Complex Analysis

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Chapter 1

Preface

1.1 Background

This course was taught by Dr. Jonathan Osborne in the spring of 2021. This class was taken as a sophomore at Thomas Jefferson High School for Science and Technology.

Chapter 2

The Basics of Complex Numbers

2.1 Complex Numbers Review

2.1.1 Definition

Complex numbers are of the form $a + bi$, $a, b \in \mathbb{R}$ and $i^2 = -1$.

2.1.2 Complex Conjugate

Definition: $\overline{a + bi} = a - bi$ If $z = a + bi$, then $z \cdot \overline{z} = a^2 + b^2 > 0$ unless $z = 0$.

2.1.3 Multiplication

The multiplication of 2 complex numbers is a complex number. Multiplication proceeds via basic algebraic distribution.

2.1.4 Division

Division of complex numbers is defined as long as the denominator is not 0. To do division operations, multiple the top and bottom by the complex conjugate of the denominator and simplify.

2.1.5 Modulus

Modulus of $z = a + ib$ is given by

$$|z| = \sqrt{a^2 + b^2}$$

$$|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = |z_1|^2 |z_2|^2$$

$$|z_1 z_2| = |z_1||z_2|$$

2.2 Geometry of the Complex Plane

2.2.1 The Plane

The complex plane is just like the xy-plane except it has a real axis "x" and a complex axis "y".

2.2.2 Distances

$|z|$ gives the distance from z to the origin. $|z_2 - z_1|$ gives the distance from z_2 to z_1 in the complex plane.

2.2.3 Polar Representation

$r = |z|$ and $\theta = \arg(z)$ "the argument of z". We use the "normal" version. $(1, \pi)$ or $(1, 3\pi)$ instead of $(-1, -\pi)$

Principle Argument:

$$-\pi \leq Arg(z) \leq \pi$$

rcis form:

$a = rcos\theta, b = rsin\theta, z = r(cos\theta + isin\theta)$

Conjugates

If $\theta = Arg(z)$, $\overline{rcis\theta} = rcis(-\theta)$

Multiplication

Suppose $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$,

$$z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$$

$$arg(z_1 z_2) = arg(z_1) + arg(z_2)$$

Conclusion: Multiplication in the complex plane consists of a rotation and a dilation.

2.2.4 Exponential Representation + Taylor Series

$$cis\theta = e^{i\theta}$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots)$$

$$e^{i\theta} = cos\theta + isin\theta$$

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$cosz = \frac{e^{iz} + e^{-iz}}{2}$$

Cosine can be used on the entire complex plane.

2.2.5 Hyperbolic Functions

$$\cos(a+ib) = \frac{e^{i(a+ib) - e^{-i(a+ib)}}}{2} = \frac{e^{ia}e^{-b} + e^{-ia}e^{b}}{2}$$

$$= \cos a \cdot \frac{e^{b} + e^{-b}}{2} - i \sin a \cdot \frac{e^{b} - e^{-b}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\tanh z = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

2.3 Exponentiation

Converting to exponential form is a lot easier than using binomial theorem. Multiplication, division, and exponentiation is a lot easier in exponential notation.

$$(1 + i\sqrt{3})^7 = (2e^{i\frac{\pi}{3}})^7 = 2^7 \cdot e^{i\frac{7\pi}{3}}$$

2.3.1 Not Bijective

$$\forall a \in \mathbb{R} : a \neq 0,\ e^{z} = a \text{ for inifinitely many z.}$$

2.3.2 Fractional Exponentiation

Using a exponential notation, $\pm$ results arise for different arguments.

Principal Value

Convention is to use the principal argument during calculations. Remeber that the principal argument is **not always positive** but the one between $-\pi$ and π.

Example:

$$\text{For } n \in \mathbb{Z},\, i = e^{i\frac{\pi}{2}} \cdot e^{i \cdot 2\pi n} \Rightarrow i^{\frac{1}{3}} = \text{cis}\left(\frac{\pi}{6} + \frac{2\pi n}{3}\right)$$

Note that these values form an equilateral triangle when polygon. Note that $i^{\frac{1}{q}}$ for $q \in \mathbb{N}$, results in an regular $n - gon$.

Calculators: TI calculators do not always return the principal value.
Phase: $e^{i\theta}$ is called a **phase factor**.

2.3.3 Raising to the i Power

Raising something to the i power "switches" the argument and modulus.

$$i^i = (e^{i\frac{\pi}{2}+2in\pi})^i = e^{-\frac{\pi}{2}-2n\pi}$$

$$2^i = (e^{\ln 2} \cdot e^{2\pi i n})^i = e^{-2\pi n} \cdot \text{cis}(\ln 2)$$

Chapter 3

Mappings

3.1 Mapping Basics

3.1.1 Visulization of Complex Functions

3.1.2 Animation

Complex functions are usually visualized by graphing the input space and then the output space with perhaps an animation transforming a grid on the complex plane. Suppose, in a linear algebra sense, that the function operates on the entire domain, transforming it into the codomain.

Color Map

A color spectrum is used to denote different points on the complex plane. The output is a 2D color plot where the color at some point denotes the mapped value of that point through the function.

3.1.3 Conformal Map

A conformal map is a map that preserves angles and orientations.

3.1.4 Anti-conformal Map

An anti-conformal map is a map that preserves angles and but reverses orientation.

3.2 Functions and "Maps"

$$f(z) = u + iv$$

3.2.1 Tangents

Curve of constant y:

$$u_x + iv_x$$

Curve of constant x:

$$u_y + iv_y$$

3.2.2 Conformal Requirements

Confromal maps requires:

$$u_y + iv_y = i(u_x + iv_x)$$

The multiplication by i (rotates by $90°$) allows angles and orientation to be preserved.

3.2.3 Cauchy-Riemann Equations

By equating the imaginary and real parts of the conformal requirement above,

$$u_x = v_y$$

$$u_y = -v_x$$

If these equations are satisfied for some function f, then f is a conformal map.

3.2.4 Substitutions

Note the following substitutions,

$$x = \frac{z + \overline{z}}{2},\ y = \frac{z - \overline{z}}{2i}$$

These allow conversion of a function from in terms of $x, y \in \mathbb{Z}$ to $z \in \mathbb{C}$.

3.2.5 Dependence on z and $\overline{z}$

For $f = u + iv$,

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} = (u_x + iv_x)\frac{1}{2} + (u_y + iv_y)\frac{1}{2i} = \frac{u_x + v_y}{2} + i\frac{v_x - u_y}{2}$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \overline{z}} = (u_x + iv_x)\frac{1}{2} + (u_y + iv_y)\frac{-1}{2i} = \frac{u_x - v_y}{2} + i\frac{v_x + u_y}{2}$$

$\therefore$ if Cauchy-Riemann is satisfied, f is only in terms of z as $\frac{\partial f}{\partial \overline{z}} = 0$.

3.3 Stereographic Projection

3.3.1 1D-2D Case

Consider mapping the real axis to a unit circle. For each point on the real axis, draw a line segment to the top of the circle i and map it to the point of intersection of that extended linesegment with the circle. Note that this works from the real axis to the circle, but this does not work from the circle to the real axis. The circle is known as **compact** (topology) while the real axis is not. But, notice that every point except the top of the circle is mapped to. Thus, by adding the **point of infinity** at i which allows for a 2-way mapping.

3.3.2 2D-3D Case

In the same way, the entire complex plane can be mapped to a unit where the point at (1, 0, 0) is the point at infinity. Since by adding one point, the plane can be compactified, this is known as **one point compactification**. Note that $i\infty$, $(1-\sqrt{3})\infty$, etc. all map to the same point of infinity. Just how the origin has no argument, the point at ∞ has no argument. This sphere is known as the **Reimann Sphere**.

3.4 Inversion

3.4.1 Function

The inversion function is defined as follows:

$$f(z) = \frac{1}{z}$$

$$f(z) = \frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

Curves of constant y has a tangent "vector":

$$\frac{\partial f}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

Curves of constant x has tangent "vector":

$$\frac{\partial f}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} + i\frac{y^2-x^2}{(x^2+y2)^2}$$

Notice $i\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}$.

3.4.2 Mappings

Inversion turns "circle/lines" into "circle/lines". Circles remain finite and don't go to infinity while lines all do. Under inversion, $0^{\pm} \to \pm\infty$ and $\pm\infty \to 0^{\pm}$. Inversion maps points with arbitrarily large modulus' to points with arbitrarily small modulus' (near origin).

Circles

Circle's are determined by any 3 non-collinear points (one-to-one). Circles that pass through the origin are mapped to lines not passing through the origin.

3.4.3 Lines

If a line doesn't pass through the origin, under inversion, the line is mapped to a circle passing through the origin and the other 2 mapped points defining the line.

3.5 Mobius Transformation

A mobius transformation is a function of the form

$$f(z) = \frac{az+b}{cz+d} \mid ad - bc \neq 0$$

This is the most general transformation that maps the whole Riemann sphere (including the point at infinity) to itself in a confromal and one-to-one manner.

3.5.1 Inversion and Identity

Note that the inversion mapping is simply when $a = 0$, $b = 1$, $c = 1$, $d = 0$. And the identity tranformation is when $a = z$, $b = 0$, $c = 0$, $d = 1$

3.5.2 $c = 0$

If $c = 0$ this is just a dilation and a translation. In this case, circles and lines are kept separate and "do not mix".

3.5.3 Composition

Begin with:

$$z$$

Rotate/dilate by c:

$$cz$$

Translate by d:

$$cz + d$$

Invert:

$$\frac{1}{cz+d}$$

Rotate/dilate by $\frac{bc-ad}{c}$:

$$\frac{bc-ad}{c(cz+d)}$$

Translate by $\frac{a}{c}$:

$$\frac{bc - ad}{c(az + d)} + \frac{a}{c}$$

Simplify:

$$\frac{az + b}{cz + d}$$

This shows that any mobius transformation is simply a composition of conformal, one-to-one, circle/line preserving functions. So, a mobius transformation must be one-to-one and conformal while preserving circle/line.

3.5.4 Finding a transformation

Mobius transformations are defined by 3 unique maps. Find a Mobious transformation that maps 2 to 0, i to ∞, and $1 + i$ to 1.

$$\frac{z - 2}{z - i} \cdot \frac{1 + i - i}{1 + i - 2} \cdot 1 = -\frac{1 + i}{2} \frac{z - 2}{z - i}$$

Does the line $2x + y = 5$ map to a circle or a line? =¿ Maps to a circle. What about the circle $x^2 + y^2 + 2y = 3$? =¿ Maps to a line.

3.5.5 Another Example

Find a mobius transformation that maps the exterior of the circle $|z| = 1$ to the region above the line $x + y = 1$. We can start by mapping the circular boundary to the line. Picking 3 points (1, -1, i) on the circle and determining orientation (clock-wise), we can choose the points that the 3 points map to. 1 maps to i, -1 maps to 1, and i maps to ∞. Since i goes to ∞, the transformation is of the form,

$$\frac{az + b}{z - i}$$

Pluggin in the two mappings we get,

$$\frac{a + b}{1 - i} = i$$

$$\frac{-a + b}{-1 - i} = 1$$

Solving the equations,

$$b = 0,\ a = 1 + i$$

$$\therefore f(z) = \frac{(1 + i)z}{z - i}$$

This checks out using the origin as a test point, so this is true. Note that there are an infinite number of mobius transformations that do this mapping.

Chapter 4

Complex Arithmetic and Elementary Functions

4.1 Utilizing Trig Functions

4.1.1 Rewriting with Trig Functions

$$\text{Simplify } \frac{e^{iz} - e^{3iz}}{e^{2iz}}$$

Factoring out the exponent average,

$$= \frac{e^{2iz}(e^{-iz} - e^{iz})}{e^{2iz}} = -(e^{iz} - e^{-iz}) = -2i\sin(z)$$

$$\text{Simplify } \frac{e^{3iz} + e^{11iz}}{e^{2iz} - e^{5iz}}$$

$$\frac{e^{7iz}(e^{-4iz} + e^{4iz})}{e^{\frac{7iz}{2}}(e^{-\frac{3iz}{2}} - e^{\frac{3iz}{2}})} = e^{\frac{7iz}{2}} \frac{\cos(4z)}{\sin(\frac{3z}{2})}$$

This allows easy isolation of the real and imaginary parts of the function.

4.1.2 A Cool Function

Find a closed form for the sum $\sum_{k=0}^{N} \sin(k\theta)$ for real values of θ. Using geometric series,

$$\sum_{k=0}^{N} e^{ik\theta} = \frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}}$$

$$= \frac{e^{i\frac{N+1}{2}\theta}}{e^{i\frac{\theta}{2}}} \cdot \frac{e^{-i\frac{N+1}{2}\theta} - e^{i\frac{N+1}{2}\theta}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} = e^{i\frac{N}{2}\theta} \cdot \frac{\sin(\frac{N+1}{2}\theta)}{\sin(\frac{\theta}{2})}$$

So for the original series,

$$\sum_{k=0}^{N}\sin(k\theta) = \Im(\sum_{k=0}^{N}e^{ik\theta})\frac{\sin(\frac{N\theta}{2})\cdot\sin(\frac{(N+1)\theta}{2})}{\sin(\frac{\theta}{2})}$$

4.2 Complex Powers

Find the principal value of $(1-i\sqrt{3})^{2+i}$.

$$= (2e^{-i\frac{\pi}{3}})^{2+i} = (e^{\ln(2)-i\frac{\pi}{3}})^{2+i} = e^{2\ln(2)-i\frac{2\pi}{3}+i\ln(2)+\frac{\pi}{3}} = e^{2\ln(2)+\frac{\pi}{3}}e^{i(\ln(2)-\frac{2\pi}{3})}$$

$$= 3e^{\frac{\pi}{3}}\operatorname{cis}(\ln(2)-\frac{2\pi}{3})$$

Note that taking complex powers mixes the modulus and the argument.

Branch Cuts and Color Maps

Branch cuts are evident in color maps for functions that show a discontinuity in argument. With some fractional powers, this exists. For pure complex powers, the discontinuity is shown in the modulus instead of the argument (concentric circles of color).

4.3 Logarithms

Find $\text{Log}(-\sqrt{3}-i)$.

$$= \text{Log}(2e^{-i\frac{5\pi}{6}}) = \ln(2) - i\frac{5\pi}{6}$$

4.3.1 Standard Definition

$$\text{Log}z = \ln|z| + i\text{Arg}(z)$$

4.3.2 Notation

Note that the capital logarithm **Log**(), is the **principal complex logarithm**. The lowercase natural logarithm ln is the **real logarithm**. The lowercase logarithm **log**() is an **arbitrary complex logarithm** that does not use the principal argument. Note that the base of all of these logarithms is e in mathematics.

4.3.3 Conundrums

$$\text{Log}(z^2) \stackrel{?}{=} 2\text{Log}(z)$$

$$\ln|z|^2 + i\text{Arg}(z^2) = 2\ln|z| + 2i\text{Arg}(z)$$

Note that $\text{Arg}(z^2)$ *only* equals $2\text{Arg}(z)$ when $-\pi < 2\text{Arg}(z) \leq \pi$. $\therefore$ the original statement is only true when $-\pi < 2\text{Arg}(z) \leq \pi$. In general, if the sides are not equal, they will differ by a multiple of $2\pi i$.

Chapter 5

Regions and Branch Cuts

5.1 Terminology

5.1.1 Neighborhood

A **neighborhood** of a point is a disk of some nonzero radius centered at that point.

5.1.2 Open

A region is said to be **open** if it doesn't contain its boundary. Every point in the region has a neighborhood also contained in the region.

5.1.3 Closed

Closed regions are not a major topic in this class as it focuses majorly on topology.

5.1.4 Connected

A region is said to be **connected** if you can get from any one point in the region to another point in the region without leaving the region. There is a curve contained in the region from initial to final.

5.1.5 Domain

A region is said to be a **domain** if it is open and connected.

5.1.6 Simply Connected

A connected region is said to be **simply connected** if its boundary is connected (only works for complex analysis - not 3D). In a more general topological sense, every closed loop in a simply connected region can be deformed to a point in that region without leaving the region. (no holes).

5.2 Modifying Regions and Branch Cuts

Consider $|z| > 0$. This by itself is a non-simply connected domain. But, suppose points along a ray from the origin to the point at infinity is deleted from the domain. This simply connects the domain. This cut is called a branch cut.

5.3 Branch Points

Consider $f(z) = \sqrt{z^2 - 1}$. Notice that the negative real axis is a branch cut. However, suppose we manipulated the function $f(z) = \sqrt{z^2 - 1} = i\sqrt{1 - z^2}$. This now has 2 branch cuts on the real axis.

5.3.1 Main Idea

Algebraic manipulation can change branch cuts but branch points always stay the same. Branch points are where endpoints of the branch cuts. Not only that, algebraic manipulation can change the function itself!

5.3.2 Technical Definition

Can't be defined properly

Chapter 6

Complex Limits and Derivatives

6.1 Limits

6.1.1 Definition

The function $f(z)$ is said to have the limit F as z approaches z_0 if given any $\epsilon > 0 \; \exists \; \delta \mid |f(z) - F| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

6.1.2 Punctured Disk

A punctured disk is a disk with an open boundary except the center. The $0 < |z - z_0| < \delta$ gives a punctured disk. $|f(z) - F| < \epsilon$ gives another disk. The limit definition essentially states that the punctured disk can be mapped inside the other disk.

6.1.3 Triangle Inequality

For any complex numbers z, w.

$$||z| - |w|| \leq |z + w| \leq |z| + |w|$$

Drawing the complex numbers, we see that this enforces that the numbers form a triangle. This is basically the complex analog of the simple triangle inequality from geometry.

6.2 Limit Laws

6.2.1 The Sum Law

Definition

If $\lim_{z\to z_0} f(z) = F$ and $\lim_{z\to z_0} g(z) = G$, then $\lim_{z\to z_0} [f(z) + g(z)] = F + G$. Given $\epsilon > 0$, we need to find a δ such that $|f(z) + g(z) - F - G| < \epsilon$ whenever $0 < |z - z_0| < \delta$. Note that we can assume the standalone limits exist.

Proof

Given any $\epsilon > 0$, we can certainly find δ_f: $|f(z)-F| < \frac{\epsilon}{2}$ whenever $0 < |z-z_0| < \delta_f$, and δ_g: $|g(z) - G| < \frac{\epsilon}{2}$ whenever $0 < |z - z_0| < \delta_g$. Now, for $0 < |z - z_0| < \min(\delta_f, \delta_g)$, we have $|f(z)+g(z)-F-G| \leq |f(z)-F|+ \leq |g(z)-G| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$. Therefore the choice $\min(\delta_f, \delta_g)$ for δ satisfies our requirements and proves the theorem.

6.2.2 The Product Law

Definition

If $\lim_{z\to z_0} f(z) = F$ and $\lim_{z\to z_0} g(z) = G$, then $\lim_{z\to z_0} [f(z)g(z)] = FG$. Given $\epsilon > 0$, we need to find a δ such that $|f(z)g(z) - FG| < \epsilon$ whenever $0 < |z - z_0| < \delta$. Note that we can assume the standalone limits exist.

Proof

$$\begin{aligned}|f(z)g(z) - FG| &= |(f(z) - F + F)g(z) - FG| = |(f(z) - F)g(z) + F(g(z) - G)| \\ &= |(f(z) - F)(g(z) - G) + (f(z) - F)G + F(g(z) - G)|\end{aligned}$$

Given $\epsilon > 0$, we can find δ_f for which $|f(z)-F| < min(\frac{\epsilon}{3}, 1)$ for $0 < |z-z_0| < \delta_f$, and δ_g for which $|g(z) - G| < min(\frac{\epsilon}{3}, 1)$ for $0 < |z - z_0| < \delta_f$. Note that there are a number of cases we need to consider.
Case 1: Suppose $F = G = 0$. Given $\epsilon > 0$, we can find δ_f such that $|f(z)| < \epsilon$ whenever $0 < |z-z_0| < \delta_f$ and δ_g such that $|g(z)| < 1$ whenever $0 < |z-z_0| < \delta_g$

Case 2: Suppose $F = 0, G \neq 0$. Given $\epsilon > 0$, we can find δ_f such that $|f(z)| < \min(\frac{\epsilon}{2|G|}, 1)$ whenever $0 < |z - z_0| < \delta_f$ and δ_g such that $|g(z) - G| < \min(\frac{\epsilon}{2}, 1)$ whenever $0 < |z - z_0| < \delta_g$.

Case 3: Suppose $FG \neq 0$. Given $\epsilon > 0$, we can find δ_f such that $|f(z) - F| < \min(\frac{\epsilon}{3|G|}, 1, \frac{\epsilon}{3})$ whenever $0 < |z - z_0| < \delta_f$ and δ_g such that $|g(z) - G| < \min(\frac{\epsilon}{3|F|}, 1, \frac{\epsilon}{3})$ whenever $0 < |z - z_0| < \delta_g$.

Finishing: Now for each of these cases, the triangle inequality guarantees that $|f(z)g(z) - FG| = |(f(z) - F)(g(z) - G) + (f(z) - F)G + F(g(z) - G)| < \epsilon$ which proves the theorem.

6.3 Derivatives

6.3.1 Definition

The function $f(z)$ is said to be differentiable at z_0 if the following exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

In this case, the limit is $f'(z_0)$. Note that the derivative of a sum = sum of the derivatives provided that the two functions are differentiable.

6.3.2 Products

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\left[f(z) - f(z_0) + f(z_0)\right] g(z) - f(z_0)g(z_0)}{z - z_0}$$
$$= \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} g(z) + f(z_0) \frac{g(z) - g(z_0)}{z - z_0}\right]$$

We know that this is equal to the sum of the limits so,

$$\boxed{\therefore \frac{d}{dz}(f(z)g(z)) = f'(z_0)g(z_0) + f(z_0)g'(z_0)}$$

6.3.3 Entire

z is **entire**, with derivative 1. All polynomials then are entire. Entire means that a function is differentiable on the whole complex plane.

6.3.4 Product Rule

As long as $z_0 \neq 0$,

$$\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \frac{z_0^n}{z_0} \cdot \frac{(\frac{z}{z_0})^n - 1}{\frac{z}{z_0} - 1}$$

Letting $w = \frac{z}{z_0}$,

$$z^{n-1} \lim_{w \to 1} \frac{w^n - 1}{w - 1} = nz^{n-1}$$

We can substitue $n = -m$ and continue to derive this for rational numbers as well.

6.3.5 Exponentials

$$\lim_{h \to 0} \frac{e^{z+h} - e^z}{h} = \lim_{h \to 0} \frac{e^z(e^h - 1)}{h} = e^z \lim_{h \to 0} \frac{e^h - 1}{h} = e^z$$

So, e^z is entire as it is defined on the whole complex plane and thus is differentiable on it as well.

6.3.6 More on Differentiability

If $f(z) = u(x, y) + iv(x, y)$, with u, v differentiable functions of x, y, then

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

For $f(z)$ to be differentiable, it has to hold the same value for an arbitrary direction of approach. Considering an approach of constant y,

$$= \lim_{z \to z_0} \frac{u(x, y_0) + iu(x, y_0) - [u(x_0, y_0) + iv(x_0, y_0)]}{x - x_0} = u_x(x_0, y_0) + iv_x(x_0, y)$$

Approaching at constant x,

$$= \lim_{z \to z_0} \frac{u(x_0, y) + iu(x_0, y) - [u(x_0, y_0) + iv(x_0, y_0)]}{i(y - y_0)} = -i(u_y + iv_y) = v_y - iu_y$$

Setting these equal to each other, we can see that if f is differentiable, then it satisfied the Cauchy-Reimann Equations and is conformal.

$$\boxed{\therefore \text{conformal} \Leftrightarrow \text{differentiable}}$$

6.3.7 An Important Statement

$f(z)$ is differentiable at z_0 iff $f(z) = f(z_0) + f'(z_0)(z - z_0) + \xi(z, z_0)$, where given any $\epsilon > 0 \; \exists \; \delta \mid |\xi(z, z_0)| < \epsilon |z - z_0|$ whenever $0 < |z - z_0| < \delta$. Note that this means the error term is "faster" than linear. If $f'(z_0) = 0$, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) \left[1 + \frac{\xi(z, z_0)}{f'(z, z_0)(z - z_0)} \right]$$

This shows that when z is close enough to z_0 this function is nothing but a translation $f(z_0)$ and a rotation. The term with the ξ goes to zero. So, locally, this function is conformal. Note that Cauchy Reimann equations are sufficient for differentiability. But for conformality, both a non-zero derivative and cauchy-reimann are needed.

6.3.8 Determining Differentiability

Determine where $f(z) = f(x + yi) = x^3 + y^2 + 3ix^2y$ is differentiable. For this to satisfy the cauchy reiman equations, $u_y = -v_x$. Solving the system gives $y = 0$ or $x = -\frac{1}{3}$.

Chapter 7

Laplace's Equation

7.1 Analytic and Harmonic

The function $f(z)$ is said to be **analytic** at z_0 if it is differentiable on an open set containing z_0. So, $f(z)$ has to satisfy the Cauchy-Reimann equations in a neighborhood of z_0. Taking derivatives and using Clauriat's Theorem,

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0$$

This partial differential equation is **Laplace's Equation**. Any function that satisfies Laplace's equation is **harmonic**. With a similar trick, we can find that the above equation holds for the imaginary part as well. The real and imaginary parts of a analytic function are harmonic. However, this is **not if and only if**. This only goes one way. If real and imaginary parts of a function are harmonic, it is not guaranteed that the function is analytic. But, this can be modified to be true. If the function $u(x, y)$ is harmonic on a *simply connected* domain D, then it is the real part of a function that is analytic on D. The imagniary part of this function is called the **harmonic conjugate** that is unique *up to a constant*.

Chapter 8

The Open Mapping Theorem

8.1 The Open Mapping Theorem

8.1.1 Statement

A **nonconstant** function analytic on domain D maps D to an open set.

8.1.2 Results

If $f(z)$ is analytic on domain D and $|f(z)|$ or $Arg(f(z))$ is constant, $f(z)$ must be constant.

8.2 The Maximum Modulus Principle

8.2.1 Statement

Every function $f(z)$ **analytic** on domain D attains the maximum modulus on every closed connection region $R \in D$ on the boundary of R.

8.2.2 Results

Suppose $f(z) = g(z)$ are both analytic on the boundary $|z| < 2$, then $f(z) = g(z) \, \forall \, z \mid |z| < 2$.

8.3 Fundamental Theorem of Algebra

8.3.1 Statement

Every nonconstant polynomial $p(z)$ has at least one zero in the complex plane.

8.3.2 Proof

Assume that polynomial $p(z)$ is never zero in the complex plane. Since all polynomials are entire an $p(z) \neq 0$, $\frac{1}{p(z)}$ is entire. So, $\frac{1}{p(z)}$ achieves its maximus modulus on every disk $|z| \leq R$. On $|z| = R$, as R grows without bound, $|\frac{1}{p(Re^{it})}| \to 0$. This is a contradiction. So, $p(z)$ must have at least one zero.

8.3.3 Generalization

Given $f(z)$ is entire, if given any $N > 0$ a value of R can be found for which $|f(Re^{it})| > N \, \forall \, t \in \mathbb{R}$. We can conclude that $f(z)$ has at least one zero in the complex plane.

8.4 A Strong Result

8.4.1 Definition

Suppose nonconstant $u(x, y)$ is harmonic on a simply connected domain D. Then, we can find a harmonic conjugate $v(x, y)$ for which $f(z) = u + iv$ is analytic on D. This can be used in conjuction with the Maximum Modulus Principle to prove that $u(x, y)$ can achieve neither a local maximum nor local minimum on D.

8.4.2 Proof

Consider $|e^{f(z)}| = |e^{u+iv}| = e^u$. Then, consider $|e^{-f(z)}| = e^{-u}$. Using MMP, this can now be proved.

Chapter 9

Integration

9.1 Heat Equation

The heat flow is proportional to the cross sectional area and is inversely proportional to the length. The thermal conductivity is this constant of proportionality.

$$\begin{aligned}
\text{heat flow } &= \kappa \frac{A(T_h - T_c)}{L} = \kappa A \frac{\partial T}{\partial x} \\
\text{net heatflow into region } &= \kappa A \left[\frac{T_2 - T}{\Delta x} - \frac{T - T_1}{\Delta x} \right] \Rightarrow \kappa A \frac{\partial^2 T}{\partial x} L \\
&= \kappa \frac{\partial^2 T}{\partial x^2} V
\end{aligned}$$

From chemistry,

$$Q = m c_p \Delta T$$

So,

$$\kappa \frac{\partial^2 T}{\partial x} V = m c_p \frac{\partial T}{\partial t}$$

Rearranging,

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

This gives us the heat equation:

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c_p} \nabla^2 T}$$

In steady-state or equilibrium temperature distributions, temperature is time independent so $\nabla^2 T = 0$. Thus, steady-state temperature distributions are harmonic functions.

9.2 Contour Integrals

Given a function f and a contour C, define the contour integral $\int_C f(z)dz$ by parameterizing $C : z(0) =$ starting point of C, $z(1) =$ ending point of C. Then,

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(z\left(\frac{k}{n}\right)\right)\left(z\left(\frac{k}{n}\right) - z\left(\frac{k-1}{n}\right)\right)$$

9.3 Integral Bounds

If $|f(z)| < M \,\forall\, z \in C$, then

$$|\int_C f(z)dz| \leq \lim_{n\to\infty} \sum_{k=1}^{n} \left|f\left(z\left(\frac{k}{n}\right)\right)\right|\left|z\left(\frac{k}{n}\right) - z\left(\frac{k-1}{n}\right)\right|$$

$$< M \cdot \lim_{n\to\infty} \sum_{k=1}^{n} \left|z\left(\frac{k}{n}\right) - z\left(\frac{k-1}{n}\right)\right| \leq ML$$

9.4 Fundamental Theorem of Contour Integrals

9.4.1 Definition

If $F(z)$ can be found for which $F'(z) = f(z) \,\forall\, z \in \mathbb{C}$, then,

$$\int_c f(z)dz = F(z_{end}) - F(z_{start})$$

9.4.2 Proof

Parameterize $C : z(t), t \in [0,1]$. Let $z_k = z\left(\frac{k}{n}\right)$

$$F(z(1)) - F(z(0)) = F(z_n) - F(z_0) = F(z_n) - F(z_1) + F(z_1) - F_{z_0}$$

$$= \sum_{k=1}^{n}(F_{z_k} - F_{z_{k-1}}) = \sum_{k=1}^{n}(F(z_{k-1}) + F'(z_{k-1})(z_k - z_{k-1}) + \xi(z_k, z_{k-1}) - F(z_{k-1}))$$

$$= \sum_{k=1}^{n} F'(z_{k-1})(z_k - z_{k-1}) + \sum_{k=1}^{n} \xi(z_k, z_{k-1})$$

As $n \to \infty$, we get the Fundamental Theorem of Contour Integerals.

9.4.3 Corollary

If $F'(z) = f(z) \,\forall\, z \in D$ where D is a domain, then $\int_c f(z)dz$ is independent of path for every controur $C \subset D$.

9.4.4 Branch Cuts

$$\int_C \frac{dz}{z^2+1} = \int_C \left(\frac{1}{z-i} - \frac{1}{z+i}\right)\frac{dz}{2i}$$

If C avoids the following branch cuts, this is valid:

$$= \frac{1}{2i}\left(\log(z-i) - \log(z+i)\right)_{\text{start}}^{\text{end}}$$

If C crosses the branch cuts, this no longer works. We just need to manipulate the branch cuts so that the function does not cross it. So if C crosses the top branch cut in a downwards facing parabola, we can do the following to move the top branch cut.

$$= \frac{1}{2i}\left(\log(-i(z-i)) + \frac{i\pi}{2} - \log(z+i)\right)_{\text{start}}^{\text{end}}$$

9.5 Path Independence

If C_1, C_2 are on C,

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = \oint_C f(z)dz = \oint_c ((udx - vdy) + i(vdx + udy))$$

Using Green's theorem (not rigorous),

$$= \iint_D ((-v_x - u_y + i(u_x - v_y))dA$$

This $= 0$ if $f(z)$ satisfies Cauchy reimann. So, whenever $f(z)$ is analytic throughout a region that allows one contour to be deformed to the other.

9.6 Contour Deformation

If C_1 and C_2 both begin at z_1 and end at z_2 and C_1 can be continuously deformed into C_2 without leaving the domain D of analyticity of $f(z)$, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

9.7 Cauchy's Integral Formula

Suppose $f(z)$ is analytic on a simply connected domain D containing the closed loop C with z_0 in its interior.

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz$$

9.8 Cauchy's Integral Formula for Derivatives

For analytic f at z_0.

$$f(z) - f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z}d\xi - \frac{1}{2\pi i}\oint \frac{f(\xi)}{\xi - z_0}d\xi$$

$$= \frac{1}{2\pi i}\oint_C f(\xi)\frac{z - z_0}{(\xi - z)(\xi - z_0)}d\xi$$

Manipulating,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)(\xi - z_0)}d\xi$$

Taking limits as $z \to z_0$,

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^2}d\xi$$

This shows that for a function that is analytic at z_0, it's derivative also exists. This differentiation can continue yeilding the following:

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^{k+1}}d\xi$$

. Thus a function that is analytic at a given point is infinitely differentiable at that given point.

9.9 Liouville's Theorem

9.9.1 Statement

A bounded entire function is constant.

9.9.2 Proof

$$f'(z_0) = \frac{1}{2\pi i}\oint_{|z-z_0|=R} \frac{f(\xi)}{(\xi - z)^2}d\xi$$

for an arbitrary point z_0 by CIFFD. Using integral bounds, suppose

$$|f(z)| < M \,\forall\, z \in \mathbb{C}$$

then,

$$|f'(z)| < \frac{1}{2\pi}\frac{M}{R^2}2\pi R = \frac{M}{R}$$

As $R \to \infty$, this statement still holds true so $f'(z) = 0$ so $f(z)$ is constant.

9.10 Another Statement

Suppose $f(z)$ is entire and $|f(z)| < 10|z|^3\sqrt{|z|}\,\forall\, z : |z| > 30$. Then $f(z)$ is a polynomial of degree 3 at most.

9.10.1 Proof

For any point z_0,

$$f^{(4)}(z_0) = \frac{4!}{2\pi i}\oint_{|z-z_0|=R} \frac{f(\xi)}{(\xi - z_0)^5}d\xi$$

is true by CIFFD. For $R > 30 + |z_0|$,

$$|f^{(4)}(z_0)| < \frac{24}{2\pi}\cdot\frac{10R^{\frac{7}{2}}}{R^5}\cdot 2\pi R = \frac{240}{R^{\frac{1}{2}}}$$

Expanding R without bound, this must still be true to $f^{(4)}(z) = 0$.

9.10.2 General Statemnt

If a entire function is bounded by any power of z, it must be a polynomial function or constant. Any non-polynomial entire functions must grow faster than any power of z (think exponential).

9.11 Poisson's Formula

$$f(z) = \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{f(\xi)}{\xi - z}d\xi$$

Adding a term,

$$f(z) = \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{f(\xi)}{\xi - z}d\xi + \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{f(\xi)\overline{z}}{R^2 - \xi\overline{z}}d\xi$$

Note that the modulus of the singularity is $|\xi| = |\frac{R^2}{z}| = \frac{R^2}{z} > R$ for $|z| < R$. This means that the added term is 0.

$$= \frac{1}{2\pi i}\oint_{|\xi|=R} f(\xi)\left(\frac{1}{\xi - z} + \frac{\overline{z}}{R^2 - \xi\overline{z}}\right)d\xi$$

Note that when this fraction is combined, the numerator is independent of z.

$$= \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{R^2 - r^2}{(Re^{it} - z)(R^2 - Re^{it}z)}$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{R^2 - r^2}{Re^{it}}\cdot\frac{1}{(Re^{it} - z)(Re^{-it} - \overline{z})}$$

$$= \frac{R^2 - r^2}{2\pi i} \oint_{|\xi|=R} \frac{f(\xi)}{Re^{it}|Re^{it} - z|^2}$$

Completing the substitution of $\xi = Re^{it}$,

$$= \frac{R^2 - r^2}{2\pi i} \int_0^{2\pi} \frac{f(Re^{it})}{Re^{it}|Re^{it} - z|^2} \cdot iRe^{it} dt$$

Simplifying,

$$= \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2} dt$$

Taking the real part,

$$u(x, y) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(R\cos t, R\sin t)}{|Re^{it} - z|^2}$$

This shows that if we know the value of a harmonic function everywhere on circle, we can use this to find information of its values inside.

9.12 Harnack's Inequality

Manipulating Poisson's formula above,

$$u(r\cos\theta, r\sin\theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(R\cos t, R\sin t)}{R^2 - 2Rr\cos(t - \theta) + r^2} dt$$

Taking bounds,

$$u(r\cos\theta, r\sin\theta) \le \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(R\cos t, R\sin t)}{(R - r)^2} dt$$

$$= \frac{R + r}{R - r} \frac{1}{2\pi} \int_0^{2\pi} u(R\cos t, R\sin t) dt = \frac{R + r}{R - r} u(0, 0)$$

Using the same manipulation on the other side,

$$\frac{R - r}{R + r} u(0, 0) \le u(r\cos\theta, r\sin\theta)$$

This yields Harnack's Inequality,

$$\frac{R - r}{R + r} u(0, 0) \le u(r\cos\theta, r\sin\theta) \le \frac{R + r}{R - r} u(0, 0) dt$$

9.13 Reimann Mapping Theorem

Any simply connected domain whose boundary consists of more than two points can be mapped in a one to one invertible analytic way to the interior of a unit disk.

Chapter 10

Working

10.1 Working Hard

Some useful content.

Chapter 11

Working

11.1 Working Hard

Some useful content.