Chapter 1

Integration

1.1 Integration Techniques

1.1.1 Auxiliary Parameters

$$\text{Evaluate } \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^4}$$

Instead consider,

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+b} = \frac{1}{a}\int_{-\infty}^{\infty} \frac{dx}{x^2+\frac{b}{a}} = \frac{1}{a}\cdot\frac{1}{\sqrt{\frac{b}{a}}}\arctan(\frac{x}{\sqrt{\frac{b}{a}}})\ |_{-\infty}^{\infty} = \pi a^{-\frac{1}{2}} b^{-\frac{1}{2}}$$

Now, look at the derivatives with respect to the parameters.

$$\frac{\partial^3}{\partial b^3}\int_{-\infty}^{\infty} \frac{dx}{ax^2+b} = -6\int_{-\infty}^{\infty} \frac{dx}{(ax^2+b)^4} = \pi\frac{-1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}a^{-\frac{1}{2}}b^{-\frac{7}{2}}$$

Now, plug in values for the parameters,

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^4} = \pi\frac{-1}{6}\cdot\frac{-1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot 1\cdot 1 = \frac{5\pi}{16}$$

1.1.2 Gaussian Integral

The gaussian integral is defined as follows:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

Let,

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

Then,

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$
$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-(x^2+y^2)}$$

Switching to polar,

$$= \int_{0}^{\infty} r dr \int^{2\pi} \theta e^{-r^2} = 2\pi \int_{0}^{\infty} r e^{-r^2}$$

Using a u-substitution with $u = -r^2$,

$$= -\pi \int_{0}^{-\infty} e^u du = \pi$$

Returning to the first integral,

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

For the original integral,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\frac{\pi}{\alpha}}$$

Now, taking derivatives,

$$\frac{\partial}{\partial \alpha} \Rightarrow -\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} = \frac{-1}{2} \sqrt{pi} \alpha^{\frac{-3}{2}}$$