Chapter 1

Complex Limits and Derivatives

1.1 Limits

1.1.1 Definition

The function $f(z)$ is said to have the limit F as z approaches z_0 if given any $\epsilon > 0 \ \exists \ \delta \mid |f(z) - F| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

1.1.2 Punctured Disk

A punctured disk is a disk with an open boundary except the center. The $0 < |z - z_0| < \delta$ gives a punctured disk. $|f(z) - F| < \epsilon$ gives another disk. The limit definition essentially states that the punctured disk can be mapped inside the other disk.

1.1.3 Triangle Inequality

For any complex numbers z, w.

$$||z| - |w|| \leq |z + w| \leq |z| + |w|$$

Drawing the complex numbers, we see that this enforces that the numbers form a triangle. This is basically the complex analog of the simple triangle inequality from geometry.

1.2 Limit Laws

1.2.1 The Sum Law

Definition

If $\lim_{z\to z_0} f(z) = F$ and $\lim_{z\to z_0} g(z) = G$, then $\lim_{z\to z_0} [f(z) + g(z)] = F + G$. Given $\epsilon > 0$, we need to find a δ such that $|f(z) + g(z) - F - G| < \epsilon$ whenever $0 < |z - z_0| < \delta$. Note that we can assume the standalone limits exist.

Proof

Given any $\epsilon > 0$, we can certainly find δ_f: $|f(z)-F| < \frac{\epsilon}{2}$ whenever $0 < |z-z_0| < \delta_f$, and δ_g: $|g(z) - G| < \frac{\epsilon}{2}$ whenever $0 < |z - z_0| < \delta_g$. Now, for $0 < |z - z_0| < \min(\delta_f, \delta_g)$, we have $|f(z)+g(z)-F-G| \leq |f(z)-F|+ \leq |g(z)-G| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$. Therefore the choice $\min(\delta_f, \delta_g)$ for δ satisfies our requirements and proves the theorem.

1.2.2 The Product Law

Definition

If $\lim_{z\to z_0} f(z) = F$ and $\lim_{z\to z_0} g(z) = G$, then $\lim_{z\to z_0} [f(z)g(z)] = FG$. Given $\epsilon > 0$, we need to find a δ such that $|f(z)g(z) - FG| < \epsilon$ whenever $0 < |z - z_0| < \delta$. Note that we can assume the standalone limits exist.

Proof

$$\begin{aligned}|f(z)g(z) - FG| &= |(f(z) - F + F)g(z) - FG| = |(f(z) - F)g(z) + F(g(z) - G)| \\ &= |(f(z) - F)(g(z) - G) + (f(z) - F)G + F(g(z) - G)|\end{aligned}$$

Given $\epsilon > 0$, we can find δ_f for which $|f(z)-F| < min(\frac{\epsilon}{3}, 1)$ for $0 < |z-z_0| < \delta_f$, and δ_g for which $|g(z) - G| < min(\frac{\epsilon}{3}, 1)$ for $0 < |z - z_0| < \delta_f$. Note that there are a number of cases we need to consider.
Case 1: Suppose $F = G = 0$. Given $\epsilon > 0$, we can find δ_f such that $|f(z)| < \epsilon$ whenever $0 < |z-z_0| < \delta_f$ and δ_g such that $|g(z)| < 1$ whenever $0 < |z-z_0| < \delta_g$

Case 2: Suppose $F = 0, G \neq 0$. Given $\epsilon > 0$, we can find δ_f such that $|f(z)| < \min(\frac{\epsilon}{2|G|}, 1)$ whenever $0 < |z - z_0| < \delta_f$ and δ_g such that $|g(z) - G| < \min(\frac{\epsilon}{2}, 1)$ whenever $0 < |z - z_0| < \delta_g$.

Case 3: Suppose $FG \neq 0$. Given $\epsilon > 0$, we can find δ_f such that $|f(z) - F| < \min(\frac{\epsilon}{3|G|}, 1, \frac{\epsilon}{3})$ whenever $0 < |z - z_0| < \delta_f$ and δ_g such that $|g(z) - G| < \min(\frac{\epsilon}{3|F|}, 1, \frac{\epsilon}{3})$ whenever $0 < |z - z_0| < \delta_g$.

Finishing: Now for each of these cases, the triangle inequality guarantees that $|f(z)g(z)-FG| = |(f(z)-F)(g(z)-G)+(f(z)-F)G+F(g(z)-G)| < \epsilon$ which proves the theorem.

1.3 Derivatives

1.3.1 Definition

The function $f(z)$ is said to be differentiable at z_0 if the following exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

In this case, the limit is $f'(z_0)$. Note that the derivative of a sum = sum of the derivatives provided that the two functions are differentiable.

1.3.2 Products

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{[f(z) - f(z_0) + f(z_0)]\, g(z) - f(z_0)g(z_0)}{z - z_0}$$
$$= \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} g(z) + f(z_0) \frac{g(z) - g(z_0)}{z - z_0} \right]$$

We know that this is equal to the sum of the limits so,

$$\boxed{\therefore \frac{d}{dz}(f(z)g(z)) = f'(z_0)g(z_0) + f(z_0)g'(z_0)}$$

1.3.3 Entire

z is **entire**, with derivative 1. All polynomials then are entire. Entire means that a function is differentiable on the whole complex plane.

1.3.4 Product Rule

As long as $z_0 \neq 0$,

$$\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \frac{z_0^n}{z_0} \cdot \frac{(\frac{z}{z_0})^n - 1}{\frac{z}{z_0} - 1}$$

Letting $w = \frac{z}{z_0}$,

$$z^{n-1} \lim_{w \to 1} \frac{w^n - 1}{w - 1} = nz^{n-1}$$

We can substitue $n = -m$ and continue to derive this for rational numbers as well.

1.3.5 Exponentials

$$\lim_{h \to 0} \frac{e^{z+h} - e^z}{h} = \lim_{h \to 0} \frac{e^z(e^h - 1)}{h} = e^z \lim_{h \to 0} \frac{e^h - 1}{h} = e^z$$

So, e^z is entire as it is defined on the whole complex plane and thus is differentiable on it as well.

1.3.6 More on Differentiability

If $f(z) = u(x,y) + iv(x,y)$, with u, v differentiable functions of x, y, then

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

For $f(z)$ to be differentiable, it has to hold the same value for an arbitrary direction of approach. Considering an approach of constant y,

$$= \lim_{z \to z_0} \frac{u(x, y_0) + iu(x, y_0) - [u(x_0, y_0) + iv(x_0, y_0)]}{x - x_0} = u_x(x_0, y_0) + iv_x(x_0, y)$$

Approaching at constant x,

$$= \lim_{z \to z_0} \frac{u(x_0, y) + iu(x_0, y) - [u(x_0, y_0) + iv(x_0, y_0)]}{i(y - y_0)} = -i(u_y + iv_y) = v_y - iu_y$$

Setting these equal to each other, we can see that if f is differentiable, then it satisfied the Cauchy-Reimann Equations and is conformal.

$$\boxed{\therefore \text{conformal} \Leftrightarrow \text{differentiable}}$$

1.3.7 An Important Statement

$f(z)$ is differentiable at z_0 iff $f(z) = f(z_0) + f'(z_0)(z - z_0) + \xi(z, z_0)$, where given any $\epsilon > 0 \ \exists \ \delta \mid |\xi(z, z_0)| < \epsilon |z - z_0|$ whenever $0 < |z - z_0| < \delta$. Note that this means the error term is "faster" than linear. If $f'(z_0) = 0$, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) \left[1 + \frac{\xi(z, z_0)}{f'(z, z_0)(z - z_0)} \right]$$

This shows that when z is close enough to z_0 this function is nothing but a translation $f(z_0)$ and a rotation. The term with the ξ goes to zero. So, locally, this function is conformal. Note that Cauchy Reimann equations are sufficient for differentiability. But for conformality, both a non-zero derivative and cauchy-reimann are needed.

1.3.8 Determining Differentiability

Determine where $f(z) = f(x + yi) = x^3 + y^2 + 3ix^2 y$ is differentiable. For this to satisfy the cauchy reiman equations, $u_y = -v_x$. Solving the system gives $y = 0$ or $x = -\frac{1}{3}$.