Chapter 1

Convergence

1.1 Cauchy's Convergence Criterion

The sequence $\{a_k\}_{k=0}^{\infty}$ of complex numbers converges iff given any $\epsilon > 0 \,\exists\, N \mid$ $|a_m - a_n| < \epsilon$ whenever $m, n > N$. If $\{a_k\}_{k=0}^{\infty}$ converges to L, then $|a_m - a_n| \leq$ $|a_m - L| + |a_n - L|$

1.2 Limit Superior

$\limsup |a_k| = L$ means given any $\epsilon > 0$, $|a_k| > L - \epsilon$ for infinitely many k, but $|a_k| < L + \epsilon$ for only finitely many k.

1.3 Laurent Series

The Taylor series is guaranteed to converge inside the disk of convergence and is guaranteed to converge outside the disk of convergence. Convergence on the bondary of the disk is unpredictable. Laurent series are similar to taylor series but negative power terms are allowed.

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z - z_0)^k : c_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}$$

Laurent series converge on an annulus: $r < |z - z_0| < R$. There can be multiple annuli of convergence with the same center. There is a Laurent series that converges on the disk of convergence of the Taylor series but without the center (punctured). The negative power terms limit the small circle and the positive power terms limit the outer radius of convergence. Note that by FTCI,

$$\oint_C f(z)dz = 2\pi i \cdot c_{-1}$$

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This is because c_{-1} is the coefficient of $\frac{1}{z}$. If a Laurent series converging in a punctured disk about z_0 has an infinite number of negative power terms, 0 is an essential singularity of the function.

1.4   Types of Singularities

z_0 is an isolated singularity of $f(z)$ if f is not analytic at z_0 but $\exists R : f(z)$ is analytic on $0 < |z - z_0| < R$. It is a pointat which the function is not analytic (singular), but the funtion is analytic on a punctured disk centered at that point. There are 3 types. Type I is if $|f(z)|$ is bounded on $0 < |z - z_0| < R$. This is a removable singularity. Type II is if $|f(z)|$ is not bounded but $|\frac{1}{f(z)}|$ is bounded. Type II singularities can be removed by multiplying by $(z - z_0)^m$ for some m. These are known as poles of order m. If $m = 1$, it is called a simple pole. After multiplying, $f(z_0) \neq 0$ otherwise the $\frac{1}{|f(z)|}$ does not exist at z_0. Type III is neither type I nor II and is known as an essential singularity.

1.5   Weirstrass Theorem

This theorem states that if a function has an essential singularity, it gets arbitrarily close to every complex number infintely many times on every punctured disk centered at the singularity.

1.6   Picard's Little Theorem

If f is entire and non-constant, f takes on all values on the complex plane with the exception of at most 1 point.

1.7   Picard's Great Theorem

If f is analytic and has an essential singularity at some point w. Then, on any punctured disk centered at w, $f(z)$ takes on all possible complex values with at most a single exception infinitely many times. This exceptional value is known as Picard's Exceptional Value.

1.8   Equality and Convergence

If there is a sequence of distinct points converging to z_0 at which $f(z)$ and $g(z)$ are both analytic and $f(z_k) = g(z_k)$, then the Taylor series centered at z_0 for these two functions is the same. By analytic continuation, for diferent series, this can be done for much of the complex plane. Note that the Taylor series has no knowledge of branch cuts. Through continutation around a singular point, Taylor series can give different values of f. From this, we can determine if there is a singular point.

1.9   Residue

If z_0 is an isolated singularity of $f(z)$, we define the **residue** of f at z_0 as

$$\text{Res}[f; z_0] = \frac{1}{2\pi i} \oint_{|z-z_0|=\epsilon} f(z)dz = C_{-1}$$

for small values of ϵ where C_{-1} is the coefficient of $(z - z_0)^{-1}$ in the Laurent Series converging to $f(z)$ on the punctured disk $0 < |z - z_0| < R$.

1.10   Residue Formula

Suppose z_0 is a pole of order m of $f(z)$. Then,

$$f(z) = \frac{c_{-m}}{(z - z_0)^m} + \cdots + \frac{c_{-1}}{z - z_0} + \cdots$$

Making this function analytic at z_0,

$$(z - z_0)^m f(z) = c_{-m} + c_{-m+1}(z - z_0) + \cdots + c_{-1}(z - z_0)^{m-1} + \cdots$$

Taking $m - 1$ derivatives, we can set take the limit as $z \to z_0$ to isolate the c_{-1} term. So,

$$\lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right] = (m - 1)!c_{-1}$$

Rearranging,

$$\boxed{\text{Res}[f; z_0] = c_{-1} = \frac{1}{(m - 1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]}$$

This is known as the **Residue Formula**.

1.11   Using Residues

Consider $\oint_{|z+2|=3} \frac{z^3 e^{\frac{3}{z}}}{(z-2)^2} dz$. This has two singularities and the contour only contains the essential singularity at 0. We could use a Laurent Series with $R = 2$ centered at 0, but this may not be easy. Instead, we can consider the integral around all the singularities and subtract the residue at 2. (Let I be the integrand)

$$= \oint_{|z|=5} I dz - 2\pi i \text{Res}[I; 2]$$

The large integral can be done with a laurent series centered at 0 with $r = 2$.

$$\Rightarrow \frac{z^3}{z^2} \frac{e^{\frac{3}{z}}}{(1 - \frac{2}{z})^2}$$

We need the $\frac{1}{z}$ term. After expanding with the binomial expansion,

$$\Rightarrow \cdots + \frac{1}{z} \cdot \frac{57}{2} + \cdots \Rightarrow 57\pi i$$

For the residue, we have a pole of order 2. So,

$$\text{Res}[I; 2] = \frac{1}{(2-1)!} \lim_{z \to 2} \frac{d}{dz} \left[(z-2)^2 I \right] = \lim_{z \to 2} \left[3z^2 e^{\frac{3}{z}} - z^3 \cdot \frac{3}{z^2} e^{\frac{3}{z}} \right] = 6e^{\frac{3}{2}}$$

So, the original integral is

$$\therefore = \pi i (57 - 12 e^{\frac{3}{2}})$$

So,

$$\text{Res}[I, 0] = \frac{57}{2} - 6e^{\frac{3}{2}}$$

This gives us a way to find the residue at an essential singularity.

1.12   Real Integrals

Consider the arctan integral $I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 9}$.

$$I = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^2 + 9}$$

Closing the loop with an arc from R to $-R$ such that $3i$ is inside, this loop C has integral

$$\oint_C \frac{dz}{z^2 + 9} = 2\pi i \text{Res}[\frac{1}{z^2 + 9}, 3i]$$

Since this is a simple pole,

$$= 2\pi i \lim_{z \to 3i} \frac{z - 3i}{z^2 + 9} = \frac{\pi}{3}$$

Calling the general integral in the limit $I(R)$,

$$\oint_C \frac{dz}{z^2 + 9} = I(R) + \int_0^{\pi} \frac{iRe^{it} dt}{(Re^{it})^2 + 9}$$

Using integral bounds, we can bound this part by:

$$\frac{R}{R^2 - \frac{R^2}{2}} \cdot \pi \to 0$$

So, the original integral is simply $\frac{\pi}{3}$.

1.12.1 Multiple Contributions

Consider $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$. Adding a semicircle, the integral of the closed loop is (let integrand be ξ)

$$\oint_C \xi dz = 2\pi i \left(\text{Res}[\xi, i] + \text{Res}[\xi, 2i] \right)$$

Using L'Hopitals,

$$= 2\pi i \left(\frac{i}{6} - \frac{i}{3} \right) = \frac{\pi}{3}$$

This loop integral is also,

$$= I(R) + \int_0^\pi \frac{(Re^{it})^2 i Re^{it} dt}{(R^2 e^{2it} + 1)(R^2 e^{2it} + 4)}$$

Using integral bounds, we see the integral part goes to 0. So,

$$I = \frac{\pi}{3}$$

1.12.2 Arc at Infinity

Consider $P \int_{-\infty}^{\infty} \frac{(2x+5)dx}{x^2+2x+10}$. This integral diverges as a log, but if we take it symmetrically, we can get a defined value. This is known as the **Cauchy Principle Value**. Using L'Hopitals and residues, the integral of the closed loop (complete with semicircle) is

$$= (1 + 2i)\pi$$

This is also

$$= I(R) + \int_0^\infty \frac{(2Re^{it} + 5)i Re^{it} dt}{(Re^{it} + 1)^2 + 9}$$

As $R \to \infty$,

$$= I(R) + \pi \cdot 2i \Rightarrow I = \pi$$

1.12.3 Auxiliary Parameters

Consider $\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2}$. We can consider instead $\int_{-\infty}^{\infty} \frac{dx}{x^2+a}$. Using a semicircle, calculating the residue, and bounding the semicircle till it's muted, we see this $= \pi a^{-\frac{1}{2}}$. We can take a derivative with respect to a on both sides giving the value for the general case.

$$-\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a)^2} = -\frac{1}{2}\pi a^{-\frac{3}{2}} \Rightarrow I = \frac{\pi}{54}$$

1.12.4 Closing Below

Previously, we only closed the loop on the upper-half of the plane. Regardless of which direction the loop is closed, one the direction of the closed loop is accounted for ($-$ for clockwise), the value will be the same.

1.12.5   Trignometric Real Integrals

Consider $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx$. If we try to close the loop, we notice that on the imaginary axis, as the hemisphere grows, $\cos(iy) \to \infty$. This is a problem. Note that $\cos(x) = \Re(e^{ix})$. So,

$$= \Re \left[\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx \right]$$

The integral over the close loop is

$$= 2\pi i \cdot \frac{e^{ii}}{2i} = \frac{\pi}{e} = I(R) + \int_0^\pi \frac{e^{iRe^{it}}}{R^2 e^{2it} + 1} iRe^{it} dt$$

Noticing $|e^{iRe^{it}}| = e^{-R\sin t}$, we bound the integral part,

$$< \left| \int_0^\infty \frac{R}{R^2 - \frac{R^2}{2}} \cdot e^{-R\sin t} dt \right|$$

In the upper half of the plane, the exponential is < 1. So,

$$< \frac{2\pi}{R} \to 0$$

So, the original integral $= \frac{\pi}{e}$ as well.

1.13   Jordan's Lemma

Let $p(z)$ and $q(z)$ be polynomials with $\text{degree}(q(z)) \geq \text{degree}(p(z)) + 1$. Then,

$$\lim_{R \to \infty} \int_{|z|=R \text{ upper half}} \frac{p(z)}{q(z)} e^{iz} dz = 0$$

In the worst case, the degree of $q(z)$ is exactly one greater than that of $p(z)$. Parameterizing the upper semicircle, we can use integral bounds. Consider $p(z) = 1, q(z) = z$. Then,

$$\left| \int_0^\pi \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it} dt \right| = \int_0^\pi e^{-R\sin t} dt = 2 \int_0^{\frac{\pi}{2}} e^{-R\sin t} dt$$

Graphing, we see $\sin t > \frac{2t}{\pi}$ (above the secant line). So,

$$< 2 \int_0^{\frac{\pi}{2}} e^{-R \cdot \frac{2t}{\pi}} dt = \frac{\pi}{R} \left[1 - e^{-R} \right] \to 0$$

1.14 Harder Real Integrals

1.14.1 Long Trig Integrals

Consider $\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^2-ix+2}dx$. Using the definition of $\sin(2x)$,

$$= \int_{-\infty}^{\infty} \frac{x \cdot \frac{e^{2ix}-e^{-2ix}}{2i}}{(x-2i)(x+i)}dx$$

Breaking apart,

$$= \frac{1}{2i}\left[\oint_{C+} \frac{ze^{2iz}}{(z-2i)(z+i)}dz - \oint_{C-} \frac{ze^{-2iz}}{(z-2i)(z+i)}dz\right]$$

Here C+ is the upper semicircle and C- is the lower one. Using residues,

$$= \pi\frac{2ie^{-4}}{3i} + \pi\frac{-ie^{-2}}{-3i} = \frac{\pi}{3}(e^{-2} + 2e^{-4})$$

Using Jordan's Lemma's Result and integral bounds, the semicircles have 0 contribution so the original integral is this as well.

1.14.2 Keyhole Contour

Consider $\int_0^{\infty} \frac{dx}{x^2+2x+2}$. A keyhole contour involves two full circles and introducing a branch cut with an arbitrary power α in hopes to get an indeterminant form and use L'Hopitals to finally evaluate.

1.14.3 Pizza Slice Contour

Consider $\int_0^{\infty} \frac{dx}{x^3+1} = I$. Closing the curve with a "pizza slice" avoiding the other singularities,

$$\oint_C \frac{dz}{z^3+1} = I - \int_0^{\infty} \frac{e^{i\frac{2\pi}{3}}dx}{(xe^{i\frac{2\pi}{3}})} = I - e^{i\frac{2\pi}{3}}I = -2ie^{i\frac{\pi}{3}}\sin\frac{\pi}{3}I$$

Using residues,

$$= 2\pi i\text{Res}[\frac{1}{z^3+1}; e^{i\frac{\pi}{3}}] = 2\pi i \cdot \frac{1}{3e^{i\frac{2\pi}{3}}} = \frac{2\pi i}{3}e^{-i\frac{2\pi}{3}}$$

Equating and rearranging,

$$I = \frac{\pi}{3\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$$

Consider this generally for

$$\int_0^{\infty} \frac{x^{\alpha-1}dx}{x^n+1} = I(\alpha)$$

Here, the integral around the "pizza slice" around one singularity is, (circular portions are 0)

$$= I(\alpha) + \int_R^\epsilon \frac{(xe^{\frac{2i\pi}{n}})e^{i\frac{2\pi}{n}}dx}{x^n + 1} = I(\alpha) - e^{i\frac{2\alpha\pi}{n}}I(\alpha) = e^{i\frac{\alpha\pi}{n}}(-2i\sin\frac{\alpha\pi}{n})I(\alpha)$$

Looking at this from the perspective of residues,

$$= 2\pi i \cdot \frac{(e^{i\frac{\pi}{n}})^{\alpha-1}}{n(e^{i\frac{\pi}{n}})^{n-1}}$$

Equating, cancelling, and rearranging,

$$\boxed{I(\alpha) = \frac{\pi}{n}\csc\frac{\alpha\pi}{n}}$$

1.14.4   Pole on the Contour

Consider $P\int_{-\infty}^\infty \frac{\sin x dx}{x^2-x-6}$. Notice this has two simple poles on the contour. Call the integrand ξ. Then,

$$= \lim_{\epsilon\to 0^+}\lim_{\delta\to 0^+}\int_{-\infty}^{-2-\epsilon}\xi dx + \int_{-2+\epsilon}^{3-\delta}\xi dx + \int_{3+\delta}^\infty \xi dx$$

Here, we can use $\Im$ to deal with sin and close around each simple pole either above or below. WLOG, if we close above, the closed loop integral is

$$= I - \frac{ie^{-2i}}{5}\int_\pi^0 dt + \frac{ie^{3i}}{5}\int_\pi^0 dt$$

So, the original integral is

$$\frac{\pi}{5}[\cos 3 - \cos 2]$$

For a simple pole, the contribution halfway aroudn the pole is simply half the residue at that pole.