Chapter 1

Integration

1.1 Heat Equation

The heat flow is proportional to the cross sectional area and is inversely proportional to the length. The thermal conductivity is this constant of proportionality.

$$\begin{aligned} \text{heat flow } &= \kappa \frac{A(T_h - T_c)}{L} = \kappa A \frac{\partial T}{\partial x} \\ \text{net heatflow into region } &= \kappa A \left[\frac{T_2 - T}{\Delta x} - \frac{T - T_1}{\Delta x} \right] \Rightarrow \kappa A \frac{\partial^2 T}{\partial x} L \\ &= \kappa \frac{\partial^2 T}{\partial x^2} V \end{aligned}$$

From chemistry,

$$Q = mc_p \Delta T$$

So,

$$\kappa \frac{\partial^2 T}{\partial x} V = mc_p \frac{\partial T}{\partial t}$$

Rearranging,

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

This gives us the heat equation:

$$\boxed{\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c_p} \nabla^2 T}$$

In steady-state or equilibrium temperature distributions, temperature is time independent so $\nabla^2 T = 0$. Thus, steady-state temperature distributions are harmonic functions.

1.2 Contour Integrals

Given a function f and a contour C, define the contour integral $\int_C f(z)dz$ by parameterizing $C : z(0) =$ starting point of C, $z(1) =$ ending point of C. Then,

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(z\left(\frac{k}{n}\right)\right)\left(z\left(\frac{k}{n}\right) - z\left(\frac{k-1}{n}\right)\right)$$

1.3 Integral Bounds

If $|f(z)| < M \,\forall\, z \in C$, then

$$|\int_C f(z)dz| \leq \lim_{n\to\infty} \sum_{k=1}^{n} \left|f\left(z\left(\frac{k}{n}\right)\right)\right|\left|z\left(\frac{k}{n}\right) - z\left(\frac{k-1}{n}\right)\right|$$

$$< M \cdot \lim_{n\to\infty} \sum_{k=1}^{n} \left|z\left(\frac{k}{n}\right) - z\left(\frac{k-1}{n}\right)\right| \leq ML$$

1.4 Fundamental Theorem of Contour Integrals

1.4.1 Definition

If $F(z)$ can be found for which $F'(z) = f(z) \,\forall\, z \in \mathbb{C}$, then,

$$\int_c f(z)dz = F(z_{end}) - F(z_{start})$$

1.4.2 Proof

Parameterize $C : z(t), t \in [0, 1]$. Let $z_k = z\left(\frac{k}{n}\right)$

$$F(z(1)) - F(z(0)) = F(z_n) - F(z_0) = F(z_n) - F(z_1) + F(z_1) - F_{z_0}$$

$$= \sum_{k=1}^{n}(F_{z_k} - F_{z_{k-1}}) = \sum_{k=1}^{n}(F(z_{k-1}) + F'(z_{k-1})(z_k - z_{k-1}) + \xi(z_k, z_{k-1}) - F(z_{k-1}))$$

$$= \sum_{k=1}^{n} F'(z_{k-1})(z_k - z_{k-1}) + \sum_{k=1}^{n} \xi(z_k, z_{k-1})$$

As $n \to \infty$, we get the Fundamental Theorem of Contour Integerals.

1.4.3 Corollary

If $F'(z) = f(z) \,\forall\, z \in D$ where D is a domain, then $\int_c f(z)dz$ is independent of path for every controur $C \subset D$.

1.4.4 Branch Cuts

$$\int_C \frac{dz}{z^2+1} = \int_C \left(\frac{1}{z-i} - \frac{1}{z+i}\right)\frac{dz}{2i}$$

If C avoids the following branch cuts, this is valid:

$$= \frac{1}{2i}\left(\log(z-i) - \log(z+i)\right)_{\text{start}}^{\text{end}}$$

If C crosses the branch cuts, this no longer works. We just need to manipulate the branch cuts so that the function does not cross it. So if C crosses the top branch cut in a downwards facing parabola, we can do the following to move the top branch cut.

$$= \frac{1}{2i}\left(\log(-i(z-i)) + \frac{i\pi}{2} - \log(z+i)\right)_{\text{start}}^{\text{end}}$$

1.5 Path Independence

If C_1, C_2 are on C,

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = \oint_C f(z)dz = \oint_c ((udx - vdy) + i(vdx + udy))$$

Using Green's theorem (not rigorous),

$$= \iint_D ((-v_x - u_y + i(u_x - v_y))dA$$

This $= 0$ if $f(z)$ satisfies Cauchy reimann. So, whenever $f(z)$ is analytic throughout a region that allows one contour to be deformed to the other.

1.6 Contour Deformation

If C_1 and C_2 both begin at z_1 and end at z_2 and C_1 can be continuously deformed into C_2 without leaving the domain D of analyticity of $f(z)$, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

1.7 Cauchy's Integral Formula

Suppose $f(z)$ is analytic on a simply connected domain D containing the closed loop C with z_0 in its interior.

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$

1.8 Cauchy's Integral Formula for Derivatives

For analytic f at z_0.

$$f(z) - f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z}d\xi - \frac{1}{2\pi i}\oint \frac{f(\xi)}{\xi - z_0}d\xi$$

$$= \frac{1}{2\pi i}\oint_C f(\xi)\frac{z - z_0}{(\xi - z)(\xi - z_0)}d\xi$$

Manipulating,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)(\xi - z_0)}d\xi$$

Taking limits as $z \to z_0$,

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^2}d\xi$$

This shows that for a function that is analytic at z_0, it's derivative also exists. This differentiation can continue yeilding the following:

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^{k+1}}d\xi$$

. Thus a function that is analytic at a given point is infinitely differentiable at that given point.

1.9 Liouville's Theorem

1.9.1 Statement

A bounded entire function is constant.

1.9.2 Proof

$$f'(z_0) = \frac{1}{2\pi i}\oint_{|z - z_0| = R} \frac{f(\xi)}{(\xi - z)^2}d\xi$$

for an arbitrary point z_0 by CIFFD. Using integral bounds, suppose

$$|f(z)| < M \,\forall\, z \in \mathbb{C}$$

then,

$$|f'(z)| < \frac{1}{2\pi}\frac{M}{R^2}2\pi R = \frac{M}{R}$$

As $R \to \infty$, this statement still holds true so $f'(z) = 0$ so $f(z)$ is constant.

1.10 Another Statement

Suppose $f(z)$ is entire and $|f(z)| < 10|z|^3\sqrt{|z|}\,\forall\, z : |z| > 30$. Then $f(z)$ is a polynomial of degree 3 at most.

1.10.1 Proof

For any point z_0,

$$f^{(4)}(z_0) = \frac{4!}{2\pi i}\oint_{|z-z_0|=R}\frac{f(\xi)}{(\xi - z_0)^5}d\xi$$

is true by CIFFD. For $R > 30 + |z_0|$,

$$|f^{(4)}(z_0)| < \frac{24}{2\pi}\cdot\frac{10R^{\frac{7}{2}}}{R^5}\cdot 2\pi R = \frac{240}{R^{\frac{1}{2}}}$$

Expanding R without bound, this must still be true to $f^{(4)}(z) = 0$.

1.10.2 General Statemnt

If a entire function is bounded by any power of z, it must be a polynomial function or constant. Any non-polynomial entire functions must grow faster than any power of z (think exponential).

1.11 Poisson's Formula

$$f(z) = \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{f(\xi)}{\xi - z}d\xi$$

Adding a term,

$$f(z) = \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{f(\xi)}{\xi - z}d\xi + \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{f(\xi)\overline{z}}{R^2 - \xi\overline{z}}d\xi$$

Note that the modulus of the singularity is $|\xi| = |\frac{R^2}{z}| = \frac{R^2}{z} > R$ for $|z| < R$. This means that the added term is 0.

$$= \frac{1}{2\pi i}\oint_{|\xi|=R} f(\xi)\left(\frac{1}{\xi - z} + \frac{\overline{z}}{R^2 - \xi\overline{z}}\right)d\xi$$

Note that when this fraction is combined, the numerator is independent of z.

$$= \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{R^2 - r^2}{(Re^{it} - z)(R^2 - Re^{it}z)}$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=R}\frac{R^2 - r^2}{Re^{it}}\cdot\frac{1}{(Re^{it} - z)(Re^{-it} - \overline{z})}$$

$$= \frac{R^2 - r^2}{2\pi i} \oint_{|\xi|=R} \frac{f(\xi)}{Re^{it}|Re^{it} - z|^2}$$

Completing the substitution of $\xi = Re^{it}$,

$$= \frac{R^2 - r^2}{2\pi i} \int_0^{2\pi} \frac{f(Re^{it})}{Re^{it}|Re^{it} - z|^2} \cdot iRe^{it} dt$$

Simplifying,

$$= \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})}{|Re^{it} - z|^2} dt$$

Taking the real part,

$$u(x, y) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(R\cos t, R\sin t)}{|Re^{it} - z|^2}$$

This shows that if we know the value of a harmonic function everywhere on circle, we can use this to find information of its values inside.

1.12 Harnack's Inequality

Manipulating Poisson's formula above,

$$u(r\cos\theta, r\sin\theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(R\cos t, R\sin t)}{R^2 - 2Rr\cos(t - \theta) + r^2} dt$$

Taking bounds,

$$u(r\cos\theta, r\sin\theta) \leq \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(R\cos t, R\sin t)}{(R - r)^2} dt$$

$$= \frac{R + r}{R - r} \frac{1}{2\pi} \int_0^{2\pi} u(R\cos t, R\sin t) dt = \frac{R + r}{R - r} u(0, 0)$$

Using the same manipulation on the other side,

$$\frac{R - r}{R + r} u(0, 0) \leq u(r\cos\theta, r\sin\theta)$$

This yields Harnack's Inequality,

$$\frac{R - r}{R + r} u(0, 0) \leq u(r\cos\theta, r\sin\theta) \leq \frac{R + r}{R - r} u(0, 0) dt$$

1.13 Reimann Mapping Theorem

Any simply connected domain whose boundary consists of more than two points can be mapped in a one to one invertible analytic way to the interior of a unit disk.