Shenzhen Winter Camp

Lecture 5

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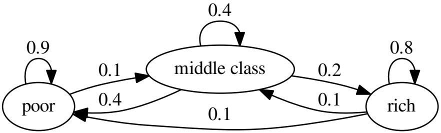
2018

Background: Finite Markov Chains



$$\mathbb{P}\{X_{t+1} = \text{poor} \mid X_t = \text{rich}\} = 0.1$$

Distributions

We start with a **finite state space** $\mathbb{X} = \{x_1, \ldots, x_n\}$

Example. $x_1 =$ poor, $x_2 =$ middle class, $x_3 =$ rich

A **distribution** on $\mathbb{X}$ is a $\phi \colon \mathbb{X} \to \mathbb{R}$ such that

- $\phi(x) \geqslant 0$ for all $x \in \mathbb{X}$
- $\sum_{x \in \mathbb{X}} \phi(x) = 1$

Example. $\phi(x_1) = 1/2$, $\phi(x_2) = 1/4$, $x_3 = 1/4$

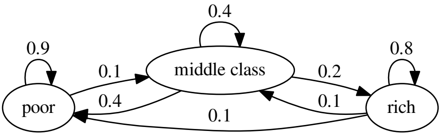
Let $\mathbb{D}$ be the set of distributions on $\mathbb{X}$

A **stochastic kernel** on $\mathbb{X}$ is a $P \colon \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ such that

$$\sum_{y \in \mathbb{X}} P(x, y) = 1 \text{ for all } x \in \mathbb{X}$$

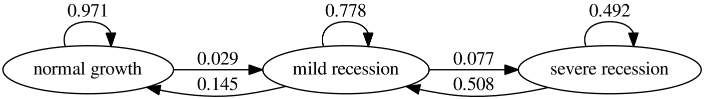
Interpretation: $P(x, y) =$ probability of moving $x \to y$ in one step

Example. $P(\text{rich}, \text{poor}) = 0.1$



Stochastic kernels can be represented by **weighted directed graphs**

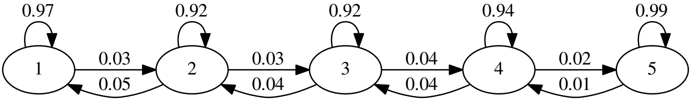
Example. (Hamilton, 2005) estimates a statistical model of the business cycle based on US unemployment data



- set of nodes is $\mathbb{X}$
- no edge means $P(x, y) = 0$

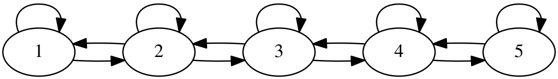
Example. International growth dynamics study of Quah (1993)

State = real GDP per capita relative to world average



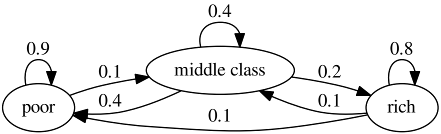
- state 1 means GDP per capita is $\leqslant 1/4$ of world ave
- state 2 means GDP per capita is $1/4 - 1/2$ of world ave
- $\ldots$

Dropping labels gives the directed graph



If P is a stochastic kernel, then

- $P(x, \cdot) \in \mathbb{D}$ for any x
- if at x today, then next period's state is drawn from $P(x, \cdot)$



If rich today, then next period is a draw from

$$P(\text{rich}, \cdot) = (0.1, 0.1, 0.8)$$

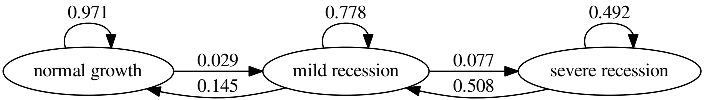
Matrix representation

We can represent any stochastic kernel P by a **Markov matrix**

$$P = \begin{pmatrix} P(x_1, x_1) & \cdots & P(x_1, x_n) \\ \vdots & & \vdots \\ P(x_n, x_1) & \cdots & P(x_n, x_n) \end{pmatrix}$$

- square
- nonnegative
- rows sum to one

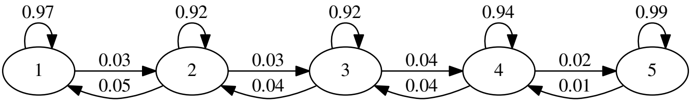
Example. (Hamilton, 2005)



Markov matrix:

$$P_H := \begin{pmatrix} 0.971 & 0.029 & 0 \\ 0.145 & 0.778 & 0.077 \\ 0 & 0.508 & 0.492 \end{pmatrix}$$

Example. Quah (1993)



$$P_Q = \begin{pmatrix} 0.97 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.05 & 0.92 & 0.03 & 0.00 & 0.00 \\ 0.00 & 0.04 & 0.92 & 0.04 & 0.00 \\ 0.00 & 0.00 & 0.04 & 0.94 & 0.02 \\ 0.00 & 0.00 & 0.00 & 0.01 & 0.99 \end{pmatrix}$$

Let ψ be in $\mathbb{D}$ and let P be a stochastic kernel on $\mathbb{X}$

The corresponding **Markov chain** on $\mathbb{X}$ is generated as follows

set $t = 0$ and draw X_t from ψ ;
while $t < \infty$ **do**
  draw X_{t+1} from the distribution $P(X_t, \cdot)$;
  let $t = t + 1$;
end

Here ψ is called the **initial condition**

Linking Marginals

By the law of total probability we have

$$\mathbb{P}\{X_{t+1} = y\} = \sum_{x \in \mathbb{X}} \mathbb{P}\{X_{t+1} = y \mid X_t = x\} \cdot \mathbb{P}\{X_t = x\}$$

Letting ψ_t be the distribution of X_t, this becomes

$$\psi_{t+1}(y) = \sum_{x \in \mathbb{X}} P(x, y) \psi_t(x) \qquad (y \in \mathbb{X})$$

In matrix form, with ψ_i as **row** vectors, this becomes

$$\psi_{t+1} = \psi_t P$$

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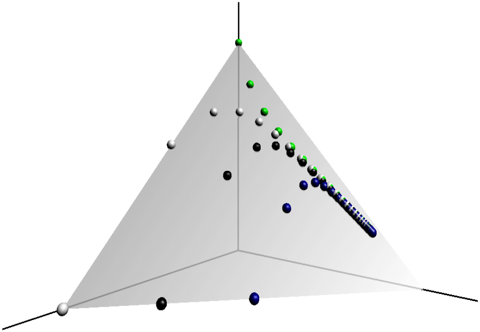
We can view $\psi_{t+1} = \psi_t P$ as a **dynamical system** $(\mathbb{D}, P)$



Trajectories in $\mathbb{D}$ under Hamilton's business cycle model

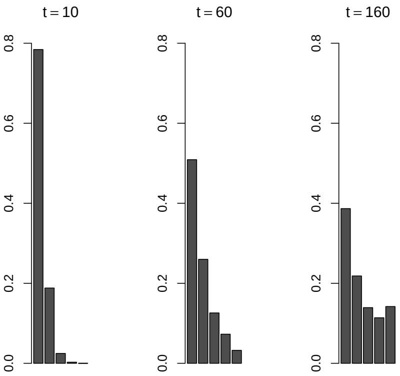


Figure: Distributions from Quah's stochastic kernel, $X_0 = 1$

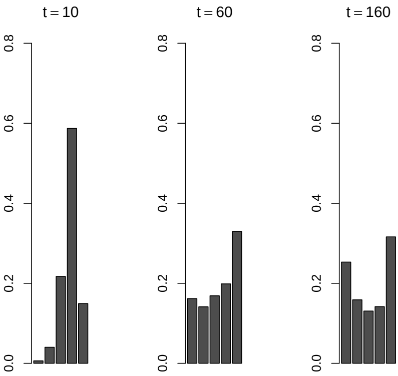


Figure: Distributions from Quah's stochastic kernel, $X_0 = 4$

Stationary Distributions

Let P be a stochastic kernel on $\mathbb{X}$

If $\psi^* \in \mathbb{D}$ satisfies

$$\psi^*(y) = \sum_{x \in \mathbb{X}} P(x, y) \psi^*(x) \quad \text{for all} \quad y \in \mathbb{X}$$

then ψ^* is called **stationary** or **invariant** for P

Equivalent: $\psi^* P = \psi^*$

Equivalent: ψ^* is a steady state of $(\mathbb{D}, P)$

Fact. Every finite state Markov chain has at least one stationary distribution (see Brouwer fixed point theorem)

Probabilistic Properties

Let P be a stochastic kernel on $\mathbb{X}$ and let x, y be states

- $P^k(x, y) =$ probability of moving $x \to y$ in k steps

We say that y is **accessible** from x if $x = y$ or

$$\exists\, k \in \mathbb{N} \text{ such that } P^k(x, y) > 0$$

Equivalent: Accessible in the induced directed graph

A stochastic kernel P on $\mathbb{X}$ is called **irreducible** if every state is accessible from any other

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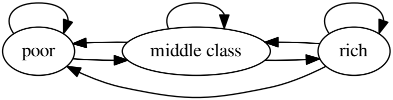
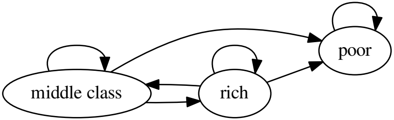
Irreducible:



Not irreducible:



Aperiodicity

Let P be a stochastic kernel on $\mathbb{X}$

State $x \in \mathbb{X}$ is called **aperiodic** under P if

$$\exists\, n \in \mathbb{N} \text{ such that } k \geqslant n \implies P^k(x, x) > 0$$

A stochastic kernel P on $\mathbb{X}$ is called **aperiodic** if every state in $\mathbb{X}$ is aperiodic under P

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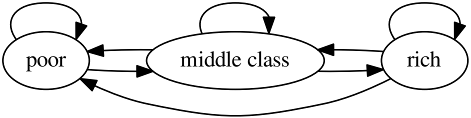
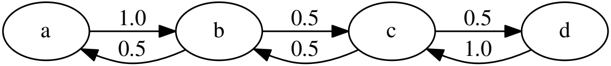
Aperiodic:

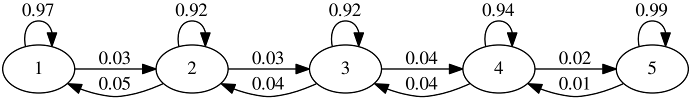


Periodic:



Stability of Markov Chains

Recall the distributions generated by Quah's model



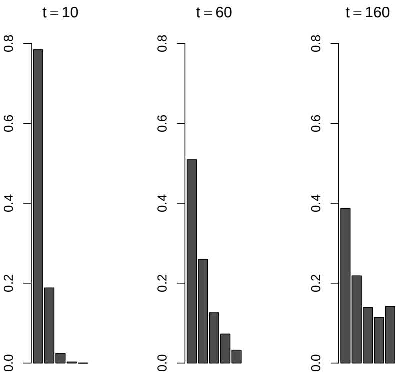


Figure: $X_0 = 1$

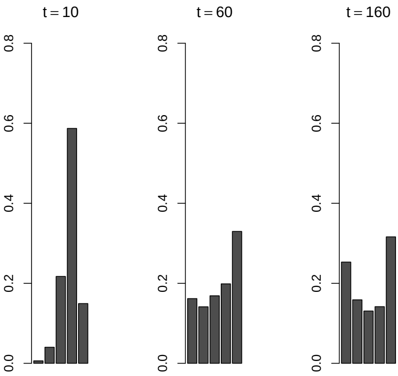


Figure: $X_0 = 4$

What happens when $t \to \infty$?

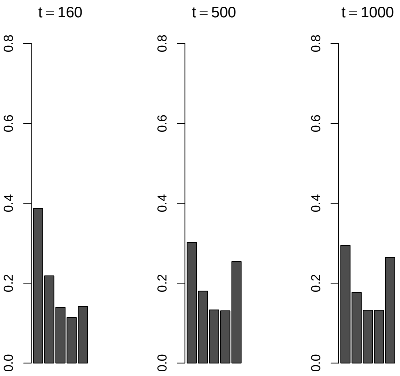


Figure: $X_0 = 1$

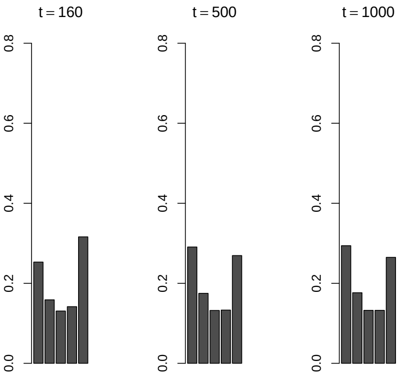


Figure: $X_0 = 4$

At $t = 1000$, the distributions are almost the same for both starting points

This suggests we are observing a form of stability

But how to define stability of Markov chains?

A stochastic kernel P on $\mathbb{X}$ is called **globally stable** if the dynamical sytem $(\mathbb{D}, P)$ is globally stable

Not all stochastic kernels are globally stable

Example. Let $\mathbb{X} = \{1, 2\}$ and consider the periodic Markov chain

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Ex. Show $\psi^* = (0.5, 0.5)$ is stationary for P

Ex. Show that

$$\delta_0 P^t = \begin{cases} \delta_1 & \text{if } t \text{ is odd} \\ \delta_0 & \text{if } t \text{ is even} \end{cases}$$

Conclude that global stability fails

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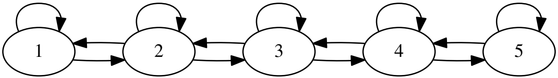
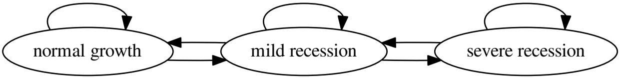
Fact. If P is aperiodic and irreducible, then $(\mathbb{D}, P)$ is globally stable

Example. Quah's stochastic kernel is globally stable



Same with Hamilton's business cycle model



```
In [1]: import quantecon as qe

In [2]: P = [[0.971 , 0.029 , 0],
   ...:       [0.145 , 0.778 , 0.077],
   ...:       [0 , 0.508 , 0.492]]

In [3]: mc = qe.MarkovChain(P)

In [4]: mc.is_aperiodic
Out[4]: True

In [5]: mc.is_irreducible
Out[5]: True

In [6]: mc.stationary_distributions
Out[6]: array([[ 0.8128 ,  0.16256,  0.02464]])
```

Discretization

We can approximate continuous state Markov processes with finite state Markov chains

This is called **discretization** of the process

A common task: discretize the Gaussian AR(1) process

$$X_{t+1} = \rho X_t + \sigma \xi_{t+1} \quad \text{where} \quad \{\xi_t\} \overset{\text{IID}}{\sim} N(0,1)$$

We need a function that maps (ρ, σ, n) to a discrete Markov chain with n states

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A common algorithm in economics is **Tauchen's** method:

```
In [10]: import quantecon as qe

In [11]: mc = qe.tauchen(0.9, 0.1, n=2)

In [12]: mc.state_values
Out[12]: array([-0.6882472,  0.6882472])

In [13]: mc.P
Out[13]:
array([[  1.00000000e+00,   2.92862845e-10],
       [  2.92862879e-10,   1.00000000e+00]])
```

Asset Pricing: An Introduction

An asset is a claim to anticipated future economic benefit

Example. Stocks, bonds, housing

Example. A friend asks if he can borrow \$100

If you agree, then you are purchasing an asset

What factors affect your evaluation of this asset?

Risk Neutral Prices

Let's consider the decisions of identical risk neutral investors

At time t, a certain payoff of G_{t+1} at $t+1$ is worth βG_{t+1} now

Here $\beta \in (0,1)$ is a common discount factor

Example. A standard calibration:

$$\beta = \frac{1}{1+r}$$

where r is a version of the risk free interest rate

If $V_t = \beta G_t$, then $(1+r)V_t = G_t$

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If G_{t+1} is stochastic and investors have rational expectations then the price at time t is

$$P_t = \beta \mathbb{E}_t G_{t+1}$$

More generally, the price of G_{t+n} at $t+n$ is

$$P_t = \beta^n \mathbb{E}_t G_{t+n}$$

Example. Under risk neutrality, European call option that expires in n periods with strike price K has price

$$P_t = \beta^n \mathbb{E}_t \max\{S_{t+n} - K, 0\}$$

- See `John/european_call_option.ipynb`

Pricing Dividend Streams

Let's now consider how to price the dividend stream $\{D_t\}$

We will price an **ex dividend** claim

- a purchase at time t is a claim to $D_{t+1}, D_{t+2}, \ldots$
- we seek P_t given β and these payoffs

For risk-neutral agents, the price satisfies

$$P_t = \beta \, \mathbb{E}_t \left(D_{t+1} + P_{t+1} \right)$$

That is, cost $=$ expected benefit, discounted to present value

A recursive expression with no natural termination point...

To solve

$$P_t = \beta \, \mathbb{E}_t \left(D_{t+1} + P_{t+1} \right)$$

let's first assume that

- $D_t = d(X_t)$ for some nonnegative function d
- $\{X_t\}$ is a finite Markov chain with stochastic matrix Q

We **guess** there is a solution of the form $P_t = p(X_t)$ for some function p

Thus, our aim is to find a p satisfying

$$p(X_t) = \beta \, \mathbb{E} \left[d(X_{t+1}) + p(X_{t+1}) \mid X_t \right]$$

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Suppose $\exists$ a p satisfying

$$p(x) = \beta \sum_{y \in \mathbb{X}} [d(y) + p(y)] \, Q(x, y) \qquad \forall \, x \in \mathbb{X}$$

This is the p we are looking for, since

$$\begin{aligned} p(X_t) &= \beta \sum_{y \in \mathbb{X}} [d(y) + p(y)] \, Q(X_t, y) \\ &= \beta \, \mathbb{E} \, [d(X_{t+1}) + p(X_{t+1}) \mid X_t] \end{aligned}$$

- Hence $P_t = \beta \, \mathbb{E}_t \, (D_{t+1} + P_{t+1})$

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Let's stack these equations:

$$p(x_1) = \beta \sum_{y \in \mathbb{X}} [d(y) + p(y)] \, Q(x_1, y)$$

$$\vdots$$

$$p(x_n) = \beta \sum_{y \in \mathbb{X}} [d(y) + p(y)] \, Q(x_n, y)$$

Treating $p = (p(x_1), \ldots, p(x_n))$ and $d = (d(x_1), \ldots, d(x_n))$ as column vectors, this is equivalent to

$$p = \beta Q d + \beta Q p$$

Does this have a unique solution and, if so, how can we find it?

Claim $r(\beta Q) < 1$

Fact. If $a \in \mathbb{R}$ and B is any square matrix, then $r(aB) = |a| r(B)$

Fact. Q a Markov matrix $\implies r(Q) = 1$

Hence $r(\beta Q) = \beta < 1$

Hence $p = \beta Q d + \beta Q p$ has a unique solution, satisfying

$$p = (I - \beta Q)^{-1} \beta Q d = \sum_{i=1}^{\infty} (\beta Q)^i d$$

- See John/markov_asset_tauchen.ipynb

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Application: LQ Risk Neutral Asset Pricing

Let's consider again the risk neutral asset pricing formula

$$P_t = \beta \mathbb{E}_t [D_{t+1} + P_{t+1}]$$

where now

$$D_t = x_t' \Delta x_t \text{ for some positive definite } \Delta \in \mathcal{M}(n \times n)$$

and

$$x_{t+1} = A x_t + C w_{t+1}$$

Assume that $\{w_t\}$ is IID with $\mathbb{E}\, w_t = 0$ and $\mathbb{E}_t [w_{t+1} w_{t+1}'] = I$

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Reminder: Neumann Series Theory

Question Under what conditions does the linear system

$$x = Ax + b$$

have a unique solution?

Here

- b is $n \times 1$
- A is in $\mathcal{M}(n \times n)$
- x is $n \times 1$ and the object we wish to solve for

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Recall that $r(A)$ is the **spectral radius** of A:

$$r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

Theorem. If $r(A) < 1$ and I is the $n \times n$ identity, then $I - A$ is invertible and

$$(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$$

Corollary The linear system $x = Ax + b$ has the unique solution

$$x^* = (I - A)^{-1} b = \sum_{i=0}^{\infty} A^i b$$

whenever $r(A) < 1$

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Preliminary: Predicting Quadratics

Fact. If $H \in \mathcal{M}(n \times n)$ and $\{x_t\}$ is as above, then

$$\mathbb{E}_t[x'_{t+1} H x_{t+1}] = x'_t A' H A x_t + \mathrm{trace}(C' H C)$$

Proof

$$\mathbb{E}_t[x'_{t+1} H x_{t+1}] = \mathbb{E}_t[(A x_t + C w_{t+1})' H (A x_t + C w_{t+1})]$$

The RHS expands to

$$\mathbb{E}_t[x'_t A' H A x_t] + 2\mathbb{E}_t[x'_t A' H C w_{t+1}] + \mathbb{E}_t[w'_{t+1} C' H C w_{t+1}]$$

$$= I + II + III$$

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Since x_t is known at t we have

$$I = \mathbb{E}_t[x_t' A' H A x_t] = x_t' A' H A x_t$$

Since $\{w_t\}$ is IID,

$$II = 2\mathbb{E}_t[x_t' A' H C w_{t+1}] = 2 x_t' A' H C \, \mathbb{E}_t[w_{t+1}] = 0$$

Finally,

$$III = \mathbb{E}_t[w_{t+1}' C' H C w_{t+1}] = \operatorname{trace}(C' H C)$$

Hence

$$\mathbb{E}_t[x_{t+1}' H x_{t+1}] = x_t' A' H A x_t + \operatorname{trace}(C' H C)$$

Predicting Dividends

Applying this to prediction of dividends gives

$$\mathbb{E}_t[D_{t+1}] = x_t' A' \Delta A x_t + \operatorname{trace}(C' \Delta C)$$

Comments

- Our time t prediction of D_{t+1} is a function of x_t
- The same true for any D_{t+j}

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Prices as Functions of the State

As before, we conjecture that

$$P_t = p(x_t) \quad \text{for some function } p$$

Another leap: guess that prices are a **quadratic** in x_t

In particular, we guess that

$$p(x) = x' \Pi x + \delta$$

for some positive definite Π and scalar δ

Substituting

$$P_t = x_t' \Pi x_t + \delta \quad \text{and} \quad D_t = x_t' \Delta x_t$$

into

$$P_t = \beta \mathbb{E}_t [D_{t+1} + P_{t+1}]$$

gives

$$\begin{aligned}
x_t' \Pi x_t + \delta &= \beta \mathbb{E}_t [x_{t+1}' \Delta x_{t+1} + x_{t+1}' \Pi x_{t+1} + \delta] \\
&= \beta \mathbb{E}_t [x_{t+1}' (\Delta + \Pi) x_{t+1}] + \beta \delta \\
&= \beta x_t' A' (\Delta + \Pi) A x_t + \beta \operatorname{trace}(C' (\Delta + \Pi) C) + \beta \delta
\end{aligned}$$

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So, we seek a pair $\Pi \in \mathcal{M}(n \times n)$, $\delta \in \mathbb{R}$ such that

$$x'\Pi x + \delta = \beta x' A'(\Delta + \Pi) A x + \beta \operatorname{trace}(C'(\Delta + \Pi) C) + \beta \delta$$

for any $x \in \mathbb{R}^n$

Suppose exists $\Pi^* \in \mathcal{M}(n \times n)$ such that

$$\Pi^* = \beta A'(\Delta + \Pi^*) A$$

Claim: If this is true and

$$\delta^* := \frac{\beta}{1 - \beta} \operatorname{trace}(C'(\Delta + \Pi^*) C)$$

then the pair Π^*, δ^* solves the above equation for any x

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Proof: By hypothesis, $\Pi^* = \beta A'(\Delta + \Pi^*)A$

$$\therefore \quad x'\Pi^* x = \beta x' A'(\Delta + \Pi^*)Ax$$

$$\therefore \quad x'\Pi^* x + \delta^* = \beta x' A'(\Delta + \Pi^*)Ax + \delta^*$$

To complete the proof, sufficies to show that

$$\delta^* = \beta \operatorname{trace}(C'(\Delta + \Pi^*)C) + \beta\delta^*$$

This is true from definition of δ^*

Last step: Find $\Pi \in \mathcal{M}(n \times n)$ that solves

$$\Pi = \beta A'(\Delta + \Pi)A \tag{1}$$

Claim: A unique solution to (1) exists whenever $\rho(A) < 1/\sqrt{\beta}$

Proof: Letting $M := \beta A' \Delta A$ and $\Lambda := \sqrt{\beta} A'$, we can express (1) as

$$\Pi = \Lambda \Pi \Lambda' + M$$

- A discrete Lyapunov equation in Π

Since $\rho(\Lambda) < 1$, a unique solution Π^* exists

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LQ Asset Pricing Summary

We have shown that

$$\rho(A) < \frac{1}{\sqrt{\beta}} \implies \Pi = \beta A'(\Delta + \Pi)A \text{ has a unique solution}$$

The solution Π^* and associated δ^* gives the pricing function

$$p^*(x) := x'\Pi^* x + \delta^*$$

This pricing function satisfies the risk neutral asset pricing equation

Ex. Show that Π is positive definite whenever A is nonsingular

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Risk Aversion

Is it appropriate to use risk neutral pricing?

Consider a two period problem

$$\max_{\alpha}\{u(C_t) + \beta \mathbb{E}_t u(C_{t+1})\}$$

$$\text{s.t. } C_t = E_t - P_t \alpha \quad \text{and} \quad C_{t+1} = E_{t+1} + \alpha G_{t+1}$$

Here

- G_{t+1} is the payoff of the asset
- α is the share purchased
- P_t is the current price
- E_t and E_{t+1} are endowments

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Rewrite as

$$\max_{\alpha}\{u(E_t - P_t\alpha) + \beta\mathbb{E}_t u(E_{t+1} + \alpha G_{t+1})\}$$

The **first order condition** is

$$u'(E_t - P_t\alpha)P_t = \beta\mathbb{E}_t u'(E_{t+1} + \alpha G_{t+1})G_{t+1}$$

Rearranging,

$$P_t = \mathbb{E}_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}G_{t+1}\right]$$

Note: reduces to the risk neutral case when u has the form $u(x) = ax + b$

To accommodate the previous case, let's write the price of a claim to payoff G_{t+1} as

$$P_t = \mathbb{E}_t \left[M_{t+1} G_{t+1} \right]$$

where M_{t+1} is called the **stochastic discount factor**, or SDF

The special case $\beta = M_{t+1}$ is the risk neutral case

The other famous special case, described above, is

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

This is the SDF derived in Lucas (1978)

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Pricing Dividend Streams with Risk Aversion

Let's apply this to pricing the dividend stream $\{D_t\}$

As before, the asset is a claim to $D_{t+1}, D_{t+2}, \ldots$

Our aim is to solve for $\{P_t\}$ given $\{M_t\}$ and $\{D_t\}$

The price now satisfies

$$P_t = \mathbb{E}_t \left[M_{t+1} (D_{t+1} + P_{t+1}) \right]$$

We can solve this recursion for P_t following our previous path

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To solve

$$P_t = \mathbb{E}_t[M_{t+1}(D_{t+1} + P_{t+1})]$$

let's assume that

- $D_{t+1} = d(X_{t+1})$ for some nonnegative function d
- $M_{t+1} = m(X_t, X_{t+1})$ for some positive function m
- $\{X_t\}$ is a finite Markov chain with stochastic matrix Q

Guessing a solution of the form $P_t = p(X_t)$, we aim to solve

$$p(X_t) = \mathbb{E}\left[m(X_t, X_{t+1})(d(X_{t+1}) + p(X_{t+1})) \mid X_t\right]$$

As before, it's sufficient to find a p satisfying

$$p(x) = \sum_{y \in \mathbb{X}} m(x, y) \left[d(y) + p(y) \right] Q(x, y)$$

for all $x \in \mathbb{X}$

Equivalently, with

$$K(x, y) := m(x, y) Q(x, y)$$

we seek a p that solves

$$p(x) = \sum_{y \in \mathbb{X}} \left[d(y) + p(y) \right] K(x, y)$$

for all $x \in \mathbb{X}$

Treating $K(x, y)$ as a matrix and using matrix algebra, we can stack the equations

$$p(x) = \sum_{y \in \mathbb{X}} [d(y) + p(y)] K(x, y)$$

to obtain

$$p = Kd + Kp$$

This equation has the unique solution

$$p = (I - K)^{-1} Kd$$

whenever $r(K) < 1$

- See John/markov_asset_tauchen.ipynb