University of Toronto at Scarborough
Department of Computer & Mathematical Sciences

NOTES:

- There are 16 numbered pages the exam. It is your responsibility to ensure that, at the start of the test, this booklet has all its pages. The last 3 pages are intentionally left blank.
- Answer all questions. Explain and justify your answers.
- **Show all your work**. Credit will not be given for numerical answers if the work is not shown. If you need more space for your answers then use the last 3 pages of the test which are intentionally left blank.
- Do NOT continue your solutions on the back of test pages (these will not be graded).
- No cell phones, smart watches, and any type of e-mail or instant messaging devices are allowed to be brought to the exam. Be sure that if you have any, that they are OFF and in your backpack away from you.
- No scrap paper or any sort of paper may be brought to your test seat. All necessary space needed for your solutions is included in the test. The backs of test papers may be used as scrap paper.
- The use of a calculator is **NOT** permitted.

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

1. **[18 marks]** **TRUE OR FALSE** Carefully read each statement. If the assertation must be true, the **circle T** (for true). Otherwise, **circle F** (for false). Justification is neither required nor rewarded, but a small workplace is given for your rough work. Each correct answer earns 2 points and each incorrect or blank answer earns 0 points.

(a) Let $a, b \in \mathbb{R}$ such that $a < b$. If f^5 is not integrable on $[a, b]$ then f is not integrable on $[a, b]$.

T or F

(b) Suppose that f is continuous everywhere. If f has a horizontal asymptote as $x \to \infty$, then $\displaystyle\int_2^{\infty} f(x)\, dx$ must converge.

T or F

(c) Let f be a real-valued function such that $[0, 2] \subset dom(f)$. Let $n \in \mathbb{Z}^+$ and let $P = \{x_i\}_{i=0}^{n}$ be a Riemann partition of $[0, 2]$. Then

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\triangle = \sum_{i=0}^{n-1} f(x_i)\triangle = R_{n-1}.$$

T or F

(d) Let f be a real-valued function with $[0, 1] \subset dom(f)$. For any partition P of $[0, 1]$, $U(f, P) \neq L(f, P)$ implies that $\displaystyle\int_0^1 f(x)\, dx$ does not exist.

T or F

(e) If f be a continuous everywhere function then $\displaystyle\int_0^x f(t)\ dt$ is an infinite family of continuous functions.

T or F

(f) If $\displaystyle\int_0^2 f(x)\ dx = 3$ and $\displaystyle\int_0^2 g(x)\ dx = 2$ then $\displaystyle\int_0^2 f\left(g\left(x\right)\right)\ dx = 6$.

T or F

(g) Suppose that f is integrable on any $[a, b] \subset \mathbb{R}$, we have that

$$\int_{-3}^{-2} f(x)\ dx = -\int_2^3 f(x)\ dx.$$

T or F

(h) For all real numbers a and b with $a < b$, we have for f integrable on [a,b],

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |\ f(x)\ |\ \ dx.$$

T or F

(i) There exist infinitely many different partitions of $[-3, -2]$.

T or F

2. **[8 marks]** Let $x > 0$. Prove that the value of the following expression does <u>not</u> depend on x:

$$\int_0^x \frac{1}{1+t^4}\,dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{\frac{4}{3}}}\,dt.$$

Fully justify your argument. Do <u>not</u> evaluate any integrals.

3. **[9 marks]** Let $a, b, c \in \mathbb{R}$ such that $a < b$. Use ONLY the Riemann definition of the definite integral to prove : if f and g are integrable on $[a, b]$ then

$$\int_a^b (f(x) - g(x) + c)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx + c(b - a).$$

4. **[9 marks]** Let function $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & , \text{if } x \in \mathbb{Q} \\ -1 & , \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that $f(x) + 2$ is not integrable on $[1, 7]$.

5. [**5 marks**] Evaluate $\lim_{n\to\infty} \dfrac{1+2+3+\cdots+n}{n^2}$. Fully justify your solution.

6. Evaluate the following integrals. Make sure to appropriately justify your solutions.

   (a) [**2 marks**] $\displaystyle\int 6\left(e^{4-3x}\right)^2\ dx$

(b) **[7 marks]** $\displaystyle\int e^{1-x}\cos(x)\,dx$

(c) **[8 marks]** $\displaystyle\int \frac{\cos(x)}{\sin^2(x) - \sin(x) - 6}\, dx$

(d) **[7 marks]** $\displaystyle\int_1^3 \frac{1}{(x+1)\sqrt{x}}\, dx$

7. **[7 marks]** Does the following integral converge or diverge? If it converges then what does it converge to? Fully justify your answer. Show all your work.

$$\int_0^{\infty} \frac{e^x}{e^{2x}+3}\, dx$$

8. **[5 marks]** Let f be a continuous everywhere function. Suppose that F is an antiderivative of f on $[-1, 1]$ and that F is an even function on $[-1, 1]$. Is $f(x)\cos^2(x)$ integrable on $[-1, 1]$? Make sure to fully justify your answer.

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