Chapter 1

The Gamma Function

1.1 Factorial and an Introduction to the Gamma Function

1.1.1 Introduction

Consider x as a positive integer and n as a large natural number.

$$(n+x)! = (n+x)(n+x-1)(n+x-2)\cdots n! \approx n^x n!$$

$$(x+n)! = (x+n)(x+n-1)(x+n-2)\cdots(x+1)x!$$

Dividing,

$$1 \approx \frac{n^x n!}{(x+n)(x+n-1)(x+n-2)\cdots(x+1)x!}$$

Rearranging,

$$x! \approx \frac{n^x n!}{(x+n)(x+n-1)(x+n-2)\cdots(x+1)}$$

Note that this is true for non-positive x.

$$(x-1)! \approx \frac{n^x n!}{(x+n)(x+n-1)(x+n-2)\cdots(x+1)(x)}$$

$$\lim_{n\to\infty} \frac{n^x n!}{(x+n)(x+n-1)(x+n-2)} = \Gamma(x)$$

This is known as the gamma function.

1.2 Establishing a General Definition

1.2.1 Recurrence Relations and the Factorial Function

$$\Gamma(x+1) = \lim_{n\to\infty} \frac{n^x n!}{(x+1+n)(x+n)\cdots(x+1)}$$

$$= \lim_{n\to\infty} \frac{nx}{x+1+n} \cdots \frac{n^x n!}{(x+n)(x+n-1)\cdots(x+1)(x)}$$

Separating the product into two limits and simplifying,

$$\boxed{\therefore \Gamma(x+1) = x\Gamma(x)}$$

Let's look at $\Gamma(1)$.

$$\Gamma(1) = \lim_{n\to\infty} \frac{n \cdot n!}{(1+n)(n)(n-1)\cdots(2)(1)} = \lim_{n\to\infty} \frac{n \cdot n!}{(n+1)n!} = 1$$

$$\Gamma(2) = 2 \cdot \Gamma(1) = 2$$

$$\vdots$$

$$\boxed{\therefore \Gamma(m) = (m-1)!}$$

The gamma is also known as the shifted factorial function.

1.2.2 An Intuitive Derivation

$$\frac{1}{\Gamma(x)} = \lim_{n\to\infty} \frac{(x+n)(x+n-1)\cdots(x+1)(x)}{n! \cdot n^x}$$

$$= x \cdot \lim_{n\to\infty} \frac{x+n}{n} \cdot \frac{x+n-1}{n-1} \cdots \frac{x+2}{2} \cdot \frac{x+1}{1} \cdot n^{-x}$$

Shifting,

$$\frac{1}{\Gamma(x+1)} = \lim_{n\to\infty} \left[(1+\frac{x}{n})(1+\frac{x}{n-1})\cdots(1+\frac{x}{2})(1+x)n^{-x}\right]$$

$$= \lim_{n\to\infty} \left[\prod_{k=1}^{n}(1+\frac{x}{k}) \cdot n^{-x}\right]$$

We want to move the n^{-x} term into the product. We know the following,

$$\lim_{n\to\infty} [H_n - \ln(n)] = \gamma$$

where γ is the Euler-Mascheroni constant. Using this, we can transform n^{-x} into a product.

$$n^{-x} = e^{-x\ln(n)} \approx e^{-x(H_n-\gamma)} = e^{\gamma x} e^{-x\sum_{k=1}^{n}\frac{1}{k}} = e^{\gamma x} \prod_{k=1}^{n} e^{\frac{-x}{k}}$$

Using this,

$$= \lim_{n\to\infty} \left[\prod_{k=1}^{n}(1+\frac{x}{k}) \cdot n^{-x}\right] = \lim_{n\to\infty} e^{\gamma x} \prod_{k=1}^{n} \left[(1+\frac{x}{k})e^{-\frac{x}{k}}\right]$$

Note that for some fixed x, we can go out far enough until the exponent is very small. So for large k,

$$(1+\frac{x}{k})e^{-\frac{x}{k}} = (1+\frac{x}{k})(1-\frac{x}{k}+\frac{x^2}{2k^2}+\cdots) = 1-\frac{x^2}{k^2}+\frac{x^2}{2k^2}-\cdots \approx 1-\frac{x^2}{k^2}$$

Since $\frac{1}{k^2}$'s sum converges, this expression seems to converge. So,

$$\frac{1}{\Gamma(1+x)} \text{ is defined for all } x \in \mathbb{C}$$

1.2.3 A Rigorous Proof

This can be rigorously proved using the limit comparison test with $\frac{x^2}{k^2}$.

$$\lim_{k\to\infty} \frac{(1+\frac{x}{k})e^{\frac{-x}{k}}-1}{\frac{x^2}{k^2}}$$

$$\overset{LH}{=} \lim_{k\to\infty} \frac{\frac{-x}{k^2}e^{\frac{-x}{k}}+(1+\frac{x}{k})(\frac{x}{k^2})e^{\frac{-x}{k}}}{-2\frac{x^2}{k^3}} \cdot \frac{k^2}{k^2}$$

$$= \lim_{k\to\infty} \frac{x^2}{-2x^2} = \frac{-1}{2}$$

1.3 The Reflection Identity

1.3.1 Continuing

$$[\Gamma(1+x)\Gamma(1-x)]^{-1} = e^{\gamma x}\prod_{k=1}^{n} e^{\frac{-x}{k}} \cdot e^{-\gamma x}\prod_{j=1}^{n} e^{\frac{x}{j}} = \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

Using the sin function,

$$= \frac{\sin(\pi x)}{\pi x}$$

Taking the reciprocal,

$$\Gamma(1+x)\Gamma(1-x) = \frac{\pi x}{\sin(\pi x)}$$

$$x\Gamma(x)\Gamma(1-x) = \frac{\pi x}{\sin(\pi x)}$$

$$\boxed{\therefore \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}}$$

This is known as the **Reflection Identity**.

1.3.2 Interesting Values

Using the reflection identity for $x = \frac{1}{2}$,

$$\Gamma^2(\frac{1}{2}) = \pi \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$$
$$\Gamma(\frac{3}{2}) = \frac{1}{2} \cdot \sqrt{\pi} = (\frac{1}{2})!$$

Now, using the gamma function, we can get the values of all kinds of exotic factorials!

1.4 Finding a Maclaurin Expansion

$$\Gamma(1+z) = e^{-\gamma z} \prod_{k=1}^{\infty} \left[(1+\frac{z}{k})^{-1} e^{\frac{z}{k}}\right]$$

$$\ln \Gamma(1+z) = -\gamma z + \sum_{k=1}^{\infty} \left[\frac{z}{k} - \ln(1+\frac{z}{k})\right] = -\gamma z + \sum_{k=1}^{\infty} \left[\frac{z}{k} - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(\frac{z}{k})^j}{j}\right]$$

Note that the first nested term is $\frac{z}{k}$ which cancels.

$$= -\gamma z + \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \frac{(-1)^j z^j}{k^j j} = -\gamma z + \sum_{j=2}^{\infty} \left(\frac{(-1)^j \zeta(j) z^j}{j}\right); \ |z| < 1$$

$$\boxed{\therefore \ln \Gamma(1+z) = -\gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} z^k}$$

1.5 Finding an Integral Representation

For large natural n, consider

$$\int_0^n t^{z-1}(1-\frac{t}{n})^n dt$$

Using repeated integration by parts,

$$= \left[\frac{t^z}{z}(1-\frac{t}{n})^n + \frac{t^{z+1}}{z(z+1)} n(1-\frac{t}{n})^{n-1}\frac{1}{n} + \frac{n(n-1)}{n^2}(1-\frac{t}{n})^{n-2}\frac{t^{z+2}}{z(z+1)(z+2)} + \cdots\right]_0^n$$

Note that all terms are 0 except for the final one evaluated at n.

$$= fracn!n \frac{n^{z+n}}{z(z+1)(z+2)\cdots(z+n)} = \frac{n! n^z}{z(z+1)(z+2)\cdots(z)}$$

This looks like the gamma function! We just need to add the limit.

$$\boxed{\therefore \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt; \ \Re(z) > 0}$$

This gives us access to a number of integrals.

1.6 Integrating with the Gamma Function

Consider this example:

$$\int_0^\infty e^{-2x^3}dx$$

Setting $t = 2x^3$ and $x = (\frac{t}{2})^{\frac{1}{3}}$.

$$= \int_0^\infty \frac{1}{2^{\frac{1}{3}}}\frac{1}{3}t^{\frac{-2}{3}}e^{-t}dt$$

$$= \frac{1}{2^{\frac{1}{3}}} \cdot \frac{1}{3}\Gamma(\frac{1}{3}) = \frac{\Gamma\left(\frac{4}{3}\right)}{\sqrt[3]{2}}$$

1.7 Generating Functions

1.7.1 $\Gamma(1+\epsilon)$

Consider $\int_0^\infty \ln x \cdot e^{-x}dx$. First, let's consider $\int_0^\infty x^\epsilon e^{-x}dx$. We know this $= \Gamma(1+\epsilon)$. So one option is differentiating the Gamma function. However, we can leverage series expansions instead.

$$\int_0^\infty e^{-x}x^\epsilon dx = \int_0^\infty e^{-x}e^{\epsilon \ln x}dx = \int_0^\infty e^{-x}\sum_{k=0}^\infty \frac{(\epsilon \ln x)^k}{k!}dx$$

$$= \sum_{k=0}^\infty \frac{\epsilon^k}{k!}\int_0^\infty e^{-x}\ln^k x dx = \Gamma(1+\epsilon)$$

We know that,

$$\Gamma(1+\epsilon) = \exp(\ln\Gamma(1+\epsilon)) = \exp\left(-\gamma\epsilon + \sum_{k=2}^\infty \frac{(-1)^k\zeta(k)}{k}\epsilon^k\right)$$

So,

$$\Gamma(1+\epsilon) = 1 + \left[-\gamma\epsilon + \sum_{k=2}^\infty \frac{(-1)^k\zeta(k)}{k}\epsilon^k\right] + \frac{1}{2!}\left[-\gamma\epsilon + \sum_{k=2}^\infty \frac{(-1)^k\zeta(k)}{k}\epsilon^k\right]^2 + \cdots$$

$$= 1 - \gamma\epsilon + (\frac{\zeta(2)-\gamma^2}{2})\epsilon^2 + \cdots$$

So, we need to find the coefficient of ϵ^1 for the original integral.

$$k = 0 : \int_0^\infty e^{-x}dx = 1$$

$$k = 1 : \frac{\epsilon^1}{1!}\int_0^\infty \ln x \cdot e^{-x}dx = -\gamma\epsilon$$

$$k = 2 : \frac{\epsilon^2}{2!}\int_0^\infty \ln^2 x \cdot e^{-x}dx = \frac{\zeta(2)+\gamma^2}{2}\epsilon^2$$

So, we call $\Gamma(1+\epsilon)$ the **generating function** for all these integrals.

1.7.2 Another Example

Consider $\int_0^\infty x^2 \ln x \cdot e^{-3x} dx$. Instead, let's consider $\int_0^\infty x^{2+\epsilon} e^{-3x} dx$. One method is to use a u-substitution for $u = 3x$,

$$\int_0^\infty x^{2+\epsilon} e^{-3x} dx = (\frac{1}{3})^{3+\epsilon} \int_0^\infty u^{2+\epsilon} e^{-u} du = \frac{1}{27} \cdot e^{-\epsilon \ln 3} \cdot \Gamma(3+\epsilon)$$

Interpreting this as a Maclaurin series like before,

$$\int_0^\infty x^{2+\epsilon} e^{-3x} dx = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \int_0^\infty x^2 \ln^k x \cdot e^{-3x} dx$$

Note we don't have an expansion for $\Gamma(3+\epsilon)$ but we can use the recurrence relation. Then, we can expand as a series as before.

$$\frac{1}{27} e^{-\epsilon \ln 3} \Gamma(3+\epsilon) = \frac{1}{27} e^{-\epsilon \ln 3} (2+\epsilon)(1+\epsilon)\Gamma(1+\epsilon)$$

$$= \frac{2}{27}(1+\frac{\epsilon}{2})(1+\epsilon) \exp\left(-(\gamma + \ln 3)\epsilon + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} \epsilon^k\right)$$

Instead of doing all the work again we can use the result from before and replace γ for $\gamma + \ln 3$.

$$= \frac{2}{27}\left(1 + \frac{3}{2}\epsilon + \frac{\epsilon^2}{2}\right)\left[1 - (\gamma + \ln 3)\epsilon + \left(\frac{\zeta(2) - (\gamma + \ln 3)^2}{2}\right)\epsilon^2\right]$$

$$= \frac{2}{27}\left[1 + \left(\frac{3}{2} - \gamma - \ln 3\right)\epsilon + \cdots\right]$$

So, our original integral is when $k = 1$ and equals the coefficient of ϵ.

$$\boxed{\int_0^\infty x^2 \ln x \cdot e^{-3x} dx = \frac{2}{27}\left(\frac{3}{2} - \gamma - \ln 3\right)}$$