Homework - inequality

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1.1 Problem 1:

the original maximization problem is equivalent to the following one:

$$\max_{s_t, b_{t+1}} \Big[\frac{(-s_t + w_t l_t + (1-\zeta) b_t + g_t)^{1-\eta}}{1-\eta} - \frac{1}{1-\eta} + \beta \frac{((1+r_{t+1}) s_t - b_{t+1})^{1-\eta}}{1-\eta} - \frac{\beta}{1-\eta} + \beta \chi \frac{[(1-\zeta) b_{t+1}]^{1-\eta}}{1-\eta} - \frac{\beta \chi}{1-\eta} \Big]$$

F.O.C.'s w.r.t. s_t and b_{t+1} give us:

$$b_{t+1}^* = \frac{(1-\zeta+\beta^{\frac{1}{\eta}})(1+r_{t+1}) b_t + g_t (1+r_{t+1})}{(1+(1+r_{t+1})\beta^{\frac{1}{\eta}})(1+(\chi(1-\zeta)^{1-\eta})^{\frac{1}{\eta}})} = \frac{(1+r_{t+1}) \quad s_t}{1+(\chi(1-\zeta)^{1-\eta})^{\frac{1}{\eta}}} \quad ①$$

$$s_t^* = \frac{b_t(1-\zeta+\beta^{\frac{1}{\eta}}) + g_t}{1+(1+r_{t+1})\beta^{\frac{1}{\eta}}} = \frac{1}{1+\tilde{\beta}_{t+1}^{\frac{1}{\eta}}} (w_t l_t + (1-\zeta) b_t + g_t) \quad ②$$

$$c_t^{y*} = w_t l_t + (1-\zeta) b_t + g_t - s_t^* \quad ③$$

$$c_t^{o*} = (1+r_{t+1}) s_t^* - b_{t+1}^*.$$

$$\tilde{\beta}_{t+1}^{\frac{1}{\eta}} = \beta \left[1 + \chi^{\frac{1}{\eta}} (1-\zeta)^{\frac{1-\eta}{\eta}}\right]^{\eta} (1+r_{t+1})^{1-\eta}.$$

1.1. Problem 2:

From ① we can derive: $b_t^* = \dfrac{(1+r_t) s_{t-1}}{1+(\chi(1-\zeta)^{1-\eta})^{\frac{1}{\eta}}}$ ⑤

Plug ⑤ ③ into ②, we have: $s_t^* = c_3 l_t + c_4 s_{t-1}^* + c_5$, where

$$c_3 = \frac{w_t}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}, \quad c_4 = \frac{(1-\zeta)(1+r_t)}{(1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}})(1+\chi^{\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-1}{\eta}})}, \quad c_5 = \frac{g_t}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}$$

and

$$\tilde{\beta}_{t+1} = \beta\left(1+\chi^{\frac{1}{\eta}}(1-\zeta)^{\frac{1-\eta}{\eta}}\right)^{\eta}(1+r_{t+1})^{1-\eta}$$

1.3. Problem 3

F.O.C.s:

$$\begin{cases} \frac{\partial \pi}{\partial K_t} = \alpha A\left(\frac{K_t}{L_t}\right)^{\alpha-1} - (r_t+\delta) = 0 \Rightarrow k_t = \left(\frac{\delta + r_t}{\alpha A}\right)^{\frac{1}{\alpha-1}}, \quad k_t := \frac{K_t}{L_t} \\ \frac{\partial \pi}{\partial L_t} = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} - w_t = 0 \Rightarrow k_t = \left(\frac{w_t}{(1-\alpha)A}\right)^{\frac{1}{\alpha}}, \quad k_t := \frac{K_t}{L_t} \end{cases}$$

$$\Rightarrow \begin{cases} r_t = \alpha A k_t^{\alpha-1} - \delta & (6) \\ w_t = (1-\alpha) A k_t^{\alpha} & (7) \end{cases}$$

1.4. Problem 4

From (2) we know:

$$s_{it} = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left[w_t l_{it} + (1-\zeta)b_{it} + g_t\right]$$

$$K_{t+1} = \int s_{it}\,di = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left(w_t\int l_{it}\,di + (1-\zeta)\int b_{it}\,di + g_t\right)$$

$$= \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left(w_t + \frac{1-\zeta}{\zeta}g_t + g_t\right)$$

$$= \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left(w_t + \int b_{it}\,di\right)$$

From (1) we know: $b_{it}^* = \frac{(1+r_t)s_{it-1}}{1+\chi^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-1}{\eta}}} = \varphi(1+r_t)s_{it-1}, \quad \varphi := \frac{1}{1+\chi^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-1}{\eta}}}$

Therefore, $K_{t+1} = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left(w_t + \varphi(1+r_{t+1})\int s_{it-1}\,di\right) = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}}\left(w_t + \varphi(1+r_t)K_t\right)$

Plug in ⑥, ⑦, we have:

$$K_{t+1} = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} \left((1-\alpha) A K_t^{\alpha} + \varphi (1+\alpha A K_t^{\alpha-1} - \delta) K_t \right), \text{ since } L_t = 1.$$

$$k_{t+1} = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} \left[(1-\alpha+\varphi\alpha) A k_t^{\alpha} + \varphi(1-\delta) k_t \right.$$

Problem 5

From ② we know $S_t = \frac{1}{1+\tilde{\beta}_{t+1}^{-\frac{1}{\eta}}} \left[w_t l_t + (1-\zeta)\varphi(1+r_t) S_{t-1} + g_t \right]$ <1>

From government's budget constraint, we know $g_t = g = \zeta\varphi(1+r)K$ <2>

$a_{t+1} = S_t$, $\therefore a_t = S_{t-1}$ <3>

From <1> <2> <3>, we obtain

$$a_{t+1} = \frac{w}{1+\tilde{\beta}^{-\frac{1}{\eta}}} l_t + \frac{(1-\zeta)\varphi(1+r)}{1+\tilde{\beta}^{-\frac{1}{\eta}}} a_t + \frac{\zeta\varphi(1+r)}{1+\tilde{\beta}^{-\frac{1}{\eta}}} K.$$

Therefore, $a_{t+1} = C_3 l_t + C_4 a_t + C_5$,

where $C_3 = \frac{w}{1+\tilde{\beta}^{-\frac{1}{\eta}}}$, $C_4 = \frac{(1-\zeta)\varphi(1+r)}{1+\tilde{\beta}^{-\frac{1}{\eta}}}$, $C_5 = \frac{\zeta\varphi(1+r)}{1+\tilde{\beta}^{-\frac{1}{\eta}}} K.$

Problem 6.

$$a_{i1}^B = C_4 a_0^B + C_3 l_{i0} + C_5$$

$$a_{i2}^B = C_4^2 a_0^B + C_4 C_3 l_{i0} + C_3 l_{i1} + C_5$$

$$a_{i3}^B = C_4^3 a_0^B + C_4^2 C_3 l_{i0} + C_4 C_3 l_{i1} + C_3 l_{i2} + C_5$$

$$a_{i4}^B = C_4^4 a_0^B + C_4^3 C_3 l_{i0} + C_4^2 C_3 l_{i1} + C_4 C_3 l_{i2} + C_3 l_{i3}$$

...

$Var(a_{i1}^B) = C_3 \sigma^2$ $\quad$ $Var(a_{i2}^B) = \sigma^2 C_3^2 (1 + (C_4+v)^2)$

$$Var(a_{it}^B) = C_3^2 \sigma^2 \sum_{t=1}^{t} \left(\sum_{s=0}^{k-1} C_4^{k-1-s} v^s \right)^2$$

$$\because \sum_{s=0}^{k-1} c_4^{k-1-s} v^s = \frac{c_4^k - v^k}{c_4 - v}$$

$$\therefore Var(a_{it}^B) = c_3^2 \sigma^2 \sum_{k=1}^{t} \left(\frac{c_4^k - v^k}{c_4 - v}\right)^2 = \frac{c_3^2 \sigma^2}{(c_4 - v)^2} \sum_{k=1}^{t} \left(c_4^{2k} + v^{2k} - 2(c_4 v)^k\right)$$

$$= \frac{c_3^2 \sigma^2}{(c_4 - v)^2} \cdot \frac{(c_4 - v)^2 (1 + c_4 v)}{(1 - c_4^2)(1 - v^2)(1 - c_4 v)}$$

$$= c_3^2 \sigma^2 \frac{(1 + c_4 v)}{(1 - c_4^2)(1 - v^2)(1 - c_4 v)}$$

~~∵ c_3 与 τ 无关，τ 越高，c_4~~

$\because$ c_3 is independent of τ and c_4 falls in τ, the higher the tax is, the smaller the $Var(a_{it}^B)$ is. which means the estate taxes reduce wealth inequality in the long run.