Chapter 1

The Zeta Function

1.1 The sin function, the Basel Problem, and more results

1.1.1 Beginning

$\sin(0) = 0$. So, let's divide by x to remove the factor of x in the expansion.

$$\frac{\sin(x)}{x} = (1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})\cdots$$

Using difference of squares,

$$= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2})\cdots$$

Note that we have not established these 2 functions are equivalent but that they have the same zeroes. This was used to say Euler's argument wasn't rigorous (the full thing took another 10 years). Without the proven rigour, let's suppose this statement. Expanding the product by powers of x,

$$= 1 - \frac{x^2}{\pi^2}(1 + \frac{1}{4} + \frac{1}{9} + \cdots) + \frac{x^4}{\pi^4}(\frac{1}{1 \cdot 4} + \frac{1}{1 \cdot 9} + \cdots + \frac{1}{4 \cdot 9} + \frac{1}{4 \cdot 16} + \cdots) - \cdots$$

$$= 1 - \frac{x^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{x^4}{\pi^4}\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{n^2m^2} + \cdots$$

1.1.2 Deriving results

Using the taylor series,

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

Note that these 2 expansions are equal. This **solves the Basel problem**.

$$\therefore \frac{-X^2}{3!} = \frac{-X^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

We can use this for other series.

$$\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right) = \frac{\pi^4}{36}$$

Also,

$$= \sum_{n=1}^{\infty}\frac{1}{n^4} + \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{m^2n^2} + \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{m^2n^2}$$

$$\therefore \frac{\pi^4}{36} = \sum_{n=1}^{\infty} + \frac{2\pi^4}{120} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Euler continued this all the way to $\sum_{n=1}^{\infty}\frac{1}{n^{26}}$.

1.1.3 Conversion

To write it in a closed form,

$$\frac{sin(x)}{x} = \prod_{k=1}^{\infty}(1 - \frac{x^2}{k^2\pi^2})$$

Writing as a series,

$$\ln\frac{sin(x)}{x} = \sum_{k=1}^{\infty} ln\left(1 - \frac{x^2}{k^2+\pi^2}\right)$$

$$= -\sum_{k=1}^{\infty}\sum_{j=1}\frac{\frac{x^2}{k^2\pi^2}}{j} \Rightarrow -sum_{j=1}^{\infty}\frac{x^{2j}}{j}\frac{\zeta(2j)}{\pi^{2j}}$$

Looking at cotangent,

$$\cot(x) - \frac{1}{x} = -\sum_{j=1}^{\infty}2\frac{\zeta(2j)}{\pi^2 j}x^{2j-1}$$

1.2 Dirichlet Series

1.2.1 Zeta Series

This function is not really a dirichlet series but it's related.

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

1.2.2 Eta Series

This series is an alternating Zeta Series.

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots$$

1.2.3 Lambda Series

This series is a Zeta Series with only odd terms.

$$\lambda(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots$$

1.2.4 Beta Series

This series is an alternating lambda series.

$$\lambda(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \cdots$$

1.2.5 Even Zeroes

Eueler has given all the even zeroes of the zeta function.

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{\pi^6}{945}$$

1.2.6 Deriving Additional Values

$$\eta(2) = \frac{\pi^2}{12}$$

$$\lambda(2) = \frac{3\pi^2}{24}$$

1.2.7 Deriving Additional Results

$$\eta(s) = \zeta(s) - 2(\frac{1}{2^s}\zeta(s)) = (1 - 2^{1-s})\zeta(s)$$

$$\lambda(s) = \zeta(s) - (\frac{1}{2^s}\zeta(s)) = (1 - 2^{-s})\zeta(s)$$

$\beta(s)$ is not related to the other functions.

1.2.8 Apery's Constant

Apery's constant is $\zeta(3)$ because the French mathematician proved that it was irrational. For zeta, the odd's are hard and the even's are known exactly.

1.2.9 Catalan's Constant

Catalan's constant is $\beta(2)$. For β the even's are hard and the odd ones are known.

1.3 Weirstrass Approximation Theorem

You can approximate an arbitrarily continuous function by an arbitrary polynomial.