

$$\begin{array}{r} x \\ 1+x^2 \overline{)\, x^3 \qquad} \\ x^3 + x \\ \hline -x \end{array}$$

(2) $\iint_D y\, dx\, dy$

$$= \int_0^1 \int_0^{2y+1} y\, dx\, dy + \int_1^2 \int_0^{-y+4} y\, dx\, dy$$

$$= \int_0^1 \left[xy \right]_0^{2y+1} dy + \int_1^2 \left[xy \right]_0^{-y+4} dy$$

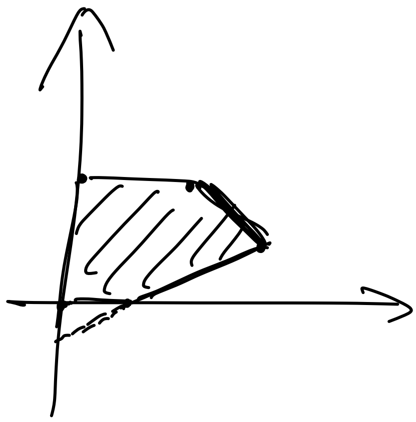
$$= \int_0^1 (2y^2 + y)\, dy + \int_1^2 (-y^2 + 4y)\, dy$$

$$= \left[\frac{2}{3} y^3 + \frac{y^2}{2} \right]_0^1 + \left[-\frac{y^3}{3} + 2y^2 \right]_1^2$$

$$= \frac{2}{3} + \frac{1}{2} + \left(-\frac{8}{3} + \frac{\cancel{8}^{24}}{3} + \frac{1}{3} - \frac{\cancel{2}^{6}}{3} \right)$$

$$= \frac{\cancel{3}^{26}}{\cancel{3}_{6}} + \frac{\cancel{1}^{3}}{\cancel{2}_{6}}$$

$$= \frac{29}{6}$$



E7-2

$$\iint_D (x-y)^2\sqrt{1-(x+y)^2}\,dx\,dy$$

$u=x+y$, $v=x-y$ とおくと

$x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$

$x_u=\frac{1}{2}$, $x_v=\frac{1}{2}$, $y_u=\frac{1}{2}$, $y_v=-\frac{1}{2}$

ヤコビアンは

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = -\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}$$

またDは

$E: |u|\leq 1$, $|v|\leq 1$

に移る。

$$\therefore \iint_D (x-y)^2\sqrt{1-(x+y)^2}\,dx\,dy$$

$$=\iint_E v^2\sqrt{1-u^2}\cdot\frac{1}{2}\,du\,dv$$

$$=\frac{1}{2}\cdot\underbrace{\int_{-1}^{1}\sqrt{1-u^2}\,du}_{\text{半円}}\cdot\int_{-1}^{1}v^2\,dv$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{2}{3}$$

$$=\frac{\pi}{6}$$

S7-2

(1)

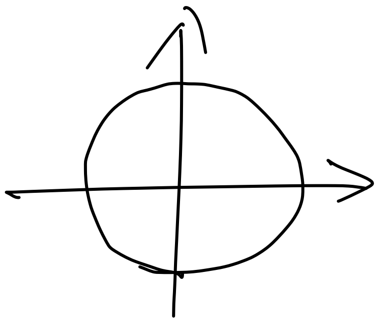
$x+y=u$, $y=v$ とおく

$x=u-v$, $y=v$

$x_u=1$, $x_v=-1$, $y_u=0$, $y_v=1$

$E: u^2+v^2\leq 1$

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = 1$$



$$\therefore \iint_D (x+y)^4 \, dx\,dy$$

$$= \iint_E u^4 \, du\,dv$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^1 r^5 \cos^4\theta \, dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{1}{6}\cos^4\theta \, d\theta$$

$$= \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{8}$$

(2)

$u = x+y,\ v = y$

$\rightarrow x = u - v,\ y = v$

$x_u = 1,\ x_v = -1,\ y_u = 0,\ y_v = 1$

また、

$D: 0 \le x \le 1,\ 0 \le y \le 1 - x$

は

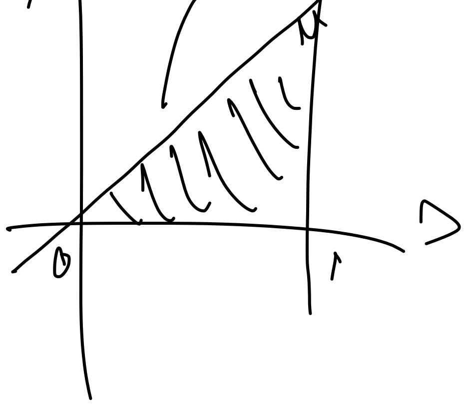
$D: 0 \le x,\ 0 \le y,\ x + y \le 1$

と同じ。

$\therefore E: 0 \le u - v,\ 0 \le v,\ u \le 1$

ヤコビアンは

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = 1$$



$$\iint_D e^{(x+y)^2} \, dx\,dy$$

$$= \int_0^1\int_0^u e^{u^2} \, dv\,du$$

$$= \int_0^1 \left[v e^{u^2} \right]_0^u du$$

$$= \int_0^1 u e^{u^2} du$$

$$= \left[\frac{1}{2} e^{u^2} \right]_0^1$$

$$= \frac{1}{2} e - \frac{1}{2} = \frac{1}{2}(e-1)$$

E7-3

(1) $$\iint_D \sqrt{\frac{1-(x^2+y^2)}{1+x^2+y^2}}\, dx\, dy$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr\, d\theta$$

ここで、

$$\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}} \cdot r\, dr$$

$$= \frac{1}{2} \int_0^1 \sqrt{\frac{1-t}{1+t}}\, dt$$

$$= \frac{1}{2} \int_0^1 \frac{1-t}{\sqrt{1-t^2}}\, dt$$

$$= \frac{1}{2} \int_0^1 \left(\frac{1}{\sqrt{1-t^2}} - \frac{t}{\sqrt{1-t^2}} \right) dt$$

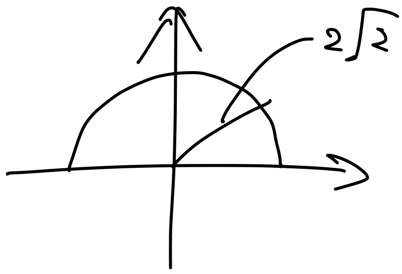
$$= \frac{1}{2} \left[\sin^{-1} t + \frac{1}{2}\sqrt{1-t^2} \right]_0^1$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - 1 \right)$$

$$\therefore \iint_D \sqrt{\frac{1-(x^2+y^2)}{1+x^2+y^2}}\, dx\, dy$$

$$= \frac{1}{2}\pi^2 - \pi$$

57-3



(1) $\iint_D \frac{x^2}{\sqrt{1+x^2+y^2}}\,dxdy$

$= \int_0^{\pi}\int_0^{2\sqrt{2}} \frac{r^3\cos^2\theta}{\sqrt{1+r^2}}\,dr\,d\theta$

$= 2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \cdot \int_0^{2\sqrt{2}} \frac{r^3}{\sqrt{1+r^2}}\,dr$

$= 2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\cdot\int_0^{2\sqrt{2}} \frac{r^3}{\sqrt{1+r^2}}\,dr$

ここで

$\int_0^{2\sqrt{2}} \frac{r^3}{\sqrt{1+r^2}}\,dr$

$= \int_0^{2\sqrt{2}} \frac{r(1+r^2)-r}{\sqrt{1+r^2}}\,dr$

$= \int_0^{2\sqrt{2}} \left(r\sqrt{1+r^2} - \frac{r}{\sqrt{1+r^2}}\right)dr$

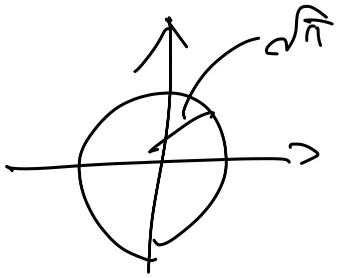
$= \left[\frac{2}{3}\cdot\frac{1}{2}(1+r^2)^{\frac{3}{2}} - \sqrt{1+r^2}\right]_0^{2\sqrt{2}}$

$= \frac{1}{3}\cdot 27 - 3 - \frac{1}{3} + 1$

$= 6 + \frac{2}{3}$

$= \frac{20}{3}$

$\therefore \iint_D \frac{x^2}{\sqrt{1+x^2+y^2}}\,dxdy = \frac{10}{3}\pi$



(2) $\iint_D x^2\sin(x^2+y^2)\,dxdy$

$= \int_0^{2\pi}\int_0^{\sqrt{\pi}} r^3\cos^2\theta\sin r^2\,dr\,d\theta$

$= 4\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \cdot \int_0^{\sqrt{\pi}} r^3\sin r^2\,dr$

$= \pi\cdot\int_0^{\sqrt{\pi}} r^3\sin r^2\,dr$

ここで、$r^2 = t$とおくと、$2r\,dr = dt$

r	$0 \to \sqrt{\pi}$
t	$0 \to \pi$

$$\therefore \int_0^{\sqrt{\pi}} r^3 \sin r^2 \, dr$$

$$= \frac{1}{2}\int_0^{\pi} t \sin t \, dt$$

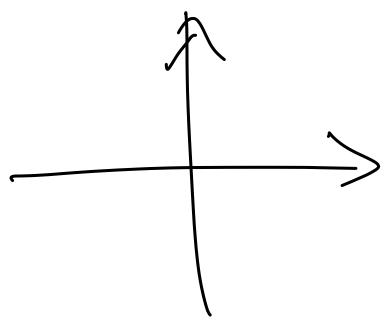
$$= \frac{1}{2}\Big[-t\cos t + \sin t\Big]_0^{\pi}$$

$$= \frac{\pi}{2}$$

$+ \quad t \quad -\cos t$

$- \quad 1 \quad -\sin t$

$$\therefore \iint_D x^2 \sin(x^2+y^2)\,dx\,dy = \frac{\pi^2}{2}$$



E7-4

$$\iint_D \ln(x^2+y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 \ln r^2 \cdot r\,dr\,d\theta$$

$$= 2\pi \int_0^1 r \ln r^2 \, dr$$

ここで

$$\int_\alpha^1 r \ln r^2 \, dr$$

$$= \int_\alpha^1 2r \ln r \, dr$$

$$= \Big[r^2 \ln r - \frac{r^2}{2}\Big]_\alpha^1$$

$$= -\frac{1}{2} - \alpha^2 \ln \alpha + \frac{\alpha^2}{2}$$

$(r^2 \ln r)'$

$= 2r \ln r + \frac{r^2}{r}$

また、

$$\lim_{\alpha \to 0} \alpha^2 \ln \alpha = \lim_{\alpha \to 0} \frac{\ln \alpha}{\alpha^{-2}}$$

$$= \lim_{\alpha \to 0} \frac{\alpha^{-1}}{-2\alpha^{-3}}$$

$$= \lim_{\alpha \to 0} \frac{\alpha^{2}}{-2}$$

$$= 0$$

$$\therefore \iint_D \ln(x^2+y^2)\,dx\,dy = 2\pi \cdot \left(-\frac{1}{2}\right) = -\pi$$

§7-4

(1) $\displaystyle \iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$

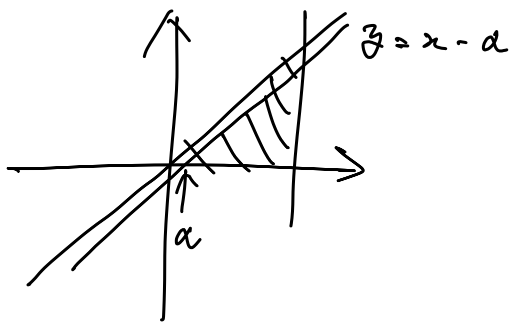
$$= \int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta$$

$$= 2\pi \cdot \lim_{\beta \to 1} \int_0^{\beta} \frac{r}{\sqrt{1-r^2}}\,dr$$

$$= 2\pi \cdot \lim_{\beta \to 1} \left[-\sqrt{1-r^2}\right]_0^{\beta}$$

$$= 2\pi \cdot \lim_{\beta \to 1} \left(-\sqrt{1-\beta^2}+1\right)$$

$$= 2\pi$$



(2) $\displaystyle \iint_D \frac{1}{\sqrt{x-y}}\,dx\,dy$

$$= \lim_{\alpha \to 0} \int_{\alpha}^{1} \int_0^{x-\alpha} \frac{1}{\sqrt{x-y}}\,dy\,dx$$

$$= \lim_{\alpha \to 0} \int_{\alpha}^{1} \left[-2\sqrt{x-y}\right]_0^{x-\alpha} dx$$

$$= \lim_{\alpha \to 0} \int_{\alpha}^{1} \left(-2\sqrt{\alpha}+2\sqrt{x}\right) dx$$

$$= \lim_{\alpha \to 0} \left[-2\sqrt{\alpha}\,x + \frac{4}{3}x^{\frac{3}{2}}\right]_{\alpha}^{1}$$

$$= \lim_{\alpha \to 0} \left(-2\sqrt{\alpha} + \frac{4}{3} + 2\alpha^{\frac{3}{2}} - \frac{4}{3}\alpha^{\frac{3}{2}}\right)$$

$$= \frac{4}{3}$$

E7-5

(1) $\iint_D e^{-x^2-y^2}\,dx\,dy$

これを無限にとばす

$D_n : x \geq 0,\ y \geq 0,\ x^2+y^2 \leq n$

$$\therefore \iint_{D_n} e^{-x^2-y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^n e^{-r^2} r\,dr\,d\theta$$

$$= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^n$$

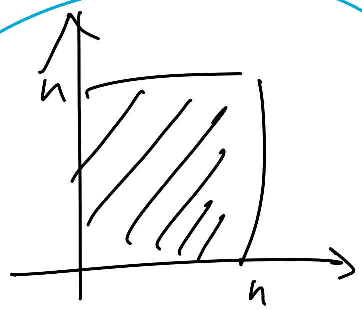
$$= \frac{\pi}{2}\left(-\frac{1}{2}e^{-n^2}+\frac{1}{2}\right)$$

$$\therefore \iint_D e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty}\iint_{D_n} e^{-x^2-y^2}\,dx\,dy$$

$$= \frac{\pi}{4}$$

(2)

$F_n : 0 \leq x \leq n,\ 0 \leq y \leq n$ とおく



無限にとばすと
(1)と同じになる

$$\iint_{F_n} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_0^n e^{-x^2}\,dx \cdot \int_0^n e^{-y^2}\,dy$$

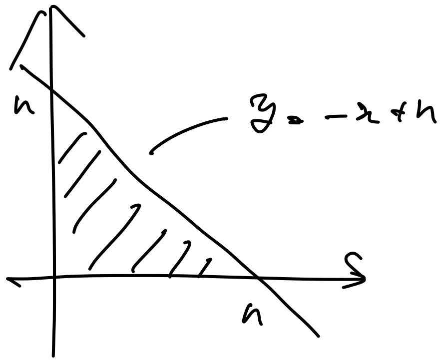
$$= \left(\int_0^n e^{-x^2}\,dx\right)^2$$

$$\therefore \left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

$$\therefore \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

S7-5

$D_n : 0 \leq x \leq n,\ 0 \leq y \leq -x+n$



$$\iint_{D_n} \frac{1}{(x+y+1)^3}\,dx\,dy$$

$$= \int_0^n \int_0^{-x+n} \frac{1}{(x+y+1)^3}\,dy\,dx$$

$$= \int_0^n \left[-\frac{1}{2}\cdot\frac{1}{(x+y+1)^2} \right]_0^{-x+n} dx$$

$$= -\frac{1}{2}\int_0^n \left(\frac{1}{(n+1)^2} - \frac{1}{(x+1)^2}\right) dx$$

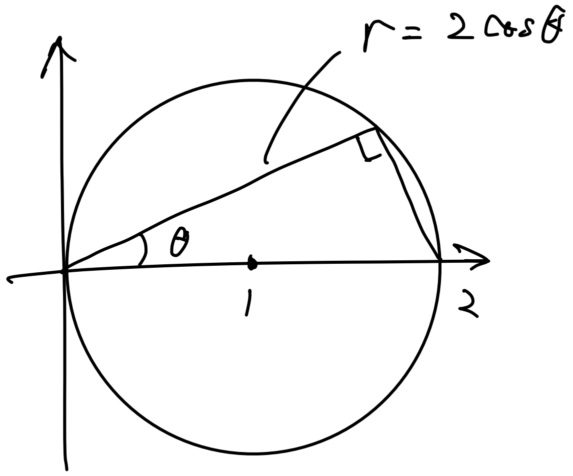
$$= -\frac{1}{2}\left[\frac{x}{(n+1)^2} + \frac{1}{x+1}\right]_0^n$$

$$= -\frac{1}{2}\left(\frac{n}{(n+1)^2} + \frac{1}{n+1} - 1\right)$$

$$= -\frac{1}{2}\cdot(-1)$$

$$= \frac{1}{2}$$

問7-6



$$z = \sqrt{4-(x^2+y^2)}$$

$$D : 0 \le r \le 2\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$V = 2\iint_D \sqrt{4-(x^2+y^2)}\, dx\, dy$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} \sqrt{4-r^2}\, r\, dr\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}}\right]_0^{2\cos\theta} d\theta$$

$$= -\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(8(1-\cos^2\theta)^{\frac{3}{2}} - 8\right) d\theta$$

$$= -\frac{32}{3}\int_0^{\frac{\pi}{2}} (\sin^3\theta - 1)\, d\theta$$

$$= -\frac{32}{3}\int_0^{\frac{\pi}{2}} (\sin\theta - \cos^2\theta\sin\theta - 1)\, d\theta$$

$$= -\frac{32}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta - \theta\right]_0^{\frac{\pi}{2}}$$

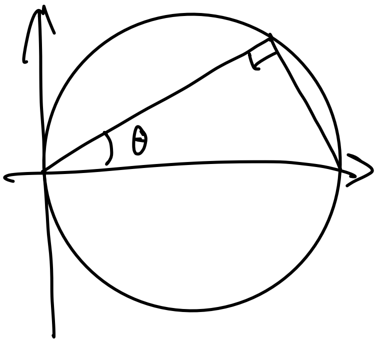
$$= -\frac{32}{3}\left(-\frac{\pi}{2} + 1 - \frac{1}{3}\right)$$

$$= \frac{32}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$$

S7-6

$f(x, y) = 2x - (x^2 + y^2)$ とする。

また, $x^2 + y^2 = 2x$ とすれば, $D: (x-1)^2 + y^2 = 1$



$$\therefore V = \iint_D f(x, y)\, dx\, dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (2r\cos\theta - r^2)\, r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3} r^3 \cos\theta - \frac{r^4}{4} \right]_0^{2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{16}{3}\cos^4\theta - \frac{12}{3}\cos^4\theta \right) d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= \frac{8}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{2}$$

E7-7

極座標変換する。

$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$

ヤコビアンは

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\cos^2\theta\sin\theta\cos^2\varphi + r^2\sin^3\theta\sin^2\varphi$$
$$+ r^2\cos^2\theta\sin\theta\sin^2\varphi + r^2\sin^3\theta\cos^2\varphi$$

$$= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta$$

$$\therefore \iiint_D (x^2+y^2)\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\sin^2\theta\cdot r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= 2\pi\int_0^{\pi}\left[\frac{r^5}{5}\sin^3\theta\right]_0^a$$

$$= \frac{2}{5}a^5\pi\cdot 2\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta$$

$$= \frac{2}{5}a^5\pi\cdot 2\cdot\frac{2}{3}$$

$$= \frac{8}{15}\pi a^5$$

57-7

$D = \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} \leq 1$ とする

$$V = \iiint_D dx\,dy\,dz$$

$x = ar\sin\theta\cos\varphi,\ y = br\sin\theta\sin\varphi,\ z = cr\cos\theta$ とおく.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} a\sin\theta\cos\varphi & ar\cos\theta\cos\varphi & -ar\sin\theta\sin\varphi \\ b\sin\theta\sin\varphi & br\cos\theta\sin\varphi & br\sin\theta\cos\varphi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix}$$

$$= abcr^2\cos^2\theta\sin\theta\cos^2\varphi + abcr^2\sin^3\theta\sin^2\varphi$$
$$+ abcr^2\cos^2\theta\sin\theta\sin^2\varphi + abcr^2\sin^3\theta\cos^2\varphi$$

$$= abc\left(r^2\cos^2\theta\sin\theta + r^2\sin^3\theta\right)$$

$$= abcr^2\sin\theta$$

$$\therefore V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\,r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= abc\cdot 2\pi\cdot\int_0^{\pi}\frac{1}{3}\sin\theta\,d\theta$$

$$= abc \cdot 2\pi \left[-\frac{1}{3} \cos\theta \right]_0^{\pi}$$

$$= \frac{2}{3} \pi abc$$