Факт. 1.39 из учебника который я использую на самом деле возникает раньше аксиоматики данной нам на лекциях, но из нее его можно вывести

Утв. $\langle\varphi|\hat{\mathcal{L}}^\dagger \overset{DC}{\longleftrightarrow} \hat{\mathcal{L}}|\varphi\rangle$

◀ $\langle\varphi|\mathcal{L}^\dagger|\varphi\rangle = \int_{\mathbb{R}} \varphi^* \mathcal{L}^\dagger \varphi\, dx = \int_{\mathbb{R}} \varphi \mathcal{L}^* \varphi^* dx = \int_{\mathbb{R}} \varphi \underbrace{(\mathcal{L}\varphi)^*}_{\xi} dx = \langle\xi|\varphi\rangle \Rightarrow$ $\boxed{\text{Если } |\varphi\rangle = \hat{\mathcal{L}}|\varphi\rangle = |\xi\rangle}$ $\Rightarrow \langle\xi|\xi\rangle = ?$

$\Rightarrow \langle\varphi|\hat{\mathcal{L}}^\dagger$ дуален $\hat{\mathcal{L}}|\varphi\rangle$

Задание 1

а) $(\hat{\mathcal{L}}^\dagger\hat{\mathcal{L}})^\dagger \Rightarrow \hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}^{\dagger\dagger} = \hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}$

б) $(\hat{\mathcal{L}} + \hat{\mathcal{L}}^\dagger)^\dagger \Rightarrow \hat{\mathcal{L}}^\dagger + \hat{\mathcal{L}}^{\dagger\dagger} = \hat{\mathcal{L}}^\dagger + \hat{\mathcal{L}} = \hat{\mathcal{L}} + \hat{\mathcal{L}}^\dagger$

в) $i\hat{\mathcal{L}} - i\hat{\mathcal{L}}^\dagger = \boxed{\hat{B}^\dagger = \hat{B},\ \hat{J}^\dagger = -\hat{J}^\dagger:\ \hat{\mathcal{L}} = \hat{B} + \hat{J}} = -i\hat{J} + i\hat{B} - i(\hat{B}+\hat{J})^\dagger = -i\hat{J} + i\hat{B} - i\hat{B} + i\hat{J} \overset{\dagger}{\Rightarrow} -i\hat{J} - i\hat{B} + i\hat{B} + i\hat{J}$

короче представил в виде эрмитова и антиэрмитова

г) $\{\hat{A}, \hat{B}\}$

д) $i[\hat{A}, \hat{B}]$

A, B – эрм.

$\{\hat{A}; \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A} \overset{\dagger}{\Rightarrow} \hat{B}^\dagger\hat{A}^\dagger + \hat{A}^\dagger\hat{B}^\dagger = \hat{B}\hat{A} + \hat{A}\hat{B}$ т.к. эрмитов

$[\hat{A}; \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \overset{\dagger}{\Rightarrow} -i\hat{B}^\dagger\hat{A}^\dagger + i\hat{A}^\dagger\hat{B}^\dagger = \hat{A}\hat{B} - i\hat{B}\hat{A}$

◀ $\hat{A}\hat{B}|\varphi\rangle = \hat{A}(\hat{B}|\varphi\rangle) \overset{DC}{\rightarrow} \langle\varphi|\hat{B}^\dagger\hat{A}^\dagger$

$= (\hat{A}\hat{B})|\varphi\rangle \overset{DC}{\rightarrow} \langle\varphi|(\hat{A}\hat{B})^\dagger$ $\Rightarrow B^\dagger A^\dagger = (AB)^\dagger$ ■

J.J. Sakurai "Modern Quantum Mechanics" 1.39

◀ Докажем что $\exists \hat{B}, \hat{J},\ \forall \hat{\mathcal{L}}:\ \hat{B}^\dagger = \hat{B},\ \hat{J}^\dagger = -\hat{J},\ \hat{\mathcal{L}} = \hat{B} + \hat{J}$

Пример $B = \frac{1}{2}(\hat{\mathcal{L}} + \hat{\mathcal{L}}^\dagger)$ $\quad J = \frac{1}{2}(\hat{\mathcal{L}} - \hat{\mathcal{L}}^\dagger)$ ■

Задание 2

$\langle\varphi|\hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}|\varphi\rangle =$ [J.J. Sakurai "Modern Quantum Mechanics" 1.39 $\langle\varphi|\hat{\mathcal{L}}^\dagger \overset{DC}{\longleftrightarrow} \hat{\mathcal{L}}|\varphi\rangle$; $|\xi\rangle = \hat{\mathcal{L}}|\varphi\rangle$] $= (\langle\varphi|\mathcal{L}^\dagger)(\mathcal{L}|\varphi\rangle) = \langle\xi|\xi\rangle \in \mathbb{R}^+$ ■

действительное оно также потому, что $\hat{\mathcal{L}}^\dagger\hat{\mathcal{L}}$ – эрмитов что мы доказали в 1.а

Зазание 3

$[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$

$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

$[\hat{A}, \hat{C}] = \hat{A}\hat{C} - \hat{C}\hat{A}$

$\Rightarrow$ $[\hat{A}, \hat{B}]\hat{C} = \hat{A}\hat{B}\hat{C} - \cancel{\hat{B}\hat{A}\hat{C}}$

$\hat{B}[\hat{A}, \hat{C}] = \cancel{\hat{B}\hat{A}\hat{C}} - \hat{B}\hat{C}\hat{A}$

$[\hat{A}, \hat{B}]\hat{C} - \hat{B}[\hat{A}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = [\hat{A}, \hat{B}\hat{C}]$

Задание 4

$$\hat{p}_i = -i\hbar \frac{\partial}{\partial x_i} \qquad \hat{r}_i = x_i \qquad \partial_i \equiv \frac{\partial}{\partial x_i}$$

а) $[\hat{p}_i, \hat{p}_j] = (-i\hbar)^2 \frac{\partial^2}{\partial x_i \partial x_j} - (-i\hbar)^2 \frac{\partial^2}{\partial x_j \partial x_i} = -\hbar \left(\frac{\partial}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_j \partial x_i} \right) = \begin{cases} 0 & \text{Если } \exists \partial_i \varphi, \partial_j \varphi \text{ и } \partial_i \varphi, \partial_j \varphi \text{ – непрерывны (Теорема Шварца)} \\ 0 & i = j \\ -\hbar \left(\frac{\partial}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_j \partial x_i} \right) & \text{в остальных случаях} \end{cases}$

б) $[\hat{p}_i, \hat{x}_j] = \cancel{-i\hbar x_j} \partial_i - i\hbar \partial_i x_j + \cancel{i\hbar x_j} \partial_i = -i\hbar \delta_{ij}$

в) $[\hat{r}_i, \hat{r}_j] = x_i x_j - x_j x_i = 0$

г) $[\hat{p}_x, f(x)] = -i\hbar f'(x) - \cancel{i\hbar f(x)} \partial_x + \cancel{i\hbar f(x)} \partial_x = -i\hbar f'(x)$

д) $[\hat{p}_x^2, f(x)] = \cancel{\hbar^2 f(x)} \partial_x^2 - \cancel{\hbar^2 f(x)} \partial_x^2 - 2\hbar^2 f'(x) \partial_x - \hbar^2 f''(x) = \hbar^2 (2 f'(x) \partial_x + f''(x))$

$$\partial_x^2 f(x) = \partial_x f'(x) + \partial_x f \partial_x = f'' + f' \partial_x + f' \partial_x + f'' \partial_x^2$$