University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA37

Assignment # 0

This assignment is based on prerequisite A31 and A67 material. It is **not** being submitted **nor** is it being quizzed. This assignment contains questions that provide a sample of what you are expected to know from the prerequisite courses. You may seek help with this material in any TA or instructor office hours should you need help refreshing your memory! *(As the topics and material in this assignments are entirely review of prerequisite material, only partial solutions to this assignment will be provided.)*

STUDY: MATA31 (see corresponding portions of your textbook) and A67 where appropriate.

EXERCISES:

1. Let $p \in \mathbb{Z}$. Prove: if p^2 is even then p is even.
   Proof structure for different types of mathematical statements and connectives is important. Knowing how to adequately justify and present your work is very important. *We care very little about the final answer, we care about 'how you got there'.*

2. Let
   $$x_1 = 3,\ x_{n+1} = \frac{1}{4 - x_n},\ \text{for } n \geq 1.$$
   Use induction to prove that prove that $0 < x_{n+1} < x_n < 4$, for all $n \in \mathbb{N}$.
   The principle of mathematical induction is a proof technique for statements about or explicitly dependent upon the natural numbers. It is a technique that will show up throughout A37 but especially in the topics of 'sequences and series'.

3. Prove the following statements :

   (a) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

   (b) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

   (c) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

   These will be useful formulas when in A37 we 'evaluate definite integrals by definition'.

4. Evaluate each of the following limits. Make sure to justify your computations.

   (a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x - 2}{1 + 4x + 5x^2}$

   (b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$

   (d) $\displaystyle\lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1}$

   (e) $\displaystyle\lim_{x\to\infty} (x - \ln(x))$

   (f) $\displaystyle\lim_{x\to\infty} x^3 e^{-x^2}$

   (g) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

   (h) $\displaystyle\lim_{x\to\infty} \frac{5^x - 4^x}{3^x - 2^x}$

   Knowing how to compute limits at infinity will be needed when dealing with certain material with the A37 topics of 'definite integrals' , '(type 1) improper integrals' and 'series (certain convergence tests)' etc.

5. Differentiate the following functions :

   (a) $f(x) = \arctan(x) + \sin(2x) + 7^x$.

   (b) $g(x) = \ln(5x+2) - e^{5x}$.

(c) $h(x) = \frac{x^2\sqrt{x} - x^{1.2}}{x^\pi} + \arcsin(x)$.

Knowing how to differentiate and know the derivatives of some standard functions of calculus will play a crucial role when we 'compute antiderivatives' (by inspection).

6. Use continuity theorems to prove that the function $f(t) = \cos(3t-4) + \frac{t^2}{t-1}$ is continuous over the interval $[2, 6]$.
   Knowing and being able to prove that a function is continuous over a certain region such as an interval will be needed in certain uses in our A37 topic of 'the fundamental theorem of calculus'.

7. Let $A = \{1 + n^{-1} \mid n \in \mathbb{N}\}$. Compute $sup(A)$ and $inf(A)$. Prove that your answers are correct.
   Supremum and infimum will show up in A37 when we study 'the Darboux definition of the definite integral' and in the topic of 'sequences'.

8. Provide a (rough) sketch and label a diagram of each region enclosed by the following set of curves :

   (a) $y^2 - \pi^2 = x$ and $x = \sin(y)$
   (b) $y = \ln(x),\ x = 0,\ y = 0$ and $y = 1$.
   (c) $f(x) = \sqrt{1-x^2} + 2$ and the x-axis.
   (d) $y = 3|x-2| + 1$, $x = -1$, $x = 3$ and the x-axis.

   Being able to sketch, and compute intersection points, with your standard functions of calculus will be needed when we study 'applications of the definite integral (namely, areas between curves)' and/or when we 'evaluate definite integrals geometrically'.

   The rest of the questions below, their rigor and their ideas, are relevant to what we will need when we study A37 topics such at 'sequences', etc.

9. Provide complete and accurate $\delta - \epsilon$ proofs of the following :

   (a) $\lim_{x\to -4} 9x + 1 = -35$
   (b) $\lim_{x\to 4} x^2 + 2x + 1 = 25$.

10. Let $a, b \in \mathbb{R}$. Let $f,\ g\colon \mathbb{R} \to \mathbb{R}$ be functions. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{n\to\infty} g(x) = b,$ then provide a complete and accurate $N - \epsilon$ proof that $\lim_{n\to\infty}(f(x) + g(x)) = a + b.$

11. Let $a, b, c \in \mathbb{R}$. If $\lim_{n\to c} f(x) = a$ and $\lim_{n\to c} g(x) = b,$ then provide a complete and accurate $\delta - \epsilon$ proof that $\lim_{n\to c}(f(x)g(x)) = ab.$

12. Consider the function

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove (i.e. provide a complete and accurate $\delta - \epsilon$ proof) that

$$\lim_{x\to 0} g(x) = 0.$$

13. Provide complete and accurate $N - \epsilon$ proofs of the following :

 (a) $\lim_{x\to\infty} \dfrac{3}{n+2} = 0.$

 (b) $\lim_{x\to\infty} \dfrac{x - \sin(x+1)}{3x+9} = \dfrac{1}{3}.$

 (c) $\lim_{x\to\infty} \dfrac{4x^2 - 7}{2x^3 - 5} = 0.$

14. Let $b,\ c,\ \ell,\ m \in \mathbb{R}$. Let $f,\ g\colon \mathbb{R} \to \mathbb{R}$ be functions. Suppose that $\lim_{x\to\infty} f(x) = \ell$ and $\lim_{x\to\infty} g(x) = m$. Give a complete and accurate $\epsilon - N$ proof that $\lim_{x\to\infty}(2bf(x) - 3cg(x))$ exists.