Факт. 1.39 из учебника который я использую на самом деле возникает раньше аксиоматики данной нам на лекциях, но из нее его можно вывести

Утв. $\langle\varphi|\hat{\mathcal{L}}^{\dagger} \overset{Dc}{\leftrightarrow} \hat{\mathcal{L}}|\varphi\rangle$

$$◀ \langle\varphi|\mathcal{L}^{\dagger}|\varphi\rangle = \int_{\mathbb{R}} \varphi^* \mathcal{L}^{\dagger}\varphi\, dx = \int_{\mathbb{R}} \varphi \mathcal{L}^* \varphi^* dx = \int_{\mathbb{R}} \varphi(\mathcal{L}\varphi)^* dx \overset{*}{\to} \int_{\mathbb{R}} \varphi^* \mathcal{L}\varphi\, dx = \langle\varphi|\mathcal{L}|\varphi\rangle$$

Задание 1

а) $\hat{\mathcal{L}}^{\dagger}\hat{\mathcal{L}}^{\dagger} \Rightarrow \hat{\mathcal{L}}^{\dagger}\hat{\mathcal{L}}^{\dagger\dagger} = \hat{\mathcal{L}}^{\dagger}\hat{\mathcal{L}}$

б) $\hat{\mathcal{L}} + \hat{\mathcal{L}}^{\dagger\dagger} \Rightarrow \hat{\mathcal{L}}^{\dagger} + \hat{\mathcal{L}}^{\dagger\dagger} = \hat{\mathcal{L}}^{\dagger} + \hat{\mathcal{L}} = \hat{\mathcal{L}} + \hat{\mathcal{L}}^{\dagger}$

в) $i\hat{\mathcal{L}} - i\hat{\mathcal{L}}^{\dagger} = \boxed{\hat{\beta}^{\dagger} = \hat{\beta},\ \hat{J}^{*} = -\hat{J}^{\dagger}:\ \hat{\mathcal{L}} = \hat{\beta} + \hat{J}} = -i\hat{J} + i\hat{\beta} - i(\hat{\beta} + \hat{J})^{\dagger} = -i\hat{J} + i\hat{\beta} - i\hat{\beta} + i\hat{J} \overset{\dagger}{\Rightarrow} -i\hat{J} - i\hat{\beta} + i\hat{\beta} + i\hat{J}$

короче представил в виде эрмитова и антиэрмитова

г) $\{\hat{A}, \hat{B}\}$

д) $i[\hat{A}, \hat{B}]$

A, B – эрм.

$$\boxed{\{\hat{A}; \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}} \overset{\dagger}{\Rightarrow} \hat{B}^{\dagger}\hat{A}^{\dagger} + \hat{A}^{\dagger}\hat{B}^{\dagger} = \hat{B}\hat{A} + \hat{A}\hat{B}$$

т.к. эрмитов

$$\boxed{[\hat{A}; \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}} \overset{\dagger}{\Rightarrow} -i\hat{B}^{\dagger}\hat{A}^{\dagger} + i\hat{A}^{\dagger}\hat{B}^{\dagger} = \hat{A}\hat{B} - i\hat{B}\hat{A}$$

$$◀ \hat{A}\hat{B}|\varphi\rangle = \hat{A}(\hat{B}|\varphi\rangle) \overset{Dc}{\to} \langle\varphi|\hat{B}^{\dagger}\hat{A}^{\dagger}$$

$$= (\hat{A}\hat{B})|\varphi\rangle \overset{Dc}{\to} \langle\varphi|(\hat{A}\hat{B})^{\dagger} \quad \Bigg| \Rightarrow B^{\dagger}A^{\dagger} = (AB)^{\dagger} \quad ■$$

J. J. Sakurai "Modern Quantum Mechanics" 1.39

◀ Докажем что $\exists \hat{\beta}, \hat{J}, \forall \hat{\mathcal{L}}: \hat{\beta}^{\dagger} = \hat{\beta},\ \hat{J}^{\dagger} = -\hat{J},\ \hat{\mathcal{L}} = \hat{\beta} + \hat{J}$

Пример $\beta = \frac{1}{2}(\hat{\mathcal{L}} + \hat{\mathcal{L}}^{\dagger}) \quad J = \frac{1}{2}(\hat{\mathcal{L}} - \hat{\mathcal{L}}^{\dagger})$ ■

Задание 2

$$\langle\varphi|\hat{\mathcal{L}}^{\dagger}\hat{\mathcal{L}}|\varphi\rangle = \boxed{\begin{array}{l}\text{J. J. Sakurai} \\ \text{"Modern Quantum Mechanics"} \\ 1.39\ \langle\varphi|\hat{\mathcal{L}}^{\dagger} \overset{Dc}{\leftrightarrow} \hat{\mathcal{L}}|\varphi\rangle \\ |\xi\rangle = \hat{\mathcal{L}}|\varphi\rangle\end{array}} = (\langle\varphi|\mathcal{L}^{\dagger})(\mathcal{L}|\varphi\rangle) = \langle\xi|\xi\rangle \in \mathbb{R}^{+} \quad ■$$

действительное оно также потому, что $\hat{\mathcal{L}}^{\dagger}\hat{\mathcal{L}}$ – эрмитов что мы доказали в 1.а

Задание 3

$$[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$$

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{A}, \hat{C}] = \hat{A}\hat{C} - \hat{C}\hat{A}$$

$$\Rightarrow \begin{array}{l} [\hat{A}, \hat{B}]\hat{C} = \hat{A}\hat{B}\hat{C} - \cancel{\hat{B}\hat{A}\hat{C}} \\ \hat{B}[\hat{A}, \hat{C}] = \cancel{\hat{B}\hat{A}\hat{C}} - \hat{B}\hat{C}\hat{A} \end{array}$$

$$[\hat{A}, \hat{B}]\hat{C} - \hat{B}[\hat{A}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = [\hat{A}, \hat{B}\hat{C}]$$

Задание 4

$\hat{p}_i = -i\hbar \frac{\partial}{\partial x_i} \qquad \hat{r}_i = x_i \qquad \partial_i \equiv \frac{\partial}{\partial x_i}$

$$\text{а) } [\hat{p}_i, \hat{p}_j] = (-i\hbar)^2 \frac{\partial^2}{\partial x_i \partial x_j} - (-i\hbar)^2 \frac{\partial^2}{\partial x_j \partial x_i} = -\hbar \left(\frac{\partial}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_j \partial x_i} \right) = \begin{cases} 0 & \text{Если } \exists\, \partial_i \varphi, \partial_j \varphi \text{ и } \partial_i \varphi, \partial_j \varphi \text{ – непрерывны (Теорема Шварца)} \\ 0 & i = j \\ -\hbar \left(\frac{\partial}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_j \partial x_i} \right) & \text{в остальных случаях} \end{cases}$$

$$\text{б) } [\hat{p}_i, \hat{x}_j] = \cancel{-i\hbar x_j} \partial_i - i\hbar \partial_i x_j + i\cancel{\hbar x_j} \partial_i = -i\hbar \delta_{ij}$$

$$\text{в) } [\hat{r}_i, \hat{r}_j] = x_i x_j - x_j x_i = 0$$

$$\text{г) } [\hat{p}_x, f(x)] = -i\hbar f'(x) - i\cancel{\hbar f(x)} \partial_x + i\hbar \cancel{f(x)} \partial_x = -i\hbar f'(x)$$

$$\text{д) } [\hat{p}_x^2, f(x)] = \hbar^2 \cancel{f(x)} \partial_x^2 - \hbar^2 \cancel{f(x)} \partial_x^2 - 2\hbar^2 f'(x) \partial_x - \hbar^2 f''(x) = \hbar^2 (2 f'(x) \partial_x + f''(x))$$

$$\partial_x^2 f(x) = \partial_x f'(x) + \partial_x f \partial_x = f'' + f' \partial_x + f' \partial_x + f'' \partial_x^2$$