Задание 1

$$\langle\varphi|\varphi\rangle = |c|^2 \int\limits_{\mathbb{R}} dx \exp\left[-\frac{(x-x_0)^2}{a^2}\right]$$

$$c^2 \int\limits_{\mathbb{R}} \exp\left[-\frac{(x-x_0)^2}{a^2}\right] dx = |c|^2 \sqrt{\pi a^2} = 1$$

$$|c| = (\pi a^2)^{-1/4}$$

а) $\varphi_2(x) = \varphi_1(x+a)$

1. $\langle\varphi_2|\hat{x}|\varphi_2\rangle = \int \varphi_1^*(x)\, x\, \varphi_1(x)\, dx = \boxed{x=\iota+a} = \int \varphi_2^*(\iota)\, \iota\, \varphi_2(\iota) d\iota + a\int \varphi^*(\iota)\varphi(\iota)\, d\iota = \langle\varphi_2|\hat{\iota}|\varphi_2\rangle + a$

$\langle x_2\rangle = \langle x_1\rangle + a$

2. $\langle\varphi_1|\hat{p}_x|\varphi_1\rangle = -i\hbar \int \varphi_2^*(x) \frac{\partial}{\partial x} \varphi_2(x)\, dx = \boxed{x=\iota+a} = -i\hbar \int \varphi_2^*(\iota) \frac{\partial}{\partial \iota} \varphi_2(\iota)\, d\iota = \langle\varphi_2|\hat{\iota}|\varphi_2\rangle$

$\langle p_2\rangle = \langle p_1\rangle$

б) $\varphi_2 = \exp\left(\frac{ip_0x}{\hbar}\right)\varphi_1$

1. $\langle\varphi_1|\hat{x}|\varphi_1\rangle = \int \varphi_1^*(x)\, x\, \varphi_1(x) dx = \int \exp^*(ia)\varphi_1^*(x)\, x \exp(ia)\varphi_1(x)\, dx = \int \varphi_2^*(x)\, x\, \varphi_2(x) dx \quad \Rightarrow \quad \langle x_1\rangle = \langle x_2\rangle$

Заметим: $\exp(ia)\cdot(\exp(ia))^* = 1$
$a \in \mathbb{R}$

2. $\underbrace{\langle\varphi_1|\hat{p}|\varphi_1\rangle}_{=\langle p_1\rangle} = -i\hbar\int \varphi_1^*(x) \frac{\partial}{\partial x}\varphi_1(x) dx = -i\hbar \int \varphi_1^*(x)\exp^*(iax)\exp(iax) \frac{\partial}{\partial x}\varphi_1(x)\, dx = \underbrace{-i\hbar \int \varphi_2^*(x) \frac{\partial}{\partial x} \varphi_2(x)\, dx}_{=\langle p_2\rangle} \underbrace{- i\hbar \int \varphi_1^*(x) \exp^*(iax)\, \varphi_1(x) \frac{\partial}{\partial x} \exp(iax)\, dx}_{=+i\hbar a = p_0} \Rightarrow \langle p_1\rangle = \langle p_2\rangle + p_0$

Так же заметим что

$$\int\limits_{\mathbb{R}} \exp\left[\frac{x^2}{2a^2}\right] x \exp\left[\frac{x^2}{2a^2}\right] = 0 = \langle x\rangle$$

т.к. функция нечетна

$$-i\hbar \int\limits_{\mathbb{R}} \exp\left[\frac{x^2}{2a^2}\right] \frac{\partial}{\partial x} \exp\left[\frac{x^2}{2a^2}\right] dx = 0 = \langle p\rangle$$

чисто мнимое число

Из всего этого следует что

$\langle x\rangle = x_0$

$\langle p\rangle = p_0$

$$\langle x^2\rangle = |c|^2 \int x^2 \exp\left[-\frac{(x-x_0)^2}{a^2}\right] dx = \underbrace{|c|^2 \int\limits_{\mathbb{R}} \iota^2 \exp\left(-\frac{\iota^2}{a^2}\right) d\iota}_{=|c|^2\Gamma\left(\frac{3}{2}\right) a^{3/2} = a^2/2} + \underbrace{2|c|^2 x_0 \int\limits_{\mathbb{R}} \iota \exp\left(-\frac{\iota^2}{a^2}\right) d\iota}_{0} + \underbrace{|c|^2 x_0^2 \int\limits_{\mathbb{R}} \exp\left(-\iota^2/a^2\right) d\iota}_{=x_0^2}$$

$$\langle p^2\rangle = |c|^2\hbar \int \exp\left[\frac{ip_0x}{\hbar} - \frac{(x-x_0)^2}{2a^2}\right] \frac{\partial}{\partial x} \exp\left[\frac{ip_0x}{\hbar} - \frac{(x-x_0)^2}{2a^2}\right] dx =$$

$$= -|c|^2\hbar^2 \int\limits_{-\infty}^{+\infty} \exp\left[-\frac{ip_0x}{\hbar} - \frac{x-x_0}{2a^2}\right]\left(\frac{ip_0}{\hbar} - \frac{x-x_0}{2a^2}\right)^2 \exp\left[\frac{ip_0x}{\hbar} - \frac{x-x_0}{2a^2}\right] dx =$$

$$= p_0^2 - \underbrace{c^2\hbar^2 \int \frac{ip_0(x-x_0)}{\hbar^2a^2} \exp\left[-\frac{(x-x_0)^2}{2a^2}\right] dx}_{\in\mathbb{C}} - \underbrace{c^2\hbar^2 \int \frac{(x-x_0)^2}{4a^4} \exp\left[-\frac{(x-x_0)^2}{2a^2}\right] dx}_{=\frac{c^2\hbar^2}{2}\int\limits_0^{+\infty} \frac{\iota}{a^4} \exp\left[-\frac{\iota}{2a^2}\right] d\iota = \frac{c^2\hbar^2}{2}} = p_0^2 - \frac{\hbar^2}{2a^2}$$

$\delta_a = \langle a^2\rangle - \langle a\rangle^2$

$\delta_x = \frac{a^2}{2} \qquad \langle x\rangle = x_0$

$\delta_p = \frac{\hbar^2}{2a^2} \qquad \langle p\rangle = p_0$

Задание 2

$$[\hat{H},\hat{p}] = \underset{①}{-i\hbar\frac{\partial}{\partial x}H_f} - i\hbar\frac{\partial}{\partial x}U(x) + \underset{②}{i\hbar H_f\frac{\partial}{\partial x}} + i\hbar U(x)\frac{\partial}{\partial x} \ominus$$

H_f – гамильтониан свободной частицы $\quad ①+② = [\hat{p},\hat{H}_f] = 0$

$$\ominus \; \cancel{-i\hbar U(x)\frac{\partial}{\partial x}} - i\hbar U'(x) + \cancel{i\hbar U(x)\frac{\partial}{\partial x}} = 0 \iff -i\hbar U'(x)=0 \Rightarrow U'(x)=0 \Rightarrow U(x)$$ – независит от x

Задание 3

$$\psi_p = \mathcal{F}[\psi_x]\left(\frac{p}{\hbar}\right) = \frac{C}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} \exp\left[\frac{ip_0x}{\hbar}\right]\exp\left[-\frac{(x-x_0)^2}{2a^2}\right]\exp\left[-\frac{ip_xx}{\hbar}\right]dx =$$

$$= \frac{C}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} \exp\left[ix\underbrace{\left(\frac{p_0-p_x}{\hbar}\right)}_{\frac{p_0-p_x}{\hbar}=\xi}\right]\underbrace{\exp\left[-\frac{(x-x_0)^2}{2a^2}\right]}_{\text{гаусиан переходит в себя}}dx = \frac{|a|C}{\sqrt{\hbar}}\exp\left[-a^2\frac{(p_0-p_x)^2}{2\hbar^2} + \frac{i(p_0-p_x)x_0}{\hbar}\right]$$

$$\langle a\rangle = \langle\varphi|\hat{A}|\varphi\rangle = \langle\varphi|\sum_{i=0}^{\infty}c_ia_i|\varphi_i\rangle = \sum_{i=0}^{\infty}c_ia_i\langle\varphi|\varphi_i\rangle$$

$$\langle\varphi| = \sum_{j=0}^{\infty}c_j\langle\varphi_j| \quad\Bigg|\Rightarrow \langle a\rangle = \sum_{i=0}^{\infty}c_i^2a_i$$