2.3 Dynamic Optimization: A Cake Eating Example

Here we will look at a very simple dynamic optimization problem. We begin with a finite horizon and then discuss extensions to the infinite horizon.[4]

Suppose that you have a cake of size W_1. At each point of time, $t = 1, 2, 3, ...T$ you can consume some of the cake and thus save the remainder. Let c_t be your consumption in period t and let $u(c_t)$ represent the flow of utility from this consumption. The utility function is not indexed by time: preferences are stationary. Assume $u(\cdot)$ is real-valued, differentiable, strictly increasing and strictly concave. Assume that $\lim_{c\to 0} u'(c) \to \infty$. Represent lifetime utility by

$$\sum_{t=1}^{T} \beta^{(t-1)} u(c_t)$$

where $0 \leq \beta \leq 1$ and β is called the **discount factor**.

For now, assume that the cake does not depreciate (melt) or grow. Hence, the evolution of the cake over time is governed by:

$$W_{t+1} = W_t - c_t \tag{2.1}$$

for $t = 1, 2, ..T$. How would you find the optimal path of consumption, $\{c_t\}_1^T$?[5]

2.3.1 Direct Attack

One approach is to solve the constrained optimization problem directly. This is called the **sequence problem** by Stokey and Lucas (1989). Consider the problem of:

$$\max_{\{c_t\}_1^T, \{W_t\}_2^{T+1}} \sum_{t=1}^{T} \beta^{(t-1)} u(c_t) \tag{2.2}$$

subject to the transition equation (2.1), which holds for $t = 1, 2, 3, ...T$. Also, there are non-negativity constraints on consumption and the cake given by: $c_t \geq 0$ and $W_t \geq 0$. For this problem, W_1 is given.

Alternatively, the flow constraints imposed by (2.1) for each t could be combined yielding:

$$\sum_{t=1}^{T} c_t + W_{T+1} = W_1. \tag{2.3}$$

The non-negativity constraints are simpler: $c_t \geq 0$ for $t = 1, 2, ..T$ and $W_{T+1} \geq 0$. For now, we will work with the single resource constraint. This is a well-behaved problem as the objective is concave and continuous and the constraint set is compact. So there is a solution to this problem.[6]

Letting λ be the multiplier on (2.3), the first order conditions are given by:

$$\beta^{t-1} u'(c_t) = \lambda$$

for $t = 1, 2, ..., T$ and

$$\lambda = \phi$$

where ϕ is the multiplier on the non-negativity constraint on W_{T+1}. The non-negativity constraints on $c_t \geq 0$ are ignored as we assumed that the marginal utility of consumption becomes infinite as consumption approaches zero within any period.

Combining equations, we obtain an expression that links consumption across any two periods:

$$u'(c_t) = \beta u'(c_{t+1}). \tag{2.4}$$

This is a necessary condition for optimality for **any** t: if it was violated, the agent could do better by adjusting c_t and c_{t+1}. Frequently, (2.4) is referred to as an **Euler**

equation.

To understand this condition, suppose that you have a proposed (candidate) solution for this problem given by $\{c_t^*\}_1^T, \{W_t^*\}_2^{T+1}$. Essentially, the Euler equation says that the marginal utility cost of reducing consumption by ε in period t equals the marginal utility gain from consuming the extra ε of cake in the next period, which is discounted by β. If the Euler equation holds, then it is impossible to increase utility by moving consumption across adjacent periods given a candidate solution.

It should be clear though that this condition may not be sufficient: it does not cover deviations that last more than one period. For example, could utility be increased by reducing consumption by ε in period t saving the "cake" for two periods and then increasing consumption in period t+2? Clearly this is not covered by a single Euler equation. However, by combining the Euler equation that hold across period t and $t+1$ with that which holds for periods $t+1$ and $t+2$, we can see that such a deviation will not increase utility. This is simply because the combination of Euler equations implies:

$$u'(c_t) = \beta^2 u'(c_{t+2})$$

so that the two-period deviation from the candidate solution will not increase utility.

As long as the problem is finite, the fact that the Euler equation holds across all adjacent periods implies that any finite deviations from a candidate solution that satisfies the Euler equations will not increase utility.

Is this enough? Not quite. Imagine a candidate solution that satisfies all of the Euler equations but has the property that $W_T > c_T$ so that there is cake left over. This is clearly an inefficient plan: having the Euler equations holding is necessary but not sufficient. Hence the optimal solution will satisfy the Euler equation for each period and the agent will consume the entire cake!

Formally, this involves showing the non-negativity constraint on W_{T+1} must bind. In fact, this constraint is binding in the above solution: $\lambda = \phi > 0$. This non-negativity constraint serves two important purposes. First, in the absence of a constraint that $W_{T+1} \geq 0$, the agent would clearly want to set $W_{T+1} = -\infty$ and thus die with outstanding obligations. This is clearly not feasible. Second, the fact that the constraint is binding in the optimal solution guarantees that cake is not being thrown away after period T.

So, in effect, the problem is pinned down by an initial condition (W_1 is given) and by a terminal condition ($W_{T+1} = 0$). The set of $(T-1)$ Euler equations and (2.3) then determine the time path of consumption.

Let the solution to this problem be denoted by $V_T(W_1)$ where T is the horizon of the problem and W_1 is the initial size of the cake. $V_T(W_1)$ represents the maximal utility flow from a T period problem given a size W_1 cake. From now on, we call this a **value function**. This is completely analogous to the indirect utility functions expressed for the household and the firm.

As in those problems, a slight increase in the size of the cake leads to an increase in lifetime utility equal to the marginal utility in any period. That is,

$$V_T'(W_1) = \lambda = \beta^{t-1} u'(c_t), t = 1, 2, ...T.$$

It doesn't matter when the extra cake is eaten given that the consumer is acting optimally. This is analogous to the point raised above about the effect on utility of an increase in income in the consumer choice problem with multiple goods.

2.3.2 Dynamic Programming Approach

Suppose that we change the above problem slightly: we add a period 0 and give an initial cake of size W_0. One approach to determining the optimal solution of this augmented problem is to go back to the sequence problem and resolve it using this longer horizon and new constraint. But, having done all of the hard work with the T period problem, it would be nice not to have to do it again!

Finite Horizon Problem

The dynamic programming approach provides a means of doing so. It essentially converts a (arbitrary) T period problem into a 2 period problem with the appropriate rewriting of the objective function. In doing so, it uses the value function obtained from solving a shorter horizon problem.

So, when we consider adding a period 0 to our original problem, we can take advantage of the information provided in $V_T(W_1)$, the solution of the T period problem given W_1 from (2.2). Given W_0, consider the problem of

$$\max_{c_0} u(c_0) + \beta V_T(W_1) \tag{2.5}$$

where

$$W_1 = W_0 - c_0; W_0 \text{ given.}$$

In this formulation, the choice of consumption in period 0 determines the size of the cake that will be available starting in period 1, W_1. So instead of choosing a sequence of consumption levels, we are just choosing c_0. Once c_0 and thus W_1 are determined, the value of the problem from then on is given by $V_T(W_1)$. This function completely summarizes optimal behavior from period 1 onwards. For the purposes of the dynamic programming problem, it doesn't matter how the cake will be consumed after the initial period. All that is important is that the agent will be acting optimally and thus generating utility given by $V_T(W_1)$. This is the **principle of optimality**, due to Richard Bellman, at work. With this knowledge, an optimal decision can be made regarding consumption in period 0.

Note that the first order condition (assuming that $V_T(W_1)$ is differentiable) is given by:

$$u'(c_0) = \beta V_T'(W_1)$$

so that the marginal gain from reducing consumption a little in period 0 is summarized by the derivative of the value function. As noted in the earlier discussion of the T period sequence problem,

$$V_T'(W_1) = u'(c_1) = \beta^t u'(c_{t+1})$$

for $t = 1, 2, ...T - 1$. Using these two conditions together yields

$$u'(c_t) = \beta u'(c_{t+1}),$$

for $t = 0, 1, 2, ...T - 1$, a familiar necessary condition for an optimal solution.

Since the Euler conditions for the other periods underlie the creation of the value function, one might suspect that the solution to the $T + 1$ problem using this dynamic programming approach is identical to that from using the sequence approach.[7] This is clearly true for this problem: the set of first order conditions for the two problems are identical and thus, given the strict concavity of the $u(c)$ functions, the solutions will be identical as well.

The apparent ease of this approach though is a bit misleading. We were able to make the problem look simple by pretending that we actually knew $V_T(W_1)$. Of course, we had to solve for this either by tackling a sequence problem directly or by building it recursively starting from an initial single period problem.

On this latter approach, we could start with the single period problem implying $V_1(W_1)$. We could then solve (2.5) to build $V_2(W_1)$. Given this function, we could move to a solution of the $T = 3$ problem and proceed iteratively, using (2.5) to build $V_T(W_1)$ for any T.

Example

We illustrate the construction of the value function in a specific example. Assume $u(c) = ln(c)$. Suppose that $T = 1$. Then $V_1(W_1) = \ln(W_1)$.

For $T = 2$, the first order condition from (2.2) is

$$1/c_1 = \beta/c_2$$

and the resource constraint is

$$W_1 = c_1 + c_2.$$

Working with these two conditions:

$$c_1 = W_1/(1+\beta) \text{ and } c_2 = \beta W_1/(1+\beta).$$

¿From this, we can solve for the value of the 2-period problem:

$$V_2(W_1) = \ln(c_1) + \beta \ln(c_2) = A_2 + B_2 \ln(W_1) \tag{2.6}$$

where A_2 and B_2 are constants associated with the two period problem. These constants are given by:

$$A_2 = \ln(1/(1+\beta)) + \beta \ln(\beta/(1+\beta)) \qquad B_2 = (1+\beta)$$

Importantly, (2.6) does not include the *max* operator as we are substituting the optimal decisions in the construction of the value function, $V_2(W_1)$.

Using this function, the $T = 3$ problem can then be written as:

$$V_3(W_1) = \max_{W_2} \ln(W_1 - W_2) + \beta V_2(W_2)$$

where the choice variable is the state in the subsequent period. The first order condition is:

$$\frac{1}{c_1} = \beta V_2'(W_2).$$

Using (2.6) evaluated at a cake of size W_2, we can solve for $V_2'(W_2)$ implying:

$$\frac{1}{c_1} = \beta \frac{B_2}{W_2} = \frac{\beta}{c_2}.$$

Here c_2 the consumption level in the second period of the three-period problem and thus is the same as the level of consumption in the first period of the two-period problem. Further, we know from the 2-period problem that

$$1/c_2 = \beta/c_3.$$

This plus the resource constraint allows us to construct the solution of the 3-period problem:

$$c_1 = W_1/(1+\beta+\beta^2), \quad c_2 = \beta W_1/(1+\beta+\beta^2), \quad c_3 = \beta^2 W_1/(1+\beta+\beta^2).$$

Substituting into $V_3(W_1)$ yields

$$V_3(W_1) = A_3 + B_3 \ln(W_1)$$

where

$$A_3 = \ln(1/(1+\beta+\beta^2)) + \beta \ln(\beta/(1+\beta+\beta^2)) + \beta^2 \ln(\beta^2/(1+\beta+\beta^2)), \qquad B_3 = (1+\beta+\beta^2)$$

This solution can be verified from a direct attack on the 3 period problem using (2.2) and (2.3).

2.4 Some Extensions of the Cake Eating Problem

Here we go beyond the T period problem to illustrate some ways to use the dynamic programming framework. This is intended as an overview and the details of the assertions and so forth will be provided below.

2.4.1 Infinite Horizon

Basic Structure

Suppose that we consider the above problem and allow the horizon to go to infinity. As before, one can consider solving the infinite horizon sequence problem given by:

$$\max_{\{c_t\}_1^\infty, \{W_t\}_2^\infty} \sum_{t=1}^{\infty} \beta^t u(c_t)$$

along with the transition equation of

$$W_{t+1} = W_t - c_t$$

for t=1,2,......

Specifying this as a dynamic programming problem,

$$V(W) = \max_{c \in [0,W]} u(c) + \beta V(W - c)$$

for all W. Here $u(c)$ is again the utility from consuming c units in the current period. $V(W)$ is the value of the infinite horizon problem starting with a cake of size W. So in the given period, the agent chooses current consumption and thus reduces the size of the cake to $W' = W - c$, as in the transition equation. We use variables with primes to denote future values. The value of starting the next period with a cake of that size is then given by $V(W - c)$ which is discounted at rate $\beta < 1$.

For this problem, the **state variable** is the size of the cake (W) that is given at the start of any period. The state completely summarizes all information from the past that is needed for the forward looking optimization problem. The **control variable** is the variable that is being chosen. In this case, it is the level of consumption in the current period, c. Note that c lies in a compact set. The dependence of the state tomorrow on the state today and the control today, given by

$$W' = W - c$$

is called the **transition equation**.

Alternatively, we can specify the problem so that instead of choosing today's consumption we choose tomorrow's state.

$$V(W) = \max_{W' \in [0,W]} u(W - W') + \beta V(W') \tag{2.7}$$

for all W. Either specification yields the same result. But choosing tomorrow's state often makes the algebra a bit easier so we will work with (2.7).

This expression is known as a **functional equation** and is often called a Bellman equation after Richard Bellman, one of the originators of dynamic programming. Note that the unknown in the Bellman equation is the value function itself: the idea is to find a function $V(W)$ that satisfies this condition for all W. Unlike the finite horizon problem, there is no terminal period to use to derive the value function. In effect, the fixed point restriction of having $V(W)$ on both sides of (2.7) will provide us with a means of solving the functional equation.

Note too that time itself does not enter into Bellman's equation: we can express all relations without an indication of time. This is the essence of **stationarity**.[8] In fact, we will ultimately use the stationarity of the problem to make arguments about the existence of a value function satisfying the functional equation.

A final very important property of this problem is that all information about the past that bears on current and future decisions is summarized by W, the size of the cake at the start of the period. Whether the cake is of this size because we initially had a large cake and ate a lot or a small cake and were frugal is not relevant. All that matters is that we have a cake of a given size. This property partly reflects the fact that the preferences of the agent do not depend on past consumption. But, in fact, if this was the case, we could amend the problem to allow this possibility.

The next part of this chapter addresses the question of whether there exists a value function that satisfies (2.7). For now, we assume that a solution exists and explore its properties.

The first order condition for the optimization problem in (2.7) can be written as

$$u'(c) = \beta V'(W').$$

This looks simple but what is the derivative of the value function? This seems particularly hard to answer since we do not know $V(W)$. However, we take use the fact that $V(W)$ satisfies (2.7) for all W to calculate V'. Assuming that this value function is differentiable,

$$V'(W) = u'(c),$$

a result we have seen before. Since this holds for all W, it will hold in the following period yielding:

$$V'(W') = u'(c').$$

Substitution leads to the familar Euler equation:

$$u'(c) = \beta u'(c').$$

The solution to the cake eating problem will satisfy this necessary condition for all W.

The link from the level of consumption and next period's cake (the controls from the different formulations) to the size of the cake (the state) is given by the **policy function:**

$$c = \phi(W), \quad W' = \varphi(W) \equiv W - \phi(W).$$

Using these in the Euler equation reduces the problem to these policy functions alone:

$$u'(\phi(W)) = \beta u'(\phi(W - \phi(W)))$$

for all W.

These policy functions are very important for applied research since they provide the mapping from the state to actions. When elements of the state as well as the action are observable, then these policy functions will provide the foundation for estimation of the underlying parameters.

An Example

In general, actually finding closed form solutions for the value function and the resulting policy functions is not possible. In those cases, we try to characterize certain properties of the solution and, for some exercises, we solve these problems numerically.

However, as suggested by the analysis of the finite horizon examples, there are some versions of the problem we can solve completely. Suppose then, as above, that $u(c) = ln(c)$. Given the results for the T-period problem, we might conjecture that the solution to the functional equation takes the form of:

$$V(W) = A + B\ln(W)$$

for all W. With this guess we have reduced the dimensionality of the unknown function $V(W)$ to two parameters, A and B. But can we find values for A and B such that $V(W)$ will satisfy the functional equation?

Taking this guess as given and using the special preferences, the functional equation becomes:

$$A + B\ln(W) = \max_{W'} \ln(W - W') + \beta(A + B\ln(W')) \tag{2.8}$$

for all W. After some algebra, the first-order condition implies:

$$W' = \varphi(W) = \frac{\beta B}{(1+\beta B)}W.$$

Using this in (2.8) implies:

$$A + B\ln(W) = \ln\frac{W}{(1+\beta B)} + \beta(A + B\ln(\frac{\beta BW}{(1+\beta B)}))$$

for all W. Collecting terms into a constant and terms that multiply $ln(W)$ and then imposing the requirement that the functional equation must hold for all W, we find that

$$B = 1/(1-\beta)$$

is required for a solution. Given this, there is a complicated expression that can be used to find A. To be clear then we have indeed guessed a solution to the functional equation. We know that because we can solve for (A, B) such that the functional equation holds for all W using the optimal consumption and savings decision rules.

With this solution, we know that

$$c = W(1-\beta), W' = \beta W.$$

Evidently, the optimal policy is to save a constant fraction of the cake and eat the remaining fraction.

Interestingly, the solution to B could be guessed from the solution to the T-horizon problems where

$$B_T = \sum_{t=1}^{T} \beta^{t-1}.$$

Evidently, $B = lim_{T\to\infty} B_T$. In fact, we will be exploiting the theme that the value function which solves the infinite horizon problem is related to the limit of the finite solutions in much of our numerical analysis.

Here are some exercises that add some interesting elements to this basic structure. Both begin with finite horizon formulations and then progress to the infinite horizon problem.

Exercise 2.1

Suppose that utility in period t was given by $u(c_t, c_{t-1})$. How would you solve the T period problem with these preferences? Interpret the first order conditions. How would you formulate the Bellman equation for the infinite horizon version of this problem?

Exercise 2.2

Suppose that the transition equation was modified so that

$$W_{t+1} = \rho W_t - c_t$$

where $\rho > 0$ represents a return from the holding of cake inventories. How would you solve the T period problem with this storage technology? Interpret the first order