$$\dot{x} + \gamma x = \varphi(t)$$

$$\dot{x} + \gamma x = 0$$

$$x = C\exp(-\gamma t)$$

$$\dot{C}\exp(-\gamma t) = \varphi(t)$$

$$\dot{C} = \varphi(t)\exp(\gamma t)$$

$$x = \int_{-\infty}^{t} \varphi(\tau)\exp(\gamma(t-\tau))\,d\tau$$ вынужденное решение

$$x = \int_{-\infty}^{t} \varphi(\tau)\underbrace{\exp(\gamma(t-\tau))}_{}\,d\tau + C_1$$ общее решение

ядро в физике всегда зависит от разности

$$x(t) = \int_{\mathbb{R}} d\tau\; G(t-\tau)\varphi(\tau)$$

$$G(t) = \Theta(t)\exp(-\gamma t)$$

$$\left(\frac{d}{dt} + \gamma\right) G = \delta(t)\exp(-\gamma t) = \delta(t)$$

$$\left(\frac{d}{dt} + \gamma\right)\int_{\mathbb{R}} d\tau\, G(t-\tau)\varphi(\tau) = \int_{\mathbb{R}} d\tau\,\delta(t-\tau)\varphi(\tau) = \varphi(t)$$

$$\ddot{x} + \nu^2 x = \varphi(t)$$

$$x = \int_{-\infty}^{t} d\tau\, G(t-\tau)\varphi(\tau)$$

$$\left(\frac{d^2}{dt^2} + \nu^2\right) G(t) = \delta(t)$$

Решаем т.к G - линейна при малых t

$$\int_{-\varepsilon}^{+\varepsilon}\left(\frac{d^2}{dt^2} + \cancelto{0}{\nu^2}\right) G\,dt = 1 \Rightarrow \frac{dG}{dt} = 1 \qquad \varepsilon \to 0$$

Составим задачу Коши $G(\varepsilon) = 0$ $\frac{dG}{dt}(\varepsilon) = 1$

$$G(t) = \frac{1}{\nu}\sin(\nu t)$$

В действительности

φ — существует при $t > t_0$

$$x(t) = \int_{t_0}^{\infty} d\tau\, G(t-\tau)\varphi(\tau)$$

Решение с нач условиями

$$x(t) = \dot{x}(0)\,G(t) + x(0)\,\dot{G}(t) + \int_0^t G(\tau - t)\varphi(\tau)\,d\tau$$

$$\mathcal{L}\left(\frac{d}{dt}\right)x = \varphi$$

$\mathcal{L}$ – многочлен n-степени

$\mathcal{L}\left(\frac{d}{dt}\right)G(t) = \delta(t)$ – очев что это решение

$$\boxed{\begin{gathered}\left(\frac{d^3}{dt^3} + a_1\frac{d^2}{dt^2} + a_2\frac{d}{dt} + a_2\right)G(t) = \delta(t) \Rightarrow G(\varepsilon) = 0 \quad \dot{G}(\varepsilon) = 0 \\ \int_{-\varepsilon}^{+\varepsilon}\left(\frac{d^3}{dt^3} + \dots\right)G(t)\,dt = \frac{d^2G}{dt^2}(\varepsilon) = 1\end{gathered}}$$

то-есть скачек при $n-1$ производной

Задача Коши

$$G(\varepsilon) = 0 \quad \ddot{G}(\varepsilon) = 0 \quad \dots \quad G^{n-2}(\varepsilon) = 0 \quad G^{n-1}(\varepsilon) = 1$$

$$\mathcal{L}\left(\frac{d}{dt}\right)x = 0$$

$$x = \sum_i b_i \exp(z_i t)$$

$$\mathcal{L}(z) = 0$$

$$\left.\begin{aligned} G(\varepsilon) &= 0 & \textstyle\sum_i b_i &= 0 \\ \ddot{G}(\varepsilon) &= 0 & \textstyle\sum z_i b_i &= 0 \\ &\dots \\ G^{n-2}(\varepsilon) &= 0 & \textstyle\sum z_i^{n-2} b_i &= 0 \\ G^{n-1}(\varepsilon) &= 0 & \textstyle\sum z_i^{n-1} b_i &= 1 \end{aligned}\right| \Rightarrow \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ z_1 & z_2 & & \dots & z_n \\ & & \dots & & \\ z_1^n & z_2^n & & \dots & z_n^n \end{vmatrix} = \prod_{j>j}(z_j - z_i) \neq 0$$

тогда решение нормальное

Если система равновесна то то решение может только затухоте

если система открыта (не в равновесии с окр. средой) то мож иметь незатух. член

Пример

$$\left(\frac{d^4}{dt^4} + 4\vartheta^2 \frac{d^2}{dt^2} + 3\vartheta^4\right) x = \varphi$$

$$\mathcal{L}(z) = z^4 + 4\vartheta^2 z^2 + 3\vartheta^4 = 0 \Rightarrow z^2 = -2\vartheta^2 \pm \sqrt{4\vartheta^4 - 3\vartheta^2}$$

$$z_{12}^2 = -\vartheta^2 \quad z_{34}^2 = -3\vartheta^2$$

$$G = \sum_i b_i \exp(z_i t) \qquad z_1 = i\vartheta \quad z_2 = -i\vartheta \quad z_3 = i\sqrt{3}\vartheta \quad z_4 = -i\sqrt{3}\vartheta$$

$$G = b_1 \sin(\vartheta t + \varphi_1) + b_2 \sin(\sqrt{3}\vartheta t + \varphi_2)$$

$$G(\varepsilon) = 0 \Rightarrow b_1 \sin(\varphi_1) + b_2 \sin(\varphi_2) = 0$$

$$\ddot{G}(\varepsilon) = 0 \Rightarrow b_1 \vartheta^2 \sin(\varphi_1) + 3 b_2 \vartheta^2 \sin(\varphi_2) = 0$$

$$\dot{G}(\varepsilon) = 0 \Rightarrow b_1 \vartheta \cos(\varphi_1) + b_2 \sqrt{3}\vartheta \cos(\varphi_2) = 0$$

$$\dddot{G}(\varepsilon) = 1 \Rightarrow b_1 \vartheta^3 \cos(\varphi_1) + b_2 3^{3/2} \cos(\varphi_2) = 0$$

$$\varphi_1 = 0 \qquad \varphi_2 = 0$$

$$\Rightarrow b_1 = -\frac{1}{2\sqrt{3}\vartheta^3} \qquad b_2 = \frac{1}{2\vartheta^3}$$

Семинар

$$\ddot{x} + 2\gamma\dot{x} + \vartheta^2 x = \varphi(t)$$

$$\mathcal{L}\left(\frac{d}{dt}\right) = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \vartheta^2 \Rightarrow \xi^2 + 2\gamma\xi + \vartheta^2 = 0 \Rightarrow (\xi + \gamma)^2 = -\vartheta^2 + \gamma^2 \Rightarrow \xi_{12} = -\gamma \pm \sqrt{\gamma^2 - \vartheta^2}$$

① $$\left.\begin{array}{l} G(t) = b_1 \exp\left(\left[\gamma + \sqrt{\gamma^2 - \vartheta^2}\right]t\right) + b_2\exp\left(\left[-\gamma - \sqrt{\gamma^2 + \vartheta^2}\right]t\right) \\ G(\varepsilon) = 0 \\ \dot{G}(\varepsilon) = 1 \end{array}\right| \Rightarrow \left.\begin{array}{l} b_1 + b_2 = 0 \\ b_1\left[-\gamma + \sqrt{\gamma^2 - \vartheta^2}\right] + b_2\left[-\gamma - \sqrt{\gamma^2 - \vartheta^2}\right] = 1 \end{array}\right| \Rightarrow$$

$$b_1 = -b_2 \qquad b_2 = -\frac{1}{2\sqrt{\gamma^2 - \vartheta^2}} \qquad b_1 = \frac{1}{2\sqrt{\gamma^2 - \vartheta^2}} \qquad \begin{pmatrix} 1 & 1 \\ -\vartheta + \sqrt{\gamma^2 - \vartheta^2} & -\vartheta - \sqrt{\gamma^2 - \vartheta^2} \end{pmatrix}$$

$$G = \left\{\frac{1}{2\sqrt{\gamma^2 - \vartheta^2}}\exp\left(\left[-\gamma + \sqrt{\gamma^2 - \vartheta^2}\right]t\right) - \frac{1}{2\sqrt{\gamma^2 - \vartheta^2}}\exp\left(\left[-\gamma - \sqrt{\gamma^2 - \vartheta^2}\right]t\right)\right\}\Theta(t) =$$

$$= \frac{\Theta(t)}{\sqrt{\vartheta^2 - \gamma^2}}\exp(-\gamma t)\sin\left(\sqrt{\vartheta^2 - \gamma^2}\,t\right)$$

② $\vartheta = \gamma \Rightarrow \xi = -\gamma$

$$G = b_1\exp(-\gamma t) + b_2 t\exp(-\gamma t) \Rightarrow -\gamma b_1\exp(-\gamma t) + b_2\exp(-\gamma t) - b_2\gamma t\exp(-\gamma t)$$

$$\left.\begin{array}{l} G(\varepsilon) = 0 \\ \dot{G}(\varepsilon) = 1 \end{array}\right| \Rightarrow \left.\begin{array}{l} b_1 = 0 \\ b_2 = 1 \end{array}\right| \Rightarrow G = t\exp(-\gamma t)$$

③ $\vartheta < \gamma$

$$G = \frac{\exp(-\gamma t)}{2\sqrt{\gamma^2 - \vartheta^2}}\left[\exp\left(\sqrt{\gamma^2 - \vartheta^2}\,t\right) - \exp\left(-\sqrt{\gamma^2 - \vartheta^2}\,t\right)\right]\Theta(t) = \frac{\exp(-\gamma t)}{\sqrt{\gamma^2 - \vartheta^2}}\sinh\left(\sqrt{\gamma^2 - \vartheta^2}\,t\right)\Theta(t)$$

Решение сингулярных

$$\mathcal{L}\left(\frac{d}{dt}\right)G(t) = 0 \Rightarrow \left(\frac{d}{dt} + \gamma\right)^2 G(t) = 0$$

$$\left(\frac{d}{dt} + \gamma\right)G(t) = 0 \Rightarrow G(t) = \exp(-\gamma t)\,C(t)$$

$$\left(\frac{d}{dt} + \gamma\right)\left(\frac{d}{dt} + \gamma\right)\exp(-\gamma t)C(t) = 0$$

$$\left(\frac{d}{dt} + \gamma\right)\exp(-\gamma t)\frac{d}{dt}C(t) = \exp(-\gamma t)\frac{d^2}{dt^2}C(t)$$

$$\left[\frac{d^2}{dt^2}+\vartheta^2\right]^2 G(t)=\delta(t)$$

$$\frac{d^4}{dt^4}+2\vartheta^2\frac{d^2}{dt^2}+\vartheta^4 = \mathcal{L}\left(\frac{d}{dt}\right)$$

$$z^2=-\vartheta^2 \Rightarrow z_{12}=i\vartheta \quad z_{34}=-i\vartheta$$

$$G(t) = \Theta(t)\left\{\left[b_1+b_2 t\right]\exp(i\vartheta t)+\left[b_3+b_4 t\right]\exp(-i\vartheta t)\right\}$$

$$b_1+b_3=0 \Rightarrow G(t)=\Theta(t)\left\{2i\sin(\vartheta t)b_1+ b_2\exp(i\vartheta t)+b_4\exp(-i\vartheta t)\right\}$$

$$b_1\vartheta+b_2=0 \Rightarrow b_2=-\vartheta b_1$$

$$b_2-b_4=0 \Rightarrow G(t)=\Theta(t)\left\{b_1\sin(\vartheta t)+b_2 t\cos(\vartheta t)\right\}$$

$$-\vartheta^2(b_1\vartheta+3b_2)=1 \Rightarrow 2\vartheta^3 b_1=1 \Rightarrow$$

$$\Rightarrow G(t)=\frac{1}{2\vartheta^3}\sin(\vartheta t)-\frac{t}{2\vartheta^2}\cos(\vartheta t)$$