Chapter 1

The Basics of Complex Numbers

1.1 Complex Numbers Review

1.1.1 Definition

Complex numbers are of the form $a + bi$, $a, b \in \mathbb{R}$ and $i^2 = -1$.

1.1.2 Complex Conjugate

Definition: $\overline{a + bi} = a - bi$ If $z = a + bi$, then $z \cdot \overline{z} = a^2 + b^2 > 0$ unless $z = 0$.

1.1.3 Multiplication

The multiplication of 2 complex numbers is a complex number. Multiplication proceeds via basic algebraic distribution.

1.1.4 Division

Division of complex numbers is defined as long as the denominator is not 0. To do division operations, multiple the top and bottom by the complex conjugate of the denominator and simplify.

1.1.5 Modulus

Modulus of $z = a + ib$ is given by

$$|z| = \sqrt{a^2 + b^2}$$

$$|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = |z_1|^2 |z_2|^2$$

$$|z_1 z_2| = |z_1||z_2|$$

1.2 Geometry of the Complex Plane

1.2.1 The Plane

The complex plane is just like the xy-plane except it has a real axis ”x” and a complex axis ”y”.

1.2.2 Distances

$|z|$ gives the distance from z to the origin. $|z_2 - z_1|$ gives the distance from z_2 to z_1 in the complex plane.

1.2.3 Polar Representation

$r = |z|$ and $\theta = \arg(z)$ ”the argument of z”. We use the ”normal” version. $(1, \pi)$ or $(1, 3\pi)$ instead of $(-1, -\pi)$

Principle Argument:

$$-\pi \leq Arg(z) \leq \pi$$

rcis form:

$a = rcos\theta, b = rsin\theta, z = r(cos\theta + isin\theta)$

Conjugates

If $\theta = Arg(z)$, $\overline{rcis\theta} = rcis(-\theta)$

Multiplication

Suppose $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$,

$$z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$$

$$arg(z_1 z_2) = arg(z_1) + arg(z_2)$$

Conclusion: Multiplication in the complex plane consists of a rotation and a dilation.

1.2.4 Exponential Representation + Taylor Series

$$cis\theta = e^{i\theta}$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots)$$

$$e^{i\theta} = cos\theta + isin\theta$$

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$cosz = \frac{e^{iz} + e^{-iz}}{2}$$

Cosine can be used on the entire complex plane.

1.2.5 Hyperbolic Functions

$$\cos(a+ib) = \frac{e^{i(a+ib) - e^{-i(a+ib)}}}{2} = \frac{e^{ia}e^{-b} + e^{-ia}e^{b}}{2}$$

$$= \cos a \cdot \frac{e^{b} + e^{-b}}{2} - i \sin a \cdot \frac{e^{b} - e^{-b}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\tanh z = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

1.3 Exponentiation

Converting to exponential form is a lot easier than using binomial theorem. Multiplication, division, and exponentiation is a lot easier in exponential notation.

$$(1+i\sqrt{3})^7 = (2e^{i\frac{\pi}{3}})^7 = 2^7 \cdot e^{i\frac{7\pi}{3}}$$

1.3.1 Not Bijective

$\forall a \in \mathbb{R} : a \neq 0$, $e^z = a$ for inifinitely many z.

1.3.2 Fractional Exponentiation

Using a exponential notation, $\pm$ results arise for different arguments.

Principal Value

Convention is to use the principal argument during calculations. Remeber that the principal argument is **not always positive** but the one between $-\pi$ and π.

Example:

$$\text{For } n \in \mathbb{Z},\ i = e^{i\frac{\pi}{2}} \cdot e^{i \cdot 2\pi n} \Rightarrow i^{\frac{1}{3}} = \text{cis}\left(\frac{\pi}{6} + \frac{2\pi n}{3}\right)$$

Note that these values form an equilateral triangle when polygon. Note that $i^{\frac{1}{q}}$ for $q \in \mathbb{N}$, results in an regular $n - gon$.

Calculators: TI calculators do not always return the principal value.
Phase: $e^{i\theta}$ is called a **phase factor**.

1.3.3 Raising to the i Power

Raising something to the i power "switches" the argument and modulus.

$$i^i = (e^{i\frac{\pi}{2} + 2in\pi})^i = e^{-\frac{\pi}{2} - 2n\pi}$$

$$2^i = (e^{\ln 2} \cdot e^{2\pi in})^i = e^{-2\pi n} \cdot \text{cis}(\ln 2)$$