STA465 Homework 1

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Question 1

Solutions:

(a)

$\because p(y|\mu) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$,

therefore the Jeffreys' Prior for μ is

$p(\mu) \propto \sqrt{I(\mu)}$

$$= \sqrt{-E_{y\sim p(y|\mu)}[\frac{d^2}{d\mu^2}\log p(y|\mu)]}$$

$$= \sqrt{-E_{y\sim p(y|\mu)}[\frac{d^2}{d\mu^2}(-\frac{(y-\mu)^2}{2\sigma^2} - \log\sqrt{2\pi}\sigma)]} = \sqrt{E_{y\sim p(y|\mu)}[\frac{d}{d\mu}(\frac{-(y-\mu)}{\sigma^2}]} = \sqrt{E_{y\sim p(y|\mu)}[\frac{1}{\sigma^2}]}$$

$$= \sqrt{\frac{1}{\sigma^2}} \propto 1$$

(b)

$\because p(\mu) \propto 1 \& \mu \in (-\infty, +\infty)$, $\therefore \int p(\mu)d\mu \neq 1$, which indicates that this is not a proper prior distribution.

As the prior $p(\mu) \propto 1$, the posterior will be the same as the likelihood.

$\because p(y|\mu)$

$= (\frac{1}{\sqrt{2\pi}\sigma})^n e^{\frac{-\sum_{i=1}^n (y_i-\mu)^2}{2\sigma^2}} = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{(\mu-\sum y_i/n)^2}{2\sigma^2/n}} e^{\frac{(\sum y_i/n)^2 - \sum y_i^2/n}{2\sigma^2/n}}$, where $y = (y_1, y_2 \ldots y_n)$

Therefore,

$p(\mu|y)$

$\propto p(y|\mu)p(\mu) = p(y|\mu)$

$\propto e^{-\frac{(\mu-\sum y_i/n)^2}{2\sigma^2/n}}$

Based on the Jeffreys' posterior pdf, $\mu|y \sim N(\sum_{i=1}^n y_i/n, \sigma^2/n)$, which is a proper posterior distribution.

(c)

$\because p(y|\mu) = (\frac{1}{\sqrt{2\pi\sigma}})^n e^{\frac{-\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma^2}}$, where $y = (y_1, y_2 \ldots y_n)$ then,

$$p(y,\mu) = p(y|\mu)p(\mu) = (\frac{1}{\sqrt{2\pi\sigma}})^n e^{\frac{-\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\tau}} e^{\frac{-\mu^2}{2\tau^2}}$$

$$= (\frac{1}{\sqrt{2\pi\sigma}})^n \frac{1}{\sqrt{2\pi\tau}} e^{-(\frac{(\mu-\frac{\sum y_i\tau^2}{(n\tau^2+\sigma^2)})^2}{2\sigma^2\tau^2/(n\tau^2+\sigma^2)})} e^{(\frac{\frac{\tau^4(\sum y_i)^2}{(n\tau^2+\sigma^2)}-\tau^2\sum y_i^2}{2\sigma^2\tau^2})}$$

$$p(y) = \int p(y,\mu)d\mu = (\frac{1}{\sqrt{2\pi\sigma}})^n \frac{1}{\sqrt{2\pi\tau}} e^{(\frac{\frac{\tau^4(\sum y_i)^2}{(n\tau^2+\sigma^2)}-\tau^2\sum y_i^2}{2\sigma^2\tau^2})} \int_{-\infty}^{+\infty} e^{-(\frac{(\mu-\frac{\sum y_i\tau^2}{(n\tau^2+\sigma^2)})^2}{2\sigma^2\tau^2/(n\tau^2+\sigma^2)})} d\mu$$

$$= (\sqrt{2\pi\sigma^2\tau^2/(n\tau^2+\sigma^2)})(\frac{1}{\sqrt{2\pi\sigma}})^n \frac{1}{\sqrt{2\pi\tau}} e^{(\frac{\frac{\tau^4(\sum y_i)^2}{(n\tau^2+\sigma^2)}-\tau^2\sum y_i^2}{2\sigma^2\tau^2})}$$

Then,

$$p(\mu|y) = \frac{p(y,\mu)}{p(y)} = \frac{(\frac{1}{\sqrt{2\pi\sigma}})^n \frac{1}{\sqrt{2\pi\tau}} e^{-(\frac{(\mu-\frac{\sum y_i\tau^2}{(n\tau^2+\sigma^2)})^2}{2\sigma^2\tau^2/(n\tau^2+\sigma^2)})} e^{(\frac{\frac{\tau^4(\sum y_i)^2}{(n\tau^2+\sigma^2)}-\tau^2\sum y_i^2}{2\sigma^2\tau^2})}}{(\sqrt{2\pi\sigma^2\tau^2/(n\tau^2+\sigma^2)})(\frac{1}{\sqrt{2\pi\sigma}})^n \frac{1}{\sqrt{2\pi\tau}} e^{(\frac{\frac{\tau^4(\sum y_i)^2}{(n\tau^2+\sigma^2)}-\tau^2\sum y_i^2}{2\sigma^2\tau^2})}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau^2/(n\tau^2+\sigma^2)}} e^{-(\frac{(\mu-\frac{\sum y_i\tau^2}{(n\tau^2+\sigma^2)})^2}{2\sigma^2\tau^2/(n\tau^2+\sigma^2)})}$$

Based on the posterior pdf above, $\mu|y \sim N(\frac{\sum_{i=1}^{n} y_i}{(n+\sigma^2/\tau^2)}, \sigma^2/(n+\sigma^2/\tau^2))$.

Both of the posterior distribution in (c) and the Jeffreys' posterior distribution in (b) are normal, however, they have different parameters. They will have the same parameter if $\tau \to \infty$ with n fixed or if $n \to \infty$ with τ fixed.

(d)

No, I don't expect an improper prior to always yield a proper posterior. As $\int p(y|\Theta)p(\Theta)d\mu$ is not always finite.