University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA37 Winter 2025

Assignment # 1

You are may wish to work on this assignment prior to your tutorial during the week of Jan. 13th. You may ask questions about this assignment in that tutorial.

STUDY: Chapter 4, Sections: 4.1, 4.2 (*omitting* Trapezoid Sums def 4.8 & Upper and Lower Sums def 4.7).

HOMEWORK:

At the beginning of your TUTORIAL during the week of Jan. 20th you may be asked to either submit the following "Homework" problems or write a quiz based on this assignment and/or related material from the lectures and textbook readings. All submissions to weekly evaluations will be done via crowdmark in your officially registered tutorial. This part of your assignment will count towards the 20% of your final mark, which is based on weekly assignments / quizzes.

1. (a) **Read the course general information sheet.**
   (b) **Make sure that you are enrolled in a tutorial that fits your schedule.**
2. Let $n \in \mathbb{Z}^+$. Prove : If a_k and b_k are real-valued functions of k then

$$\sum_{k=1}^{n}(a_k^2 + 2b_k - 6) = -6n + \sum_{k=1}^{n} a_k^2 + 2\sum_{k=1}^{n} b_k.$$

Do NOT use any sigma notation properties in your proof. Do NOT use any pre-requisite formulus. Make sure to fully justify your work.

3. Let $n \in \mathbb{Z}^+$. Let f be a real-valued function such that $[0, 2] \subseteq dom(f)$. Let $P = \{x_i\}_{i=0}^n$ be a Riemann partition of $[0, 2]$. Prove or disprove (by exhibiting a counter-example) each of the following :

   (a) $L_n = \sum_{i=1}^{n} f(x_{i-1}) \bigtriangleup x = \sum_{i=0}^{n-1} f(x_i) \bigtriangleup x.$

   (b) $\sum_{i=1}^{n} x^2 = \frac{x(x+1)(2x+1)}{6}$, any $x \in \mathbb{R}$.

4. For each of the following, graph the function $y = f(x)$ and "geometrically" find the exact signed area between the graph of f over the specified interval $[a, b]$, $a, b \in \mathbb{R}$ $a < b$. That is, provide a sketch of the graph of $f(x)$ with adequate justification as to how it was obtained/graphed, shade in the area to be computed in your diagram, and interpret that area as a sum of simple known geometric shapes, e.g. semi-circles, triangles, rectangles, etc.

   (a) $f(x) = \sqrt{1 - (x-1)^2} + 4$; $[0, 1]$

   (b) $f(x) = 5 - |x - 1|$; $[0, 3]$

5. Let $a, b, c \in \mathbb{R}$. Use $\sum$-notation properties, useful sum formulas and limit laws, to evaluate

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{a(i+1) - bi^2 - (ci)^3}{n^4}.$$

   Make sure to justify your work.

6. Let $n \in \mathbb{Z}^+$. Suppose that g is a continuous everywhere function. For the sum

$$\sum_{k=1}^{n} \frac{3}{n} \left(2 + \frac{3k}{n}\right) g\left(4 + \frac{6k}{n}\right)$$

   use "reverse engineering" to decide and show whether it is either a right Riemann sum for some function on some interval or a left Riemann sum for some function on some interval. What is the function? What is 'your interval of interest' $[a, b]$? *Hint : Start by choosing/identifying* $\bigtriangleup x$*, what does this tell us about* a *and* b*? What would you pick/choose to be* x_i *or* x_{i-1}*? etc.*

7. Let $n \in \mathbb{Z}^{+}$. Let $y = f(x)$ be a continuous real-valued function on $[0, 1]$. Let $P = \{x_i\}_{i=0}^{n}$ be a Riemann partition of $[0, 1]$, i.e., define $\triangle\ x = \frac{1}{n}$ and $x_i = 0 + i \triangle x,\ i = 0,\ 1,\ \ldots,\ n$. Let R_n be the Right (Riemann) sum for f over $[0, 1]$ with n subintervals. Let M_n denote the Midpoint (Riemann) sum for f over $[0, 1]$ with n subintervals.

   (a) Write down a formula for M_n. Express your formula in sigma notation with initial value of the index equal to 1. Make sure to clearly define any expressions involved in your formula.

   (b) Prove : If f is continuous, negative and strictly increasing on $[0, 1]$ then which Riemann sum, M_n or R_n is strictly smaller? Prove your claim. Make sure to fully justify your arguments.

   (c) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined on $[0, 1]$ by $f(x) = 1$ if x is a rational number and $f(x) = -1$ if x is an irrational number. Compute L_n and R_n for f over $[0, 1]$. Make sure to fully justify your work.

   (d) (Not graded) The Cantor set C is a subset of $[0, 1]$ constructed as follows. Begin with $[0, 1]$ and remove the middle third $(\frac{1}{3},\ \frac{2}{3})$. Then remove the middle third of each of the two remaining intervals. Continue in this way removing the middle third of each remaining interval from the previous step. Note that the Cantor set is a fractal. It has a self-similarity property : tje forst tjord pf C lying in $[0, \frac{1}{3}]$, is the same as the whole Cantor set, only scaled by a factor of $\frac{1}{3}$. Similarly, the last third is the whole Cantor set scaled by $\frac{1}{3}$. Let f be the function defined on $[0, 1]$ by $f(x) = 1$ if $x \in C$ and $f(x) = 0$ if $x \notin C$. Compute M_{3^n} for any $n \in \mathbb{Z}^{\geq 0}$. Make sure to fully justify your work. What definite integral does this Riemann sum estimate?

The study of mathematics, like the Nile, begins in minuteness but ends in magnificence. – Charles Caleb Colton