Chapter 1

The Beta Function

1.1 A Derivation

$$\Gamma(\alpha)\Gamma(\beta) = \left(\int_0^\infty x^{\alpha-1}e^{-x}dx\right)\left(\int_0^\infty y^{\beta-1}e^{-y}dy\right)$$

$$= \int_0^\infty dx \int_0^\infty dy x^{\alpha-1}y^{\beta-1}e^{-x-y}$$

We can instead write this integral in terms of $x+y$ as it happens over the first quadrant. So, let $u = x+y$.

$$= \int_0^\infty du \int_0^u dx \cdot x^{\alpha-1}(u-x)^{\beta-1}e^{-u}$$

Now, let's use scaling substitutions. $t = \frac{x}{u}$.

$$= \int_0^\infty du \int_0^1 u \cdot dt \cdot (ut)^{\alpha-1}(u-ut)^{\beta-1}e^{-u}$$

$$= \int_0^\infty du \cdot uu^{\alpha-1}u^{\beta-1}e^{-u} \int_0^1 dt \cdot t^{\alpha-1}(1-t)^{\beta-1} = \Gamma(\alpha+\beta) \cdot \int_0^1 dt \cdot t^{\alpha-1}(1-t)^{\beta-1}$$

$$\boxed{\int_0^1 dt \cdot t^{\alpha-1}(1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha,\beta)}$$

1.1.1 Alternate Form

Substituting $t = \frac{u}{u+1}$,

$$B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(u+1)^{\alpha+\beta}}du$$

1.2 Using Integral Forms

1.2.1 Nice Result

Using a substitution for x^n, we can eventually see that

$$\int_0^\infty \frac{x^{m-1}dx}{x^n+1} = \frac{\pi}{n}\csc\frac{m\pi}{n}\;\forall\, m,n \mid 0 < \frac{m}{n} < 1$$

1.2.2 Derivatives

Taking derivatives with respect to m,

$$\int_0^\infty \frac{x^{m-1}\ln x}{x^4+1}dx = -\frac{\pi^2}{n^2}\csc\frac{m\pi}{n}\cot\frac{m\pi}{n}$$

This gives us access to "natural logs" without worrying about expansions. Differentiating again,

$$\int_0^\infty \frac{x^{m-1}\ln^2 x}{x^n+1}dx = \frac{\pi^3}{n^3}\csc\frac{m\pi}{n}\left(2\csc^2\frac{m\pi}{n}-1\right)$$

1.2.3 More Derivations

Expanding and rearranging $\Gamma(\alpha)\Gamma(\beta)$ results in

$$\int_0^{\frac{\pi}{2}} \cos^{2\alpha-1}\theta\sin^{2\beta-1}\theta d\theta = \frac{1}{2}B(\alpha,\beta)$$