Факт. 1.39 из учебника который я использую на самом деле возникает раньше аксиоматики данной нам на лекциях, но из нее его можно вывести

Утв. $\langle\varphi|\hat{\mathcal{L}}^\dagger \overset{Dc}{\longleftrightarrow} \hat{\mathcal{L}}|\varphi\rangle$

◀ $\langle\varphi|\mathcal{L}^\dagger|\varphi\rangle = \int_{\mathbb{R}} \varphi^* \mathcal{L}^\dagger \varphi \, dx = \int_{\mathbb{R}} \varphi \mathcal{L}^* \varphi^* dx = \int_{\mathbb{R}} \varphi \underbrace{(\mathcal{L}\varphi)^*}_{=\xi} dx = \langle\xi|\psi\rangle \Rightarrow$ $\boxed{\text{Если } |\psi\rangle = \hat{\mathcal{L}}|\varphi\rangle = |\xi\rangle}$ $\Rightarrow \langle\xi|\xi\rangle = ?$

$\Rightarrow \langle\varphi|\hat{\mathcal{L}}^\dagger$ дуален $\hat{\mathcal{L}}|\varphi\rangle$

Задание 1

а) $\hat{\mathcal{L}}^\dagger\hat{\mathcal{L}} \overset{\dagger}{\Rightarrow} \hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}^{\dagger\dagger} = \hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}$

б) $\hat{\mathcal{L}} + \hat{\mathcal{L}}^\dagger \overset{\dagger}{\Rightarrow} \hat{\mathcal{L}}^\dagger + \hat{\mathcal{L}}^{\dagger\dagger} = \hat{\mathcal{L}}^\dagger + \hat{\mathcal{L}} = \hat{\mathcal{L}} + \hat{\mathcal{L}}^\dagger$

в) $i\hat{\mathcal{L}} - i\hat{\mathcal{L}}^\dagger = \boxed{\hat\beta^\dagger = \hat\beta,\ \hat J^* = -\hat J^\dagger : \hat{\mathcal{L}} = \hat\beta + \hat J} = -i\hat J + i\hat\beta - i(\hat\beta + \hat J)^\dagger = -i\hat J + i\hat\beta - i\hat\beta + i\hat J \overset{\dagger}{\Rightarrow} -i\hat J - i\hat\beta + i\hat\beta + i\hat J$

кароче представил в виде эрмитова и антиэрмитова

г) $\{\hat A, \hat B\}$

д) $i[\hat A, \hat B]$

A, B – эрм.

$\{\hat A; \hat B\} = \hat A\hat B + \hat B\hat A \overset{\dagger}{\Rightarrow} \hat B^\dagger\hat A^\dagger + \hat A^\dagger\hat B^\dagger = \hat B\hat A + \hat A\hat B$ т.к. эрмитов

$[\hat A; \hat B] = \hat A\hat B - \hat B\hat A \overset{\dagger}{\Rightarrow} -i\hat B^\dagger\hat A^\dagger + i\hat A^\dagger\hat B^\dagger = \hat A\hat B - i\hat B\hat A$

◀ $\hat A\hat B|\varphi\rangle = \hat A(\hat B|\varphi\rangle) \overset{Dc}{\rightarrow} \langle\varphi|\hat B^\dagger\hat A^\dagger$

$\quad = (\hat A\hat B)|\varphi\rangle \overset{Dc}{\rightarrow} \langle\varphi|(\hat A\hat B)^\dagger$ $\Rightarrow B^\dagger A^\dagger = (AB)^\dagger$ ■

J. J. Sakurai "Modern Quantum Mechanics" 1.39

◀ Докажем что $\exists \hat\beta, \hat J,\ \forall \hat{\mathcal{L}} : \hat\beta^\dagger = \hat\beta,\ \hat J^\dagger = -\hat J,\ \hat{\mathcal{L}} = \hat\beta + \hat J$

Пример $\beta = \frac{1}{2}(\hat{\mathcal{L}} + \hat{\mathcal{L}}^\dagger)$ $\quad J = \frac{1}{2}(\hat{\mathcal{L}} - \hat{\mathcal{L}}^\dagger)$ ■

Задание 2

$\langle\varphi|\hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}|\varphi\rangle =$ [J. J. Sakurai "Modern Quantum Mechanics" 1.39 $\langle\varphi|\hat{\mathcal{L}}^\dagger \overset{Dc}{\longleftrightarrow} \hat{\mathcal{L}}|\varphi\rangle$, $|\xi\rangle = \hat{\mathcal{L}}|\varphi\rangle$] $= (\langle\varphi|\mathcal{L}^\dagger)(\mathcal{L}|\varphi\rangle) = \langle\xi|\xi\rangle \in \mathbb{R}^+$ ■

действительное оно также потому, что $\hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}$ – эрмитов что мы доказали в 1.а

Задание 3

$[\hat A, \hat B\hat C] = \hat A\hat B\hat C - \hat B\hat C\hat A$

$[\hat A, \hat B] = \hat A\hat B - \hat B\hat A$

$[\hat A, \hat C] = \hat A\hat C - \hat C\hat A$

$\Rightarrow$ $[\hat A, \hat B]\hat C = \hat A\hat B\hat C - \cancel{\hat B\hat A\hat C}$

$\hat B[\hat A, \hat C] = \cancel{\hat B\hat A\hat C} - \hat B\hat C\hat A$

$[\hat A, \hat B]\hat C - \hat B[\hat A, \hat C] = \hat A\hat B\hat C - \hat B\hat C\hat A = [\hat A, \hat B\hat C]$

Задание 4

$$\hat{p}_i = -i\hbar \frac{\partial}{\partial x_i} \qquad \hat{r}_i = x_i \qquad \partial_i \equiv \frac{\partial}{\partial x_i}$$

а) $$[\hat{p}_i, \hat{p}_j] = (-i\hbar)^2 \frac{\partial^2}{\partial x_i \partial x_j} - (-i\hbar)^2 \frac{\partial^2}{\partial x_j \partial x_i} = -\hbar \left(\frac{\partial}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_j \partial x_i} \right) = \begin{cases} 0 & \text{Если } \exists\, \partial_i \varphi, \partial_j \varphi \text{ и } \partial_i\varphi, \partial_j\varphi \text{ – непрерывны (Теорема Шварца)} \\ 0 & i = j \\ -\hbar \left(\frac{\partial}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_j \partial x_i} \right) & \text{в остальных случаях} \end{cases}$$

б) $$[\hat{p}_i, \hat{x}_j] = \cancel{-i\hbar x_j \partial_i} - i\hbar \partial_i x_j \cancel{+ i\hbar x_j \partial_i} = -i\hbar\, \delta_{ij}$$

в) $$[\hat{r}_i, \hat{r}_j] = x_i x_j - x_j x_i = 0$$

г) $$[\hat{p}_x, f(x)] = -i\hbar f'(x) \cancel{- i\hbar f(x)\partial_x} \cancel{+ i\hbar f(x)\partial_x} = -i\hbar f'(x)$$

д) $$[\hat{p}_x^2, f(x)] = \cancel{\hbar^2 f(x)\partial_x^2} \cancel{- \hbar^2 f(x)\partial_x^2} - 2\hbar^2 f'(x)\partial_x - \hbar^2 f''(x) = \hbar^2 (2 f'(x)\partial_x + f''(x))$$

$$\partial_x^2 f(x) = \partial_x f'(x) + \partial_x f \partial_x = f'' + f'\partial_x + f'\partial_x + f''\partial_x^2$$