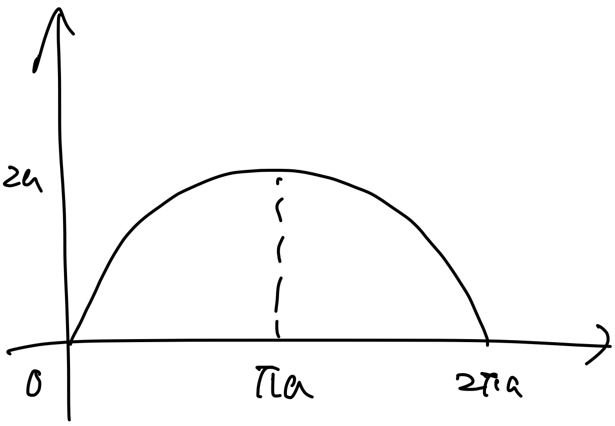
E4-1

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \qquad (0 \leq t \leq 2\pi)$$

$$\frac{dx}{dt} = a(1 - \cos t), \quad \frac{dy}{dt} = a \sin t$$

t	0	$\cdots$	π	$\cdots$	2π
$\frac{dx}{dt}$		$+$	$+$	$+$	
$\frac{dy}{dt}$		$+$	0	$-$	
x	0	$\nearrow$	πa	$\nearrow$	$2\pi a$
y	0	$\nearrow$	$2a$	$\searrow$	0



$$\therefore S = \int_0^{2\pi a} y \, dx$$

$$= \int_0^{2\pi} a(1 - \cos t) \cdot a(1 - \cos t) \, dt$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) \, dt$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt$$

$$= a^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right) dt$$

$$= a^2\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= a^2(3\pi - 0)$$

$$= 3\pi a^2$$

S4-1

$$\begin{cases} x = \cos 2t \\ y = \sin 3t \end{cases} \quad \left(0 \leq t \leq \frac{\pi}{3}\right)$$

$2t = n\pi$

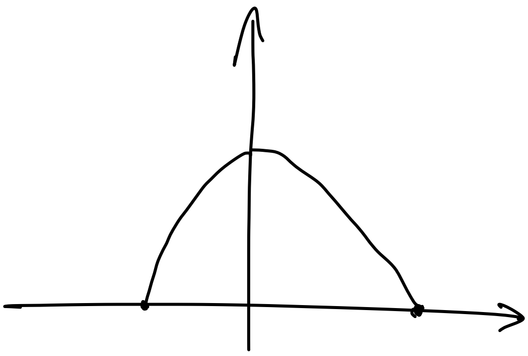
$t = \frac{\pi}{2}n$

$3t = \frac{\pi}{2}n$

$t = \frac{\pi}{6}n$

$\frac{dx}{dt} = -2\sin 2t \quad \frac{dy}{dt} = 3\cos 3t$

t	0	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{\pi}{3}$
$\frac{dx}{dt}$		$-$	$-$	$-$	
$\frac{dy}{dt}$		$+$	0	$-$	
x	1	$\searrow$	$\frac{1}{2}$	$\searrow$	$-\frac{1}{2}$
y	0	$\nearrow$	1	$\searrow$	0



$$\therefore S = \int_{-\frac{1}{2}}^{1} y\,dx$$

$$= -\int_0^{\frac{\pi}{3}} \sin 3t \cdot (-2\sin 2t)\,dt$$

$$= \int_0^{\frac{\pi}{3}} (\cos(3-2)t - \cos(3+2)t)\,dt$$

$$= \left[\sin t - \frac{1}{5}\sin 5t\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{5}\cdot\left(-\frac{\sqrt{3}}{2}\right) - 0\right)$$

$$= \frac{6\sqrt{3}}{10} = \frac{3\sqrt{3}}{5}$$

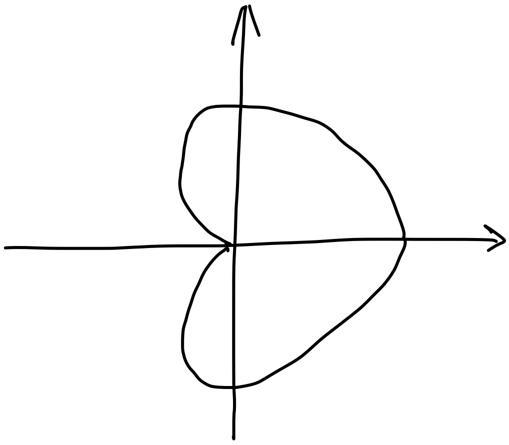
E4-2

$r = a(1+\cos\theta) \quad (a>0)$

$\pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$

$\frac{dr}{d\theta} = -a\sin\theta$

θ	0	$\cdots$	π	$\cdots$	2π
$\frac{dr}{d\theta}$		$-$	0	$+$	
r	$2a$	$\searrow$	0	$\nearrow$	$2a$



$\therefore$

$$S = 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi} a^2 (1+\cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

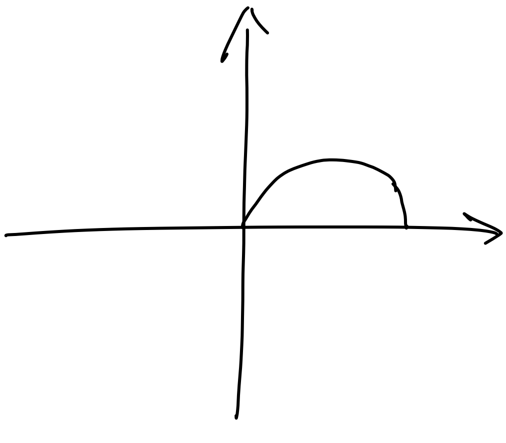
$$= a^2 \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

$$= a^2 \left(\frac{3}{2}\pi - 0\right)$$

$$= \frac{3}{2}\pi a^2$$

S4-2

$r^2 = 2a^2 \cos 2\theta \quad (a>0)$



$$S = 4\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta$$

$$= 4a^2 \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2a^2$$

E4-13

(1)

立体を平面 $x=t$ でカット

$y=\pm\sqrt{1-t^2}$

$z=\pm\sqrt{1-t^2}$

切り口の面積 $S(t)$ は（$S(t)$ は yz 平面の面積）

$S(t)=2\sqrt{1-t^2}\cdot 2\sqrt{1-t^2}$

$=4(1-t^2)$

$\therefore V=\int_{-1}^{1}S(t)\,dx$

$x=t$　　$dx=dt$　　$x\ 0\to 1$　$t\ 0\to 1$

$=2\int_0^1 4(1-t^2)\,dt$

$=8\left[t-\frac{t^3}{3}\right]_0^1$

$=\frac{16}{3}$

(2)

y 軸を中心軸とする半径 x の円筒の切り口面積は

$S(x)=2\pi x\cdot\sin x$

$+$	x	$-\cos x$
$-$	1	$-\sin x$

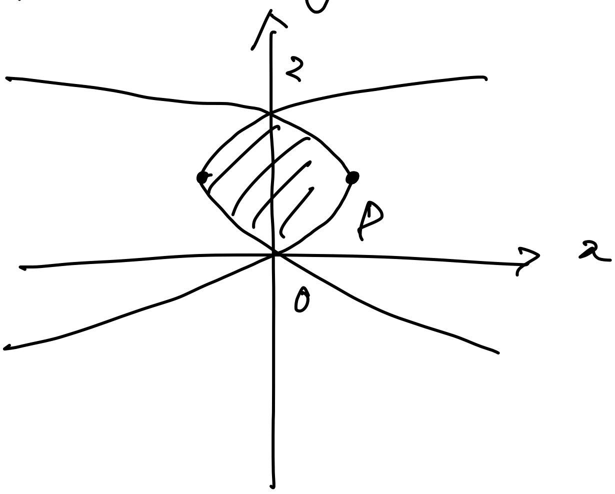
$\therefore V=\int_0^{\pi}2\pi x\sin x\,dx$

$=2\pi\left[-x\cos x+\sin x\right]_0^{\pi}$

$=2\pi(-\pi\cdot(-1)-0)$

$=2\pi^2$

S4-3



(1) $V_y = \int_0^2 \pi \times \left((y-1)^2 - 1 \right)^2 \cdot dy$

$= \int_0^2 \pi \left((y-1)^4 - 2(y-1)^2 + 1 \right) dy$

$= \pi \left[\frac{1}{5}(y-1)^5 - \frac{2}{3}(y-1)^3 + y \right]_0^2$

$= \pi \left(\frac{1}{5} - \frac{2}{3} + 2 - \left(-\frac{1}{5} + \frac{2}{3} \right) \right)$

$= \pi \left(\frac{\cancel{2}^{6}}{\cancel{5}_{15}} - \frac{\cancel{4}^{20}}{\cancel{3}_{15}} + \cancel{2}^{30}_{15} \right)$

$= \frac{16}{15}\pi$

(2) $V_x = -\int_0^2 2\pi y \cdot 2\left((y-1)^2 - 1 \right) dy$

$= -4\pi \int_0^2 y\left((y-1)^2 - 1 \right) dy$

$= -4\pi \int_0^2 \left(y^3 - 2y^2 + y - y \right) dy$

$= -4\pi \left[\frac{y^4}{4} - \frac{2}{3}y^3 \right]_0^2$

$= -4\pi \left(\frac{\cancel{4}^{12}}{3} - \frac{16}{3} \right) = \frac{16}{3}\pi$

E4-4

(1) $\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \end{cases} \quad (0 \le t \le \pi)$

$$L = \int_C \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_C \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\, dt$$

$$= \int_0^{\pi} t\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}$$

$$\frac{dx}{dt} = -\sin t + \sin t + t\cos t$$
$$= t\cos t$$

$$\frac{dy}{dt} = \cos t - \cos t + t\sin t$$
$$= t\sin t$$

(2) $y = \dfrac{e^x + e^{-x}}{2} \quad (0 \le x \le 1)$

$$L = \int_C \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_C \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \frac{e^x + e^{-x}}{2}\, dx$$

$$= \left[\frac{e^x - e^{-x}}{2}\right]_0^1$$

$$= \frac{e - e^{-1}}{2}$$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} + 1$$

$$= \frac{4 - 2 + e^{2x} + e^{-2x}}{4}$$

$$= \frac{(e^x + e^{-x})^2}{2^2}$$

S4-4

(1) $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases} \quad (0 \le t \le 2\pi)$

$$\frac{dx}{dt} = 1 - \cos t \quad , \quad \frac{dy}{dt} = \sin t$$

$$L = \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \, dt$$

$$= \int_0^{2\pi} \sqrt{2(1 - \cos t)} \, dt$$

$$= \int_0^{2\pi} \sqrt{4 \cdot \frac{1 - \cos t}{2}} \, dt$$

$$= \int_0^{2\pi} 2 \cdot \sin \frac{t}{2} \, dt$$

$$= \left[-4 \cos \frac{t}{2} \right]_0^{2\pi}$$

$$= \left(-4 \cdot (-1) - (-4) \cdot 1 \right)$$

$$= 8$$

(2) $y = \frac{2}{3} x \sqrt{x} = \frac{2}{3} x^{\frac{3}{2}} \quad (0 \le x \le 1)$

$$\frac{dy}{dx} = x^{\frac{1}{2}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^2}$$

$\therefore$

$$L = \int_0^1 \sqrt{1 + x} \, dx$$

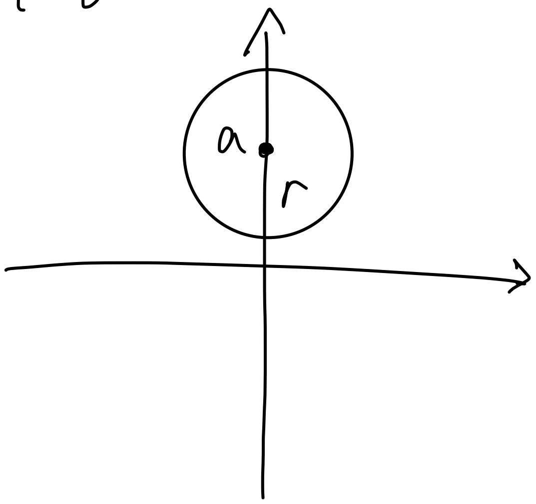
$$= \left[\frac{2}{3} (1 + x)^{\frac{3}{2}} \right]_0^1$$

$$= \frac{2}{3} \cdot 2\sqrt{2} - \frac{2}{3} \cdot 1$$

$$= \frac{4\sqrt{2} - 2}{3}$$

E4-5

(1)



$$x^2 + (y-a)^2 = r^2$$

$$y = a \pm \sqrt{r^2 - x^2}$$

$$\frac{dy}{dx} = \cancel{a} \pm \frac{-2x}{2\sqrt{r^2-x^2}}$$

$$= \cancel{a} \mp \frac{x}{\sqrt{r^2-x^2}}$$

$$S = \int_{-r}^{r} 2\pi y \cdot \sqrt{(dx^2) + (dy)^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{r^2-x^2}{r^2-x^2} + \frac{x^2}{r^2-x^2}}$$

$$= \frac{r}{\sqrt{r^2-x^2}}$$

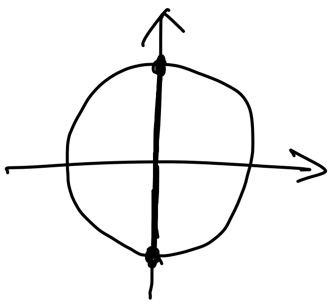
$$= \int_{-r}^{r} 2\pi\left(a + \sqrt{r^2-x^2}\right) \cdot \frac{r}{\sqrt{r^2-x^2}}\, dx$$

$$+ \int_{-r}^{r} 2\pi\left(a - \sqrt{r^2-x^2}\right) \cdot \frac{r}{\sqrt{r^2-x^2}}\, dx$$

$$= \int_{-r}^{r}\left(\frac{2\pi a r}{\sqrt{r^2-x^2}} + 2\pi r\right) dx + \int_{-r}^{r}\left(\frac{2\pi a r}{\sqrt{r^2-x^2}} - 2\pi r\right) dx$$

$$= 4\pi \int_{-r}^{r} a \frac{r}{\sqrt{r^2-x^2}}\, dx$$

$$= 4\pi \int_{-r}^{r} a \frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}\, dx$$



$$= 4\pi a \left[r \sin^{-1} \frac{x}{r} \right]_{-r}^{r}$$

$$= 4\pi a \left(r \cdot \frac{\pi}{2} - r \cdot \left(-\frac{\pi}{2}\right)\right)$$

$$= 4\pi^2 a r$$

(2) $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \quad (0 \leq t \leq 2\pi)$

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = a(1 - \cos t) \quad , \frac{dy}{dt} = a \sin t$$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$$

$$= a\sqrt{2(1 - \cos t)}$$

$$= a\sqrt{4 \cdot \frac{1 - \cos t}{2}}$$

$$= 2a \sin\frac{t}{2}$$

$$S = \int_0^{2\pi} 2\pi \cdot a \underbrace{(1 - \cos t)}_{2\sin^2\frac{t}{2}} \cdot 2a \sin\frac{t}{2}\, dt$$

$$= \int_0^{2\pi} 8\pi a^2 \sin^3\frac{t}{2}\, dt$$

$$= 8\pi a^2 \int_0^{2\pi} \left(1 - \cos^2\frac{t}{2}\right) \sin\frac{t}{2}\, dt$$

$$= 8\pi a^2 \left[-2\cos\frac{t}{2} + \frac{2}{3}\cos^3\frac{t}{2}\right]_0^{2\pi}$$

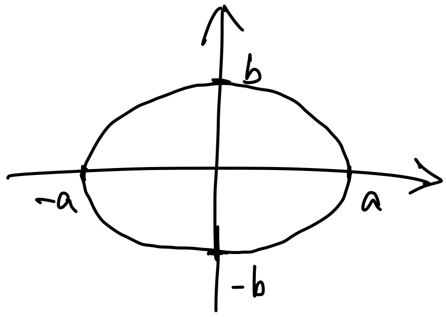
$$= 8\pi a^2 \left(-2 \cdot (-1) + \frac{2}{3} \cdot (-1) - \left(-2 + \frac{2}{3}\right)\right)$$

$$= 8\pi a^2 \left(\frac{\overset{12}{\cancel{4}}}{3} - \frac{4}{3}\right)$$

$$= \frac{64}{3}\pi a^2$$

S4-5

(1) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad (a>b>0)$



$$b^2x^2+a^2y^2=a^2b^2$$

$$a^2y^2=a^2b^2-b^2x^2$$

$$y^2=b^2-\frac{b^2}{a^2}x^2$$

$$=\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2} \qquad \frac{dy}{dx}=\pm\frac{b}{a}\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{\frac{a^2(a^2-x^2)}{a^2(a^2-x^2)}+\frac{b^2x^2}{a^2(a^2-x^2)}}$$

$$=\sqrt{\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}}$$

$\therefore$

$$S=\int_{-a}^{a}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_{-a}^{a}2\pi\frac{b}{a^2}\sqrt{a^4-(a^2-b^2)x^2}\,dx$$

$$=\int_{-a}^{a}2\pi b\sqrt{1-\frac{a^2-b^2}{a^4}x^2}\,dx$$

$\dfrac{a^2-b^2}{a^4}=k^2$ とおくと

$$S=2\pi b\int_{-a}^{a}\sqrt{1-k^2x^2}\,dx$$

ここで,

$$I=\int\sqrt{1-k^2x^2}\,dx$$

$$=x\sqrt{1-k^2x^2}-\int x\cdot\frac{-\cancel{2}k^2x}{\cancel{2}\sqrt{1-k^2x^2}}\,dx$$

$$=x\sqrt{1-k^2x^2}+\int\frac{k^2x^2}{\sqrt{1-k^2x^2}}\,dx$$

$$= x\sqrt{1+k^2x^2} + \int \frac{1-(1-k^2x^2)}{\sqrt{1-k^2x^2}}\,dx$$

$$= x\sqrt{1+x^2} + \frac{1}{k}\sin^{-1}kx - \underbrace{\int \sqrt{1-k^2x^2}\,dx}_{=I}$$

$$\therefore\ 2I = x\sqrt{1+x^2} + \frac{1}{k}\sin^{-1}kx + C$$

$$I = \frac{1}{2}\left(x\sqrt{1-k^2x^2} + \frac{1}{k}\sin^{-1}kx\right) + C$$

$$\therefore\ S = \pi b\left[x\sqrt{1-k^2x^2} + \frac{1}{k}\sin^{-1}kx\right]_{-a}^{a}$$

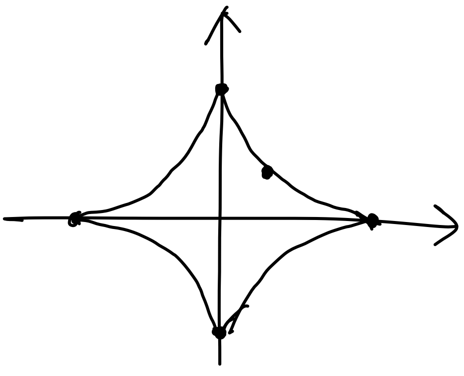
$$= \pi b\left(a\sqrt{1-k^2a^2} + \frac{1}{k}\sin^{-1}ka - \left(-a\sqrt{1-k^2a^2} + \underbrace{\frac{1}{k}\sin^{-1}k(-a)}_{=-\frac{1}{k}\sin^{-1}ka}\right)\right)$$

$$= 2\pi b\left(a\sqrt{1-k^2a^2} + \frac{1}{k}\sin^{-1}ka\right)$$

(2) $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $\quad (a>0,\ 0 \le t \le 2\pi)$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t,\quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

t	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	π	$\cdots$	$\frac{3}{2}\pi$	$\cdots$	2π
dx/dt			0		0		0		
dy/dt			0		0		0		
x	a	$\searrow$	0	$\searrow$	$-a$	$\nearrow$	0	$\nearrow$	a
y	0	$\nearrow$	a	$\searrow$	0	$\searrow$	$-a$	$\nearrow$	0



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}$$

$$= 3a\cos t\sin t\sqrt{\cos^2 t+\sin^2 t}$$

$$= 3a\cos t\sin t$$

$$S = 2\int_0^{\frac{\pi}{2}} 2\pi y\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= 2\int_0^{\frac{\pi}{2}} 2\pi\cdot a\sin^3 t\cdot 3a\cos t\sin t\,dt$$

$$= 12\pi a^2\left[\frac{1}{5}\sin^5 t\right]_0^{\frac{\pi}{2}}$$

$$= \frac{12}{5}\pi a^2$$