

Chapter 1

Linear Algebra

1.1 Basics

1.1.1 Scalars & Fields

A **scalar** is a number like 3 or $-2i$. Scalars have to elements of a **field**. Fields need an addition and multiplication operations that are both commutative and associative property. Fields also need an additive identity, an additive inverse, a multiplication operation, a multiplicative identity (1), and a multiplicative inverse. 0 is exempted from the multiplicative inverse condition.

1.1.2 Vectors & Vector Spaces

A **vector** is a mathematical object that can be added to other vectors or multiplied by a scalar. A **vector space** is a collection of vectors having two operations, addition and scalar multiplication, under which it is **closed**. Addition is **commutative** and associative. Every vector space has an **additive identity** $0: |a\rangle + 0 = |a\rangle$ ("ket" notation). Every vector $|a\rangle$ has an **additive inverse** with another vector $|-a\rangle$ such that $|-a\rangle + |a\rangle = 0$.

1.1.3 Linear Combinations

A **linear combination** of a set of vectors is a sum of scalar multiples of that set. The set of all linear combinations of a set of vectors is called the **span** of that set. The $\text{span}\{|v_k\rangle\}_{k=1}^r$ is always a vector space. The set $\{|v_k\rangle\}_{k=1}^r$ is a **spanning set** for the space. A **trivial** linear combination is where $0|v_1\rangle + 0|v_2\rangle = 0$. A **non-trivial** linear combination is $k_1|v_1\rangle + k_2|v_2\rangle = 0 : k_1, k_2 \neq 0$. A set is **linearly independent** if the only linear combination giving 0 is the trivial combination. If a set has a non-trivial linear combination giving 0, the set is **linearly dependent**. So, a **minimal** spanning set must be the same size as a linearly independent spanning set. A linearly independent spanning set is a

basis. The **dimension** of a vector space is the number of elements in a basis for that space.

1.2 Matrices

Consider matrix A . $\text{col}(A)$ is the span of the columns of A . $\text{row}(A)$ is the span of the rows of A . $\text{Nul}(A)$ is the set of all vectors $|n\rangle$ such that $A|n\rangle = 0$. The basis for the column space is given by the columns of the original matrix that have pivots. The row space stays the same after RREF, so the rows of the reduced matrix with pivots give a basis for the row space. If we give the free variables names and solve the RREF equations, we get the null space.

1.2.1 Rank Theorem

$$\text{rank}(A) + \text{nullity}(A) = \# \text{ of columns in } A$$

Thus, the dimension of the column, and row space is the same and known as the rank. The dimension of the null space is the number of free variables which is different from the rank or number of pivot variables.

1.2.2 Transpose

The **transpose** of A, A^T is the matrix obtained by exchanging the rows and columns of A . Note that under transpose, the rank does not change but the nullity does.

1.2.3 Inversion

A is **invertible** if its action can be undone. There cannot be more than one solution to $A\vec{x} = \vec{b}$ and there cannot be no solutions. This means that A has a pivot in every column and in every row which means it is a square matrix or $m = n$.

1.2.4 Determinant

The **determinant** of a square matrix is a number that can be used to characterize whether or not the matrix is invertible. A is invertible iff $\det(A) \neq 0$.

Requirements

The determinant is unchanged when a row is replaced with itself plus a multiple of another row. $\det(A) = 0$ if two rows are the same.

Properties

$\det(A)$ is separately linear in each of its rows. $\det(A)$ changes sign when two rows are exchanged. Notice that this imposes the requirements described above.

1.2.5 Definition

The determinant of an $n \times n$ matrix is the sum of all products containing exactly one element from each row and one element from each column, taken with a sign given by the permutation of the rows when the columns are taken in order from 1 to n .

$$\det(A) = (-1)^{\text{swaps}} \prod (\text{pivots})$$

Using this,

$$\det(AB) = \det(A) \det(B)$$

for any two square matrices A and B .

1.3 Basis

Consider two basis where e is the standard cartesian basis and B is some arbitrary basis. We can think of a vector with some basis as the matrix of the basis vectors times the target vector. The basis matrix takes in some base and outputs the vector in some other base. The change of basis matrix P has columns given by the new basis representation of the old basis vectors. To change from the new to the old basis, we simply multiply by P^{-1} .

1.4 Similarity Transformations

Just like how vectors depend on a basis, matrices do as well. Suppose matrix M is matrix A in the b basis instead of the e basis. Then,

$$M[\bar{x}]_B = [\bar{y}]_B \Rightarrow MP^{-1}[\bar{x}]_e = P^{-1}[\bar{y}]_e$$

Multiplying by P ,

$$PMP^{-1}[\bar{x}]_e = [\bar{y}]_e = A[\bar{x}]_e$$

So,

$$A = PMP^{-1} \text{ and } M = P^{-1}AP$$

This relationship between A and M is called a **Similarity Transformation**. Note that there are an infinite number of possible M dependent on the basis chosen.

1.4.1 Invariants

The determinant is invariant under a Similarity Transformation. The trace is also invariant.

1.5 Eigenvalues and Eigenvectors

The number λ is said to be an **eigenvalue** of square matrix A if there exists a nonzero vector $|v\rangle$ such that $A|v\rangle = \lambda|v\rangle$. In this case, $|v\rangle$ is said to be an eigenvector of A associated with eigenvalue λ .