Chapter 1

Sequences, Series, and Convergence

1.1 Sequences

1.1.1 Convergence

$\{a_k\}_{k=1}^{\infty}$ is said to converge to L if given any $\epsilon > 0 \exists K | |a_k - L| < \epsilon \forall k > K$

1.1.2 Limit Point

Definition

L is a limit point if given any $\epsilon < 0$ there are infinitely many k such that:

$$|a_k - L| < \epsilon$$

Example 1

$$a_n = \frac{(-1)^n n}{n+1}$$

does not converge but has limit points ± 1.

Alternative definition

L is a limit point if there exists a subsequence that converges to L

Completeness

The real numbers are the rational numbers + all limit points that the rational numbers can reach.

1.1.3 Bounded

Definition: $\{a_k\}_{k=1}^{\infty}$ is said to be bounded if $\exists m||a_k| < M\forall k$

Lemma: A sequence that is bounded implies that there exists at least one limit point. But, it does not imply convergence.

Example:

$$a_n = cos(n^2)$$

is bounded and thus guaranteed to have a limit point. But, we do not know certainly where the limit point is. Note that just because *cos* is periodic, that does not mean that all $L \in [-1, 1]$ is a limit point without invoking the transcendental nature of pi. (imagine the period was 5)

1.1.4 Monotonicity

A sequence is monotonic if it changes in only one direction (includes staying the same - 3, 4, 4, 5 is valid). If a sequence is bounded + monotonic, it is said to converge.

1.1.5 Example

Sequence: $\{a_n\}_{n=1}^{\infty}, a_1 = 1, a_{n+1} = 7 - \frac{5}{a_n}$
Suppose $1 < a_k < 7$,

$$a_k < 7$$

$$7 - \frac{5}{a_{k+1}} < 7 - \frac{5}{7}$$

$$a_{k+1} < 7 - \frac{5}{7} < 7$$

The sequence is bounded by 7. Suppose $a_k < a_{k+1}$,

$$7 - \frac{5}{a_k} < 7 - \frac{5}{a_{k+1}}$$

$$a_{k+1} < a_{k+2}$$

The limit L satisfies the recurrence.

$$L = 7 - \frac{5}{L} \Rightarrow L^2 - 7L + 5 = 0$$

$$L = \frac{7 + \sqrt{29}}{2}$$

(eliminate negative case)

Note: This sequence of rational numbers converges to an irrational number. This technique is used for approximations of irrational numbers.

1.1.6 Basel's Problem

Sequence: a_1, $a_{n+1} = a_n + \frac{1}{(n+1)^2}$ This is clearly monotonic.

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} \cdots$$

$\frac{1}{2^2} + \frac{1}{3^2} < 2 \cdot \frac{1}{2^2}$ $\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} < 2 \cdot \frac{1}{4^2}$ Continue this process to get,

$$a_n < 1 + \frac{1}{2} + \frac{1}{4} \cdots = 2$$

Euler proved this to converge to $\frac{\pi^2}{6}$.

1.1.7 Harmonic Series

Sequence: a_1, $a_{n+1} = a_n + \frac{1}{n+1}$ This is clearly monotonic.

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$$

$\frac{1}{3} + \frac{1}{4} > 2 \cdot \frac{1}{4}$...

$$a_n > 1 + \frac{1}{2} + \frac{1}{2} \cdots$$

. Thus, the series is unbounded.

1.2 Series

1.2.1 Cauchy's Convergence Criterion

For Series: The series $\{S_k\}_{k=1}^{\infty}$ converges iff given any $\epsilon > 0$,

$$\exists N \mid |S_m - S_n| < \epsilon \text{ for } m, n > N.$$

For Partial Series: $S_n = \sum_{k=1}^{n} a_k \Rightarrow |\sum_{N+1}^{N+P} a_k| < \epsilon \,\forall\, P \in \mathbb{N}$
Note: Not only do the individual terms have to approach zero, but also, the sum of some number of terms approaches zero.

Example:

$$\frac{1}{2^n+1} + \frac{1}{2^n+2} \cdots + \frac{1}{2^n+2^n} > 2^n \frac{1}{2^n+2^n} = \frac{1}{2}$$

Thus, for $\epsilon = \frac{1}{2}$ this series breaks Cauchy's Convergence Criterion and does not converge.

1.2.2 An Interesting Result

Suppose $\{S_n\}_{n=1}^{\infty}$ diverges, where $S_n = \sum_{k=1}^{n} a_k$ with $a \geq 0 \forall k$. Note that this is monotonic. Consider $\sum_{k=1}^{n} \frac{a_k}{S_k}$,

$$\sum_{k=N+1}^{N+P} \frac{a_k}{S_k} = \frac{a_{N+1}}{S_{N+1}} + \frac{a_{N+2}}{S_{N+2}} + \cdots + \frac{a_{N+P}}{S_{N+P}} \geq \frac{a_{N+1}}{S_{N+P}} + \frac{a_{N+2}}{S_{N+P}} + \cdots + \frac{a_{N+P}}{S_{N+P}}$$

$$\sum_{k=N+1}^{N+P} \frac{a_k}{S_k} \geq \frac{S_{N+P} - S_N}{S_{N+P}} = 1 - \frac{S_N}{S_{N+P}} \to 1$$

So, $\sum_{k=1}^{n} \frac{a_k}{S_k}$ also diverges. Since each of its elements is smaller, it diverges slower than the inital one. Thus, there is **no slowest diverging series**.

1.2.3 Harmonic Series:

Suppose $a_k = 1 \forall k$. $\sum_{k=1}^{\infty} 1$ clearly diverges.

$$S_n = \sum_{k=1}^{n} = n$$

$$\sum_{k=1}^{\infty} \frac{a_k}{S_k} = \sum_{k=1}^{\infty}$$

or the harmonic series, diverges as well.

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \ln n$$

$$\sum_{k=1}^{\infty} \frac{\frac{1}{k}}{H_k} \approx \sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

diverges as well.

1.2.4 Divergence Test

If $\lim_{k \to \infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges. This is deriven directly from Cauchy's Convergence Criterion.

1.2.5 Absolute Convergence

$\sum_{k=1}^{\infty} a_k$ is said to converge **absolutely** if $\sum_{k=1}^{\infty} |a_k|$ converges.
Statement: An absolutely convergent series is convergent.
Proof:

$$|\sum_{k=N+1}^{N+P} a_k| \leq \sum_{k=N+1}^{N+P} |a_k|$$

Lemma: If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then given any $\epsilon > 0 \; \exists N$: $\sum_{k=N+1}^{N+P} |a_k| < \epsilon \forall P \in \mathbb{N}$

1.2.6 Direct Comparison Test

Given $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are both sequences of positive numbers, if $a_k \geq b_k \forall k > K$, then

$$\sum_{k=1}^{\infty} b_k \text{ diverges then } \sum_{k=1}^{\infty} a_k \text{ diverges as well.}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges then } \sum_{k=1}^{\infty} b_k \text{ converges as well.}$$

1.2.7 Limit Comparison Test

Given $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are two sequences of positive terms, if

$$\lim_{k \to \infty} \frac{a_k}{b_k} = L \neq 0$$

then the sequences both converge or both diverge.

Proof

$$\lim_{k \to \infty} \frac{a_k}{b_k} = L \Rightarrow \text{ given any } \epsilon > 0 \exists K |$$

$$|\frac{a_k}{b_k} - L| < \epsilon \forall k > K$$

$$L - \epsilon < \frac{a_k}{b_k} < L + \epsilon$$

$$b_k(L - \epsilon) < a_k < b_k(L + \epsilon)$$

Let $N > K$,

$$(L - \epsilon) \sum_{N+1}^{N+P} b_k < \sum_{N+1}^{N+P} a_k < (L + \epsilon) \sum_{N+1}^{N+P} b_k$$

As long as $L > 0$, then the LHS can be positive. Taking cases of convergence and divergence for $\sum_{N+1}^{N+P} b_k$, the Limit Comparison Test is proven.

1.2.8 Integral Comparison Test

Suppose $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive, monotonic, decreasing terms and $f(x)$ is a continuous, monotonic, decreasing function satisfying $f(k) = a_k \forall K$.

$$\int_{n+1}^{m+1} f(k)dx < \sum_{n+1}^{m} a_k < \int_{n}^{m} f(k)dx$$

This not only shows both the integral and the sequence converging or diverging together, but it helps approximate the value.

Approximation of the Harmonic Series

Using Mathematica, $H_{100} = 5.1873775$. Estimate H_{10^6}.

$$\int_{101}^{10^6+1} \frac{1}{x}dx < \sum_{k=101}^{10^6} \frac{1}{k} < \int_{100}^{10^6} \frac{1}{x}dx$$

$$\ln \frac{10^6+1}{101} < \sum_{k=101}^{10^6} \frac{1}{k} < \ln(10^4)$$

$$H_{100} + \ln \frac{10^6+1}{101} < H_{10^6} < H_{100} + 4\ln 10$$

For some n,

$$H_{100} + \ln \frac{n+1}{101} < H_n < H_{100} + \ln \frac{n}{100}$$

The error of the bounds is,

$$\ln \frac{n}{100} - \ln \frac{n+1}{101} = \ln\left(\frac{101}{100} \cdot \frac{n}{n+1}\right)$$

This approaches $\ln 101 - \ln 100$ for large n. Note that the error doesn't change (that much) for significant changes in n.

Error of Harmonic Series

$$H_n - \ln(n+1)$$

is bounded and converges to the **Euler-Mascheroni constant**.

1.2.9 Geometric Series

$$S_n = 1 + r + r^2 + \cdots + r^n$$

$$rS_n = r + r^2 + \cdots + r^n + r^{n+1}$$

$$rS_n - S_N = r^{n+1} - 1$$

$$\therefore S_n = \frac{r^{n+1}-1}{r-1}\,\forall\, r \in \mathbb{C} \mid r \neq 1,\; S_n = n+1 \text{ for } r = 1$$

Note that this works for complex numbers as well.

$$1 + r + r^2 + \cdots = \frac{1}{1-r}; |r| < 1$$

1.2.10 Using the Taylor Series

Find a series for $\frac{1}{5-2x}$ centered at $x = 0$.

$$\frac{1}{5-2x} = \frac{1}{5} \cdot \frac{1}{1-\frac{2x}{5}} = \frac{1}{5} \cdot \sum_{k=0}^{\infty} \left(\frac{2x}{5}\right)^k ; |x| < \frac{5}{2}$$

Note that this is a disk in the complex plane. Find a series for $\frac{1}{5-2x}$ centered at $x = -4$. (powers of $x + 4$)

$$\frac{1}{5-2x} = \frac{1}{13-2(x+4)} = \frac{1}{13} \cdot \sum_{k=0}^{\infty} \left(\frac{2(x+4)}{13}\right)^k ; |x+4| < \frac{13}{2}$$

Singularities

Singularities at which a series is undefined can pose a problem in convergence.

1.2.11 Root Test

' Consider $\sum_{k=0}^{\infty} a_k$.

$$\text{If } \lim_{k\to\infty} \sqrt[k]{|a_k|} = r, \text{ then given any } \epsilon > 0,\, \exists K :$$

$$r - \epsilon < \sqrt[k]{|a_k|} < r + \epsilon \Rightarrow (r-\epsilon)^k < |a_k| < (r+\epsilon)^k$$

$$\sum_{k=N+1}^{N+P} (r-\epsilon)^k < \sum_{k=N+1}^{N+P} |a_k| < \sum_{k=N+1}^{N+P} (r+\epsilon)^k \text{ for } N > K.$$

$$\text{If } r < 1, \sum a_k \text{ converges absolutely.}$$

$$\text{If } r > 1, \sum a_k \text{ diverges absolutely.}$$

$$\text{If } r = 1, \text{ the test is inconclusive.}$$

1.2.12 Power Series

$$\sum_{k=0}^{\infty} c_k (z-z_0)^k \text{ converges on a disk, } |z - z_0| < \frac{1}{\lim_k \to \sqrt[k]{|c_k|}} = R$$

z_0 is the center of the series. R is the radius of convergence or the distance from the center to the nearest singular point.

Example

Find the Maclauring series for $\frac{2}{3-2x+x^2}$.

$$\frac{2}{3-2x+x^2} = \frac{2}{3} \cdot \frac{1}{1-\frac{2x-x^2}{3}} = \frac{2}{3} \cdot \sum_{k=0}^{\infty} \left(\frac{2x-x^2}{3}\right)^k$$

Rearranging,

$$= \frac{2}{3}\left[1 + \left(\frac{2x-x^2}{3}\right) + \left(\frac{2x-x^2}{3}\right)^2 + \left(\frac{2x-x^2}{3}\right)^3 + \cdots\right]$$

Expanding (can change region of convergence),

$$= \frac{2}{3}\left[1 + \frac{2}{3}x + x^2\left(-\frac{1}{3}+\frac{4}{9}\right) + x^3\left(-\frac{2\cdot 2}{9}+\frac{8}{27})\right) + \cdots\right]$$

$$= \frac{2}{3}\left[1 + \frac{2}{3}x + \frac{1}{9}x^2 - \frac{4}{27}x^3 + \cdots\right]$$

Note the power series in terms of x only converges in a disk while the original sum converges in a different region specifically $|2x-x^2| < 3$. Finding singularity points,

$$x^2 - 2x + 3 = 0 \Rightarrow \cdots \Rightarrow x = 1 \pm i\sqrt{2}$$

Thus,

$$R = |1 \pm i\sqrt{2}| = \sqrt{3}$$

1.2.13 Ratio Test

1.2.14 Alternating Series Test

The terms are monotonic decreasing in absolute value and the terms alternate signs. Then, the original sequence can be separated into 2 convergent subsequences. If the limit point of these 2 convergent subsequences is the same, $lim_{n\to\infty} a_n = 0$, then the original sequence also converges.

1.2.15 Complication of Conditional Convergence

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

Argument 1:

The first term is 1, the next term has magnitude less than 1, so the sum is between $\frac{1}{2}$ and 1.

Argument 2:

Add the positive terms until it exceeds 3, then subtract $\frac{1}{2}$. Next add positive terms to 3, then subtract $\frac{1}{4}$, etc. So, the sequence converges to 3.

A Word of Caution

Rearranging terms in a conditionally convergent series can change the sum. The Alternating Series Test depends on the fact that the terms are added in order. Argument 2 destroys the meaning of the word converge.

1.2.16 Cauchy's Condensation Test

Suppose $\{a_n\}_{n=1}^{\infty}$ is decreasing and $a_n \geq 0$ for all $n \in \mathbb{N}$. Then, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=0}^{\infty} 2^n a_n$ converge or diverge together.

1.3 Taylor Series

1.3.1 Lagrange Error

Let,

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)dt$$

Doing repeated integration by parts tabularly,

$$= \left[f'(t)(t-x) - \frac{1}{2}f''(t)(t-x)^2 + \frac{1}{3!}f^{(3)}(t)(t-x)^3 \right]_{x_0}^{x} - \int_{x_0}^{x} \frac{1}{3!}f^{(4)}(t-x)^3 dt$$

$$= f'(t)(x-x_0) + \frac{1}{2}f''(t)(x-x_0)^2 + \frac{1}{3!}f^{(3)}(x-x_0)^3 - \int_{x_0}^{x} \frac{1}{3!}f^{(4)}(t-x)^3 dt$$

So,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(x-t)^n dt$$

The integral part of this expression is the lagrange error when approximating a function with a set number of taylor series terms.

1.3.2 Building Block Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k,\ |x| < 1.$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k},\ |x| < 1.$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1},\ |x| < 1.$$

Note the radius of convergence for this series is 1 as it is not defined for $x = \pm i$ even though it is defined for all reals.

$$(1+x)^n = 1 + nx + \frac{n(n+1)}{2}x^2 + \frac{n(n-1)(n-2)}{3}x^3 \cdots = \sum_{k=0}^{\infty} \binom{n}{k} x^k,\ |x| < 1.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!},\ |x| < \infty.$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},\ |x| < \infty.$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!},\ |x| < \infty.$$

1.3.3 Derived Series

Example 1

Consider $f(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}$; $f(0) = 1$.

$$= \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{x})^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k+1)!}$$

This can be used to find arbitrary derivatives.

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k+1)!} = \sum_{y=0}^{\infty} \frac{f^{(y)}(0)}{y!} x^y$$

To compute, $f^{(10)}(0)$ look at the x^{10} term.

$$\frac{(-1)^{10} x^{10}}{21!} = \frac{f^{(10)(0)} x^{10}}{10!} \Rightarrow f^{(10)}(0) = \frac{10!}{21!}$$

Another Example

Consider $g(x) = \frac{\ln(3+2x^2) - \ln(3)}{x}$; $g(0) = 0$. Find $g^{(6)}(0)$.

$$\ln(3 + 2x^2) = \ln\left[3(1 + \frac{2}{3}x^2)\right] = \ln 3 + \ln(1 + \frac{2}{3}x^2)$$

$$\therefore g(x) = \frac{\ln(1 + \frac{2}{3}x^2)}{x} = \frac{-\sum_{k=0}^{\infty} \frac{(-\frac{2}{3}x^2)^k}{k}}{x} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (\frac{2}{3})^k x^{2k-1}}{k},\ |x| < \sqrt{\frac{3}{2}}$$

$$= \sum_{y=0}^{\infty} \frac{g^{(y)}(0)}{y!} x^y$$

Since we are looking for x^6 but that makes $k = \frac{7}{2} \notin \mathbb{N}$. This means that a x^6 term is absent.

$$\therefore g^{(6)}(0) = 0$$

An easier method is noticing $g(x)$ is odd which means all the even derivatives are 0. To find $g^{(7)}(0)$, observe $k = 4$...

$$\frac{g^{(7)}(0)}{7!} x^7 = \frac{(-1)^5 \left(\frac{2}{3}\right)^4 x^7}{4} \Rightarrow g^{(7)}(0) = -\frac{2^2}{3^4} 7!$$

A Cooler One

Consider $h(x) = \int_0^x \frac{1-\cos(2t^3)}{t^3} dt$. Find $h^{(13)}(0)$.

$$\frac{1-\cos(2t^3)}{t^3} = \frac{1 - \sum_{k=0}^{\infty} \frac{(-1)^k (2t^3)^{2k}}{(2k)!}}{t^3} = \frac{\sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2t^3)^{2k}}{(2k)!}}{t^3}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^{2k} t^{6k-3}}{(2k)!} \Rightarrow h(x) = \sum_{\infty} k = 1 \frac{(-1)^{k+1} 2^{2k} x^{6k-2}}{(2k)!(6k-3)} = \sum_{y=0}^{\infty} \frac{h^{(y)}(0)}{y!} x^y$$

Since there is no natural k that makes this work, $h^{(13)}(0) = 0$.