Shenzhen Winter Camp

Lecture 2

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Dynamics

Essential to almost all areas of economics and finance

- Can't price an asset today without considering what it could be sold for tomorrow
- Can't analyze viability of a pension system without considering time paths for income, savings, etc.
- Central banks can't choose interest rates without considering future inflation, unemployment and output

Introductory Example: Solow–Swan

We start with a simple example: Solow–Swan growth

1. Agents save some of their current income
2. Those savings are used to increase capital stock
3. Capital is combined with labour to produce output
4. Output is income (wages, rent on capital)
5. Return to step 1

What happens to output / capital / etc. over time?

In the model, output in each period is

$$Y_t = F(K_t, L_t) \qquad (t = 0, 1, 2, \ldots)$$

Here

- K_t = capital
- L_t = labor
- Y_t = output
- F is the aggregate production function

F assumed to be **homogeneous of degree one** (HD1), meaning

$$F(\lambda K, \lambda L) = \lambda F(K, L) \quad \text{for all } \lambda \geqslant 0$$

Examples.

Cobb-Douglas:

$$F(K, L) = AK^{\alpha} L^{1-\alpha}$$

CES:

$$F(K, L) = \gamma \{\alpha K^{\rho} + (1 - \alpha) L^{\rho}\}^{1/\rho}$$

QuantEcon

Closed economy:

$$\text{current domestic investment} = \text{aggregate domestic savings}$$

The savings rate is a positive constant s, so

$$\text{investment} = \text{savings} = sY_t = sF(K_t, L_t)$$

Depreciation means that 1 unit of capital today becomes $1 - \delta$ units next period

Thus, capital stock evolves according to

$$K_{t+1} = sF(K_t, L_t) + (1 - \delta)K_t$$

QuantEcon

We simplify $K_{t+1} = sF(K_t, L_t) + (1 - \delta)K_t$ as follows

Assume that $L_t =$ some constant L

Set $k_t := K_t / L$ and use HD1 to get

$$
\begin{aligned}
k_{t+1} &= s\frac{F(K_t, L)}{L} + (1 - \delta)k_t \\
&= sF(k_t, 1) + (1 - \delta)k_t
\end{aligned}
$$

Setting $f(k) := F(k, 1)$, the final expression is

$$k_{t+1} = sf(k_t) + (1 - \delta)k_t$$

QuantEcon

In summary, we can write

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) := sf(k) + (1 - \delta)k$$

This kind of equation is called a (scalar) **difference equation**

Question: What are the implied properties of $\{k_t\}$?

More generally, given

- difference equation $x_{t+1} = g(x_t)$
- initial condition x_0,

what are the properties of $\{x_t\}$?

45 Degree Diagrams

Useful for one dimensional dynamic systems

Equally helpful for both linear and nonlinear systems

Let's look at some examples, starting with the difference equation

$$x_{t+1} = g(x_t) \quad \text{when} \quad g(x) = 2 + 0.5x$$

We want to be able to take any x_0 and map out the sequence

$$x_0, \quad x_1 = g(x_0), \quad x_2 = g(x_1), \quad \ldots$$

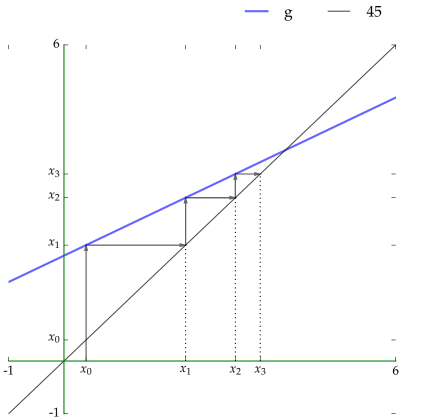


Figure: $g(x) = 2 + 0.5x$ with $x_0 = 0.4$

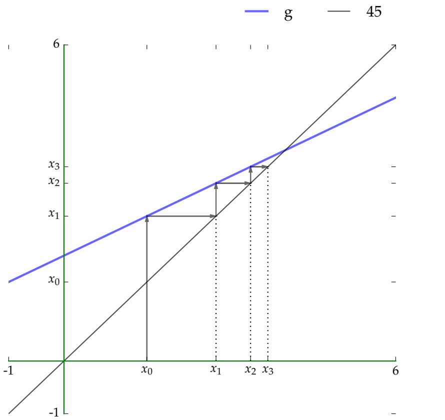


Figure: $g(x) = 2 + 0.5x$ with $x_0 = 1.5$

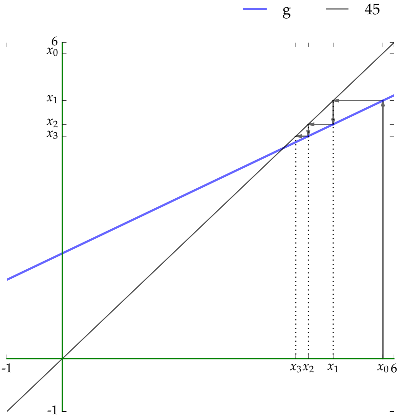


Figure: $g(x) = 2 + 0.5x$ with $x_0 = 5.8$

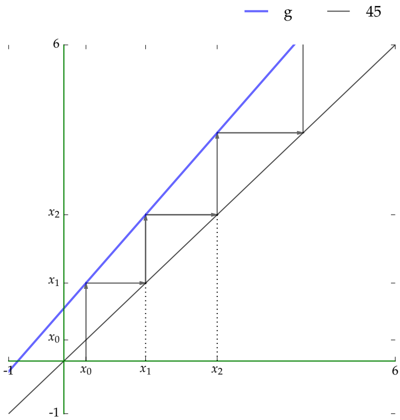


Figure: $g(x) = 1 + 1.2x$ with $x_0 = 0.4$

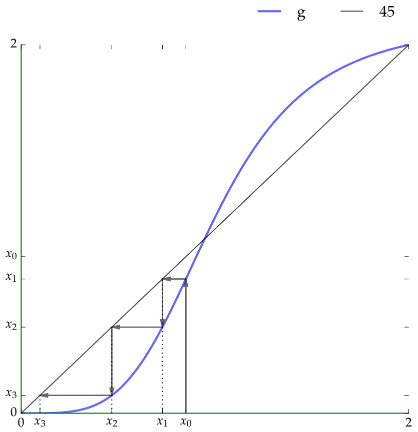


Figure: $g(x) = 2.125/(1 + x^{-4})$ with $x_0 = 0.85$

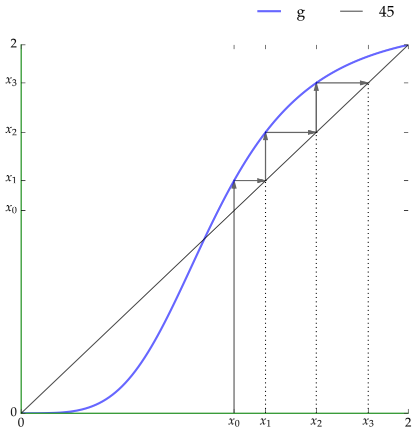
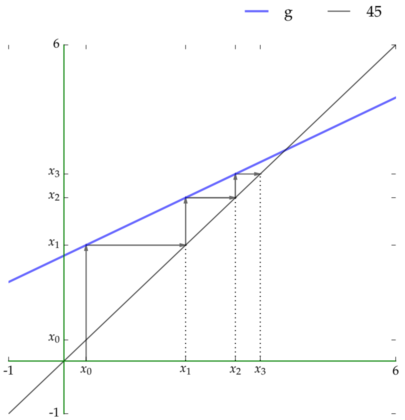


Figure: $g(x) = 2.125/(1 + x^{-4})$ with $x_0 = 1.1$

Let's compare

- 45 degree diagrams
- corresponding time series plots



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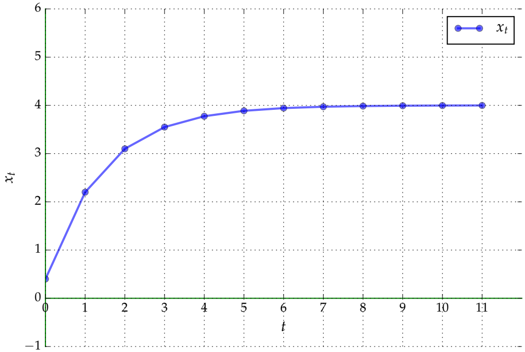

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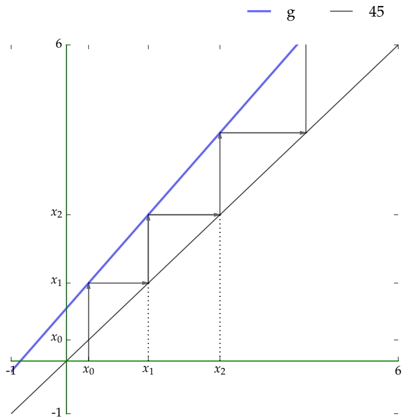
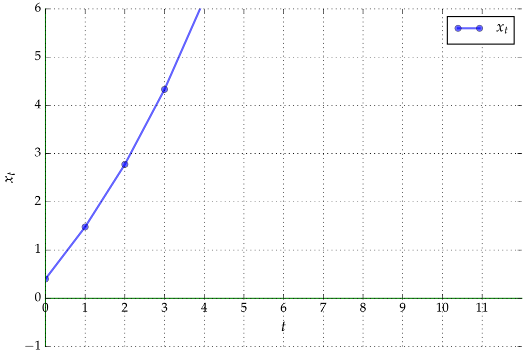


Figure: $g(x) = 1 + 1.2x$ with $x_0 = 0.4$



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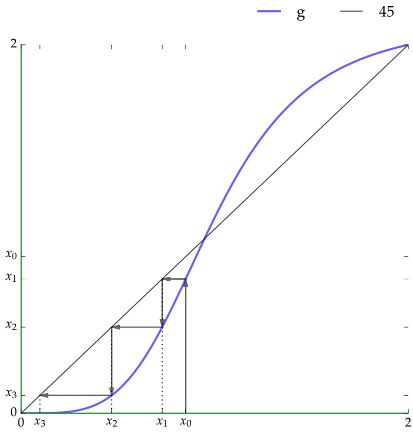

Figure: $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$ with $x_0 = 0.85$

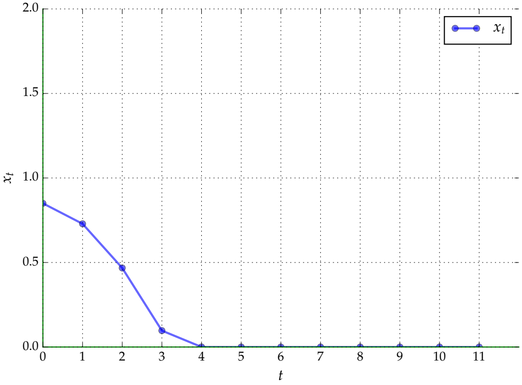

Figure: $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$ with $x_0 = 0.85$

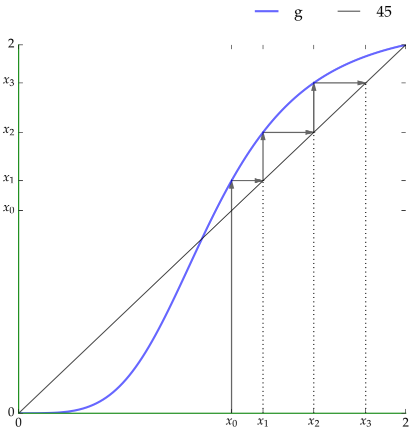

Figure: $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$ with $x_0 = 1.1$

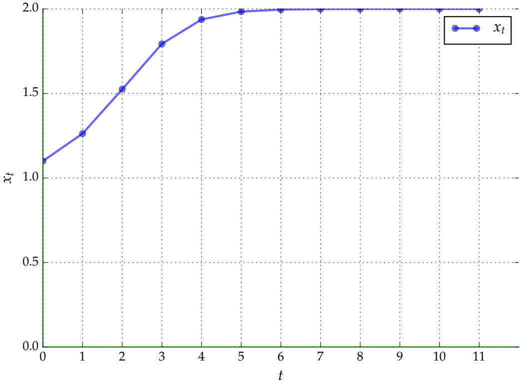

Figure: $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$ with $x_0 = 1.1$

See John/scalar_dynamics.ipynb

Back to Solow-Swan

Let's return to the model

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) := sf(k) + (1 - \delta)k$$

Let's assume that

- $f(k) = Ak^{\alpha}$ where $A = 1$ and $\alpha = 0.6$
- $s = 0.3$ and $\delta = 0.1$

The dynamics can be seen graphically

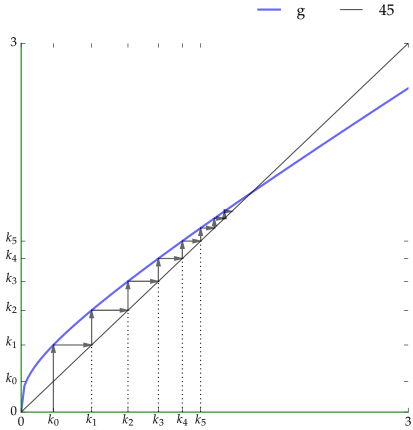

Figure: Solow-Swan dynamics, low initial capital

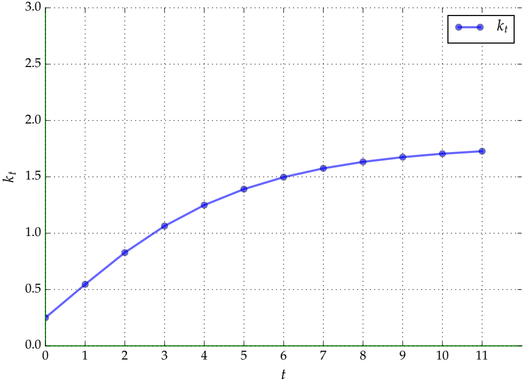

Figure: Solow-Swan dynamics, low initial capital

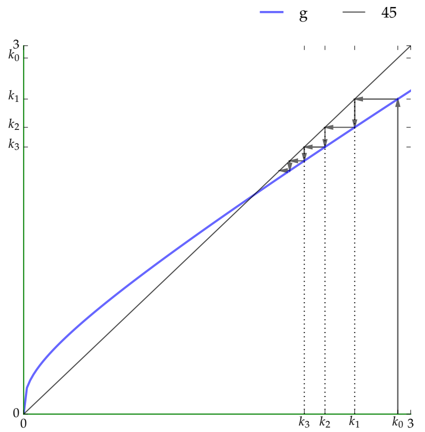

Figure: Solow-Swan dynamics, high initial capital

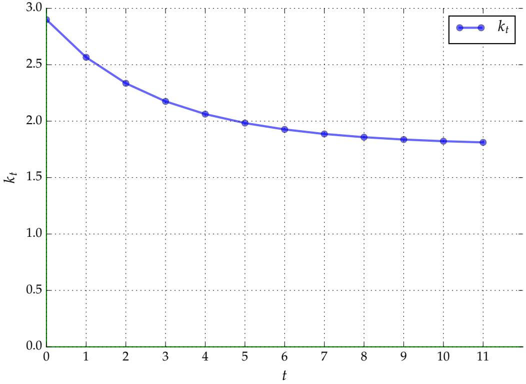

Figure: Solow-Swan dynamics, high initial capital

Graphical analysis of the model suggests that

- k_t increases over time if k_0 is small
- k_t decreases over time if k_0 is large
- k_t converges to the same point regardless of k_0

Definitions

Formally, a **dynamical system** is a pair $(\mathbb{X}, g)$, where

1. $\mathbb{X}$ is a nonempty subset of $\mathbb{R}^K$
2. g is a function mapping $\mathbb{X}$ into itself (a **self-mapping** on $\mathbb{X}$)

These objects are used to represent the difference equation

$$x_{t+1} = g(x_t) \quad \text{where} \quad g \colon \mathbb{X} \to \mathbb{X}$$

The set $\mathbb{X}$ is called the **state space**

The function g is called the **transition rule** or **law of motion**

Example. Let $g(k) = sAk^{\alpha} + (1 - \delta)k$ with

- $A > 0$
- $0 < s, \alpha, \delta < 1$

The pair $([0, \infty), g)$ is a dynamical system

The pair $((0, \infty), g)$ is a dynamical system

Example. Let $g \colon x \mapsto 2x$

The pair $([0, 1], g)$ is not a dynamical system

For example, $g(1) = 2 \notin [0, 1]$

(Hence g is not a self-mapping on $[0, 1]$)

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Let $(\mathbb{X}, g)$ be a dynamical system and consider the sequence generated recursively by

$$x_{t+1} = g(x_t), \quad \text{where } x_0 = \text{ some given point in } \mathbb{X}$$

Not that for this sequence we have

$$x_2 = g(x_1) = g(g(x_0)) =: g^2(x_0)$$

and, more generally,

$$x_t = g^t(x_0) \quad \text{where} \quad g^t = \underbrace{g \circ g \circ \cdots \circ g}_{t \text{ compositions of } g}$$

The sequence $\{g^t(x_0)\}_{t \geqslant 0}$ is called the **trajectory** of $x_0 \in \mathbb{X}$

We will also call it a **time series**

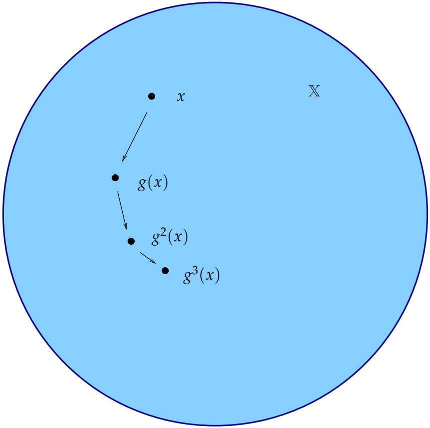


Figure: The trajectory of x under g

Fact. If g is increasing on $\mathbb{X}$ and $\mathbb{X} \subset \mathbb{R}$, then every trajectory is monotone (either increasing or decreasing)

Proof: Pick any $x \in \mathbb{X}$

Either $x \leqslant g(x)$ or $g(x) \leqslant x$ — let's treat the first case

Since g is increasing and $x \leqslant g(x)$ we have $g(x) \leqslant g^2(x)$

Putting these inequalities together gives

$$x \leqslant g(x) \leqslant g^2(x)$$

Continuing in this way gives

$$x \leqslant g(x) \leqslant g^2(x) \leqslant g^3(x) \leqslant \cdots$$

Steady States

Let $(\mathbb{X}, g)$ be a dynamical system

Suppose that x^* is a fixed point of g, so that

$$g(x^*) = x^*$$

Then, for any trajectory $\{x_t\}$ generated by g,

$$x_t = x^* \quad \implies \quad x_{t+1} = g(x_t) = g(x^*) = x^*$$

In other words, if we ever get to x^* we stay there

As a result, in this context, a fixed point of g in $\mathbb{X}$ is also called a **steady state**

- Just a fixed point, not a new concept mathematically

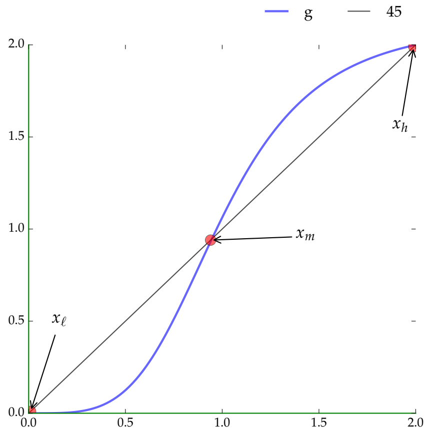


Figure: Steady states of $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$

Example. Recall the Solow-Swan growth model

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) := sAk^\alpha + (1-\delta)k$$

Assume that

1. $\mathbb{X} = (0, \infty)$
2. $A > 0$ and $0 < s, \alpha, \delta < 1$

The system $(\mathbb{X}, g)$ has a steady state given by the solution to

$$k = sAk^\alpha + (1-\delta)k$$

Ex. Solve this equation for k to get steady state

$$k^* := \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

QuantEcon

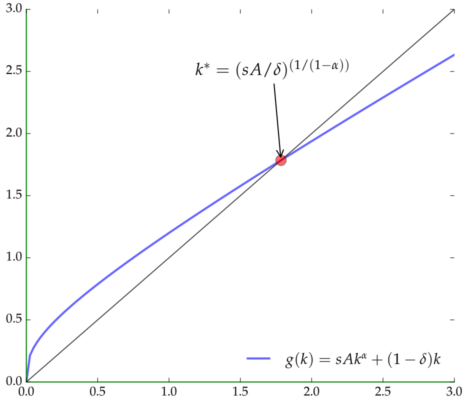


Figure: Steady state of the Solow model

Example. Let's modify the Solow-Swan model to

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) = sA(k)k^\alpha + (1-\delta)k$$

In the Azariadis-Drazen growth model A takes the form

$$A(k) = \begin{cases} A_1 & \text{if } 0 < k < k_b \\ A_2 & \text{if } k_b \leqslant k < \infty \end{cases}$$

The value k_b is a "threshold" value of capital stock

- Assume $0 < A_1 < A_2$, so more productive above k_b
- As usual, $0 < s, \alpha, \delta < 1$

This is a dynamical system with

- $\mathbb{X} = (0, \infty)$
- $g(k) = sA(k)k^\alpha + (1 - \delta)k$

Let

$$k_i^* := \left(\frac{sA_i}{\delta} \right)^{1/(1-\alpha)} \quad \text{for} \quad i = 1, 2$$

Suppose that $k_1^* < k_b < k_2^*$

Ex. Show that $(\mathbb{X}, g)$ has two steady states, given by k_1^* and k_2^*

QuantEcon

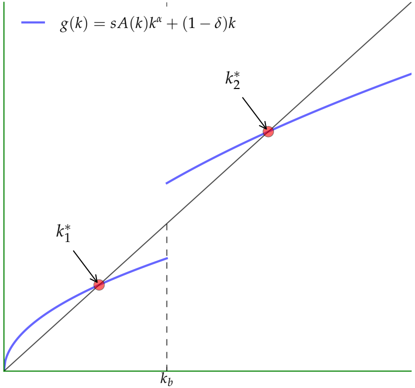


Figure: The threshold model when $k_1^* < k_b < k_2^*$

Stability: Intuition

In some settings trajectories converge

Example. Graphical analysis suggests all trajectories converge for the Solow-Swan model (see above)

Let's look at some more pictures illustrating stability

We focus on the system $(\mathbb{X}, g)$ where $\mathbb{X} = [0, 2]$ and

$$g(x) = \begin{cases} 2.125/(1 + x^{-4}) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

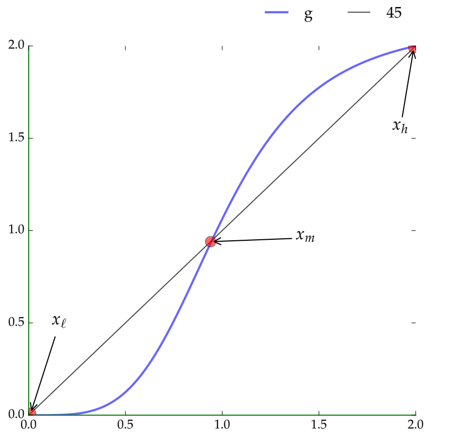


Figure: Steady states of $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$

These steady states appear to have different stability properties

1. x_ℓ is "locally stable"
    - nearby points converge to it
2. x_m is "unstable"
    - nearby points diverge from it
3. x_h is "locally stable"
    - nearby points converge to it

The "basin of attraction" for

- x_ℓ is $[x_\ell, x_m)$
- x_h is $(x_m, x_h]$

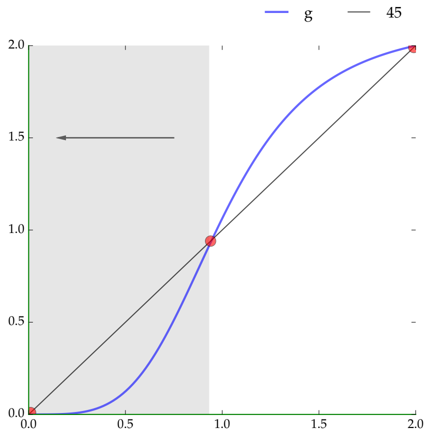


Figure: Basin of attraction for x_ℓ

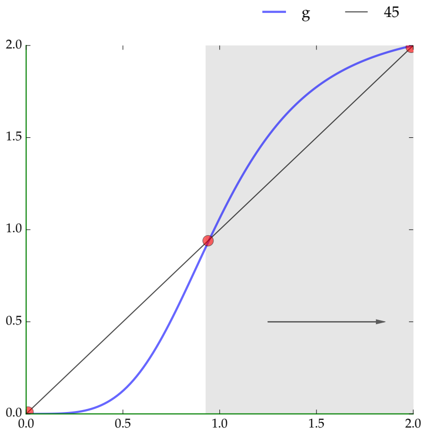


Figure: Basin of attraction for x_h

Let's try to formalize these ideas...

Local Stability

Let x^* be a steady state of $(\mathbb{X}, g)$

The **stable set** of x^* is

$$\mathscr{O}(x^*) := \{x \in \mathbb{X} : g^t(x) \to x^* \text{ as } t \to \infty\}$$

This set is nonempty (why?)

The steady state x^* called **locally stable** or an **attractor** if there exists an $\epsilon > 0$ such that

$$x \in \mathbb{X} \text{ and } \|x - x^*\| < \epsilon \implies x \in \mathscr{O}(x^*)$$

QuantEcon

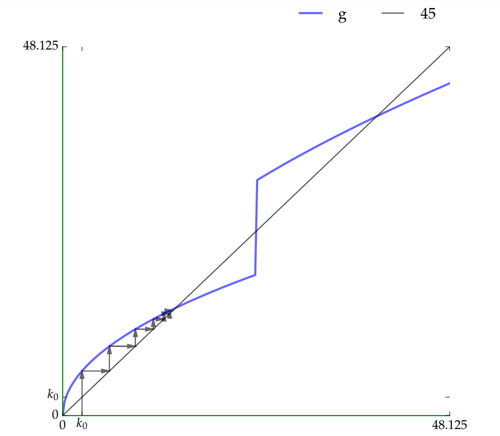
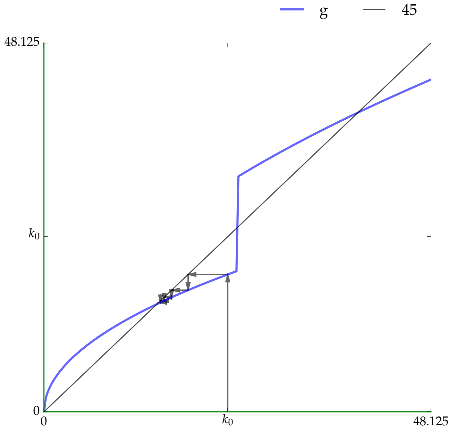


Figure: A poverty trap in the Azariadis–Drazen threshold model



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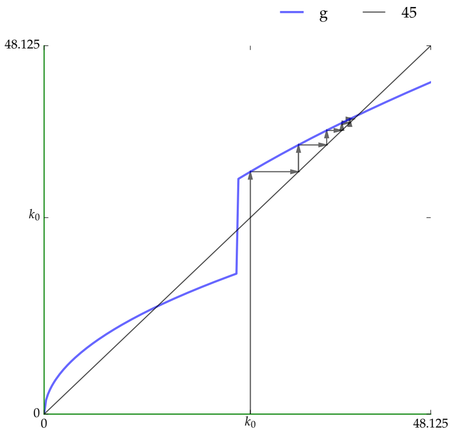

Figure: The higher steady state is also an attractor

Let $\mathbb{X} \subset \mathbb{R}$ and let $x^* \in \mathbb{X}$ be a steady state of $(\mathbb{X}, g)$

Fact. If g is continuously differentiable at x^* and $|g'(x^*)| < 1$, then x^* is locally stable for $(\mathbb{X}, g)$

Proof (omitted) shows that g is "locally a contraction" near x^* under this condition

Ex. Recall the Azariadis-Drazen growth model with steady states

$$k_i^* := \left(\frac{sA_i}{\delta} \right)^{1/(1-\alpha)} \quad \text{for} \quad i = 1, 2$$

Under the assumptions given above, show that k_1^* and k_2^* are both locally stable

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Global Stability

Dynamical system $(\mathbb{X}, g)$ is called **globally stable** if

1. g has a fixed point x^* in $\mathbb{X}$
2. x^* is the only fixed point of g in $\mathbb{X}$
3. $g^t(x) \to x^*$ as $t \to \infty$ for all $x \in \mathbb{X}$

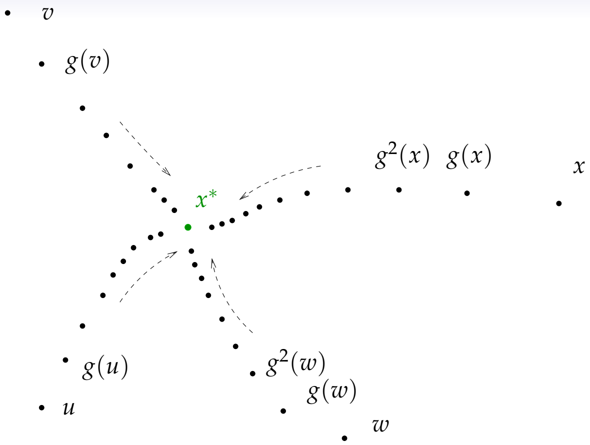

Figure: Visualizing global stability in $\mathbb{R}^2$

Example. Recall the Solow-Swan growth model where

$$k_{t+1} = g(k_t) \quad \text{for} \quad g(k) = sAk^{\alpha} + (1-\delta)k$$

with

1. $\mathbb{X} = (0, \infty)$
2. $A > 0$ and $0 < s, \alpha, \delta < 1$

The system $(\mathbb{X}, g)$ is globally stable with unique fixed point

$$k^* := \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

QuantEcon

Proof: Simple algebra shows that for $k > 0$ we have

$$k = sAk^{\alpha} + (1 - \delta)k \quad \iff \quad k = \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

Hence $(\mathbb{X}, g)$ has unique steady state k^*

It remains to show that $g^t(k) \to k^*$ for every $k \in \mathbb{X} := (0, \infty)$

Let's show this for any $k \leqslant k^*$, leaving $k^* \leqslant k$ as an exercise

Since calculating $g^t(k)$ directly is messy, let's try another strategy

QuantEcon

Claim: If $0 < k \leqslant k^*$, then $\{g^t(k)\}$ is increasing and bounded

Proof increasing: Since g increasing $\{g^t(k)\}$ is monotone

From $k \leqslant k^*$ and some algebra (exercise) we get

$$k \leqslant \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)} \implies g(k) \geqslant k \implies \{g^t(k)\} \text{ increasing}$$

Proof bounded: From $k \leqslant k^*$ and the fact that g is increasing,

$$g(k) \leqslant g(k^*) = k^*$$

Applying g to both sides gives $g^2(k) \leqslant k^*$ and so on

Hence both bounded and increasing

To complete the proof we use the following fact

Fact. If $g^t(k) \to \hat{k}$ for some $k, \hat{k} \in \mathbb{X}$ and g is continuous at $\hat{k}$, then $\hat{k}$ is a fixed point of g

Now fix $k \leqslant k^*$ and recall that $\{g^t(k)\}$ is bounded, increasing

Hence $g^t(k) \to \hat{k}$ for some $\hat{k} \in \mathbb{X}$

Because g is continuous, we know that $\hat{k}$ is a fixed point

But k^* is the only fixed point of $k = g(k)$ as discussed above

Hence $\hat{k} = k^*$

In other words, $g^t(k) \to k^*$ as claimed

Example. Consider again the Solow-Swan growth model

$$k_{t+1} = g(k_t) \quad \text{for} \quad g(k) := sAk^\alpha + (1 - \delta)k$$

where parameters are as before

If $\mathbb{X} = [0, \infty)$ then the same model $(\mathbb{X}, g)$ is not globally stable

- We showed above that g has a fixed point k^* in $(0, \infty)$
- However, 0 is also a fixed point of g on $[0, \infty)$
- Hence $(\mathbb{X}, g)$ has two steady states in $\mathbb{X} = [0, \infty)$

Moral: The state space matters for dynamic properties

Periodic Points and Cycles

If x^* is a steady state of $(\mathbb{X}, g)$ then

$$g^k(x^*) = x^* \quad \text{for all } k \in \mathbb{N}$$

However, some $(\mathbb{X}, g)$ have points x^* such that

$$g^k(x^*) = x^* \quad \text{for some but not all } k \in \mathbb{N}$$

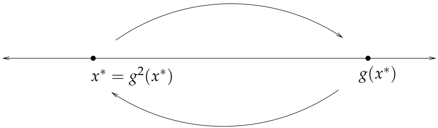

Figure: Here $g(x^*) \neq x^*$ but $g^2(x^*) = x^*$

A point $x^* \in \mathbb{X}$ is called **periodic** for dynamical system $(\mathbb{X}, g)$ if

$$g^k(x^*) = x^* \quad \text{for some } k \in \mathbb{N}$$

Example. Every steady state of $(\mathbb{X}, g)$ is periodic (set $k = 1$)

Example. If $\mathbb{X} = \mathbb{R}$ and $g(x) = -x$ then 1 is periodic because

$$g^2(1) = g(g(1)) = -(-1) = 1$$

The **period** of x^* is the smallest $k \in \mathbb{N}$ such that $g^k(x^*) = x^*$

Example. In the previous example, 1 has period 2

Example. Let $\mathbb{X} = [0,1]$ and let g be the **logistic** map

$$g(x) = 3.5x(1-x)$$

The second composition g^2 has the form

$$\begin{aligned} g^2(x) &= 3.5g(x)(1-g(x)) \\ &= 3.5^2 x(1-x)(1-3.5x(1-x)) \end{aligned}$$

It has two fixed points that are not fixed points of g

These points are periodic with period 2

QuantEcon

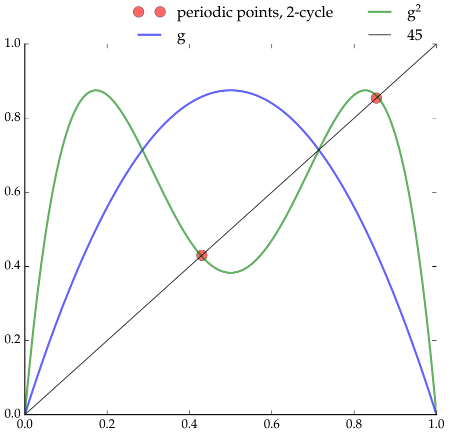

Figure: Logistic map $g(x) = 3.5x(1-x)$ and second iterate g^2

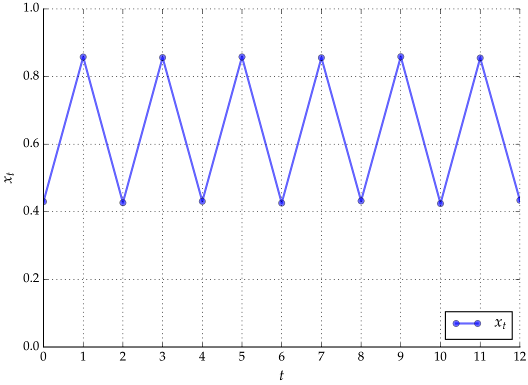

Figure: Time series of logistic map $g(x) = 3.5x(1 - x)$

Chaotic Dynamics

Some simple systems generate complicated time series

Classic example is (some of) the logistic maps

These are systems of the form $(\mathbb{X}, g)$ where $\mathbb{X} := [0, 1]$ and

$$g(x) = rx(1 - x), \qquad r \in [0, 4] \tag{1}$$

Arise mainly in biological models

Let's consider the case $r = 4$

Then almost all starting points generate “complicated" trajectories

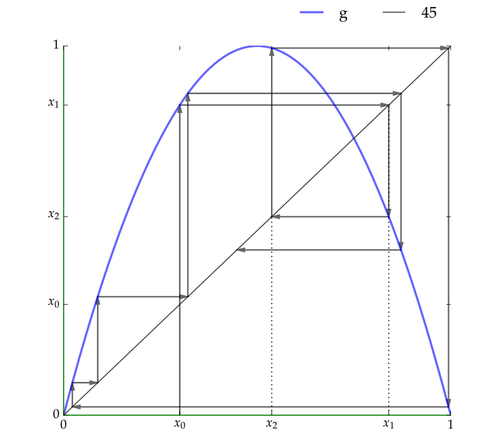

Figure: Logistic map $g(x) = 4x(1-x)$ with $x_0 = 0.3$

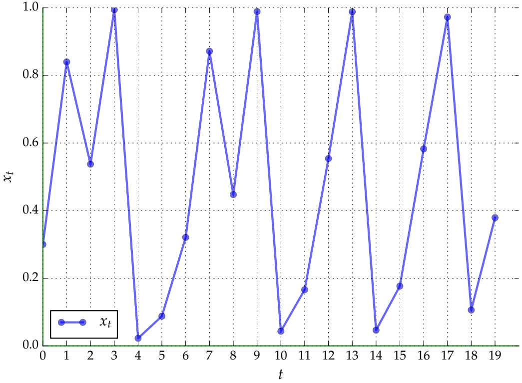

Figure: The corresponding time series

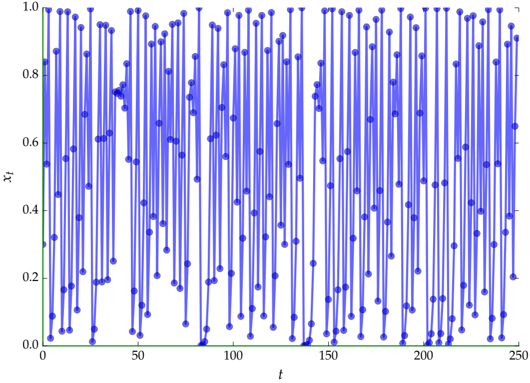

Figure: A longer time series

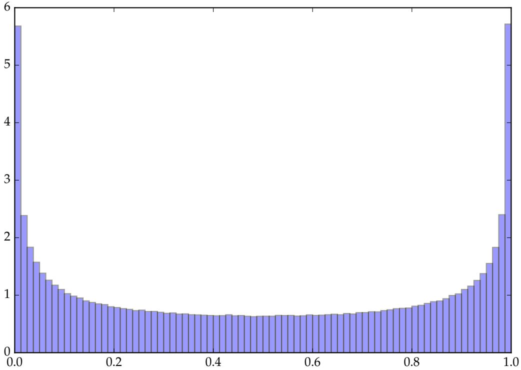

Figure: A long time series, histogram of values