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AA203 HW3

Problem 1

$$\dot{x}(t) = u(t)$$

cost function

$$J = (x(T))^2 + \int_0^T (u(t))^2\, dt$$

HJB equation:

$$0 = J_t^*(x(t),t) + \min_{u(t)} \{ \underbrace{c(x(t),u(t),t)}_{(u(t))^2} + (J_x^*(x(t),t))\, \underbrace{f(x(t),u(t),t)}_{u(t)} \}$$

$\Rightarrow$ if $x > 1+T-t$

$$J_x^*(t,x) = 2(x-T+t)$$
$$J_t^*(t,x) = 2(x-T+t) - 1$$

$$\min_{u(t)} (u(t))^2 + 2(x-T+t)\cdot u(t)$$

$$\frac{\partial}{\partial u} = \cancel{2}u + \cancel{2}(x-T+t) \overset{set}{=} 0$$

$$u^* = -(x-T+t) \rightarrow x > 1+T-t \Rightarrow u^* \leq -1 \Rightarrow u^* = -1$$

$|u(t)| < 1$

HJB eqn:

$$J_t^*(t,x) + (u^*(t))^2 + J_x^*(x(t),t)\, u^*(t)$$
$$= 2(x-T+t) - 1 + \underbrace{(u^*(t))^2}_{1} + 2(x-T+t)\underbrace{u^*(t)}_{-1} = 0 \checkmark$$

$\Rightarrow$ if $x < -(1+T-t)$

$$J_x^*(t,x) = 2(x+T-t)$$
$$J_t^*(t,x) = 2(x+T-t) - 1$$

$$\min_{u(t)} (u(t))^2 + 2(x+T-t)\, u(t)$$

$$\frac{\partial}{\partial u} = \cancel{2}u + \cancel{2}(x+T-t) \overset{set}{=} 0$$

$$u^* = -(x+T-t)$$
$$\geq -(-(1+T-t)+T-t)$$
$$\geq 1$$

but $|u(t)| \leq 1$ so $u^*(t) = 1$

HJB eqn:

$$J_t^* + (u^*(t))^2 + J_x^*\, u^*(t)$$
$$= 2(x+T-t) - 1 + 1 + 2(x+T-t) = 0 \checkmark$$

$\Rightarrow$ if $|x| \leq 1+T-t$

$$J_x^*(t,x) = \frac{2}{(1+T-t)}x \qquad J_t^*(t,x) = \frac{x^2}{(1+T-t)^2}$$

$$\min_u u(t)^2 + \frac{2x}{(1+T-t)}u(t)$$

$$\frac{\partial}{\partial u} = \cancel{2}u + \frac{\cancel{2}x}{(1+T-t)} \overset{set}{=} 0$$

$$u^* = \frac{-x}{1+T-t}$$

HJB eqn:

$$J_t^* + (u^*(t))^2 + J_x^* \, u^*(t)$$

$$\frac{x^2}{(1+T-t)^2} + \frac{x^2}{(1+T-t)^2} - \frac{2x^2}{(1+T-t)^2} = 0 \checkmark$$

$$\boxed{u^* = \begin{cases} -1 & \text{if } x > 1+T-t \\ +1 & \text{if } x < -(1+T-t) \\ \dfrac{-x}{1+T-t} & \text{if } |x| \leq 1+T-t \end{cases}}$$

Problem 2

Cost function

$$Q_f = \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \qquad R = r$$

$$Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$$

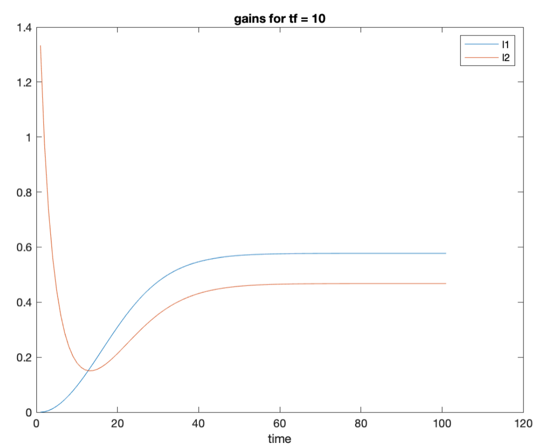
Dynamics

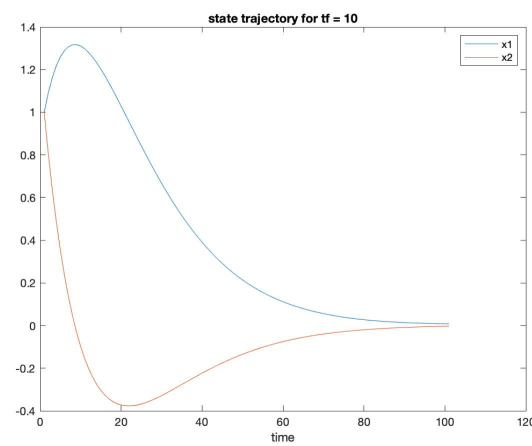
$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

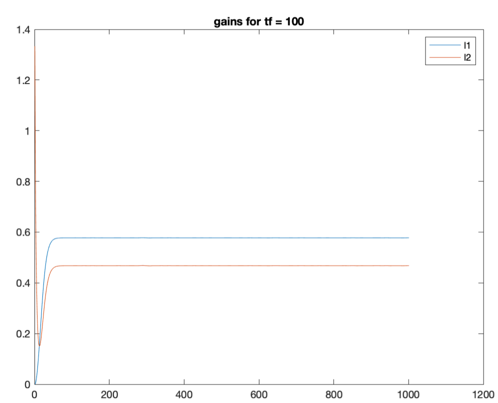
Matrix ODE

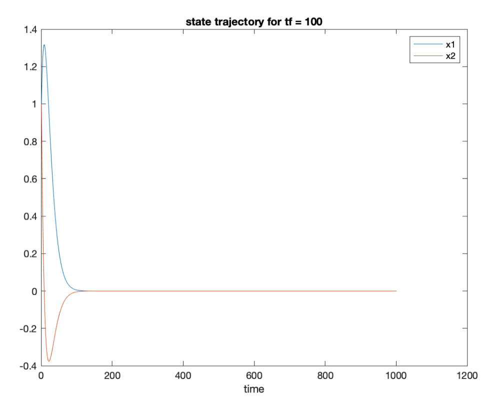
$$\frac{\partial V}{\partial t} = -\Big(\underset{(2\times1)}{Q} - V(t)\cdot \underset{(2\times1)}{B}\cdot \underset{(1\times1)}{R^{-1}}\cdot \underset{(1\times1)}{B^T} V(t) + V(t)A + A^T V(t)\Big)$$

boundary condition: $V(t_f) = Q_f$









Problem 3 — LQG control

Plant dynamics:

$\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = x_1(t) + u(t) + w(t)$ $\quad$ $w(t) \sim N(0, 4)$ ← process noise

$y(t) = x_2(t) + v(t)$ $\quad$ $v(t) \sim N(0, 0.5)$ ← measurement noise

Because of separation principle, we can jointly optimize the estimator and the controller.

To design the controller, find gain using continuous time LQR without any process noise.

Dynamics

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{X} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

Cost: $J = \mathbb{E}\left\{ \frac{1}{2} \int_0^{t_f} \left([x_1 \ x_2] \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}_{Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{1}_{R} u^2(t) \right) dt \right.$

$$Q_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Measurements:

$$y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)$$

Continuous time Kalman filter:

$\Sigma_0 = \quad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\hat{X}_0 =$

$\dot{\Sigma}(t) = A\Sigma + \Sigma A^T + \Sigma_w - \Sigma C^T \Sigma_v C \Sigma$

$\dot{\hat{X}}(t) = A\hat{X} + Bu + \underbrace{\Sigma C^T \Sigma_v^{-1}}_{K(t)} (y - C\hat{X})$

Kalman gain $\quad K(t) = \Sigma(t) C^T \Sigma_v^{-1}$

b) $u(t) = -F\hat{x}(t) \xrightarrow{F \text{ is cont. LQR gain}} C_c = F = R^{-1}B^TV$

$\dot{x}_c(t) = \dot{\hat{x}}(t) = A\hat{x} + Bu + K(t)(y - C\hat{x})$ (cont. Kalman filter update)

$u(t) = -C_c\hat{x}(t) = -R^{-1}B^TV\hat{x}(t)$

$= A\hat{x}(t) - BR^{-1}B^TV\hat{x}(t) + K(t)y(t) - K(t)C\hat{x}(t)$

rearranging terms

$= \underbrace{(A - BR^{-1}B^TV - K(t)C)}_{A_c}\hat{x}(t) + \underbrace{K(t)}_{B_c}y(t)$

$$\boxed{\begin{aligned} A_c &= A - BR^{-1}B^TV - K(t)C \\ B_c &= K(t) \\ C_c &= R^{-1}B^TV \end{aligned}}$$

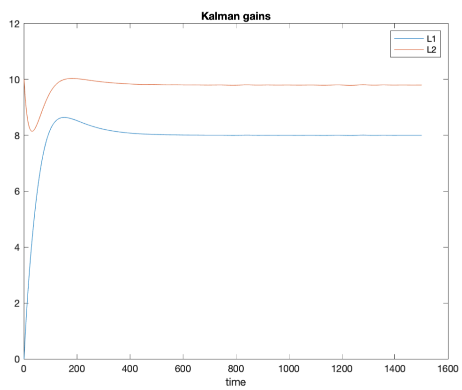
$\hat{x} = x_c$

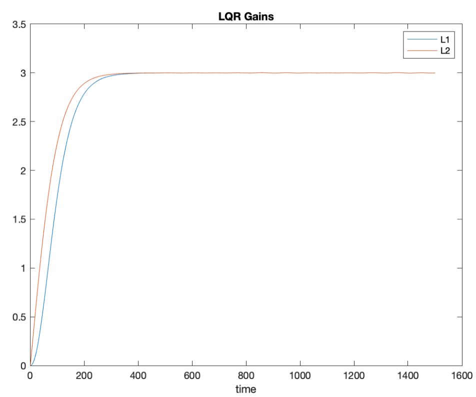
c) $y\left\{\begin{bmatrix} x \\ x_c \end{bmatrix}\right.$

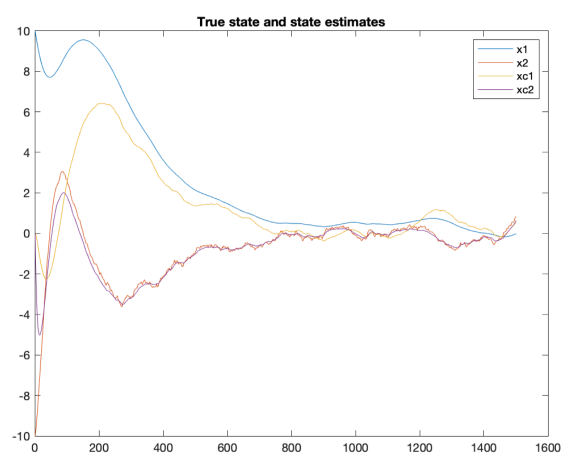
$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} Ax + B\underset{\sim}{u(t)} + \begin{bmatrix}0\\1\end{bmatrix}w(t) \\ A_cx_c(t) + B_c\underset{\sim}{y(t)} \end{bmatrix}$$

$u(t) = -C_c x_c(t)$
$y(t) = Cx(t) + v(t)$

$$= \begin{bmatrix} Ax(t) - BC_c x_c(t) + \begin{bmatrix}0\\1\end{bmatrix}w(t) \\ A_cx_c(t) + B_c[Cx(t) + v(t)] \end{bmatrix}$$







Problem 5

a) cost $= -y(\text{end})$ y value at last timestep.

$$\dot{x} = -x(t)\,u(t) + y(t)\,u^2(t)$$

$$x(t+1) - x(t) = \frac{\dot{x}(t) + \dot{x}(t+1)}{2}(\Delta t)$$

Similarly

$$\dot{y} = x(t)\,u(t) - 3y(t)\,u^2(t)$$

$$y(t+1) - y(t) = \frac{\dot{y}(t) + \dot{y}(t+1)}{2}(\Delta t)$$

$$x(0) = x0$$
$$y(0) = y0$$

See code.

b) The optimal value for the final amount of substance y is

$\boxed{0.29}$