Shenzhen Winter Camp

Lecture 6

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Asset Pricing Extensions

Making our models more realistic

- Nonstationary dividends
- Large state spaces

Asset Pricing with Nonstationary Dividends

In reality dividends are typically nonstationary

A standard model is

$$\ln \frac{D_{t+1}}{D_t} = \kappa(X_t, \eta_{t+1})$$

where

- $\{X_t\}$ is a stationary Markov process
- $\{\eta_t\} \stackrel{\text{IID}}{\sim} \phi$

Now prices are nonstationary, so we solve instead for the price dividend ratio

Start with

$$P_t = \mathbb{E}_t \left[M_{t+1} (D_{t+1} + P_{t+1}) \right]$$

The price-dividend ratio is

$$\frac{P_t}{D_t} = \mathbb{E}_t \left[M_{t+1} \frac{D_{t+1}}{D_t} \left(1 + \frac{P_{t+1}}{D_{t+1}} \right) \right]$$

With $V_t := P_t / D_t$ we have

$$V_t = \mathbb{E}_t \left[M_{t+1} \exp(\kappa(X_t, \eta_{t+1})) \, (1 + V_{t+1}) \right]$$

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In solving

$$V_t = \mathbb{E}_t \left[M_{t+1} \exp(\kappa(X_t, \eta_{t+1})) \left(1 + V_{t+1}\right) \right]$$

let's assume that

- $M_{t+1} = m(X_t, \eta_{t+1})$ for some positive function m
- $\{X_t\}$ is a finite Markov chain with stochastic matrix Q

It then suffices to find a function v such that

$$v(x) = \sum_{y \in \mathbb{X}} \int m(x, \eta) \exp(\kappa(x, \eta)) \phi(d\eta) \left[1 + v(y)\right] Q(x, y)$$

for all $x \in \mathbb{X}$

To repeat, we seek a v that solves

$$v(x) = \sum_{y \in \mathbb{X}} \int m(x, \eta) \exp(\kappa(x, \eta)) \phi(d\eta) \, [1 + v(y)] \, Q(x, y)$$

for all $x \in \mathbb{X}$

Equivalently, with

$$A(x, y) := Q(x, y) \int m(x, \eta) \exp(\kappa(x, \eta)) \phi(d\eta)$$

we seek a v that solves

$$v(x) = \sum_{y \in \mathbb{X}} [1 + v(y)] \, A(x, y)$$

Treating

- A as a matrix with i, j-th element $A(x_i, x_j)$ and
- v as a column vector with i-th element $v(x_i)$

this becomes

$$v = A\mathbb{1} + Av$$

Here $\mathbb{1}$ is an $n \times 1$ column vector of ones

This equation has the unique solution

$$v = (I - A)^{-1} A\mathbb{1}$$

whenever $r(A) < 1$

Example. Consider the dividend process

$$\ln \frac{D_{t+1}}{D_t} = \kappa(X_t, \eta_{t+1}) = \mu_d + X_t + \sigma_d \eta_{d,t+1}$$

Here $\{\eta_{d,t}\} \overset{\text{IID}}{\sim} N(0, 1)$

The state process $\{X_t\}$ obeys

$$X_{t+1} = \rho X_t + \sigma \xi_{t+1}$$

where $\{\xi_t\}$ is IID and standard normal

We can discretize it using **Tauchen's method**

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Consumption is also nonstationary, obeying

$$\ln \frac{C_{t+1}}{C_t} = \mu_c + X_t + \sigma_c \eta_{c,t+1} \quad \text{where} \quad \{\eta_{c,t}\} \overset{\text{IID}}{\sim} N(0,1)$$

We use the Lucas SDF

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

The utility function is $u(c) = c^{1-\gamma}/(1-\gamma)$

Hence

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} = \beta \exp(-\gamma(\mu_c + X_t + \sigma_c \eta_{c,t+1}))$$

Recall the defintion

$$A(x, y) := Q(x, y) \int m(x, \eta) \exp(\kappa(x, \eta)) \phi(d\eta)$$

In our case this is

$$A(x, y) = \beta \exp \left(-\gamma \mu_c + \mu_d + (1 - \gamma) x + \frac{\gamma^2 \sigma_c^2 + \sigma_d^2}{2} \right) Q(x, y)$$

Now check $r(A) < 1$ and solve via $v = (I - A)^{-1} A \mathbb{1}$

- See `John/asset_nonstationary_discretized.ipynb`

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Large State Spaces

Recall: the price-dividend ratio is a v that solves

$$v(x) = \sum_{y \in \mathbb{X}} \int m(x, \eta) \exp(\kappa(x, \eta)) \phi(d\eta) \, [1 + v(y)] \, Q(x, y)$$

for all $x \in \mathbb{X}$

Equivalently, with

$$A(x, y) := Q(x, y) \int m(x, \eta) \exp(\kappa(x, \eta)) \phi(d\eta)$$

we seek a v that solves

$$v = (I - A)^{-1} A \mathbb{1}$$

But what if the state process has more dimensions, as in, say **Schorfheide, Song and Yaron, ECMA, 2018**?

$$\ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_{c,t}\, \eta_{c,t+1},$$

$$\ln(D_{t+1}/D_t) = \mu_d + \alpha z_t + \delta \sigma_{c,t}\, \eta_{c,t+1} + \sigma_{d,t}\, \eta_{d,t+1}$$

where

$$z_{t+1} = \rho\, z_t + (1 - \rho^2)^{1/2}\, \sigma_{z,t}\, v_{t+1},$$

$$\sigma_{i,t} = \varphi_i\, \bar{\sigma} \exp(h_{i,t}),$$

$$h_{i,t+1} = \rho_{h_i} h_i + \sigma_{h_i} \xi_{i,t+1}, \quad i \in \{z, c, d\}$$

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The state can be represented as the four dimensional vector

$$X_t := (z_t, h_{z,t}, h_{c,t}, h_{d,t})$$

Suppose that we discretize as follows:

$$z \to z^1, \ldots, z^k, \qquad h_z \to h_z^1, \ldots, h_z^k, \text{ etc.}$$

That means X_t can take k^4 different values

If $k = 25$, then $A = A(x, y)$ **is** $25^4 \times 25^4$

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If A is $25^4 \times 25^4$, then it contains 25^8 floating point numbers

Each requires 8 bytes, so total memory consumption is

$$8 \times 25^8 = 1220703125000 = 1.2 \text{ terabytes}$$

Inverting it requires in the order of $25^{12} = 59604644775390625$ floating point opertions

6622 hours at 2.5 GHz

If we add another state variable then it becomes 103480286 hours $= 11812$ years . . .

This is the **curse of dimensionality**

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A Simulation-Based Approach

Recall that we are aiming to solve for the price-dividend ratio

$$V_t = \mathbb{E}_t \left[M_{t+1} \frac{D_{t+1}}{D_t} (1 + V_{t+1}) \right]$$

With $A_{t+1} = M_{t+1} \frac{D_{t+1}}{D_t}$,

$$V_t = \mathbb{E}_t \left[A_{t+1}(V_{t+1} + 1) \right]$$

Let's think about solving this using simulation

First rewrite our eq as

$$V_t = \mathbb{E}_t A_{t+1} + \mathbb{E}_t A_{t+1} V_{t+1}$$

Substitution gives

$$\begin{aligned} V_t &= \mathbb{E}_t A_{t+1} + \mathbb{E}_t A_{t+1} (\mathbb{E}_{t+1} A_{t+2} + \mathbb{E}_{t+1} A_{t+2} V_{t+2}) \\ &= \mathbb{E}_t A_{t+1} + \mathbb{E}_t A_{t+1} A_{t+2} + \mathbb{E}_t A_{t+1} A_{t+2} V_{t+2} \end{aligned}$$

Substituting again gives

$$\begin{aligned} V_t = \mathbb{E}_t A_{t+1} + \mathbb{E}_t A_{t+1} A_{t+2} + \mathbb{E}_t A_{t+1} A_{t+2} A_{t+3} \\ + \mathbb{E}_t A_{t+1} A_{t+2} A_{t+3} V_{t+3} \end{aligned}$$

The limit is

$$V_t = \mathbb{E}_t A_{t+1} + \mathbb{E}_t A_{t+1} A_{t+2} + \mathbb{E}_t A_{t+1} A_{t+2} A_{t+3} + \mathbb{E}_t A_{t+1} A_{t+2} A_{t+3} A_{t+4} + \cdots$$

Consolidating, the **forward solution** is

$$V_t^* = \mathbb{E}_t \left[\sum_{n=1}^{\infty} \prod_{i=1}^{n} A_{t+i} \right]$$

Exists if

$$\limsup_{n \to \infty} \mathbb{E}_t \left[\prod_{i=1}^{n} A_{t+i} \right]^{1/n} < 1$$

Note that

$$V_t^* = \mathbb{E}_t \left[\sum_{n=1}^{\infty} \prod_{i=1}^{n} A_{t+i} \right] = \mathbb{E}_{X_t} \left[\sum_{n=1}^{\infty} \prod_{i=1}^{n} A_{t+i} \right]$$

Written state by state, this becomes

$$v(x) = \mathbb{E}_x \left[\sum_{n=1}^{\infty} \prod_{i=1}^{n} A_i \right].$$

How can we calculate the right hand side?

Our proposal to calculate

$$v(x) = \mathbb{E}_x \left[\sum_{n=1}^{\infty} \prod_{i=1}^{n} A_i \right]$$

1. Fix large integers N and M
2. Generate M independent paths

$$A_1^{(m)}, \dots A_N^{(m)},$$

and estimate $v(x)$ via

$$v_M(x) := \frac{1}{M} \sum_{m=1}^{M} \Lambda(x, N, m) \quad \text{where} \quad \Lambda(x, N, m) := \sum_{n=1}^{N} \prod_{i=1}^{n} A_i^{(m)}$$

Disadvantages of this method:

- slow relative to discretization **if** the state space is small

Advantages of this method:

- works in high dimensions
- “lazy” evaluation
- highly parallelizable

- See John/asset_pricing_simulation.ipynb