University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA37 Winter 2025

Assignment # 3

You are may wish to work on this assignment prior to your tutorial during the week of Jan. 27th. You may ask questions about this assignment in that tutorial.

STUDY: Chapter 4, Sections: 4.4 and Supplementary material from Integration Module (Chap 13 Spivak): lower and upper (Darboux) sum definitions and the Darboux definition & 'The Integrability ϵ-Reformulation' (Thm 2).

HOMEWORK:

At the beginning of your TUTORIAL during the week of Feb. 3rd you may be asked to either submit the following "Homework" problems or write a quiz based on this assignment and/or related material from the lectures and textbook readings. All submissions to weekly evaluations will be done via crowdmark in your officially registered tutorial. This part of your assignment will count towards the 20% of your final mark, which is based on weekly assignments / quizzes.

1. Calculate $L(f, P)$ and $U(f, P)$ for each of the following. Simplify as much as possible. Make sure to justify your work, especially your m_i and M_i computations.
   (a) $f(x) = e^{x^2}$, $x \in [0, 1]$; an arbitrary partition $P = \{x_i\}_{i=0}^{n}$. Do not simplify your upper and lower sums.
   (b) $f(x) = -x^2$, $x \in [-1, 7]$; $P = \{-1,\ 0,\ \frac{1}{4},\ 2.4,\ 7\}$
2. Let $a, b \in \mathbb{R},\ a < b$. Let P be any partition of $[a, b]$. Prove : If f is a bounded real-valued function on $[a, b]$, then $L(f, P) \le U(f, P)$. Make sure to justify your work.

3. Let $f(x) = x$. Prove that for every partition P of $[0, 1]$,

$$U(f, P) + L(f, P) = 1.$$

4. Let $n \in \mathbb{Z}^+$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -\sqrt{2} & \text{, if } x \in \mathbb{Q} \\ 60 & \text{, if } x \notin \mathbb{Q} \end{cases}$$

   (a) Use the Integrability Reformulation to prove that f is not integrable on $[a, b + 1]$.

   (b) Use the Darboux definition of the definite integral to prove f is not integrable on $[a, b + 1]$

   Make sure to justify your work, especially your m_i and M_i computations.

5. Prove or disprove (by providing a counterexample) each of the following :

   (a) Suppose that f is a bounded real-valued function. If f is not integrable on $[0, 1]$ then f^2 is not integrable on $[0, 1]$.

   (b) Suppose that f and g are bounded real-valued functions. If $f(x) + g(x)$ is not integrable $[1, 4]$ then both f and g are not integrable on $[1, 4]$

   (c) Suppose that f and g are bounded real-valued functions. If $f(x) + g(x)$ is not integrable $[-5, 5]$ then at least one of f and g are not integrable on $[-5, 5]$

6. Let $A \in \mathbb{R}, A \neq 2$. Let function $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} A & \text{, if } x \in \mathbb{Q} \\ 2 & \text{, if } x \notin \mathbb{Q} \end{cases}$$

   Prove that $\displaystyle\int_0^2 g(x)dx$ does not exist for $A < 2$.

7. Evaluate the following indefinite integrals by definition. That means that you must make an **educated guess-and-check** to find some antiderivative F of the integrand f, then "$\displaystyle\int f(x)dx = F(x) + C$" where C is any

real constant. Hint : you may find it helpful to use algebra or other allowable manipulations to simplify the integrand, and/or recognize an integrand as the result of a product or quotient calculation, perhaps use indefinite integral properties etc., so as to assist in 'guessing F'.

(a) $\displaystyle\int (1-x)^3 \, dx$

(b) $\displaystyle\int 6(e^{4-3x})^2 \, dx$

(c) $\displaystyle\int \frac{2}{1-3t} \, dt$

(d) $\displaystyle\int (1+6\sec^2(x+1)) \, dx$

(e) $\displaystyle\int \sin^2(Ax)dx$ where A is any real constant

(f) $\displaystyle\int \frac{2x(1-x^2)(\pi x - x^3)}{\sqrt{x}} \, dx$

(g) $\displaystyle\int \csc(x)(\sin(x)+\cot(x)) \, dx$

(h) $\displaystyle\int \sqrt{1-\sin(x)} \, dx$

(i) $\displaystyle\int \frac{5}{x\ln(x^{15})} \, dx$

(j) $\displaystyle\int \left(\frac{1}{1-2u} + \frac{5}{u^2+1}\right) du$

(k) $\displaystyle\int \frac{1}{\tan^2(x)+1} \, dx$

(l) $\displaystyle\int \frac{4}{(x+2)^5} \, dx$

CHALLENGE PROBLEMS

These question are <u>NOT</u> being graded or evaluated in any fashion. These are only for your interest's sake.

1. Let f be defined on $[-1, 4]$ by

$$f(x) = \begin{cases} 1 & , \text{if } x < 1 \\ -8 & , \text{if } x > 1 \end{cases}$$

   Use the Integrability Reformulation (Theorem 2 of Chapter 13 of Suppl. Notes) to prove that f is integrable on $[0, 2]$.

2. Let f be defined on $[0, 2]$ by $f(x) = 1$ if $x \neq 1$ and $f(1) = 0$. Use the Integrability Reformulation to prove that f is integrable on $[0, 2]$.

3. Let $a, b \in \mathbb{R}, a < b$. If f is a constant function on $[a, b]$, then use the Darboux definition to prove that f is integrable on $[a, b]$. What is the value of this integral?

4. Let $a, b, c \in \mathbb{R}, a < b$. Use the Darboux definition of the definite integral to prove : If f is integrable on $[a, b]$, then $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

All things are difficult before they are easy — *Thomas Fuller*