$$L = \frac{(C_t^y)^{1-\eta}-1}{1-\eta} + \beta\left[\frac{(C_{t+1}^o)^{1-\eta}-1}{1-\eta} + \chi\frac{[(1-\xi)b_{t+1}]^{1-\eta}-1}{1-\eta}\right]$$

$$-\lambda\left[C_t^y + S_t - w_t l_t - (1-\xi)b_t - g_t\right]$$

$$-\mu\left[C_{t+1}^o + b_{t+1} - (1+r_{t+1})S_t\right]$$

$$\begin{cases} \dfrac{\partial L}{\partial C_t^y} = [C_t^y]^{-\eta} - \lambda = 0 \\ \dfrac{\partial L}{\partial S_t} = -\lambda + \mu(1+r_{t+1}) = 0 \\ \dfrac{\partial L}{\partial C_{t+1}^o} = \beta[C_{t+1}^o]^{-\eta} - \mu = 0 \\ \dfrac{\partial L}{\partial b_{t+1}} = \beta\chi\cdot\dfrac{(1-\xi)^{1-\eta}}{b_{t+1}^{\eta}} - \mu = 0 \end{cases}$$

$$\Rightarrow \begin{cases} C_t^y = \dfrac{1}{1+\tilde{\beta}_{t+1}^{\frac{1}{\eta}}}\left[w_t l_t + (1-\xi)b_t + g_t\right] & (1) \\ S_t = \dfrac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_t + (1-\xi)b_t + g_t\right] & (2) \\ C_{t+1}^o = \dfrac{1}{1+\chi^{\frac{1}{\eta}}(1-\xi)^{\frac{1-\eta}{\eta}}}(1+r_{t+1})S_t & (3) \\ b_{t+1} = \dfrac{1}{1+\chi^{-\frac{1}{\eta}}(1-\xi)^{\frac{1-\eta}{\eta}}}(1+r_{t+1})S_t & (4) \end{cases}$$

where $\tilde{\beta}_{t+1} = \beta\left[1+\chi^{-\frac{1}{\eta}}(1-\xi)^{\frac{1-\eta}{\eta}}\right]^{\eta}(1+r_{t+1})^{1-\eta}$

Observing (2) and (4), we can derive the following equation, representing relationship between s_t and s_{t-1} :

$$s_t = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_t + (1-\xi)\cdot\frac{(1+r_t)s_{t-1}}{1+\chi^{-\frac{1}{\eta}}(1-\xi)^{\frac{\eta-1}{\eta}}} + g_t\right] \quad (5)$$

Problem 3 :

$$\pi = \max_{K_t, L_t}\left\{A K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - (r_t+\delta)K_t\right\}$$

$$\begin{cases} \dfrac{\partial \pi}{\partial K_t} = A\alpha k_t^{\alpha-1} - (r_t+\delta) = 0 \\ \dfrac{\partial \pi}{\partial L_t} = A(1-\alpha)k_t^{\alpha} - w_t = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} r_t = A\alpha k_t^{\alpha-1} - \delta \\ w_t = A(1-\alpha)k_t^{\alpha} \end{cases}$$

where $k_t = \frac{K_t}{L_t}$ means capital per worker

Problem 4:

$$k_{t+1} = \int s_t\, di = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t\int l_t\, di + (1-\xi)\int b_t\, di + \int g_t\, di\right]$$

since $\int l_t\, di = 1$, and $g_t = \xi\int b_t\, di$, we can obtain the following equation:

$$k_{t+1} = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t + \int b_t\, di\right]$$
$$= \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t + \varphi(1+r_t)k_t\right]$$

where $\varphi = \frac{1}{1+x^{-\frac{1}{\eta}}(1-\xi)^{\frac{1-\eta}{\eta}}}$.

Problem 5:

In problem 2, we derived equation (5):

$$s_t = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_t + (1-\xi)\varphi(1+r_t)s_{t-1} + g_t\right]$$

Take into account the equation that $a_{t+1} = s_t$, it can be transfered into:

$$a_{t+1} = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_t + (1-\xi)\varphi(1+r_t)a_t + \xi\int b_t \, di\right]$$

$$= \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_t + (1-\xi)\varphi(1+r_t)a_t + \xi\varphi(1+r_t)\int s_{t-1} \, di\right]$$

since $\int s_{t-1} \, di = K_t$, the equation above can be rewritten as:

$$a_{t+1} = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_t + (1-\xi)\varphi(1+r_t)a_t + \xi\varphi(1+r_t)K_t\right]$$

In the steady-state aggregate economy, $w_t = w$, $r_t = r$, $K_t = K$,

then: $a_{t+1} = C_3 l_t + C_4 a_t + C_5$.

where $C_3 = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} w$. $C_4 = \frac{(1-\xi)\varphi(1+r)}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}$. $C_5 = \frac{\xi\varphi(1+r)}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} K$.

Problem 6:

$$var(a_{it}) = c_3^2 \sigma^2 \sum_{k=1}^{t} \left(\sum_{s=0}^{k-1} c_4^{k-1-s} v^s \right)^2$$

$$E(a_{it}) = \frac{c_3}{1-c_4}$$

$$\therefore CV(a_{it}) = \frac{\sqrt{var(a_{it})}}{E(a_{it})} = \sigma(1-c_4)\sqrt{\sum_{k=1}^{t}\left(\sum_{s=0}^{k-1} c_4^{k-1-s} v^s\right)^2}$$

Assume $c_4 \neq v$, we have:

$$\sum_{s=0}^{k-1} c_4^{k-1-s} v^s = \frac{c_4^k - v^k}{c_4 - v}$$

$$CV(a_{it}) = \sigma(1-c_4)\sqrt{\frac{1}{(c_4-v)^2}\sum_{k=1}^{t}\left(c_4^{2k} + v^{2k} - 2(c_4 v)^k\right)}$$

$$= \sigma \cdot \frac{1-c_4}{c_4-v}\sqrt{c_4^2 \frac{1-c_4^{2t}}{1-c_4^2} + v^2 \frac{1-v^{2t}}{1-v^2} - 2c_4 v \frac{1-(c_4 v)^t}{1-c_4 v}}$$

For $t \to \infty$:

$$CV(a_{i\infty}) = \sigma \frac{1-c_4}{c_4-v}\sqrt{\frac{c_4^2}{1-c_4^2} + \frac{v^2}{1-v^2} - \frac{2c_4 v}{1-c_4 v}}$$

$$= \sigma \frac{1-c_4}{c_4-v}\sqrt{\frac{(c_4-v)^2(1+c_4 v)}{(1-c_4^2)(1-v^2)(1-c_4 v)}}$$

$$= \sigma\sqrt{\frac{1-c_4}{1+c_4}}\sqrt{\frac{1+c_4 v}{(1-v^2)(1-c_4 v)}}$$

~~If there is no tax, then c~~

$$var(a_{i\infty}) = \left[CV(a_{i\infty}) \cdot E(a_{i\infty})\right]^2$$

$$= c_3^2 \frac{1}{(1-c_4^2)(1-v^2)} \frac{1+c_4 v}{1-c_4 v} \sigma^2$$

① If $v=0$, $var(a_{i\infty}) = \frac{c_3^2 \sigma^2}{1-c_4^2}$ since $c_4 = \frac{c_4^{nt}}{1+\tau} < c_4^{nt}$

so $var(a_{i\infty}^{wt}) < var(a_{i\infty}^{nt})$

② If $v>0$, when tax increases, the numerator decreases and dominator increases, so $var(a_{i\infty})$ decreases