Lecture Notes for Advanced Math Techniques

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1 Kinematics

Kinematics lies a multiple fundemantal definitions regarding velocity, acceleration, and position. Additionally, there are a number of helpful quick formulas that are nice to memorize but easy to derive. Note that motion in different coordinate directions are independent.

$$h_{max} = \frac{v_0^2 \sin^2 \theta}{2g};\ t_{top} = \frac{v_0 \cos \theta}{g};\ d_{range} = \frac{v_0 \sin 2\theta}{g}$$

2 Reference Frames

2.1 Inertial Frames

An **intertial frame** is a frame of reference that is not accelerating or rotating. If A is an intertial frame, and B is moving at a constant velocity with respect to A, then B's frame is also inertial. Note that if 2 objects are rotating relative to each other in space, only one can be inertial.

2.2 Rotated Frames

Rotating the frame of reference is very useful and can simplify many calculations.

3 Collisions

Remember to account for frictional impulse during collisions. This stems from the normal impulse that slows an object to rest. See Problem 3.2 in *Relatvitiy and Collsion Handout PhysicsWOOT 20-21*.

$$j_f \leq \mu j_N$$

In 1D elastic collisions, the relative velocity between the objects changes sign, so

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

If the masses are equal, the objects simply switch velocities.

3.1 Elastic Collisions

Elastic collisions conserve energy and happen in a "perfect world".

3.2 Inelastic Collisions

This is when energy is not conserved but is dissapatted as heat and other forms of energy.

3.3 Perfectly Inelastic Collisions

These collisions disapate the most kinetic energy and causes the colliding objects to

4 Differential Equations

5 Sinusoidal Motion

The following differential equation results in sinusoidal motion with angular velocity ω.

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

The potential energy function is parabolic.

$$U(x) = ax^2$$

The solution to this is a function of the following form,

$$x(t) = \frac{v_0}{\omega}\sin(\omega(x - x_0) + \phi) + x_0$$

Note that $A = \frac{v_0}{\omega}$, ϕ is the angular offset and x_0 is the "middle" value.

6 Forces

The normal force resists compression. The tension force resists stretching. The frictional force can apply in different directions (static friction).

7 Random

7.1 Morse Potential

In chemistry, double bonds can be thought of as a spring with variable bond length that vibrates back and forth. The potential energy in the spring is known as the morse potential.

8 Statics

Statics are when both torque and forces sum to 0. Draw the forces in properly and quickly balance using F=ma. Cancel out the horizontal and vertical forces to see if that gives any easy substitutions. Then, try picking a pivot point and analyzing the torques. Pick a pivot with a lot of forces on it so that all their torques are zero. For some connected problems, try analyzing the torque of the entire system. Remember that pivots can exert forces in the "frictional" direction and into the wall as well. Remember that at pivots, Newton's 3rd law applies so if object 1 exerts a force on object 2, an opposite force is exerted on object 1.

9 Gravity

Newtons law states that the force due to gravity is

$$\vec{F} = \frac{Gm_1m_2}{R^2}\hat{r}$$

Then,

$$U = -\int_0^\infty F dr = -\frac{Gm_1m_2}{R}$$

Note that this is with respect to ∞ at which the potential energy is 0. The main idea is that potential energy should decrease as objects get closer together.

10 Orbits

Orbits conserve angular momentum. From centripetal acceleration it is evident that,

$$v_{circ} = \sqrt{\frac{GM}{R}};\ E = -K = \frac{U}{2};$$

We can also use conservation of energy to find the escape speed.

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

Using this we can classify orbits by tangential velocity. At **apogee**, the object is at a maximum distance away. Note that if $0 < v_{tan} < v_{circ}$ then the orbit must be elliptical and R is the max distance from earth (apogee). If $v_{tan} = v_{circ}$, the orbit is circular with radius R. If $v_{circ} < v_{tan} < v_{esc}$, the orbit is elliptical and R is the minimum distance from Earth (perigee). If $v_{tan} = v_{esc}$, the object escapes the orbit parabolically. But if $v_{tan} > v_{esc}$, then the object exits the orbit but hyperbolically.

11 Fluids

Archimedes Principle states that the force that a fluid exerts on an object equals the weight of the water it displaces or

$$F_{bouyant} = m_{disp}g = \rho V_{disp}g$$

Pascal's Principle states that when a pressure is applied to a confined fluid, the fluid exerts that pressure everywhere. This can be applied to pistons showing that

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}$$

Burnoulli's Principle is simply conservation of energy applied to fluids.

$$P + \frac{1}{2}\rho v^2 + \rho gh = C \in \mathbb{R}$$

When you multiply both sides by V, we see that this is simply energy required to apply a pressure on the fluid volume in "front" plus the kinetic energy due to velocity plus the gravitational potential energy. Dividing by volume allows us to to not worry about volume but instead density which is easier to deal with. Remember that the pressure energy is the atmospheric pressure when exposed to air.

12 Rotation

When you have a ball that is rolling and moving, the center of mass moves with velocity v, but the point on the ball has to rotate AND move meaning it has velocity v + v = 2v (second v from centripetal motion). **Roling without slipping** implies that $v = rw$. If an object takes time to roll, find the time it takes for that equation to be true. Note that all points on the boundary do **not** rotate with the same speed (break-up components). The Earth rotates counter clockwise when looking down at the north pole.

13 Ficticious Forces

Translational forces are if you reference frame is linearly accelerating. Add a translational force in the opposite direction.

$$F_{trans} = -\frac{d^2\vec{R}}{dt^2}$$

Centrifugal forces should be added if you have a rotating reference frame with a static object. Then add the centrifugal force in the opposite direction of the "typical" centripital force in the outside reference frame.

$$F_{cf} = -m\omega \times (\omega \times r)$$

Coriollis forces are added if you have a rotating reference frame and the object has a velocity. Then, the coriollis force is

$$F_{cor} = -2m(\vec{\omega} \times \vec{v})$$

Azimuthal forces are added when the reference frame itself is rotating with an angular acceleration. The azimuthal force is in the opposite direction of a typical tangential acceleration.

$$F_{az} = -m\alpha \times r$$

Remember that during centripetal motion, there vector $\vec{r}$ points from the center along the radius to the arc/path.

14 Graphs

Check limiting cases especially on graph questions. Often times, you don't even need to solve completely. You just need to take some limits and maybe derivatives.

15 Uncertainty

Remember that quadrature is a Guassian approximation. Don't jump into quadrature right away but see if you can use a mathematical exploit with a log or something to make relative uncertainty easier to work with for example. Note that we don't always have to use quadrature. Try plugging in some delta's to figure changes.

16 Power

Power is the integral of force with respect to velocity or the derivative of work with respect to time. Note that this means power is proportional to energy.

17 Work

For work, remember the linear and rotational analog $W = Fd$ and $W = \tau\theta$. Work is related to energy via the work-kinetic energy theorem.