Задание 1

$\langle\varphi|\varphi\rangle = |C|^2 \int_{\mathbb{R}} dx \exp\left[-\frac{(x-x_0)^2}{a^2}\right]$

$C^2 \int_{\mathbb{R}} \exp\left[-\frac{(x-x_0)^2}{a^2}\right] dx = |C|^2 \sqrt{\pi a^2} = 1$

$|C| = (\pi a^2)^{-1/4}$

а) $\varphi_2(x) = \varphi_1(x+a)$

1. $\langle\varphi_2|\hat{x}|\varphi_2\rangle = \int \varphi_1^*(x)\, x\, \varphi(x)\, dx = \boxed{x = \iota + a} = \int \varphi_2^*(\iota)\, \iota\, \varphi_2(\iota) d\iota + a\int \varphi(\iota)\varphi(\iota) d\iota = \langle\varphi_2|\hat{\iota}|\varphi_2\rangle + a$

$\langle x_2\rangle = \langle x_1\rangle + a$

2. $\langle\varphi_1|\hat{p}_x|\varphi_1\rangle = -i\hbar \int \varphi_2^*(x) \frac{\partial}{\partial x} \varphi_2(x)\, dx = \boxed{x = \iota + a} = -i\hbar \int \varphi_2^*(\iota) \frac{\partial}{\partial \iota} \varphi_2(\iota) d\iota = \langle\varphi_2|\hat{\iota}|\varphi_2\rangle$

$\langle p_2\rangle = \langle p_1\rangle$

б) $\varphi_2 = \exp\left(\frac{ip_0 x}{\hbar}\right)\varphi_1$

1. $\langle\varphi_1|\hat{x}|\varphi_1\rangle = \int \varphi_1^*(x)\, x\, \varphi_1(x) dx = \int \exp^*(ia)\varphi_1^*(x)\, x\, \exp(ia)\varphi_1(x)\, dx = \int \varphi_2^*(x)\, x\, \varphi_2(x) dx \Rightarrow \langle x_1\rangle = \langle x_2\rangle$

Заметим: $\exp(ia)\cdot(\exp(ia))^* = 1$
$a \in \mathbb{R}$

2. $\underbrace{\langle\varphi_1|\hat{p}|\varphi_1\rangle}_{\langle p_1\rangle} = -i\hbar \int \varphi_1^*(x) \frac{\partial}{\partial x}\varphi(x) dx = -i\hbar \int \varphi_1^*(x)\exp^*(iax)\exp(iax)\frac{\partial}{\partial x}\varphi_1(x)\, dx = \underbrace{-i\hbar \int \varphi_2^*(x)\frac{\partial}{\partial x}\varphi_2(x)\, dx}_{\langle p_2\rangle} - \underbrace{i\hbar \int \varphi_1^*(x)\exp^*(iax)\, \varphi_1(x) \frac{\partial}{\partial x}\exp(iax)\, dx}_{+i\hbar a = p_0} \Rightarrow \langle p_1\rangle = \langle p_2\rangle + p_0$

Так же заметим что

$\int_{\mathbb{R}} \exp\left[\frac{x^2}{2a^2}\right] x \exp\left[\frac{x^2}{2a^2}\right] = 0 = \langle x\rangle$

т.к. функция нечетна

$-i\hbar \int_{\mathbb{R}} \exp\left[\frac{x^2}{2a^2}\right]\frac{\partial}{\partial x}\exp\left[\frac{x^2}{2a^2}\right] dx = 0 = \langle p\rangle$

чисто мнимое число

Из всего этого следует что

$\langle x\rangle = x_0$

$\langle p\rangle = p_0$

$|C|^2\,\Gamma\left(\frac{3}{2}\right) a^{3/2} = a^2/2$

$\langle x^2\rangle = |C|^2 \int x^2 \exp\left[-\frac{(x-x_0)^2}{a^2}\right] dx = |C|^2 \int_{\mathbb{R}} \iota^2 \exp\left(-\frac{\iota^2}{a^2}\right) d\iota + \cancel{2|C|^2 x_0 \int_{\mathbb{R}} \iota \exp\left(-\frac{\iota^2}{a^2}\right) d\iota}_{\,0} + |C|^2 x_0^2 \int_{\mathbb{R}} \exp(-\iota^2/a^2) d\iota$

x_0^2

$\langle p^2\rangle = |C|^2\hbar \int \exp\left[\frac{ip_0 x}{\hbar} - \frac{(x-x_0)^2}{2a^2}\right]\frac{\partial}{\partial x}\exp\left[\frac{ip_0 x}{\hbar} - \frac{(x-x_0)^2}{2a^2}\right] dx =$

$= -|C|^2\hbar^2 \int_{-\infty}^{+\infty} \exp\left[-\frac{ip_0 x}{\hbar} - \frac{x-x_0}{2a^2}\right]\left(\frac{ip_0}{\hbar} - \frac{x-x_0}{2a^2}\right)^2 \exp\left[\frac{ip_0 x}{\hbar} - \frac{x-x_0}{2a^2}\right] dx =$

$= p_0^2 - C^2\hbar^2 \int \underbrace{\frac{ip_0(x-x_0)}{\hbar^2 a^2}\exp\left[-\frac{(x-x_0)^2}{2a^2}\right] dx}_{\in \mathbb{C}} - C^2\hbar^2 \int \frac{(x-x_0)^2}{4a^4}\exp\left[-\frac{(x-x_0)^2}{2a^2}\right] dx = p_0^2 - \frac{\hbar^2}{2a^2}$

$\frac{C^2\hbar^2}{2}\int_0^{+\infty}\frac{\iota}{a^4}\exp\left[-\frac{\iota}{2a^2}\right] d\iota = \frac{C^2\hbar^2}{2}$

$\delta_a = \langle a^2\rangle - \langle a\rangle^2$

$\delta_x = \frac{a^2}{2}$ $\quad\langle x\rangle = x_0$

$\delta_p = \frac{\hbar^2}{2a^2}$ $\quad\langle p\rangle = p_0$

Задание 2

$$\left[\hat{H}, \hat{p}\right] = \underset{①}{-i\hbar \frac{\partial}{\partial x} H_f} - i\hbar \frac{\partial}{\partial x} U(x) + \underset{②}{i\hbar H_f \frac{\partial}{\partial x}} + i\hbar U(x) \frac{\partial}{\partial x} \ominus$$

H_f — гамильтониан свободной частицы $① + ② = [\hat{p}, \hat{H}_f] = 0$

$$\ominus \ \cancel{-i\hbar U(x) \frac{\partial}{\partial x}} - i\hbar U'(x) + \cancel{i\hbar U(x) \frac{\partial}{\partial x}} = 0 \iff -i\hbar U'(x) = 0 \Rightarrow U'(x) = 0 \Rightarrow U(x) \text{ — независит от } x$$

Задание 3

$$\psi_p = \mathcal{F}\left[\psi_x\right]\left(\frac{p}{\hbar}\right) = \frac{C}{\sqrt{2\pi\hbar}} \int\limits_{\mathbb{R}} \exp\left[\frac{i p_0 x}{\hbar}\right] \exp\left[-\frac{(x-x_0)^2}{2a^2}\right] \exp\left[-\frac{i p_x x}{\hbar}\right] dx =$$

$$= \frac{C}{\sqrt{2\pi\hbar}} \int\limits_{\mathbb{R}} \exp\left[ix\underbrace{\left(\frac{p_0 - p_x}{\hbar}\right)}_{\frac{p_0 - p_x}{\hbar} = \xi}\right] \underbrace{\exp\left[-\frac{(x-x_0)^2}{2a^2}\right]}_{\text{гаусиан переходит в себя}} dx = \frac{|a|C}{\sqrt{\hbar}} \exp\left[-a^2\frac{(p_0 - p_x)^2}{2\hbar^2} + \frac{i(p_0 - p_x)x_0}{\hbar}\right]$$

Заметим что $\sum\limits_n^{\infty} |\varphi_n\rangle\langle\varphi_n|$ — это единичный оператор поэтому

$$\langle\psi|\hat{A}|\psi\rangle = \sum_n \langle\psi|\varphi_n\rangle\langle\varphi_n|\hat{A}|\varphi_n\rangle\langle\varphi_n|\psi\rangle$$

Следовательно во всех представлениях $\langle\psi|\hat{A}|\psi\rangle$ равны тогда:

$\langle x\rangle = x_0$

$\langle p\rangle = p_0$

$\sigma_x = \frac{a^2}{2}$

$\sigma_p = \frac{\hbar^2}{2a^2}$