Chapter 1

Laplace's Equation

1.1 Analytic and Harmonic

The function $f(z)$ is said to be **analytic** at z_0 if it is differentiable on an open set containing z_0. So, $f(z)$ has to satisfy the Cauchy-Reimann equations in a neighborhood of z_0. Taking derivatives and using Clauriat's Theorem,

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0$$

This partial differential equation is **Laplace's Equation**. Any function that satisfies Laplace's equation is **harmonic**. With a similar trick, we can find that the above equation holds for the imaginary part as well. The real and imaginary parts of a analytic function are harmonic. However, this is **not if and only if**. This only goes one way. If real and imaginary parts of a function are harmonic, it is not guaranteed that the function is analytic. But, this can be modified to be true. If the function $u(x, y)$ is harmonic on a *simply connected* domain D, then it is the real part of a function that is analytic on D. The imagniary part of this function is called the **harmonic conjugate** that is unique *up to a constant*.